STRENGTH OF MATERIALS

(IN MKS/SI UNITS)

[A Text Book for Engineering Students of all Disciplines]

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Dedicated

to the loving memory of my son

PANKAJ

Preface

This book on Strength of Materials covers firstly the introductory course on the subject for the engineering students of all disciplines *i.e.* Mechanical, Production, Civil, Electrical, Electronic Engg. and Computer Sciences in the Engg. Colleges as well as in Polytechnics and secondly the advanced course on the subject for the students of Mechanical and Civil Engg. disciplines. This book will act as a faithfull companion to the students studying a course on Machine Design and computing stresses in machine members and to engineers serving in design offices of various Research and Development Organisations.

The author is teaching the subject for the last 23 years and is fully conversent with the difficulties experienced by the students. Therefore, while preparing the text of the book, the point of view of the students was constantly kept in mind. The contents of the book have been designed in a manner to help all grades of the students. For the relatively mediocre students unable to attend classes regularly, there are simple examples and exercises, a thorough study of which would impart confidence and a clear understanding of the subject. For the brighter students, there are complicated problems and exercises, the understanding and solution of which will help them go a long way in securing exceptionally good marks and in assuring a place of distinction in any competitive examination.

In brief, the book contains the following features :—

(1) A rigorous treatment given to the subject to meet the current requirements of the students.

(2) Providing a clear understanding of the basic principles of the subject through the worked examples which are more than 500.

(3) Thought-provoking and self-testing objective type questions which are more than 200 in number.

(4) Information provided about testing the mechanical properties of the materials in the laboratory.

(5) Solution of examples and problems both in MKS and SI units.

The advanced chapters on Bending of Curved Bars, Rotational Stresses, Energy Methods, Unsymmetrical Bending, Shear Centre and Torsion of Non Circular Shafts are no doubt available in many books but either the treatment given is too elementary or the examples given are insufficient. As a result, the students are apprehensive of these chapters when they appear in the examinations. Therefore, these topics have been thoroughly explained and a large number of solved examples are given so that the students can very well understand these advanced topics.

Constructive suggestions for improvement of the book are always solicited.

The development of this book has been strongly influenced by the author's colleagues, students and the numerous books on the subject published in India and abroad.

The author is deeply indebted to the inspiration received initially from his brilliant and genius son and this book is dedicated to his loving memory.

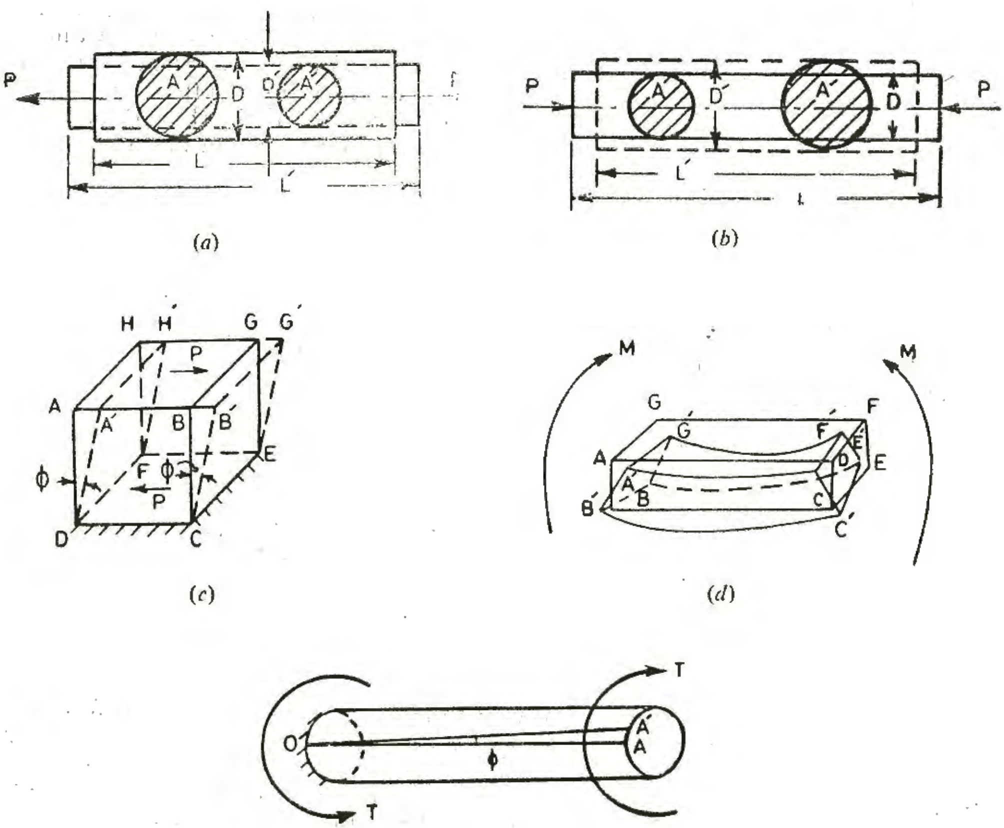
Dr. U. C. Jindal

Contents

1

Simple Stresses and Strains

A machine member or a structural member is deformed when it is subjected to a force or a moment. A force can extend or contract the member or it may distort the shape of the member while a moment can bend or twist the member, depending upon the type of the force or the moment applied.



(a) (b) (c) (d) (e)

Fig. 1·1

Fig. 1·1 (*a*) shows that a cylindrical bar of section A and length L gets extended under the action of the force P. Its length increases to L' and area of cross section decreases to A'.

Fig. 1·1 (*b*) shows that a cylindrical bar gets contracted under the action of the force P Its length decreases to L' while its area of cross-section increases to A',

Fig. 1·1 (*c*) shows a rectangular block fixed at the lower surface *DCEF* and at its top surface a force *P* acts tangential to the surface *ABGH*. The shape of the rectangular block is distorted to *A'B'G'H' FECD*.

Fig. 1·1 (*d*) shows that a rectangular bar initially straight is bent under the action of bending moment *M*. Straight bar *ABCDEFG* is deformed into *A'B'C'D'E'F'G'*.

Fig. 1·1 (*e*) shows that a circular bar fixed at one end gets twisted under the action of a twisting moment *T* applied at the other end. A line *OA* initially drawn on the surface of the bar gets deformed to *OA'*.

In this chapter we will analyse the effect of the force which extends, contracts or distorts the machine member. The effect of bending moment will be discussed in Chapter 7, while the effect of twisting moment will be analysed in Chapter 13.

1·1. NORMAL AND SHEAR FORCES

Consider a body subjected to a number of forces F_1, F_2, F_3, F_4 F_5, F_6, F_7 etc., as shown in Fig. 1·2. Say the resultant of these forces on a section *aa'* is F_R inclined at an angle θ to the plane of the section. There are two components of F_R *i.e.*, F_n and F_t. Component F_n is perpendicular to the section *aa'* while the component F_t lies in the plane of the section *aa'* and is tangential to it. The force F_n is called the Normal force on the section *aa'*. When F_n is pointing away from the plane *aa'* it is called the Normal Force (Fig. 1·2) and when it is pointing towards the plane, it is called a Compressive Force (as shown in Fig. 1·3).

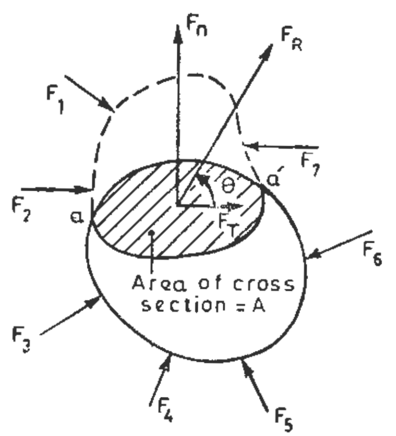


Fig. 1·2

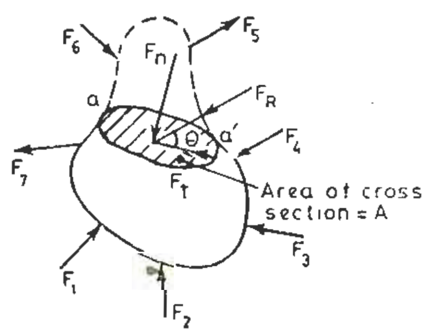


Fig. 1·3

The Force F_t on the section *aa'* is called the shear force. The shear force tending to rotate the body in the clockwise direction is taken as a positive shear force (Fig. 1·2) while the shear force tending to rotate the body in the anticlockwise direction is taken as a negative shear force The normal force per unit area is called the normal stress and the shear force per unit area is called the shear stress. Fig. 1·2 shows the tensile force F_n and positive shear force F_t on the section *aa'*.

$$f\text{, Tensile stress}=\frac{F_n}{A}\quad\text{(Tensile)}$$

$$q\text{, Shear stress}\ =\frac{F_t}{A}\quad(+\text{ve})$$

Fig. 1·3 shows compressive stress and negative shear stress on the area *A*.

Example 1·1-1 A bar of rectangular section 20 mm×30 mm carries an axial force of 10 kN. Determine the normal and shear force on a plane inclined at an angle of 30° to the axis of the bar. Determine also the magnitude and nature of the normal and shear stresses on this inclined plane.

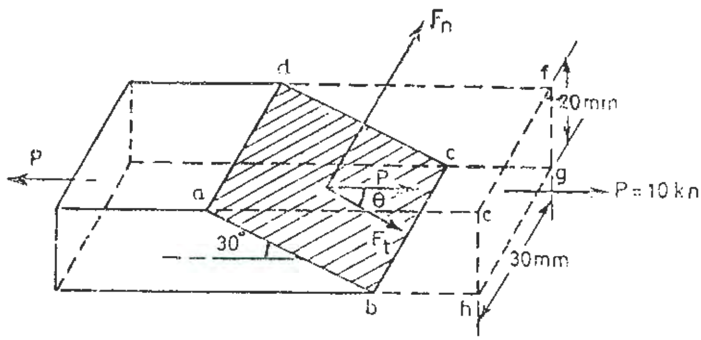


Fig. 1·4

Solution. Fig. 1·4 shows a bar of rectangular section 30 mm×20 mm, carrying an axial force $P=10$ kN. Now this force is perpendicular to a section *efgh* or to any section parallel to the plane *efgh*. Since this force P is wholly normal to such planes, these planes carry only the normal stress.

Consider a plane *abcd* inclined at an angle $\theta=30°$ to the axis of the bar. The force P is not perpendicular to the plane *abcd* but is inclined at an angle of 30°. There are two components of this force *i.e.*, $P \cos \theta$ tangential to the plane and $P \sin \theta$ normal to the plane *abcd*.

So $F_n=P \sin \theta$, a normal force pointing away from the plane, so a tensile force.

$F_t=P \cos \theta$, a shear force tending to rotate the body in the clockwise direction, so a positive shear force.

$F_n=10\times\sin 30°=5 \text{ kN}=5000 \text{ N}$

$F_t=10\times\cos 30°=8·66 \text{ kN}=8660 \text{ N}.$

Area of the inclined plane, $A=ad\times ab$

$$=30\times\frac{20}{\sin 30°}$$

$$=30\times\frac{20}{0·5}=1200 \text{ mm}^2$$

Normal stress on the plane,

$$f=\frac{F_n}{A}=\frac{5000}{1200} \text{ N/mm}^2$$

$$=4·167 \text{ N/mm}^2 \text{ (Tensile)}$$

Shear stress on the plane,

$$q=\frac{F_t}{A}=\frac{8660}{1200} \text{ N/mm}^2$$

$$=7·216 \text{ N/mm}^2 \text{ (+ve)}.$$

Example 1·1-2. Fig. 1·5 shows a stepped bar of diameters 10 mm and 20 mm respectively. An axial compressive force of 1000 kg acts on the bar. Determine the minimum and maximum normal stress in the bar.

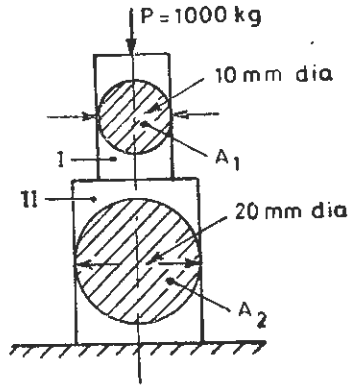


Fig. 1·5

Solution. The bar has two portions I and II.

Area of cross-section of portion I,

$A_1=\frac{\pi}{4}(1)^2=0·7854$ cm².

Area of cross-section of portion II,

$A_2=\frac{\pi}{4}(2)^2=3·1416$ cm².

Force P is the normal compressive force on all sections perpendicular to the axis. Maximum normal stress is developed in portion I with minimum area of cross section while minimum normal stress will be developed in the portion II with maximum area of cross section.

So maximum normal stress,

$$f_{max}=\frac{P}{A_1}=\frac{1000}{0·7854}=1273·23 \text{ kg/cm}^2 \text{ (compressive)}$$

Minimum normal stress

$$=\frac{P}{A_2}=\frac{1000}{3·1416}=318·31 \text{ kg/cm}^2 \text{ (compressive)}.$$

Exercise 1·1-1. A cylindrical rod of diameter 1·6 cm is subjected to a ı axial tensile force of 500 kg. Determine the normal and shear stresses on a plane inclined at a n angle of 30° to the axis of the bar.

[**Ans.** 62·17 kg/cm² (tensile), 107·68 kg/cm² (+ve)]

Note. In this case the inclined plane will be an ellipse, with major axis equal to $\frac{d}{\sin 30°}=3·2$ cm, minor axis$=d=1·6$ cm. Area of of the ellipse$=\frac{\pi\times\text{major axis}\times\text{minor axis}}{4}$.

Exercise 1·1-2. A cylindrical tapered bar of 12 mm diameter at one end and 20 mm diameter at the other end is subjected to an axial tensile force of 4000 N. Determine the maximum and minimum direct stresses developed in the bar.

Ans. [35·367 N/mm² (tensile), 12·732 N/mm² (tensile)]

Note. A normal stress is also called the direct stress.

1·2. NORMAL STRAIN

Fig. 1·1 (*a*) shows a bar of circular cross-section subjected to a tensile force P. [In practice, one end of the bar is fixed while force P is applied at the other end. To maintain equilibrium an equal and opposite force P acts as a reaction at the fixed end].

Due to this force P, the bar elongates and its original length L increases to L' and at he same time its diameter is reduced from D to D'.

Normal strain or the linear strain is defined as the change in length per unit length along the direction of the normal force. Lateral strain is defined as the change in diameter (or a dimension lateral to the axial length) per unit diameter or change in lateral dimension per unit lateral dimension as in the case of a rectangular section.

Normal strain, (due to tensile force)

$$\epsilon=\frac{\text{Change in length}}{\text{Original length}}=\frac{\text{Final length}-\text{Initial length}}{\text{Original length}}$$

$$=\frac{L'-L}{L}=\frac{\delta L}{L}\text{ (positive)}$$

Lateral strain, $$\epsilon'=\frac{\text{Change in diameter}}{\text{Original diameter}}$$

$$=\frac{\text{Final diameter}-\text{Initial diameter}}{\text{Original diameter}}$$

$$=\frac{D'-D}{D}=\frac{\delta D}{D}\text{ (negative)}$$

(as the diameter is reduced)

Similarly Fig. 1·1 (*b*) shows a bar of circular cross-section subjected to an axial compressive force P, the length of the bar is reduced and its diameter is increased.

Normal strain (due to compressive force)

$$\epsilon=\frac{\text{Final length}-\text{Initial length}}{\text{Original length}}$$

$$=\frac{L'-L}{L}=\frac{\delta L}{L}\text{ (negative)}$$

Lateral strain, $$\epsilon'=\frac{\text{Final diameter}-\text{Initial diameter}}{\text{Initial diameter}}$$

$$=\frac{D'-D}{D}=\frac{\delta D}{D}\text{ (positive)}$$

$$\frac{\text{Lateral strain}}{\text{Normal strain}}=\frac{\epsilon'}{\epsilon}=\text{a negative ratio}=\frac{1}{m}$$

The ratio of lateral strain to normal strain is called Poisson's Ratio and is denoted by $1/m$.

In the first case, normal stress

$$=\frac{P}{A}=f$$

where P=axial tensile force

$A=\frac{\pi}{4}D^2$, cross-sectional area

f is the tensile stress or a positive direct stress

ϵ=normal strain, a positive strain.

In the second case, normal stress

$$=\frac{P}{A}=f$$

where P=axial compressive force

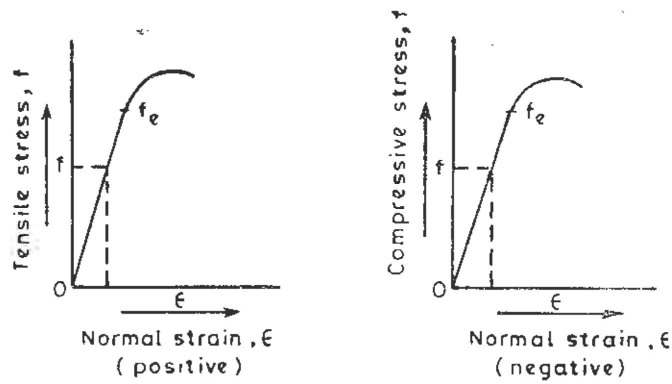
$A=\frac{\pi}{4}D^2$, cross-sectional area

f is the compressive stress or a negative direct stress.

ϵ=normal strain, a negative strain.

In both the cases of tensile and compressive forces, as the force gradually increases, the normal stress and normal strain also gradually increase and stress is proportional to strain but only upto the elastic limit, as shown in Fig. 1·6.

Fig. 1·6 shows the variation of normal strain with respect to the normal stress developed in the bar.



(a) (b)

Fig. 1·6

Upto the elastic limit, if force is removed from the bar, the bar will return to its original dimensions. Beyond the elastic limit, the graph between f and ϵ is no longer a straight line, but it is curved and once this limit f_e is crossed, the bar is subjected to a permanent deformation (or strain) after the removal of the load.

So within the elastic limit $f \propto \epsilon$ *i.e.*, the material obeys Hooke's law.

$$f \propto \epsilon$$

$$=E\epsilon, \text{ where } E \text{ is the constant of proportionality}$$

or $$E=\frac{f}{\epsilon}=\frac{\text{Normal stress}}{\text{Normal strain}} \quad ...(1)$$

This ratio of stress and strain within the elastic limit is called the Young's modulus of Elasticity and is denoted by E. Since strain is only a ratio, the units of E are the same as those of stress, f.

In order words $f=\frac{P}{A}$, $\epsilon=\frac{dL}{L}$, then

Young's modulus of elasticity,

$$E=\frac{PL}{A\delta L} \quad ...(2)$$

If E for the material is given, then

Strain, $\epsilon=\dfrac{f}{E}=\dfrac{P}{AE}$

Change in length, $dL \quad =\epsilon L=\dfrac{PL}{AE}$ ...(3)

Example 1·2-1. A circular steel bar of 10 mm diameter and 100 mm gauge length is tested under tension. A tensile force of 10 kN increases its length by 0·06 mm while the diameter is decreased by 0·0018 mm. Determine (*i*) Young's modulus of elasticity, (*ii*) Poisson's ratio for the material of the bar.

Solution. Fig. 1·7 shows a tensile test specimen. Collars are provided at the ends so that the specimen can be properly gripped in the testing machine. The central portion

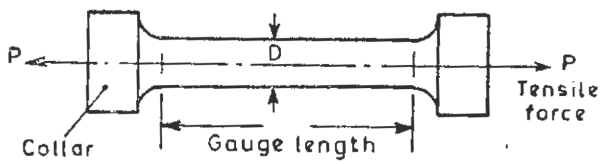


Fig. 1·7. Tensile Test Specimen.

along which the cross-section is uniform is called the gauge length as shown.

Tensile force, $P=10{,}000$ N

Area of cross-section, $A=\dfrac{\pi}{4}(10)^2$

$=78{\cdot}54 \text{ mm}^2$

Change in length, $dL=0{\cdot}06$

Original length, $L=100$ mm

Young's modulus of elasticity,

$$E=\frac{PL}{A\delta L}=\frac{10000\times 100}{78{\cdot}54\times{\cdot}06}$$

$$=\frac{1000\times 1000}{78{\cdot}54\times 0{\cdot}06}=212\times 10^3 \text{ N/mm}^2$$

Normal strain, $\epsilon=\dfrac{\delta L}{L}=\dfrac{0{\cdot}06}{100}$

Change in diameter, $\delta D=0{\cdot}0018$ mm

Original diameter, $D=10$ mm

Lateral strain, $\epsilon'=\dfrac{\delta D}{D}=\dfrac{0{\cdot}0018}{10}$

Poisson's ratio, $\dfrac{1}{m}=\dfrac{\epsilon'}{\epsilon}=\dfrac{0{\cdot}0018}{10}\times\dfrac{100}{0{\cdot}06}=\dfrac{0{\cdot}018}{0{\cdot}06}=0{\cdot}3.$

Example 1·2-2. A circular brass bar of 12 mm diameter is tested under tension. If the increase in the gauge length of 100 mm is 0·12 mm, determine the stress developed in the bar. What is the change in its diameter ?

Given E for brass $=102\times10^3$ N/mm²

$\frac{1}{m}$ for brass $=0·32.$

Solution. $E=102\times10^3$ N/mm²

Change in length, $\delta L=0·12$ mm

Gauge length, $L=100$ mm

Normal strain, $\epsilon=\frac{\delta L}{L}=\frac{0·12}{100}=0·0012$

Normal stress, $f=\epsilon\ E$

$=0·0012\times102\times10^3$ N/mm²

$=122·4$ N/mm²

Poisson's ratio, $\frac{1}{m}=\frac{\text{lateral strain}}{\text{normal strain}}=0·32$

Lateral strain $=\frac{\delta D}{D}=0·32\times·0012$

$=0·384\times10^{-3}$

Change in diameter, $\delta D=0·384\times10^{-3}\times12$ mm

$=4·608\times10^{-3}$ mm

$=0·004608$ mm.

Exercise 1·2-1. An aluminium round bar of diameter 15 mm and gauge length 150 mm is tested under tension. A tensile force of 2 tonnes produces an extension of 0·253 mm, while its diameter decreases by 0·0083 mm. Determine the Young's modulus and Poisson's ratio of aluminium. [**Ans.** 671·15 tonnes/cm², 0·328].

Exercise 1·2-2. A steel bar of rectangular cross-section 10 mm$\times$15 mm and length 100 mm subjected to a compressive force of 3 kN. If E for steel$=210\times10^3$ N/mm² and $1/m$ for steel$=0·3$, determine :

(*a*) Change in length.

(*b*) Change in 10 mm side.

(*c*) Change in 15 mm side.

[**Ans.** (*a*) ·009524 mm (*b*) $0·2857\times10^{-3}$ mm (*c*) $0·428\times10^{-3}$ mm].

1·3. BARS OF VARYING CROSS SECTIONS

Fig. 1·8 shows a bar with different diameters D_1, D_2 and D_3 with lengths along the axis equal to L_1, L_2 and L_3 respectively. Say E is the Young's modulus and $1/m$ is the Poisson's ratio of the material.

This bar is subjected to an axial compressive force P, which will produce contraction in the length of the bar and its diameters will increase.

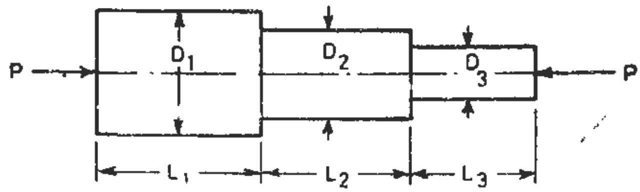


Fig. 1·8.

Force P is the normal force on all sections perpendicular to the axis. Each portion of the bar is subjected to same compressive force but the normal stress developed in each portion will be different.

Stresses

Normal stress in portion I, $f_1=-\frac{P}{A_1}$ (Compressive)

Normal stress in portion II, $f_2=-\frac{P}{A_2}$ (Compressive)

Normal stress in portion III, $f_3=-\frac{P}{A_3}$ (Compressive)

Where Areas of cross-sections are

$$A_1=\frac{\pi}{4}D_1^2,\quad A_2=\frac{\pi}{4}D_2^2,\quad A_3=\frac{\pi}{4}D_3^2$$

The normal strain in each portion is

$$\epsilon_1=-\frac{f_1}{E},\ \epsilon_2=-\frac{f_2}{E}\ \text{and}\ \epsilon_3=-\frac{f_3}{E}\ \text{(negative)}$$

Change in length in each portion

$$\delta L_1=\epsilon_1 L_1,\ \delta L_2=\epsilon_2 L_2,\ \delta L_3=\epsilon_3 L_3.$$

Total change in length, $\delta L=\delta L_1+\delta L_2+\delta L_3$

$$=\epsilon_1 L_1+\epsilon_2 L_2+\epsilon_3 L_3$$

$$=-\frac{f_1}{E}L_1-\frac{f_2 L_2}{E}-\frac{f_3 L_3}{E}$$

$$=-\frac{PL_1}{A_1E}-\frac{PL_2}{A_2E}-\frac{PL_3}{A_3E}$$

$$=-\frac{P}{E}\left[\frac{L_1}{A_1}+\frac{L_2}{A_2}+\frac{L_3}{A_3}\right]$$

or

$$\delta L=-\frac{4P}{\pi E}\left[\frac{L_1}{D_1^2}+\frac{L_2}{D_2^2}+\frac{L_3}{D_3^2}\right]$$

Showing the decrease in length, each diameter will increase due to the lateral strain.

Lateral strain in each portion will be

$$\epsilon_1' = -\frac{1}{m}\epsilon_1, \quad \epsilon_2' = -\frac{1}{m}\epsilon_2, \quad \epsilon_3' = -\frac{1}{m}\epsilon_3$$

$$\epsilon_1' = +\frac{f_1}{mE}, \quad \epsilon_2' = +\frac{f_2}{mE}, \quad \epsilon_3' = +\frac{1}{m}\frac{f_3}{E}$$

The change in diameters will be

$$\delta D_1 = \epsilon_1' D_1 = \frac{f_1 D_1}{mE}, \quad \delta D_2 = \epsilon_2' D_2 = \frac{f_2 D_2}{mE}, \quad \delta D_3 = \epsilon_3' D_3 = \frac{f_3 D_3}{mE}$$

Showing the increase in diameters.

Example 1·3-1. A stepped circular bar having diameters 20 mm, 15 mm and 10 mm over axial lengths of 100 mm, 80 mm and 60 mm is subjected to an axial tensile force of 5 kN. If $E = 100 \times 10^3$ N/mm² and $1/m = 0{\cdot}32$ for the material of the bar, determine

(*a*) Total change in length.

(*b*) Change in each diameter.

Solution. As per the data given

Lengths $l_1 = 100$ mm, $l_2 = 80$ mm, $l_3 = 60$ mm.

Diameters $D_1 = 20$ mm, $D_2 = 15$ mm, $D_3 = 10$ mm.

Since the axial force is tensile, there will be increase in length and decrease in diameters.

Total change in length $= +\dfrac{4P}{\pi E}\left[\dfrac{l_1}{D_1^2} + \dfrac{l_2}{D_2^2} + \dfrac{l_3}{D_3^2}\right]$

where $P = 5\text{ kN} = 5000\text{ N}$

$E = 100 \times 1000\text{ N/mm}^2$

or $dl = \dfrac{4 \times 5000}{100 \times 1000}\left[\dfrac{100}{20^2} + \dfrac{80}{15^2} + \dfrac{60}{10^2}\right]$

$= 0{\cdot}2\,[0{\cdot}250 + 0{\cdot}355 + 0{\cdot}600] = 0{\cdot}241$ mm.

Change in diameters $\delta D_1 = -\dfrac{f_1 D_1}{mE} = -\dfrac{4P}{\pi D_1^2} \times \dfrac{D_1}{mE} = -\dfrac{4P}{\pi D_1 mE}$

$= -\dfrac{4 \times 5000 \times 0{\cdot}32}{\pi \times 20 \times 100 \times 1000} = -1{\cdot}018 \times 10^{-3}$ mm.

Similarly $\delta D_2 = -\dfrac{4P}{\pi D_2 mE} = -\dfrac{4 \times 5000 \times 0{\cdot}32}{\pi \times 15 \times 100 \times 1000}$

$= -1{\cdot}358 \times 10^{-3}$ mm

$\delta D_3 = -\dfrac{4P}{\pi D_3 mE} = -\dfrac{4 \times 5000 \times 0{\cdot}32}{\pi \times 10 \times 100 \times 1000}$

$= -2{\cdot}037 \times 10^{-3}$ mm.

Exercise 1·3-1. A straight stepped bar of steel is of square section throughout with sides 10 mm, 12 mm, 16 mm with axial lengths of 8 cm, 10 cm and 12 cm respectively.

The bar is subjected to an axial tensile force of 3600 N.

If $E=200\times10^3$ N/mm² and $1/m=0.29$ determine

(*a*) Total change in length.

(*b*) Change in the side of the each square section.

[**Ans.** (*a*) 0·0353 mm, (*b*) 0.522×10^{-3}, 0.435×10^{-3}, 0.326×10^{-3} mm]

1·4. TAPERED BARS

Consider a *tapered round bar* of length L and diameter D_1 at one end continuously increasing to diameter D_2 at the other end, as shown in Fig. 1·9. The bar is subjected to an axial tensile force P. Say E is the Young's modulus of the material of the bar. In this case maximum stress occurs at end A and minimum stress occurs at end B, or in other words there is continuous variation of stress along the length. Consider a small element of length dx at a distance of x from the end A.

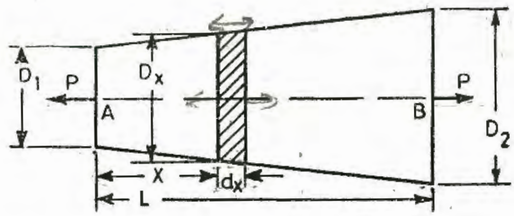


Fig. 1·9

Diameter, $D_x=D_1+\dfrac{D_2-D_1}{L}\times x$

$=D_1+Kx$

where $K=\dfrac{D_2-D_1}{L}$ a constant

Area of cross-section, $A_x=\dfrac{\pi}{4}(D_x)^2$

Stress at the section, $f_x=\dfrac{P}{A_x}=\dfrac{4P}{\pi D_x^2}$

Strain at the section, $\epsilon_x=\dfrac{f_x}{E}=\dfrac{4P}{\pi E D_x^2}$

Change in length over dx *i.e.*,

$$\delta dx=\frac{4Pdx}{\pi E D_x^2}$$

Total change in length,

$$\delta L=\int_0^L \frac{4Pdx}{\pi E D_x^2}=\int_0^L \frac{4Pdx}{\pi E(D_1+Kx)^2}$$

$$=\frac{4P}{\pi E}\left|-\frac{(D_1+Kx)^{-1}}{K}\right|_0^L=-\frac{4P}{\pi EK}\left[(D_1+KL)^{-1}-(D_1)^{-1}\right]$$

$$=-\frac{4P}{\pi KE}\left[\frac{1}{D_1+\dfrac{D_2-D_1}{L}\times L}-\frac{1}{D_1}\right]$$

$$= -\frac{4P}{\pi EK}\left[\frac{1}{D_2}-\frac{1}{D_1}\right]$$

$$= -\frac{4P}{\pi EK}\left[\frac{D_1-D_2}{D_1D_2}\right]=\frac{4P}{\pi EK}\left[\frac{D_2-D_1}{D_1D_2}\right]$$

Substituting the value of $K=\frac{D_2-D_1}{L}$

Total change in length, $\delta L=\frac{4PL}{\pi ED_1D_2}$.

Tapered Flat. Consider a flat of constant thickness t but breadth varying uniformly from B_1 at one end to B_2 at the other end. Length of the flat is L.

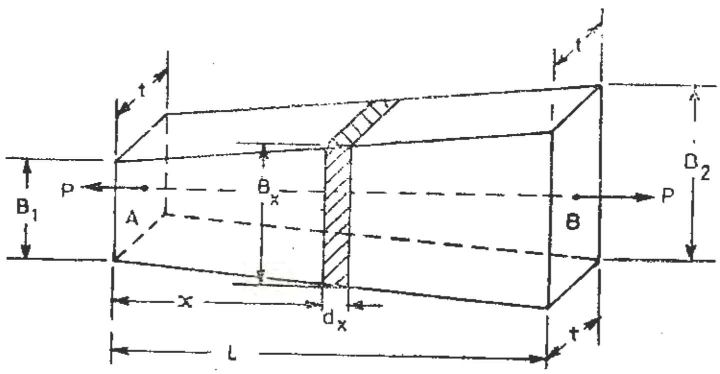


Fig. 1·10

The flat is subjected to an axial tensile force P as shown in Fig. 1·10. Say the modulus of elasticity of the material of the flat is E.

Again consider an elementary strip of thickness dx at a distance of x from end A.

Breadth $B_x=B_1+\frac{B_2-B_1}{L}\,.\,x=B_1+Kx$

where $K=\frac{B_2-B_1}{L}$, a constant

Area of cross-section, $A_x=B_x\,.\,t=(B_1+Kx)\,t$

Stress, $f_x=\frac{P}{A_x}=\frac{P}{t(B_1+Kx)}$

Strain, $\epsilon_x=\frac{f_x}{E}=\frac{P}{Et\,(B_1+Kx)}$

Change in length over dx,

$$dx=\frac{Pdx}{Et\,(B_1+Kx)}$$

$$\text{Total change in length, } \delta L = \int_0^L \frac{P\,dx}{Et\,(B_1+Kx)}$$

$$= \frac{P}{Et}\left|\frac{1}{K}\ln\,(B_1+Kx)\right|_0^L$$

$$= \frac{P}{EtK}\Big[\ln\,(B_1+KL) - \ln\,B_1\Big]$$

$$= \frac{P}{EtK}\ln\frac{B_2}{B_1} = \frac{PL}{Et(B_2-B_1)}\ln\frac{B_2}{B_1}.$$

Example 1·4-1. A bar of square section throughout, of length 1 metre tapers from an area of 20 mm × 20 mm to the area 10 mm × 10 mm. $E=200\times10^3$ N/mm², determine the change in length of the bar, if the axial force on the bar is 10 kN compressive.

Solution. Since there is compressive force acting on the bar, there will be contraction in its length.

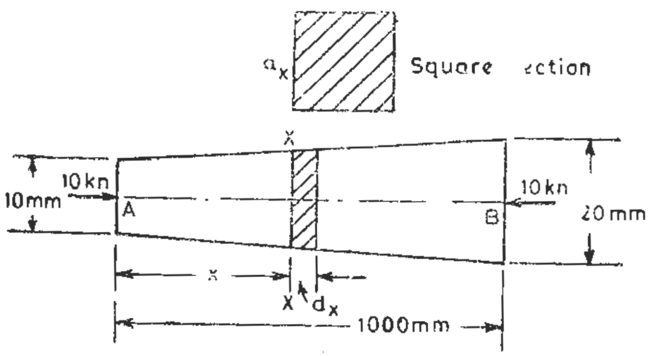


Fig. 1·11

Consider a section X-X at a distance of x from end A.

$$\text{Side of the square,} \quad a_x = 10 + \frac{20-10}{1000}\times x$$

$$= (10 + \cdot01\,x)$$

$$\text{Area of cross-section, } A_x = a_x^2 = (10+0\cdot01\,x)^2$$

$$\text{Stress,} \quad f_x = -\frac{10\times1000}{a_x^2} = -\frac{10{,}000}{(10+\cdot01x)^2}$$

$$\text{Strain,} \quad \epsilon_x = -\frac{f_x}{E} = -\frac{10{,}000}{E(10+\cdot01x)^2}$$

Change in length over dx,

$$\delta dx = -\frac{10000\,dx}{E(10+\cdot01x)^2}$$

Total change length, $$\delta l=-\int_0^{1000}\frac{10000\,dx}{200\times 10^3(10+\cdot 01x)^2}$$

$$=-\frac{1}{20}\left|-\frac{1}{(\cdot 01)}\,(10+\cdot 01x)^{-1}\right|_0^{1000}$$

$$=+\frac{1}{0\cdot 2}\left[\frac{1}{20}-\frac{1}{10}\right]$$

$$=-\frac{1}{0\cdot 2}\times\frac{10}{200}=-0\cdot 25\ \text{mm.}$$

Example 1·4-2. In a bar of rectangular section, the width tapers from 25 mm to 15 mm, while the thickness tapers from 12 mm to 8 mm over a length of 500 mm. The bar is subjected to a tensile force of 8 kN. If $E=1\times 10^5$ N/mm², determine the change in the length of the bar.

Solution. Consider that the axis of the bar is passing through CG of the section at both the ends. The Fig. 1·12 shows the tapered bar with section of 25 mm×12 mm uniformly tapering to the section 15 mm×8 mm.

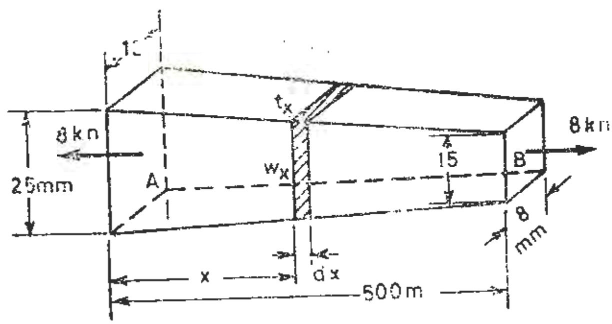


Fig. 1·12

Again consider an element of length dx, at a distance of x from the end A.

Width, $$w_x=25-\frac{(25-15)}{500}\,.\,x=(25-\cdot 02\,x)$$

Thickness, $$t_x=12-\frac{(12-8)\,x}{500}=(12-\cdot 008\,x)$$

Area of cross-section, $A_x=w_x\,t_x=(25-0\cdot 02x)(12-\cdot 008x)$

Stress, $$f_x=\frac{8\times 1000}{A_x}=\frac{8000}{(25-\cdot 02x)(12-\cdot 008x)}$$

Strain, $$\epsilon_x=\frac{8000}{E(25-\cdot 02x)(12-\cdot 008x)}$$

Change in length over dx,

$$\delta dx=\frac{8000\,dx}{E\,(25-\cdot 02x)(12-\cdot 008x)}\,.$$

Total change in length,

$$\delta l=\int_0^{500} \frac{8000\, dx}{10^5(25-0{\cdot}02x)(12-{\cdot}00\ldots}$$

$$={\cdot}08 \int_0^{500} \frac{2{\cdot}5\, dx}{(25-{\cdot}02x)(30-{\cdot}02x)}$$

$$=0{\cdot}2 \int_0^{500} \frac{1}{5}\left(\frac{1}{25-{\cdot}02x}-\frac{1}{30-{\cdot}02x}\right) dx$$

$$=04 \left| \frac{1}{(-{\cdot}02)} \ln (25-{\cdot}02x)-\frac{1}{(-{\cdot}02)} \ln (30-{\cdot}02x) \right|_0^{500}$$

$$=-2 \left| \ln (25-{\cdot}02x)-\ln (30-0{\cdot}02x) \right|_0^{500}$$

$$=-2 \left[\ln \frac{15}{25} - \ln \frac{20}{30} \right]$$

$$=+2 \left[\ln \frac{20}{30} \times \frac{25}{15} \right]$$

$$=2 \ln 1{\cdot}111=2\times{\cdot}1052={\cdot}2104 \text{ mm.}$$

Exercise 1·4-1. A tapered round bar of length 150 cm, has a diameter of 2 cm at one end which uniformly increases to a diameter of 3 cm at the other end. If $E=2000$ tonnes/cm², what load is required to produce an extension of 1 mm in the bar.

[**Ans.** 6·2832 tonne]

Exercise 1·4-2. A flat of rectangular section has area of cross-section 3 cm × 1·2 cm at one end which uniformly decreases to area of cross-section 2 cm × 1·2 cm. The length of the flat is 80 cm. If an axial tensile load of 2·5 tonnes is applied on the flat, what will be extension in its length. $E=1000$ tonnes/cm².

[**Ans.** ·676 mm]

1·5. BAR SUBJECTED TO VARIOUS FORCES

A bar of uniform section is subjected to force P_1 (compressive) at section A are and to force P_2 (tensile) at section D. Then at sections B and C, the forces applied at are P_3 and P_4 respectively as shown in Fig. 1·13. Consider the three portions I, II and III *i.e.*, AB, BC and CD of lengths L_1, L_2 and L_3 respectively.

Portion AB will be under a compressive force P_1 and the net force available at B for portion BC is P_3+P_1. The portion CD will be under a tensile force P_2 and the net force at C for the portion BC will be P_4-P_2.

To maintain equilibrium of the central portion BC, $P_3+P_1=P_4-P_2$

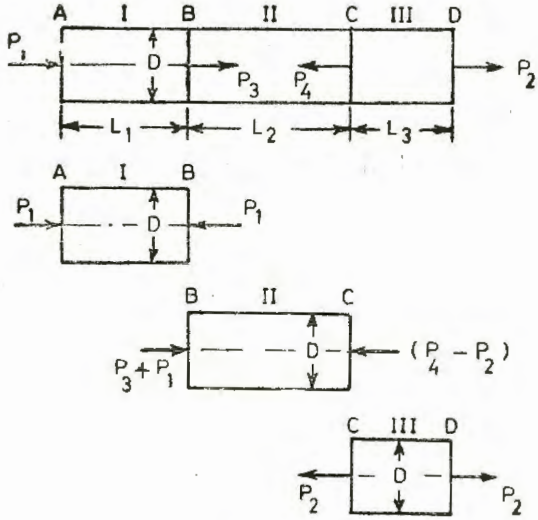


Fig. 1·13

If E is the Young's modulus of elasticity of the material. Contraction in the length AB,

$$\delta l_1 = \frac{4P_1}{\pi D^2} \times \frac{l_1}{E}$$

Contraction in the length BC,

$$\delta l_2 = \frac{4(P_3+P_1)}{\pi D^2} \times \frac{l_2}{E} = \frac{4(P_4-P_2)l_2}{\pi D^2 E}$$

Extension in the length CD,

$$\delta l_3 = \frac{4P_2}{\pi D^2} \times \frac{l_3}{E}$$

Total change in length, $\delta l = -\delta l_1 - \delta l_2 + \delta l_3$

$$= \frac{4}{\pi D^2 E} \left[-P_1\, l_1 - (P_3+P_1)\, l_2 + P_2\, l_3 \right]$$

The bar in this article can also be considered with different stepped diameters.

Example 1·5-1. A stepped circular bar 150 mm long with diameters 20 mm, 15 mm and 10 mm along the lengths AB, BC and CD respectively is subjected to various forces as shown in the Fig. 1·14. There is a tensile force on section A (as the force is pointing away from the plane) and there is a compressive force on section D (as the force is pointing towards the plane). Determine the change in length if $E = 2 \times 10^6$ kg/cm².

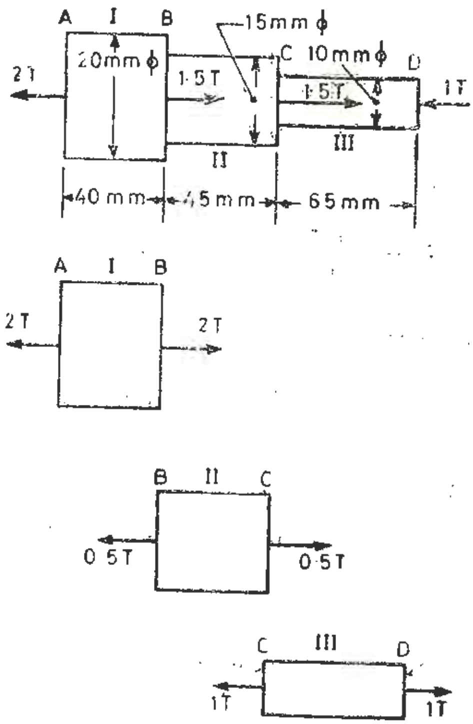


Fig. 1·14

Solution. Considering portion AB, a reaction of 2 tonnes is acting on the plane BC, so for portion BC the the force available is $1\cdot5-2=-0\cdot5$ tonne or the tensile force of $\overleftarrow{0\cdot5}$ tonne is acting on the plane B considering the portion BC. Again a compressive force of 1 tonne is acting on the plane D, there will be equal reaction of 1 tonne on plane C for portion CD. But the force at C is 1·5 tonnes, so a force of $1\cdot5-1=\overrightarrow{0\cdot5}$ tonne is acting on the plane C for portion BC. Or the portion BC is under a tensile force of 0·5 tonne.

Areas of cross sections

$$A_1=\frac{\pi}{4}\times2^2=3\cdot1416 \text{ cm}^2$$

$$A_2=\frac{\pi}{4}\times1\cdot5^2=1\cdot767 \text{ cm}^2$$

$$A_3=\frac{\pi}{4}\times1^2=0\cdot7854 \text{ cm}^2$$

Extension in portion AB,

$$\delta l_1 = \frac{2\times 4}{3{\cdot}1416\times 2000}, \quad E=2\times 10^6 \text{ kg/cm}^2=2000 \text{ tonnes/cm}^2$$

$$=1{\cdot}273\times 10^{-3} \text{ cm.}$$

Extension in portion BC,

$$\delta l_2 = \frac{0{\cdot}5\times 4{\cdot}5}{1{\cdot}767\times 2000}=0{\cdot}636\times 10^{-3} \text{ cm}$$

Contraction in portion CD,

$$\delta l_3 = \frac{1\times 6{\cdot}5}{0{\cdot}7854\times 2000}=4{\cdot}138\times 10^{-3} \text{ cm}$$

Total change in length, $\delta l=\delta l_1+\delta l_2-\delta l_3$

$$=(1{\cdot}273+0{\cdot}636-4{\cdot}138)\times 10^{-3} \text{ cm}$$

$$=-2{\cdot}229\times 10^{-3} \text{ cm.}$$

Exercise 1·5-1. A bar of steel of diameter 16 mm and length 600 mm is subjected to axial forces as shown in Fig. 1·15. If E for steel$=210\times 10^3$ N/mm². Determine the change in its length. [**Ans.** 0·00]

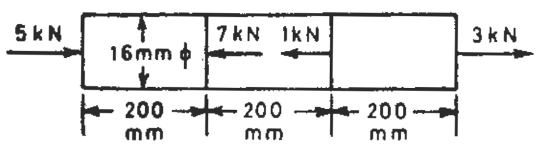


Fig. 1·15

Exercise 1·5-2. A round tapered brass bar of length 500 mm is subjected to loads as shown in Fig. 1·16. Determine the change in length if $E=1\times 10^6$ kg/cm²

[See Article 1·4] [**Ans.** −0·0655 mm]

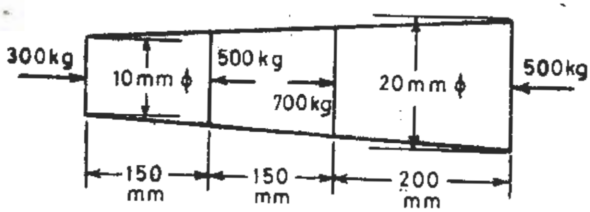


Fig. 1·16

1·6. EXTENSION UNDER SELF WEIGHT

A circular bar of uniform area of cross-section A throughout is fixed at its upper end. The length of the bar is h and its weight density is w per unit volume. The weight of the bar will stretch it.

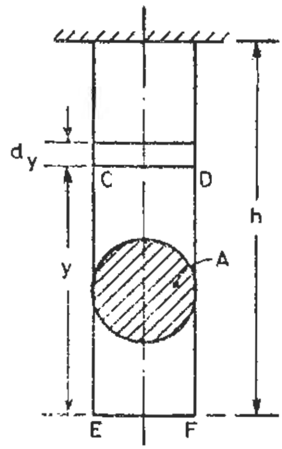


Fig. 1·17

Let us consider an elementary length dy at a distance of y from the lower edge. The weight of the portion $CDFF$ will act as a tensile force to extend the element of length dy.

Weight of the portion $CDEF=Ayw$

f_y, stress on the section

$$CD=\frac{Ayw}{A}=wy$$

Strain, $$\epsilon_y=\frac{f_y}{E}=\frac{wy}{E}$$

Change in length over dy,

$$\delta dy=\frac{wy}{E}\,dy$$

Total change in length, $$\delta h=\int_0^h \frac{wy}{E}\,dy=\frac{wh^2}{2E}$$

Total weight of the bar, $W=wAh$

So change in length $$=\frac{wh^2A}{2EA}=\frac{Wh}{2EA}.$$

Example 1·6-1. A steel bar 1 m long is suspended vertically as shown in the Fig. 1·18. The bar is solid and of diameter 12 mm for half of its length and is hollow with a hole of 6 mm diameter for the remainder of its length. If $E=2\times10^6$ kg/cm², determine the change in the length of the bar due to its own weight. Given $w=\cdot0078$ kg/cm³

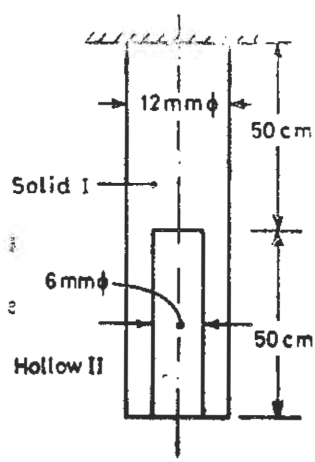


Fig. 1·18

Solution. Let us consider the bar in two portions I and II. For the hollow portion

δl_{II}=extension due to its own weight

$$=\frac{wh^2}{2E}=\frac{\cdot0078\times50\times50}{2\times2\times10^6}$$

$$=4\cdot875\times10^{-6}\text{ cm.}$$

For the portion I, solid bar, the weight of the hollow portion will act as a tensile force

$$\delta l_l = \delta l_l' + \delta l_l'' = \text{extension due to its own weight}$$
$$+\text{extension due to the weight of the lower portion}$$

$$\delta l_l' = \frac{wh^2}{2E} = \frac{0{\cdot}0078 \times 50 \times 50}{2 \times 2 \times 10^6}$$
$$= 4{\cdot}875 \times 10^{-6} \text{ cm}$$

Weight of the hollow portion

$$= W_h = \frac{\pi}{4}(1{\cdot}2^2 - 0{\cdot}6^2) \times 50 \times 0{\cdot}0078 \text{ kg} = 0{\cdot}330 \text{ kg}$$

$$\delta l_l'' = \frac{W_h}{A_s} \times \frac{50}{E}$$

where A_s = area of cross section of solid portion

$$= \frac{\pi}{4}(1{\cdot}2)^2 = 1{\cdot}131 \text{ cm}^2$$

So $$\delta l_l'' = \frac{0{\cdot}330 \times 50}{1{\cdot}131 \times 2 \times 10^6} = 7{\cdot}294 \times 10^{-6} \text{ cm}$$

Total elongation of the bar

$$= (4{\cdot}875 + 4{\cdot}875 + 7{\cdot}294) \times 10^{-6}$$
$$= 17{\cdot}044 \times 10^{-6} \text{ cm.}$$

Exercise 1·6-1. A circular steel bar diameter 10 mm and length 1500 mm is fixed at its upper end. If the weight density of steel is 0·0078 kg/cm^5 determine :

(*a*) maximum stress developed in bar,

(*b*) elongation of bar under its own weight. Given $E = 2 \times 10^6$ kg/cm^2.

[**Ans.** (*a*) 1·17 kg/cm^2 (*b*) $0{\cdot}43875 \times 10^{-3}$ mm]

1·7. BAR OF UNIFORM STRENGTH

A bar of varying section is shown in Fig. 1·19 such that the stress developed in the bar at any section is the same. Say w is the weight per unit volume of the bar. The area of cross section at the bottom edge is A_1 and area of cross section at top fixed edge is A_2 and length of the bar is H.

Let us consider a small elementary strip $ABCD$ of length dy at a distance of y from the bottom edge. Say the stress developed in the bar throughout its length is f.

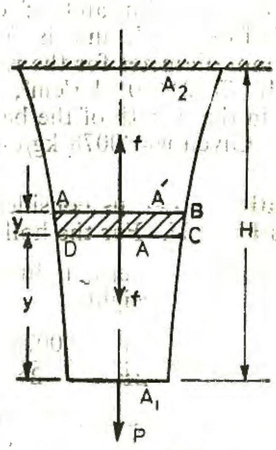


Fig. 1·19

Say area at CD $= A$

Area at AB $= A' = A + dA$

Weight of strip $= wA \, . \, dy$

For equilibrium

$$f \,.\, A+w \,.\, A\, dy=f\,(A+dA)$$

or
$$wA\, dy=f \,.\, dA$$

or
$$\frac{dA}{A}=\frac{w}{f}\cdot dy$$

Integrating both the sides

$$\int_{A_1}^{A_2}\frac{dA}{A}=\frac{w}{f}\int_0^H dy$$

or
$$\Big|\ln A\Big|_{A_1}^{A_2}=\frac{w}{f}\cdot H$$

or
$$\ln\frac{A_2}{A_1}=\frac{w}{f}\,H$$

or
$$A_2=A_1\, e^{\frac{wH}{f}}$$

or at any distance,
$$A=A_1\, e^{\frac{wy}{f}}.$$

Example 1·7-1. A vertical circular bar 150 cm high is subjected to a uniform stress of 2 6 kg/cm² throughout its length. The diameter at the bottom edge is 6 cm, determine the diameter at the top edge if it is fixed in the ceiling. Given the weight density =0·0078 kg/cm³.

Solution. Area of cross section at the bottom edge, $A_1=\frac{\pi}{4}6^2=9\pi$ cm² ; Weight density, $w=\cdot 0078$ kg/cm³ ; Height, $H=150$ cm ; Uniform stress, $f=2\cdot 6$ kg/cm².

Area of cross section at the top fixed edge

$$A_2=A_1\, e^{\frac{wH}{f}}$$

$$=9\pi\, e^{\frac{\cdot 0078\times 150}{2\cdot 6}}=9\pi\times e^{\cdot 45}$$

$$=9\pi\times 1\cdot 5683=\frac{\pi}{4}\,D_2^2$$

Diameter at the top edge,

$$D_2=\sqrt{\frac{9\pi\times 1\cdot 5683\times 4}{\pi}}$$

$$=\sqrt{56\cdot 4588}=7\cdot 514 \text{ cm.}$$

Exercise 1·7-1. A copper bar of uniform strength, 2 metres long is suspended vertically with its top edge fixed in the ceiling. The uniform stress developed in the bar is 17·8 kg/cm². If the diameter at the bottom edge is 8 mm, what will be the diameter at the top edge. Given weight density of copper$=8{\cdot}9\times10^{-3}$ kg/cm³. **[Ans.** 8·41 mm]

1·8. SHEAR STRESS AND SHEAR STRAIN

Fig. 1·1 (*c*) shows a rectangular block distorted under the action of the force P acting tangentially to the top surface $ABGH$. At the bottom surface which is fixed to the ground there is equal and opposite reaction P.

This force which is parallel to the plane is called shear force (article 1·1). This shear force per unit area of the plane on which it is applied is called the shear stress q. The shear stress tending to rotate the body in the clockwise direction is taken to be positive. The angular displacement of the vertical side AD by an angle ϕ, is called the shear angle.

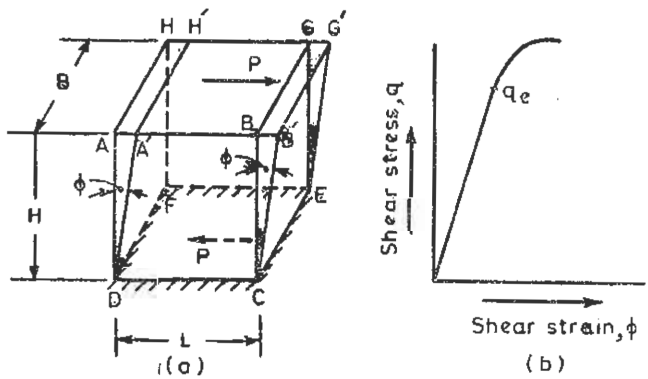


Fig. 1·20

Fig. 1·20 shows a block of length L, breadth B and height H subjected to a force P at the top surface while the bottom surface $DCEF$ is fixed. Under the action of this force, the block is distorted to a new shape $A'B'G'H'\ DCEF$.

$$\text{Shear stress}=\frac{\text{shear force, } P}{\text{area of the plane, } ABGH}=\frac{P}{B\times L}$$

$$=q \qquad \text{(generally denoted by this letter)}$$

$$\text{Shear strain}=\frac{\text{displacement } AA'}{AD}=\tan\phi.$$

But the angle ϕ is very very small within the elastic limit of the material.

$$\text{Shear strain} \simeq \tan\phi \simeq \sin\phi \simeq \phi.$$

So the shear strain is given by the angle of displacement ϕ.

If the force P is gradually increased, the shear stress q also gradually increases and the angle of displacement ϕ changes with q. Fig. 1·20 (*b*) shows the variation of ϕ with respect to q. Upto a particular limit shear stress q is directly proportional to shear strain ϕ. This limit is called the elastic limit. Within the elastic limit, if the shear force is removed from the block, the block returns to its original shape and original dimensions. But if the shear force is removed after the elastic limit stress (q_e), there will be permanent deformation or distortion left in the block.

Within the elastic limit $q \propto \phi$

or $q = G\phi$ where G is the proportionality constant

$$G = \frac{\text{shear stress, } q}{\text{shear strain, } \phi}.$$

Ratio of $\frac{q}{\phi}$ is defined as the Modulus of rigidity, G.

Complementary shear stress. Fig. 1·21 shows a block subjected to shear stress q at the top surface, and a shear stress q (due to the reaction) at the bottom surface. The shear

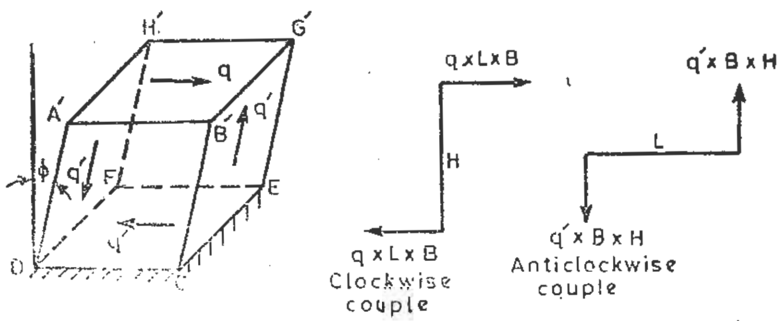


Fig. 1·21

angle ϕ is very small (and not so large as shown in the figure) the length DC, breadth $A'H'$ and height $A'D'$ can be considered as negligibly changed from their original dimensions L, B and H respectively.

Shear force at the top surface $= q \times \vec{L} \times B$

Shear force at the bottom surface $= q \times \overleftarrow{L} \times B$.

These two forces constitute a couple of arm H, tending to rotate the body in the clockwise direction.

Moment of the couple $= q \times L \times B \times H \circlearrowright$

For equilibrium this applied couple has to be balanced by the internal resistance developed in the body. Say the resisting shear stress on the vertical faces is q' as shown in the figure.

Shear force on the surface $CEG'B' = q' \times B \times H \uparrow$

Shear force on the surface $DFH'A' = q' \times B \times H \downarrow$.

These two forces constitute an anticlockwise couple of arm L, resisting the applied couple.

Moment of the resisting couple $= q' \times B \times H \times L. \circlearrowleft$

For equilibrium $q' \times BHL = q \times LBH$

or $q' = q$.

This resisting stress q' is called the complementary shear stress and always acts at an angle of 90° to the applied shear stress. Moreover if the applied stress is positive *i.e.*, tending to rotate the body in the clockwise direction, then the complementary shear stress will be negative *i.e.*, tending to rotate the body in the anticlockwise direction.

The use of the concept of complementary shear stress will be made in the chapter 3 on principal stresses.

Example 1·8-1. Fig. 1·22 shows a rivet joining two plates of thickness $t=1·2$ cm and width$=6$ cm. The plates are subjected to force $F=1200$ kg. If the diameter of the rivet is

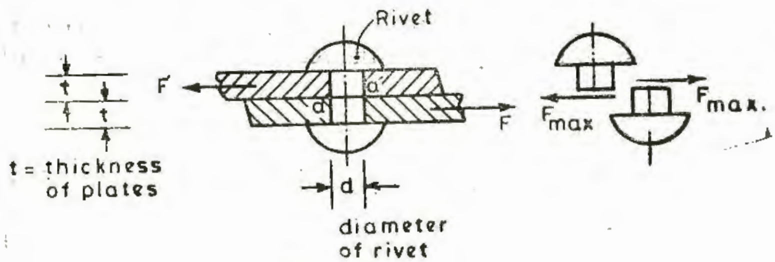


Fig. 1·22

15 mm, determine the shear stress developed in the rivet. The ultimate shear strength of the material of the rivet is 2·8 tonnes/cm². How much maximum load the plates can carry.

Solution. The plates carry the force F which acts as a shear force on the circular plane aa' of the rivet.

This plane aa' is along the contact surface between the two plates.

Force $F=1200$ kg

Area of cross section under shear force,

$$A=\frac{\pi}{4}\ (d)^2=\frac{\pi}{4}\ (1·5)^2=1·767\ \text{cm}^2$$

Shear stress in the rivet, $q=\dfrac{F}{A}=\dfrac{1200}{1·767}=679·12$ kg/cm².

If one section of the rivet is subjected to shear force, the rivet is said to be in single shear.

Now the ultimate shear stress of the material of the rivet

$$q_{ult}=2800\ \text{kg/cm}^2$$

Maximum shear force $=q_{ult}\times A$

$$=2800\times 1·767=4947·6\ \text{kg.}$$

Under this maximum shear force, the rivet will break along the section aa' as shown in the Fig. 1·22.

Example 1·8-2. A hole of 50 mm diameter is to be punched in a mild steel sheet of 1·6 mm thickness. If the ultimate shear strength of mild steel is 290 N/mm², determine the force required to punch the hole.

Solution. Fig. 1·23 shows a plate of 1·6 mm thickness with a punched hole of 50 mm diameter.

Area of cross section under shear

$=\pi Dt$

where D=diameter of punch

t=thickness of plate

So $A=\pi\times 50\times 1\cdot 6$

$=80\ \pi\ \text{cm}^2$

Ultimate shear strength of plate,

$q_{ult}=290\ \text{N/mm}^2$

Force required to punch the hole

$=q_{ult}\times A$

$=290\times 80\pi\ \text{N}=72885\cdot 12\text{N}=72\cdot 885\ \text{kN}.$

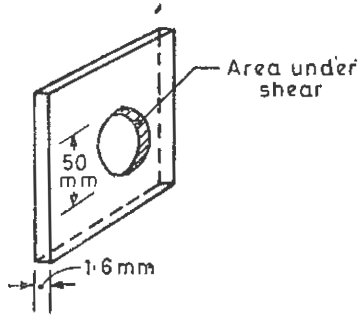


Fig. 1·23

Example 1·8-3. Fig. 1·24 shows two tie rods joined through a pin of diameter d. Tie rods are transmitting a pull of 10 kN. Determine the diameter of the pin, if the shear stress in the pin is not to exceed 100 N/mm².

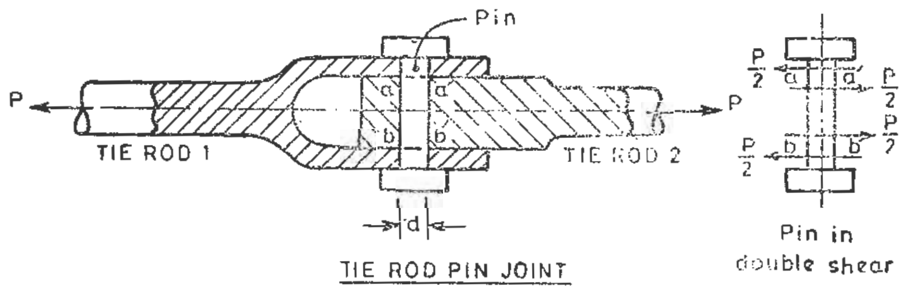


Fig. 1·24

Solution. The pull P transmitted by the tie rods acts as shear force on two planes aa' and bb' of the pin. The pull is divided equally on both the planes. This force P acts as shear force on these planes as it is parallel to the circular planes of the pin.

Shear force on each section $=\dfrac{P}{2}=\dfrac{10}{2}=5\ \text{kN}.$

Allowable shear stress in pin, $q=100\ \text{N/mm}^2$

Area of cross section, $A=\dfrac{\pi}{4}d^2$

So $$q.\frac{\pi}{4}d^2=\frac{P}{2}$$

$$100\times\frac{\pi}{4}\ d^2=5000$$

or $$d=\sqrt{\frac{5000\times 4}{100\ \pi}}$$

Diameter of the pin, $d=7\cdot 98$ mm.

Exercise 1·8-1. A rivet of diameter 25 mm joins two plates transmitting a pull of 10 kN. Determine the shear stress developed in the rivet. [**Ans.** 20·37 N/mm^2]

Exercise 1·8-2. A hole of 5 cm diameter is to be punched in a brass plate of 2 mm thickness. If the ultimate shear strength of brass is 210 N/mm^2, determine the amount of force required to punch the hole. [**Ans.** 65·97 kN]

Exercise 1·8-3. Two tie rods transmitting 2 tonnes of pull are connected by a pin. The diameter of the pin is 2·4 cm, determine the magnitude of the shear stress developed in the pin. [**Ans.** 0·221 tonne/cm^2]

1·9. VOLUMETRIC STRESS AND VOLUMERIC STRAIN

A body subjected to a stress equal in all the directions, is said to have volumetric stress or the hydrostatic stress. Consider a body at a depth h from the free surface of the liquid in

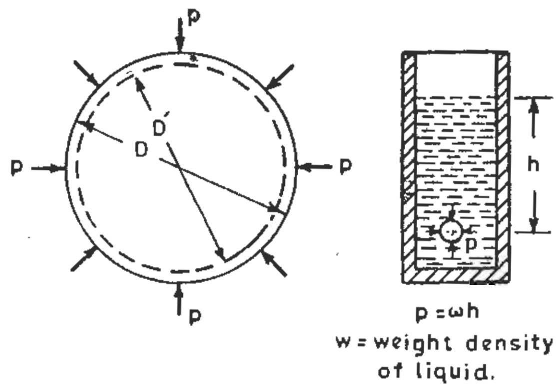


Fig. 1·25

a container. If w is the weight density of the liquid, then hydrostatic pressure $p=wh$. According to Pascal's law in Fluid Mechanics, the liquid transmits pressure equally in all the directions. *i.e.*, the intensity of the pressure on the body remains the same. Say the body is a spherical ball of volume V. When it is subjected to volumetric stress, its volume V is reduced to V'. The change in volume, $\delta V=V'-V$, (reduction in volume). The change in volume per unit volume *i.e.*, $\delta V/V$ is termed as volumetric strain, ϵ_v.

As the depth of the body increases, the magnitude of the pressure p increases and the volumetric strain also increases. Within the elastic limit, volumetric stress is directly proportional to volumetric strain.

$$p \propto \epsilon_v$$

$p=K\epsilon_v$ where K is the constant of proportionality

or $K=\dfrac{p}{e_v}$ is called the Modulus of compressibility or Bulk modulus.

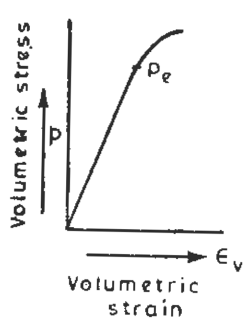


Fig. 1·26

Example 1·9-1. A spherical ball of a material 100 mm in diameter goes down to a depth of 500 metres in sea water. If the weight density of sea water $=1040$ kg/m³ and the Bulk modulus of the material is 16×10^5 kg/cm², determine the change in the volume of the ball.

Solution. Weight density, $w=1040$ kg/m³

$$=1040\times10^{-6} \text{ kg/cm}^3$$
$$=\cdot00104 \text{ kg/cm}^3$$

Depth of water upto the ball, $h=500$ metres

$$=500\times100 \text{ cm}$$

Hydrostatic pressure or volumetric stress,

$$p=wh$$
$$=\cdot00104\times500\times100=52 \text{ kg/cm}^2$$

Bulk modulus, $K=16\times10^5$ kg/cm²

Volumetric strain (reduction in volume),

$$\epsilon_v=\frac{p}{K}=\frac{52}{16\times10^5}=3\cdot25\times10^{-5}.$$

$$\epsilon_v=\frac{\delta V}{V} \quad \text{or } \delta V, \text{ change in volume}=\epsilon_v \,.\, V$$

Original volume, $$V=\frac{\pi D^3}{6}=\frac{\pi\times(100)^3}{6}=0\cdot5236\times10^6 \text{ mm}^3$$

$$\delta V=V\,.\,\epsilon_v=0\cdot5236\times10^6\times3\cdot25\times10^{-5}$$
$$=17\cdot017 \text{ mm}^3=\cdot017 \text{ cm}^3.$$

Exercise 1·9-1. A cube of 200 mm side of a material is subjected to a volumetric stress of 50 N/mm². What will be the change in its volume if the Bulk modulus of the material is 100×10^3 N/mm². [**Ans.** 4000 mm³]

1·10. TENSILE TEST ON MILD STEEL

Mild steel is the material most commonly used in machine members and in structural applications. A specimen of circular section and of the shape shown in Fig. 1·27 is clamped in the fixtures of a testing machine. Collars are provided at both the ends so that the specimen

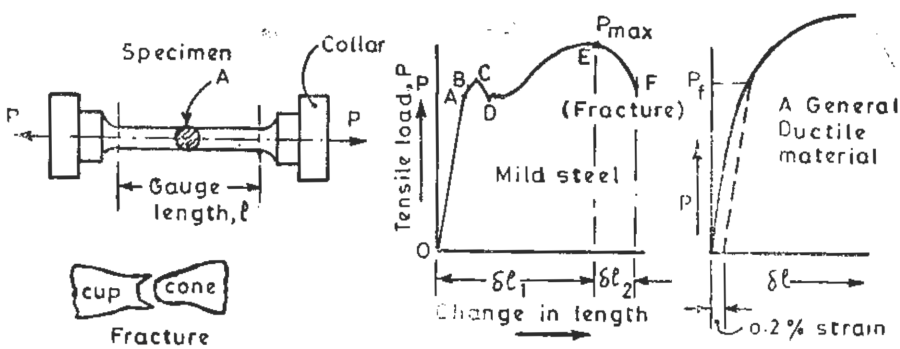


Fig. 1·27

is firmly fixed in the fixtures of the machine. The central portion, where section is uniform is called the guage length. Then the specimen is gradually extended and the internal resistance offered by the specimen (*i.e.*, tensile load) gradually increases. During the initial stage *i.e.*, when $P \propto \delta l$, extensometer is used to measure very small changes in length. After this stage, vernier scale on the machine is used to measure extension. Load and extension are simultaneously recorded till the specimen breaks into two pieces. P and δl are now plotted on a graph taking suitable scales. O to A is a straight line ; stress at A is called the limit of proportionality. The material obeys Hooke's law *i.e.*,

$$P \propto \delta l$$

or

$$\frac{P}{A} \propto \frac{\delta l}{l}$$

$$\text{stress, } f \propto \epsilon, \text{ strain}$$

$$f = E\epsilon$$ where E is the Young's modulus of elasticity.

B is the *elastic limit*, *i.e.*, if the load is removed at this stage, strain will also return to zero When the material is loaded beyond this stage, plastic deformation will occur in the material, *i.e.*, after the removal of the load, strain is not fully recovered and the residual strain (or deformation) remains in the material. From O to B, there is elastic stage and from B to the point of fracture is the plastic stage.

Beyond B *i.e.*, at the point C, there is considerable extension with decrease in internal resistance. This is called the upper yield point and the stress at this point is called the upper yield strength. At D again the internal resistance of the material increases upto the point E, *i.e.*, the maximum load point. At this point, necking takes place in the specimen and further extension takes place in the vicinity of this neck. This point is also called the point of plastic instability.

The stress at the maximum load *i.e.*, P_{max}/A is called ultimate tensile strength of the material. At the point F, the test piece breaks making a cup and cone type of fracture as shown in the figure which is a typical fracture for a ductile material. The two pieces can be joined together to find out the diameter at the neck where the specimen has broken, say the area at the neck is a.

Nominal breaking strength

$$= \frac{\text{Load at fracture}}{A \text{ (original area of cross section)}}$$

$$\text{Actual breaking strength} = \frac{\text{Load at fracture}}{a \text{ (area of neck)}}$$

Percentage reduction in areas

$$= \frac{A-a}{A} \times 100$$

$$\text{Percentage elongation} = \frac{\delta l}{l} \times 100$$

where δl is the elongation upto fracture.

The specimen of mild steel suffers a considerable increase in length till it breaks. This type of material is called a ductile material. On the contrary, a brittle material like cast iron fails after very small elongation.

In some ductile materials, which do not exhibit a definite yield point, a proof stress (at 0·2% strain) is determined to find the onset of yielding (as shown in the figure).

Percentage elongation and percentage reduction in area give estimate about the ductility of the material.

Now for different gauge lengths and different areas of cross-section the percentage elongation will be different and to draw a comparison between the ductility of various materials becomes difficult. To avoid this difficulty, similar test pieces must be tested for all the materials if the test pieces of same gauge length l and area A cannot be tested.

Barba has shown that extension upto the maximum load *i.e.*, δl_1 is proportional to the length of the specimen and extension beyond maximum load and upto the point of fracture *i.e.*, δl_2 is proportional to $\sqrt{A}$, where A is the area of cross section

$$\delta l_1 \propto l$$

or $$=bl$$

$$\delta l_2 \propto \sqrt{A}$$

$$=c\sqrt{A}$$

Total elongation, $\delta l = \delta l_1 + \delta l_2 = bl + c\sqrt{A}$

Similar test pieces will have the same ratio of $\frac{l}{\sqrt{A}} = \frac{l}{D\sqrt{0{\cdot}25\pi}}$ or $\frac{l}{D}$ ratio will be the same for similar test pieces.

The constants b and c are called the Barba's constants.

Example 1·10-1. An aluminium alloy specimen 12·5 mm diameter and 5 cm gauge length was tested under tension. During the first part of the test, following readings were recorded

Load (kg)	0	750	1000	1250	1500	1750	2000
Extension (mm)	0	·0327	·0450	·0568	·0756	·120	·216

Plot the load extension diagram and determing the following values :

1. Young's modulus of elasticity ;
2. Limit of proportionality ;
3. 0·1% proof stress.

Solution. Diameter of the specimen

$$=12{\cdot}5 \text{ mm} = 1{\cdot}25 \text{ cm}$$

Area of cross section, $A = \frac{\pi}{4}(1{\cdot}25)^2 = 1{\cdot}227 \text{ cm}^2$.

Graph is plotted between load in kg and extension in mm (taking suitable scales for both). From the graph OA is a straight line and beyond the point A load-extension curve is not a straight line. So the point A represents the limit for proportionality.

Taking a point C along the straight line

Load at C $=500$ kg

Stress $= \frac{P}{A} = \frac{500}{1{\cdot}227}$ kg/cm²

δl, extension at C $=\cdot 022$ mm

Strain $= \frac{0.022}{50}$

Young's modulus, E $= \frac{500}{1.227} \times \frac{50}{.022} = \mathbf{9.26 \times 10^5\ kg/cm^2}$.

Limit of proportionality $= \frac{1250}{1.227} = \mathbf{1018.75\ kg/cm^2}$. (See the graph)

Guage length $= 50$ mm

0·1 percent extension $= \frac{50 \times 0.1}{100} = 0.05$ mm.

At 0·05 mm extension draw a line EB parallel to the straight line OA, intersecting the load extension curve at the point B.

Load at point B $= 1838$ kg (from graph of Fig. 1·28)

So 0·1% proof stress $= \frac{1838}{1.227} = \mathbf{1498\ kg/cm^2}$.

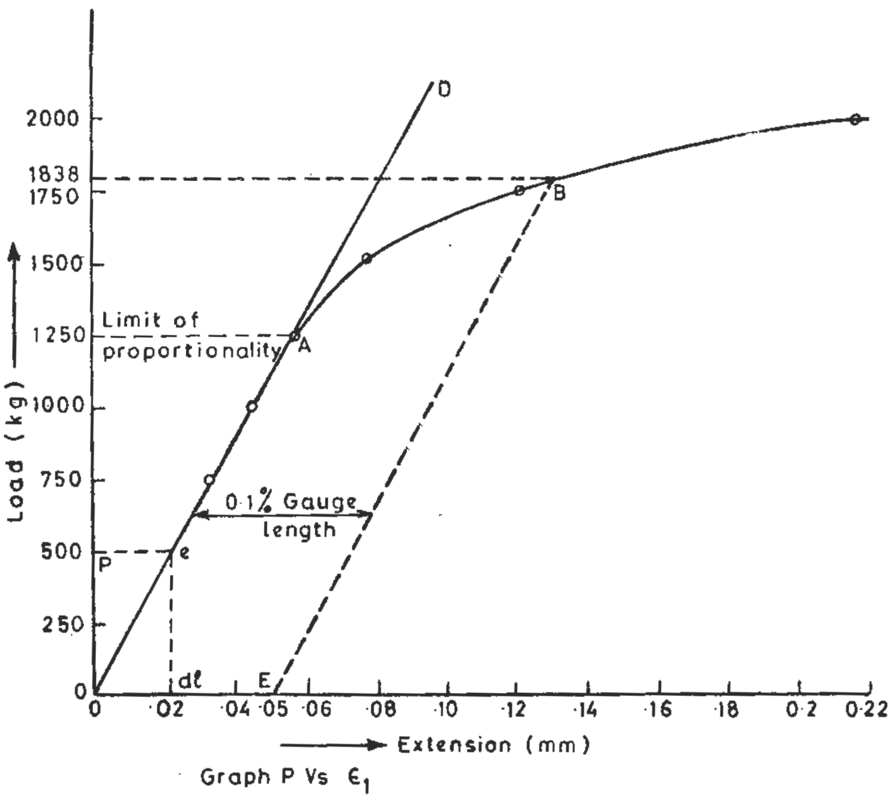


Fig. 1·28

Example 1·10-2. A specimen of an aluminium alloy 1·2 cm in diameter was tested under tension. The linear strains parallel to the application of the force P and the lateral strains measured with the help of strain gages, were as follows :

P in kg	400	800	1200	1600	2000	2400	2800	3200	3400	3600	3700	3800	3900	3950
Linear Strain ϵ_1 (1×10^{-3})	0·343	0 692	1·03	1·39	1·72	2·07	2·40	2·734	3·53	4·30	4·90	5·65	6·95	8·40
Lateral Strain ϵ_2 (1×10^{-3})	0·108	0·217	0·322	0·435	0·541	0·650	0·754	0·857	—	—	—	—	—	—

Plot graphs : (*i*) P Vs. ϵ_1 (*ii*) ϵ_1 Vs. ϵ_2 and determine (*a*) Young's modulus of elasticity, (*b*) Poisson's ratio, and (*c*) 0·1% Proof stress.

Solution. Graphs have been plotted as P Vs. ϵ_1 and ϵ_1 Vs. ϵ_2 as shown in the Fig. 1·29.

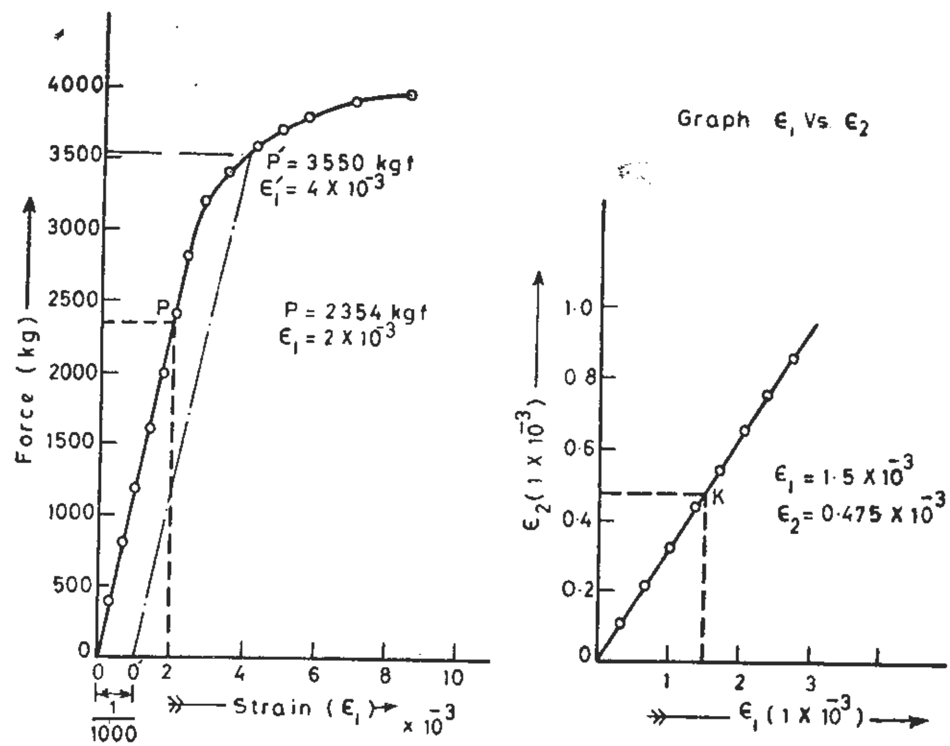


Fig. 1·29

(*a*) On the straight portion of the curve choose a point P,

where force $P=2354$ kg

Strain, $\varepsilon_1=2\times10^{-3}$

Diameter of the test piece $=1{\cdot}2$ cms

Area of cross section, $A=\frac{\pi}{4}(1{\cdot}2)^2=1{\cdot}13\ \text{cm}^2$

Young's modulus of elasticity,

$$E=\frac{P}{A}\times\frac{1}{\epsilon_1}$$

$$=\frac{2354}{1{\cdot}13}\times\frac{10^3}{2}$$

$$=\mathbf{1{\cdot}05\times10^6\ kg/cm^2}.$$

(*b*) The slope of the curve between ϵ_1 and ε_2 will give us the value of Poisson's ratio.

Choose any point K on the straight graph, where

Lateral strain, $\epsilon_2=0{\cdot}475\times10^{-3}$

Linear strain, $\epsilon_1=1{\cdot}5\times10^{-3}$

Poisson's ratio. $\frac{1}{m}=\frac{0{\cdot}475}{1{\cdot}5}\times\frac{10^{-3}}{10^{-3}}=\mathbf{0{\cdot}317}.$

(*c*) Take OO' equal to 0·1% strain or 1/1000 strain and from O' draw a line $O'P'$ parallel to the straight portion of the graph (P Vs. ϵ_1) intersecting the graph at the point P' where force $P'=3550$ kgf

Area of cross section, $A=1{\cdot}13\ \text{cm}^2$

0·1% Proof stress $=\frac{P'}{A}=\frac{3550}{1{\cdot}13}=\mathbf{3160\ kg/cm^2}.$

Example 1·10-3. A mild steel test piece of 20 cm gauge length is marked off in 2 cm length and tested to destruction. The extension in each marking measured from one end is equal to 0·32 cm, 0·36 cm, 0·46 cm, 0·50 cm, 0·96 cm, 0·84 cm, 0·52 cm, 0·44 cm, 0·36 cm and 0·32 cm.

By using gauge lengths of 8 cm, 12 cm, 16 cm and 20 cm, respectively, explain the effect of gauge length on the percentage elongation.

Solution. Gauge length, $l_1=20$ cm

Change in length, $\delta l_1=5{\cdot}08$ cm

Percent elongation, $=\frac{5{\cdot}08}{20}\times100=25{\cdot}4\%$

Gauge length, $l_2=16$ cm

Change in length, $\delta l_2=5{\cdot}08-0{\cdot}32-0{\cdot}32=4{\cdot}44$ cm (considering the middle portion)

Percent elongation $=\frac{4\cdot44}{16}\times100=\mathbf{27\cdot75\%}$.

Gauge length, $l_3=12$ cm

Change in length, $\delta l_3=4\cdot44-0\cdot36-0\cdot36=3\cdot72$ cm.

Percent elongation $=\frac{3\cdot72}{12}\times100=\mathbf{31\%}$.

Gauge length, $l_4=8$ cm

Change in length, $\delta l_4=3\cdot72-\cdot46-\cdot44=2\cdot82$ cm

Percent elongation $=\frac{2\cdot82}{8}\times100=\mathbf{35\cdot25\%}$.

It can be concluded while viewing the effect of gauge length on the percentage elongation, that the percentage elongation goes on increasing as the gauge length is decreased.

Example 1·10-4. A round specimen of wrought iron, diameter 1·25 cm and gage length equal to 10 cm was tested in tension upto fracture. The observations taken were as follows :

Load at the yield point	=2·95 tonnes
Maximum load	=4·40 tonnes
Load at the time of fracture	=3·70 tonnes
Diameter at the neck	=0·92 cm
Extension upto the maximum load point	=2·1 cm
Total extension	=2·85 cm

From the above data determine (*a*) yield strength (*b*) ultimate strength (*c*) actual breaking strength (*d*) percent elongation for a test piece of gauge length 15 cm and diameter 2 cm.

Solution. Diameter of the test piece, $d=1\cdot25$ cm.

Area of cross section, $A=\frac{\pi}{4}\times(1\cdot25)^2=1\cdot226$ cm².

$$\text{Yield strength}=\frac{\text{Load at the yield point}}{A}$$

$$=\frac{2\cdot95}{1\cdot226}=2\cdot41\ \text{tonnes/cm}^2.$$

$$\text{Ultimate strength}=\frac{\text{Maximum load}}{A}$$

$$=\frac{4\cdot40}{1\cdot226}=3\cdot59\ \text{tonnes/cm}^2.$$

$$\text{Actual breaking strength}=\frac{\text{Load at the time of fracture}}{\text{Area at the neck}}$$

$$=\frac{3\cdot7}{\frac{\pi}{4}(0\cdot92)^2}$$

$$=5\cdot57\ \text{tonnes/cm}^2.$$

Extension in the test piece $= bl + c\sqrt{A}$

$=$ extension upto the point of the maximum load + further extension

$2.85 = 2.1 + 0.75$

$\therefore$ Constants, $b = \frac{2.1}{10} = 0.21$

and $c = \frac{0.75}{\sqrt{A}}$

$= \frac{0.75}{\sqrt{1.226}} = 0.68.$

(*b*) Results in another piece, whose gauge length is 15 cm and diameter is 2 cm.

Since yield strength, ultimate strength, actual breaking strength and constants *b*, *c* are the properties of the material irrespective of the dimensions of the test piece, therefore

Yield strength $= 2.41$ tonnes/cm²

Ultimate strength $= 3.59$ tonnes/cm²

Actual breaking strength $= 5.57$ tonnes/cm²

$$\text{Percentage elongation} = \left(\frac{bl + c\sqrt{A}}{l}\right) \times 100$$

$$= \left(0.21 + \frac{0.68\sqrt{\pi}}{15}\right) \times 100 \qquad \text{Since } A = \pi \text{ cm}^2.$$

$$= \mathbf{29.03\%}.$$

Exercise 1.10-1. (*a*) In a tensile test on a specimen 12.5 mm diameter and 200 mm gauge length, the following readings are observed :—

Force in kg	500	1000	1500	2000	2500	3000	3500	4000
Extension in $.001 \times$ mm	38	75	112.5	149	188	226	264	305

Determine the Young's modulus of elasticity.

(*b*) Afterwards the specimen was tested to destruction and the maximum load recorded was 6000 kg. The diameter at the neck was 7.5 mm and the length between the gauge marks was 260 mm. Determine the ultimate strength, percentage reduction in area and percentage elongation. [**Ans.** (*a*) 2.16×10^6 kg/cm² (*b*) 4880 kg/cm², 63.9%, 30%]

Exercise 1.10-2. In a tensile test upto destruction on the specimen of mild steel of rectangular section 5 cm × 1 cm, the extensions measured on successive 1 cm length on 16 cm gauge length are as follows :—

0.17, 0.17, 0.18 ; 0.18 ; 0.19, 0.21, 0.27, 0.29 ; 0.63, 0.64, 0.28, 0.26 ; 0.19, 0.19, 0.17, 0.17 cm.

Estimate the percentage elongation of a round bar of the same material having a length 10 times the diameter. [**Ans.** Barba's constants $b = 0.17$, $c = 0.653$; 22.82%]

Exercise 1·10-3. A metallic specimen of diameter 15 mm and gauge length 100 mm is tested under tension upto fracture. The load at upper yield point is 6020 kg and at breaking point it is 6400 kg while the maximum load is 9300 kg. Distance between the gauge marks after the test is 136·7 mm, and the minimum diameter at fracture is 9·15 mm. Determine the following :—

(*a*) Yield strength.

(*b*) Ultimate tensile strength.

(*c*) Percentage elongation.

(*d*) Percentage reduction in area.

(*e*) Nominal and actual stress at fracture.

[**Ans.** (*a*) 3406·9 kg/cm² (*b*) 5263·16 kg/cm² (*c*) 36·7% (*d*) 62·81% (*e*) 3621·96 kg/cm², 9741·25 kg/cm²]

Exercise 1·10-4. A test piece 10 mm diameter was marked with 5 cm and 10 cm gauge lengths and after fracture (which occured near the middle section) the two lengths measured were 6·5 cm and 12·50 cm. Calculate the probable elongation for a test piece of the same material with gauge length 15 cm and diameter 20 mm. [**Ans.** 3·999 cm]

1·11. STRAIN ENERGY AND RESILIENCE

In the previous articles we have studied about the normal stress and normal strain ; shear stress and shear strain ; volumetric stress and volumetric strain. With in the elastic limit, stress is directly proportional to strain.

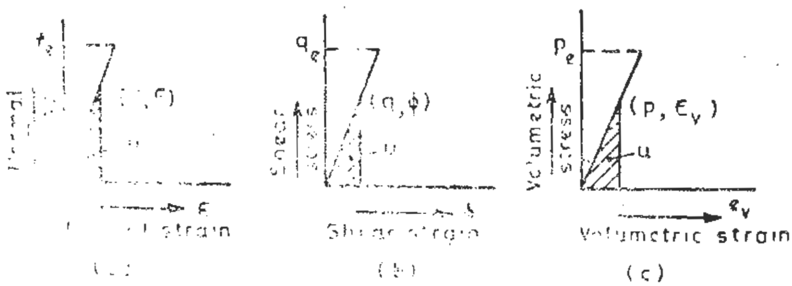


Fig. 1·30

Say a bar carries a normal stress f (tensile or compressive) producing a normal strain ϵ (extension or contraction). Then

Strain energy per unit volume, $u = \frac{1}{2} f \cdot \epsilon$

$$\text{but } \epsilon = \frac{f}{E} = \frac{f^2}{2E}$$

Total strain energy, $U = \frac{f^2}{2E} \times$ volume of the specimen.

(has the units of work done—**N** mm or **kg** cm etc. etc.).

The strain energy, U absorbed by the specimen is also called Resilience. Because if a material is loaded producing stress within the elastic limit, then whatever energy is absorbed during loading, same energy is recovered during unloading. Machine members like springs possess the property of resilience. As in the case of an internal combustion engine, during the opening of inlet or exhaust valve, spring on the stem of the valve is compressed and when the spring releases its energy the valve is closed.

The maximum strain energy absorbed by the body upto its elastic limit is termed as Proof Resilience.

Proof resilience $= \dfrac{f_e^2}{2E} \times \text{Volume} = \dfrac{f_e^2 \,.\, A \,.\, L}{2E} = \dfrac{1}{2}(f_e A)\left(\dfrac{f_e \,.\, L}{E}\right)$

where $f_e \,.\, A$ = Load at the elastic limit ; P_e

$\dfrac{f_e \,.\, L}{E}$ = change in length upto the elastic limit, δL_e

Proof resilience $= \frac{1}{2} P_e \,.\, \delta L_e$.

Proof resilence per unit volume is called the Modulus of resilience.

i.e. Modulus of resilience $= \dfrac{f_e^2}{2E}$ where f_e is the stress at the elastic limit as shown in Fig. 1·30 (*a*).

Similarly if one studies the relationship between shear stress q and shear strain ϕ ; between volumetric stress p and volumetric strain ϵ_v.

Shear strain energy per unit volume,

$$u_s = \frac{1}{2}\, q\, \phi = \frac{q^2}{2G}$$

Total shear strain energy,

$$U_s = \frac{q^2}{2G} \times V.$$

Where q is the shear stress within the elastic limit and G is the Modulus of rigidity or Shear modulus.

Volumetric strain energy per unit volume,

$$u_v = \frac{1}{2}\, p \,.\, \epsilon_v = \frac{p^2}{2K}$$

Total volumetric strain energy,

$$U_v = \frac{p^2}{2K} \,.\, V.$$

Where p is the pressure on the body within its elastic limit and K is the Bulk modulu and V is the volume of the body.

Example 1·11-1. A round bar of 15 mm diameter is subjected to a tensile force o 10 kN. If length of the bar is 200 mm determine.

(*i*) Strain energy per unit volume.

(*ii*) Total strain energy.

Given elastic limit stress for the material

$= 260$ N/mm²

Young's modulus of elasticity, $E = 210 \times 1000$ N/mm²

Solution. Diameter of the bar $=15$ mm

Area of cross section,

$$A=\frac{\pi}{4}\,(15)^2$$

$$=176{\cdot}715\ \text{mm}^2$$

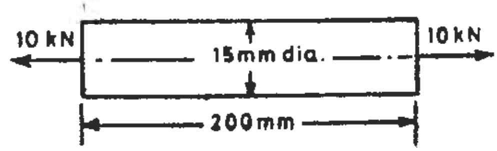


Fig. 1·31

Length of the bar,
$L=200$ mm

Volume of the bar, $V=176{\cdot}715\times200=35343\ \text{mm}^3$

Stress in the bar, $=\dfrac{10\times1000}{176{\cdot}715}=56{\cdot}59\ \text{N/mm}^2$ (less than the elastic limit stress of 260 N/mm²)

Young's modulus, $E=210\times10^3\ \text{N/mm}^2$

Strain energy per unit volume, $u=\dfrac{f^2}{2E}=\dfrac{(56{\cdot}59)^2}{2\times210\times10^3}$

$$=7{\cdot}62\times10^{-3}\ \text{N/mm}^2 \text{ per unit volume}$$

Total strain energy absorbed by the bar,

$$U=\frac{f^2}{2E}\times\text{volume}$$

$$=7{\cdot}62\times10^{-3}\times35343=269{\cdot}3\ \text{Nmm}=0{\cdot}269\ \text{Nm.}$$

Example 1·11-2. A rectangular block of 80 mm$\times$60 mm$\times$40 mm fixed at the bottom edge is subjected to a shear force of 240 kN at the top surface as shown in Fig. 1·32. Considering that shear stress is proportional to shear strain, determine.

(1) Shear angle ϕ.

(2) Shear strain energy for unit volume.

(3) Total shear strain energy for the block.

Given G, modulus of rigidity $=840\times10^2\ \text{N/mm}^2$

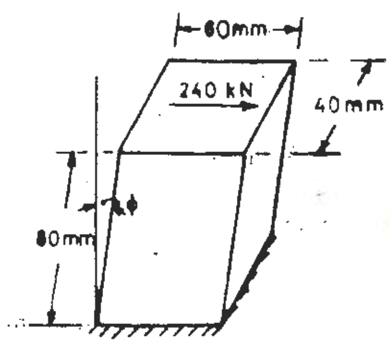


Fig. 1·32

Solution. A, Area of the surface on which shear force is applied

$$=60\times40=2400\ \text{mm}^2$$

Shear force, $F=240$ kN

Shear stress, $q=\dfrac{F}{A}=\dfrac{240\times1000}{2400}=100\ \text{N/mm}^2$

Modulus of rigidity $=840\times10^2\ \text{N/mm}^2$

Shear strain, $\phi=\frac{q}{G}=\frac{100}{840\times 100}$

$=\frac{1}{840}$ radian $=\frac{1}{840}\times\frac{180}{\pi}$ degree

$=\cdot 068°$.

One can realise that the shear angle ϕ is very very small.

Shear strain energy per unit volume,

$$u_s=\frac{q^2}{2G}=\frac{100\times 100}{2\times 840\times 100}$$

$=\cdot 059$ N/mm² per unit volume.

Volume of the block, $V=80\times 60\times 40=192\times 10^3$ mm³

Total shear strain energy,

$$U_s=\frac{q^2}{2G}\times V=\cdot 0590\times 192\times 10^3$$

$=11\cdot 424$ k Nmm $=11\cdot 424$ Nm.

Example 1·11-3. A spherical ball of aluminium of diameter 100 mm is subjected to a hydrostatic pressure of 90 N/mm². It has been observed that pressure is directly proportional to the change in volume. Determine

(*i*) Volumetric strain

(*ii*) Volumetric strain energy per unit volume

(*iii*) Total volumetric strain energy.

Given Bulk modulus for aluminium $=65\times 10^3$ N/mm²

Solution. Pressure,

$p=90$ N/mm²

Bulk modulus, K

$=65\times 10^3$ N/mm²

Volume of the sphere,

$$V=\frac{\pi D^3}{6}=\frac{\pi\times(100\text{ mm})^3}{6}$$

$=0\cdot 5236\times 10^6$ mm³.

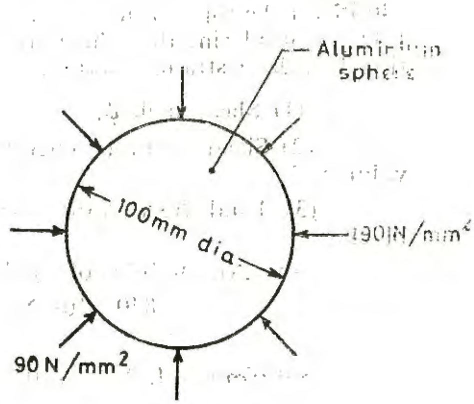


Fig. 1·33

(*i*) Volumetric strain

$$\epsilon_v=\frac{p}{K}$$

$$=\frac{90}{65\times 10^3}=1\cdot 3846\times 10^{-3}$$

(*ii*) Volumetric strain energy per unit volume,

$$u_v=\frac{p^2}{2K}=\frac{90\times 90}{2\times 65\times 10^3}$$

$=\cdot 0623$ N/mm² per unit volume.

(*iii*) Total volumetric strain energy,

$$U_v = \frac{p^2}{2K} \times V = {\cdot}0623 \times {\cdot}5236 \times 10^6$$
$$= 32620 \text{ Nmm} = 32{\cdot}62 \text{ Nm}.$$

Exercise 1·11-1. A circular bar of copper 16 mm diameter and length 200 mm is subjected to a compressive force such that its length is reduced by 0·1 mm. Determine

(*i*) Stress developed in bar

(*ii*) Strain energy per unit volume

(*iii*) Total strain energy for the bar.

Given $E = 100 \times 10^3$ N/mm²

[**Ans.** (*i*) 50 N/mm² (*ii*) $12{\cdot}5 \times 10^{-3}$ N/mm² (*iii*) 502·650 **Nmm**]

Exercise 1·11-2. A rectangular block of aluminium fixed on one plane is subjected to a shear stress q on the opposite plane. If the shear angle is 1/20 degree, determine

(*i*) Shear stress q

(*ii*) Shear strain energy per unit volume

Given $G = 500 \times 10^2$ N/mm²

[**Ans.** (*i*) 43·633 N/mm² (*ii*) 0·190 N/mm²]

Exercise 1·11-3. A spherical ball of volume $0{\cdot}5 \times 10^6$ mm³ is subjected to pressure p such that its volume is reduced by 200 mm³. If the bulk modulus $= 170 \times 10^3$ N/mm², determine

(*i*) Pressure p

(*ii*) Strain energy per unit volume

(*iii*) Total volumetric strain energy.

[**Ans.** (*i*) 68 N/mm² (*ii*) 13·6 N/mm² (*iii*) 6800 N mm]

1·12. SUDDEN LOAD

When a force P on a body is gradually increased, its change in length δl also gradually increases and the strain energy absorbed $= 1/2\ P dl$. But when whole of the load suddenly acts, change in the length δl also suddenly takes place. This type of load is called a sudden load. As an example say a load W is being lowered on to a platform and at the instant when the load is very near the platform, suddenly the rope breaks, then whole of the load W suddenly acts on the platform producing δl change in its length.

The load may act suddenly but the internal resistance of the body gradually increases from zero to the maximum value

Work done on the body by sudden load $= W.\delta l$

Strain energy of the body $= 1/2\ R.\delta l$ where R is the internal resistance

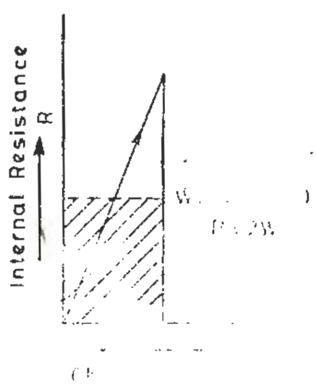


Fig. 1·34

For equilibrium $1/2\ R.\delta l = W.\delta l$

Internal resistance, $R = 2\ W$

Stress developed due to sudden load

$$f_{sudden} = \frac{R}{A} = \frac{2W}{A}$$

where A is the area of cross section

$$f_{sudden} = 2\ f_{gradual}$$

where $\frac{W}{A}$ is the stress when load W is gradually applied.

Example 1·12-1. Water under a pressure of 6 N/mm² is suddenly admitted on a plunger of diameter 100 mm. The plunger is attached to a connecting rod 25 mm diameter and 5 metres long. Determine the value of the stress developed suddenly and deformation of the rod. $E = 210 \times 10^3$ N/mm².

Solution.

Water pressure, $p = 6$ N/mm³

Plunger diameter, $D = 100$ mm

Area of cross-section of plunger,

$$A = \frac{\pi}{4}\ (100)^2 = 7854 \text{ mm}^2$$

Sudden load on plunger,

$$W = pA = 6 \times 7854 = 47124 \text{ N}$$

Diameter of connecting rod

$= 25$ mm

Area of cross section of connecting rod

$$A' = \frac{\pi}{4}\ (25)^2 = 490{\cdot}875 \text{ mm}^2$$

Stress developed in rod due to sudden load W

$$f_{sudden} = \frac{2W}{A'} = \frac{47124 \times 2}{490{\cdot}875} = 192 \text{ N/mm}^2$$

Length of the rod $= 5000$ mm

Sudden deformation in rod

$$= \frac{f_{sudden}}{E} \times l$$

$$= \frac{192}{210 \times 10^3} \times 5000 = 4{\cdot}57 \text{ mm.} \quad \textbf{Ans.}$$

Exercise 1·12-1. An axial load of 20 tonnes is suddenly applied on a bar of 8 cm diameter. Find

(*i*) Maximum instantaneous stress

(*ii*) Maximum instantaneous elongation if bar is 2m long.

Given $E = 2080$ tonnes/cm²

[**Ans.** (*i*) 0·7958 tonne/cm² (*ii*) ·0765 cm]

1·13. IMPACT LOADS

Whenever a load with some velocity acts on a body, it is said to be an impact load or a shock load. In other words the load possesses some kinetic energy which is utilised in the deformation of the body. As an example, a blacksmith gives blow or the shock load to the hot iron with the help of a hammer striking the iron with some velocity.

Consider a rod of area of cross section A and length L fixed at one end and having a collar at the other end. A weight W sliding freely on the bar is dropped on to the collar, through a height h as shown in the Fig. 1·35. The weight arrested by the collar produces an instantaneous elongation δl in the bar. Say at this instant stress developed in the bar is f_i and E is the Young's modulus of elasticity of the material of the bar.

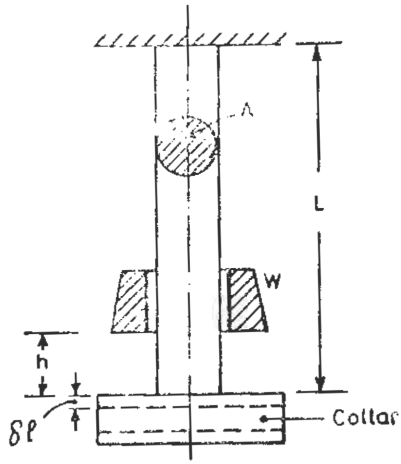


Fig. 1·35

Potential energy lost by the weight

$$=W(h+\delta l)$$

Strain energy absorbed by the bar

$$=\frac{f_i^2}{2E}\times\text{Volume of the bar}$$

$$=\frac{f_i^2}{2E}\times AL$$

For equilibrium

$$W(h+\delta l)=\frac{f_i^2}{2E}\times AL$$

$$Wh+W\delta l=\frac{f_i^2}{2E}\times AL. \quad \ldots(1)$$

Instantaneous elongation can be expressed in terms of instantaneous stress as follows :

$$\delta l=\text{instantaneous strain }(\epsilon_i)\times l$$

$$=\frac{f_i}{E}\times L$$

Equation (1) can now be written as

$$Wh+\frac{Wf_iL}{E}=\frac{f_i^2}{2E}\times AL \quad \ldots(2)$$

Multiplying throughout by $\frac{2E}{AL}$ we get

$$\frac{2WhE}{AL}+\frac{2Wf_i}{A}=f_i^2 \quad \ldots(3)$$

The value of f_i will be

$$f_i = +\frac{W}{A} \pm \sqrt{\frac{W^2}{A^2} + \frac{2hEW}{AL}}$$

$$= \frac{W}{A} + \sqrt{\frac{W^2}{A^2} + \frac{W}{A} \times \frac{2hE}{L}}$$

(Negative sign is inadmissible as the stress can not be compressive when the bar gets elongated)

$$f_i = \frac{W}{A} + \frac{W}{A}\sqrt{1 + \frac{2EAh}{WL}}$$

Instantaneous stress, $f_i = \frac{W}{A}\left[\sqrt{1 + \frac{2EAh}{WL}}\right]$

Instantaneous elongation

$$\delta l_i = \frac{f_i}{E} \times L$$

In case $h=0$, it becomes a sudden load

$$f_{sudden} = \frac{W}{A}\left[1 + \sqrt{1+0}\right] = \frac{2W}{A} \text{ as proved in the article 1·12.}$$

Example 1·13-1. A steel bar 10 mm in diameter and 150 cm long is stressed by a weigth of 12 kg dropping freely through a height of 5 cm before commencing to stretch the bar. Find the maximum instantaneous stress and instantaneous elongation.

$E = 2100$ tonnes/cm².

Solution. Diameter of the bar,

$d = 10$ mm $= 1$ em

Area of cross section, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times (1)^2 = 0·7854 \text{ cm}^2$

Length of the bar, $L = 150$ cm

Height through which load falls,

$h = 5$ cm

Falling load, $W = 12$ kg

Young's modulus, $E = 2100 \times 1000 \text{ kg/cm}^2$

Maximum instantaneous stress,

$$f_i = \frac{W}{A}\left[1 + \sqrt{1 + \frac{2EAh}{WL}}\right]$$

$$= \frac{12}{0·7854}\left[1 + \sqrt{1 + \frac{2 \times 2100 \times 1000 \times ·7854 \times 5}{12 \times 150}}\right]$$

$$= 15·279\,[1 + \sqrt{1 + 9163}]$$

$$= 15·279\,[1 + 95·73] = 1477·93 \text{ kg/cm}^2$$

Elongation in the bar, $\delta l_t = \frac{f_t}{E} \times L$

$$= \frac{1477 \cdot 93}{2100 \times 1000} \times 150 = \cdot 1055 \text{ cm.}$$

Exercise 1·13-1, A brass bar of cross sectional area 80 mm² and 2000 mm long is fixed at one end and at the other end of the bar, a collar is provided. A weight 50 N drops freely through a height of 25 mm on to the collar. Determine the instantaneous stress and instantaneous elongation developed in the bar.

$E_{brass} = 100 \times 1000$ N/mm². [**Ans.** 40·158 N/mm², ·803 mm]

1·14. STRESS CONCENTRATION IN MEMBERS UNDER TENSILE FORCE

Uptil now we have studied that stress is defined as the load per unit area and we have assumed that stress is uniform. But if a rapid change in cross section occurs along the length of the member as shown in Fig. 1·36, the stress will no longer be uniform. The Fig.

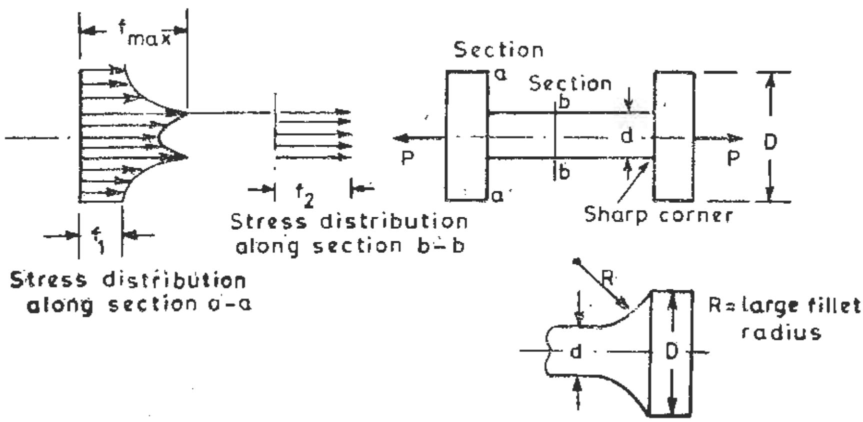


Fig. 1·36

shows a bar of two different diameters D and d subjected to a tensile force P. As per the definition, the stress in section 1 is $4P/\pi D^2$ and the stress in section 2 is $4P/\pi d^2$ but what about the stress along the section a-a where the area of cross section has abruptly changed from $\pi/4\ D^2$ to $\pi/4\ d^2$. The stress distribution along this section is not uniform and is of the shape shown in the Fig. There are various methods such as photoelasticity, strain gauges, theory of elasticity, finite element method which can be employed to determine this stress distribution. But the treatment of these topics is beyond the scope of this book. The maximum stress occurs at sharp corners as shown or in other words stress is concentrated in sharp corners, fillets etc. The bar will tend to fail along this sharp corner under tensile loading.

Stress concentration factor

$$= \frac{\text{Maximum stress}}{\text{Average stress at minimum section}}$$

$$= \frac{f_{max}}{f_2} \text{ (as shown)}$$

To avoid or to reduce the effect of stress concentration, large fillet radius should be provided at the corner so that the stress gradually increases from f_1 to f_2.

(*a*) **Small elliptical hole in a plate.** Stress distribution along the axis of the ellipse has been theoretically determined using the principles of theory of elasticity. Maximum stress would occur at the ends of the semi major axis a, as shown in the Fig. 1·37.

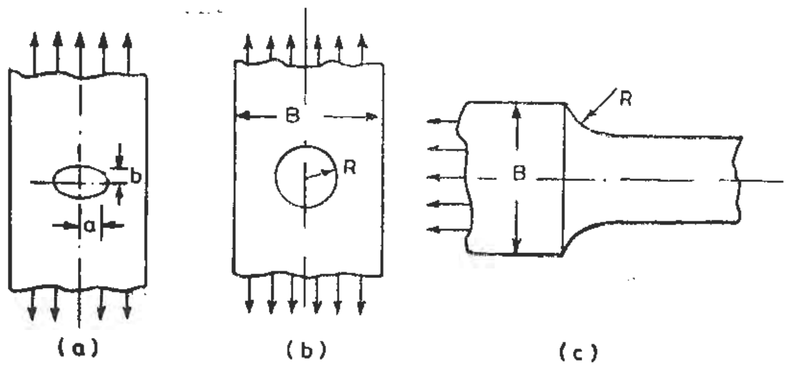


Fig. 1·37

Stress concentration factor,

$$SCF=1+\frac{2a}{b}$$

where a=Semi major axis

b=Semi minor axis of the ellipse.

Note that SCF increases rapidly as b goes on decreasing *i.e.*, when instead of an elliptical hole, there is a fine crack, SCF becomes very high. Similarly as a decreases, SCF goes on decreasing *i.e*, a longitudinal crack (along the direction of load) has no stress concentration.

(*b*) **Circular hole at the centre of the plate.** Again the results have been obtained theoretically by theory of elasticity and experimentally by photoelasticity for the SCF at the edge of a circular hole in a plate. Fig. 1·37 (*b*).

R/B	0·167	0·1	·05
SCF	2·25	2·46	2·97

R is the radius of the hole, B is the width of the plate. In this case the limiting value of SCF is 3 *i.e.*, for a large plate having a small circular hole.

(*c*) **Edge fillets.** Theory of photoelasticity has been used to determine SCF in the case of a bar having a fillet radius R at the corner. Fig. 1·37 (*c*).

R/B	0·333	0·222	0·143	0·083
SCF	1·25	1·5	1·65	1·8

These values of *SCF* for various cases have been obtained by considering that the material is loaded within the elastic region. But for a ductile material, if the yielding occurs then redistribution of stresses takes place and the *SCF* is reduced.

1·15. FACTOR OF SAFETY

We have learnt uptil now that the stress in a member is calculated on the basis of (*i*) type, magnitude and position of the applied load (*ii*) dimensions of the member and (*iii*) properties of the material. But in practice, all these factors are not accurately known and as a result correct value of stress occuring on a member is not determined.

There are various types of loads such (*a*) static load or dead load or a gradually and slowly applied load (*b*) sudden load (*c*) impact load (*d*) alternating load, when the magnitude of the load changes cyclically. Many a times the load is assumed to be concentrated at a point and while in some applications load is assumed to be distributed over an area.

In some machine members there may be abrupt change in dimension, sometimes under the compulsion of design details and it is not possible to correctly find out the effect of stress concentration.

Moreover the material is assumed to be isotropic and homogeneous while the material may be containing internal defects, sharp cuts etc. Such as in casting, the material may have internal defects such as blow holes which may act as areas of stress concentration.

The material may not be obeying Hooke's law. Brittle materials like cast iron, concrete etc., if they are assumed to obey Hooke's law, then serious errors may be introduced into the calculations for the stresses.

All the factors explained above influence the stress-strain behaviour of the material and in many applications determination of the correct value of stress occuring at a critical section becomes somewhat complicated. Under these circumstances a factor of safety is taken and an allowable stress on the material is decided such thateven when all these factors have not been accounted for, the material is not going to fail.

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Working stress or allowable stress}}.$$

For general engineering applications, the factor of safety is taken from 3 (for dead loads) to 12 (for shock loads). The value of the ultimate stress is determined through a test on the specimen of the material.

Many a times the factor of safety is decided on the basis of yield stress because at the yield point, the material ceases to obey Hooke's law and on the removal of the load, irrecoverable strains remain in the material. Moreover the material is assumed to fail at the yield point because of considerable strain without much increase in load.

Example 1·15-1. The ultimate tensile strength of mild steel is 450 N/mm². A tie bar of equal-angle section has to carry an axial load of 200 kN. The mean thickness of angle sector is 10 mm. Taking factor of safety as 5, determine the dimensions of the angle-section.

Solution. Ultimate tensile strength

$= 450 \text{ N/mm}^2$

Factor of safety $= 5$

Working stress $= \dfrac{450}{5}$

$= 90 \text{ N/mm}^2$

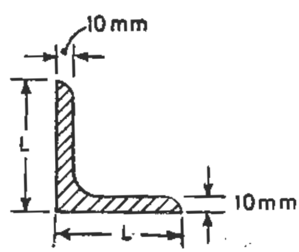


Fig. 1·38

Say the length of equal angle section

$= L$ mm(as shown in Fig. 1·38)

A, Area of cross section

$$= L\times 10+(L-10)\times 10 \text{ mm}^2 \quad \text{(neglecting rounding of corners)}$$
$$= 20\,L-100 \text{ mm}^2$$

P, Axial load on tie bar

$$= 200 \text{ kN} = 200\times 1000 \text{ N}$$

Working stress $= \dfrac{P}{A} = \dfrac{200\times 1000}{20L-100} = 90$ (as given)

or ; $200{,}000 = 1800\,L - 90{,}00$

$$1800\,L = 200{,}000 - 9{,}000 = 191{,}000$$

$$L = \frac{191{,}000}{1800} = 106{\cdot}11 \text{ mm say } 107 \text{ mm.}$$

The equal angle section is 107 mm × 107 mm × 10 mm.

Exercise 1·15-1. A short strut is of section 150 mm × 80 mm × 10 mm. If it carries a load of 150 kN, find the compressive stress in the strut. If the ultimate crushing stress for the material is 500 N/mm², what is the factor of safety ? The section is shown in the Fig. 1·39.

[**Ans.** 68·18 N/mm² ; 7·33]

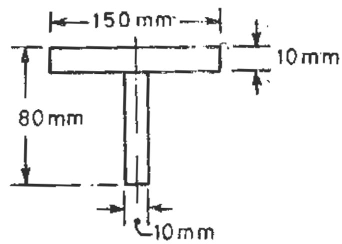


Fig. 1·39

Problem 1·1. Two parts of a certain machine component are joined by a rivet of 2 cm diameter. Determine the shear and normal stresses in the rivet if the axial force $P = 1$ tonne, and the angle of the joint is 50° to the axis of the load (See Fig. 1·40).

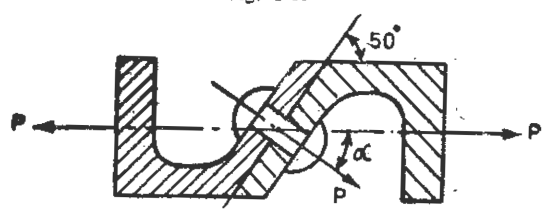


Fig. 1·40

Solution. Force, $P = 1000$ kg

Angle $\alpha = 90° - 50° = 40°$

Normal force on the rivet,

$$P_n = P\cos\alpha = 1000\times\cos 40° = 765 \text{ kg}$$

Shear force on the rivet,

$$P_s = P\sin\alpha = 1000\times\sin 40° = 642{\cdot}5 \text{ kg.}$$

Diameter of the rivet, $d = 2$ cm.

Area of cross section of the rivet,

$$A = \frac{\pi}{4}\,d^2 = 3{\cdot}14 \text{ cm}^2$$

Normal stress in the rivet,

$$f_n = \frac{P_n}{A} = \frac{765}{3{\cdot}14} = \mathbf{243\ kg/cm^2}$$

Shear stress in the rivet, (in single shear)

$$f_s = \frac{P_s}{A} = \frac{642{\cdot}5}{3{\cdot}14} = \mathbf{204{\cdot}5\ kg/cm^2}.$$

Problem 1·2. A solid circular shaft of diameter 120 mm has a collar of 20 mm thickness as shown in Fig. 1·41. The shaft is subjected to a compresssive force of 250 kN. Determine

(*i*) Compressive stress developed in the shaft.

(*ii*) Shear stress developed in collar section.

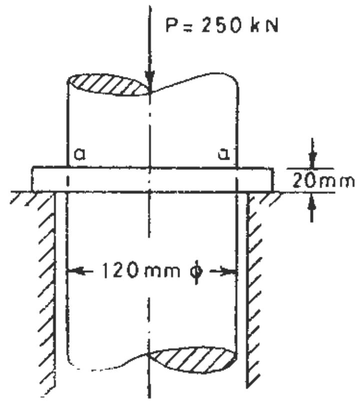


Fig. 1·41

Solution. Diameter of the shaft

$$= 120 \text{ mm}$$

Area of cross section of shaft

$$= \frac{\pi}{4} (120)^2$$

$$= 11309{\cdot}76 \text{ mm}^2$$

Compressive force $= 250 \times 1000$ N

Compressive stress in shaft

$$= \frac{250000}{11309{\cdot}76} = 22{\cdot}10 \text{ N/mm}^2.$$

This force P acts as a shear force along the circular section aa of diameter 120 mm and thickness 20 mm.

Area of cross section of collar under shear

$$= \pi \times 120 \times 20 = 7539{\cdot}8 \text{ mm}^2$$

Shear force $= 250 \times 1000$ N

Shear stress in collar section

$$= \frac{250000}{7539{\cdot}8} = 33{\cdot}15 \text{ N/mm}^2.$$

Problem 1·3. The Fig. 1·42 shows a tie bar 25 mm in diameter carrying a load which causes a stress of 120 N/mm². The tie bar is attached to a cast iron bracket with the help of 4 bolts which can be stressed only upto 90 N/mm² ? Determine the diameter of the bolt.

Solution. Tie bar is attached to the cast iron bracket with the help of four bolts as shown.

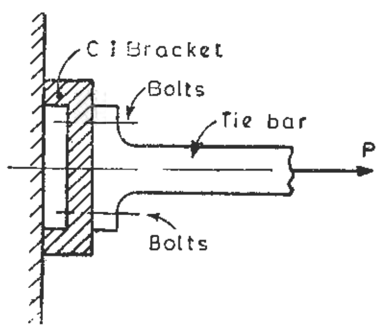


Fig. 1·42

Stress in tie bar $= 120$ N/mm²

Diameter of tie bar

$$= 25 \text{ mm}$$

Area of cross section of tie bar

$$= \frac{\pi}{4} (25)^2$$

$$= 490{\cdot}875 \text{ mm}^2$$

Tensile force in tie bar,

$$P = 120 \times 490{\cdot}875$$

$$= 58905 \text{ N}$$

Tensile force on each bolt

$$=\frac{P}{4}=\frac{58905}{4}=14726{\cdot}25 \text{ N.}$$

Allowable stress in each bolt

$$=90 \text{ N/mm}^2$$

So area of cross section of each bolt

$$=\frac{14726{\cdot}25}{90}=163{\cdot}625 \text{ mm}^2$$

Say diameter of each bolt $=d$

So $\frac{\pi}{4}\,d^2=163{\cdot}625$

$$d^2=\frac{163{\cdot}625\times 4}{\pi}=208{\cdot}333$$

Diameter of each bolt,

$$d=14{\cdot}434 \text{ mm.}$$

Problem 1·4. Three wooden pieces of square cross section 4 cm×4 cm are glued together as shown in Fig. 1·43. The outer surfaces of the assembly are glued to the foundation. What will be the average shearing stress in the glued joints if the horizontal force P=4000 kg.

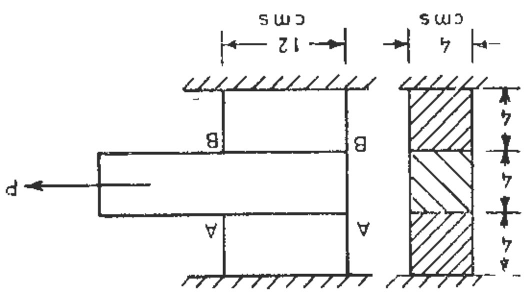


Fig. 1·43

Solution The cross sections *A-A* and *B-B* carry the shear stresses due to the load P.

Area of cross section of the glued joints

$$=12\times 4=48 \text{ cm}^2$$

Shear force, $P=4000$ kg

Average shear stress in the glued joints

$$=\frac{4000}{2\times 48}=\mathbf{41{\cdot}67 \text{ kg/cm}^2}.$$

Problem 1·5. A round brass bar as shown in the Fig. 1·44 is subjected to a tensile force of 50 kN. What must be the diameter at the middle portion if the stress there is not to exceed 160 N/mm². What should be the length of this middle portion if the total extension in the bar is 2·36 mm. $E=100\times 1000$ N/mm²,

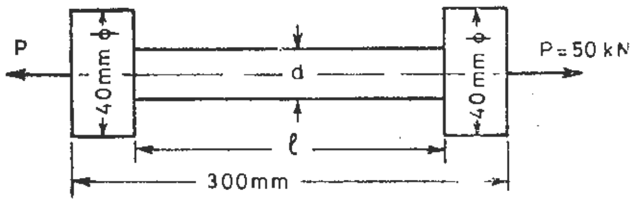


Fig. 1·44

Solution. Say the diameter at the middle portion $= d$ mm.

Area of cross section $= \frac{\pi}{4} d^2 = 0\cdot7854\ d^2\ \text{mm}^2$

Stress in the middle portion

$= 160\ \text{N/mm}^2$

So $160 \times 0\cdot7854\ d^2 = 50 \times 1000$

$$d^2 = \frac{50000}{160 \times 0\cdot7854} = 397\cdot886$$

Diameter of middle portion,

$d = 19\cdot947$ mm

Area of cross section $= 312\cdot5\ \text{mm}^2$

Say the length of middle portion

$= l$ mm.

Area of cross section of outer portion

$$= \frac{\pi}{4} (40)^2 = 1256\cdot64\ \text{mm}^2$$

Stress in the outer portion $= \frac{50 \times 1000}{1256\cdot64} = 39\cdot788\ \text{N/mm}^2$

Total change in length $= \frac{160 \times l}{E} + \frac{39\cdot788 \times (300 - l)}{E} = 0\cdot36$

$$160\ l + 11936\cdot4 - 39\cdot788\ l = 100 \times 1000 \times 0\cdot36 = 36000$$

$$120\cdot212\ l = 24063\cdot4$$

$$l = \frac{24063\cdot4}{120\cdot212}$$

Length of the middle portion

$= \mathbf{200\cdot17\ mm}$,

Problem 1·6. A rigid plate 1 m² is supported on 4 equal elastic legs *A*, *B*, *C* and *D* as shown in the Fig. 1·45. A load of 100 kg is applied on the plate at a distance of 40 cm from edge *AD* and 25 cm from edge *AB* as shown. Determine the magnitude of the compressive forces in each leg.

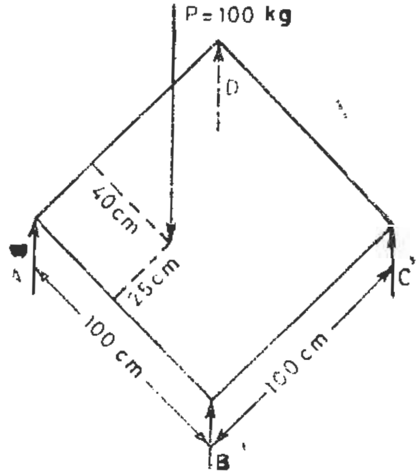


Fig. 1·45

Solution. Say the compressive force in the legs is R_A, R_B, R_C and R_D respectively.

Then total vertical load

=total reactions from supports

$$R_A+R_B+R_C+R_D=100 \text{ kg.} \qquad \ldots(1)$$

Taking moments of the forces about the edge *AB*

$$100\times25=100\ R_D+100\ R_C$$

or
$$25=R_D+R_C. \qquad \ldots(2)$$

Taking moments of the forces about the edge *AD*

$$100\times40=100\ R_B+100\ R_C$$

or
$$40=R_B+R_C. \qquad \ldots(3)$$

Now the plate *ABCD* is a rigid plate and it is not going to bend, *i.e.* it will remain in one plane. Or in other words

Mean change in lengths of *A* and *C* legs

=Mean change in lengths of *B* and *D* legs

Say a=area of cross section of each leg.

E=Young's modulus of elasticity of legs

l=length of each leg

Then
$$\frac{R_A l}{aE}+\frac{R_C l}{aE}=\frac{R_B l}{aE}+\frac{R_D l}{aE}$$

or
$$R_A+R_C=R_B+R_D. \qquad \ldots(4)$$

or
$$R_A+R_C=50 \text{ kg.} \qquad \ldots(5)$$

$$R_B+R_D=50 \text{ kg.} \qquad \ldots(6)$$

From equations (2) and (3) substracting equation (2) from equation (3)

$$R_B-R_D=15. \qquad \ldots(7)$$

From equations (6) and (7)

Compressive force in

leg *B*, $R_B=32·5$ kg

leg *D*, $R_D=17·5$ kg

leg *C*, $R_C=25-R_D=7·5$ kg

leg *A*, $R_A=59-7·5=42·5$ kg.

Problem 1·7. Two rigid yokes P & Q are connected by three elastic rods A, B and C made of the same material as shown in the Fig. 1·46. The area of cross section of bars A and C is a, while the area of cross section of the bar B is $2a$. A load of 1200 kg hangs from the lower yoke. Find the magnitude of the forces in the bars A and C, and in two portions of bar B. The frame is symmetrical about the central rod B, which is passing through a horizontal bearing as shown.

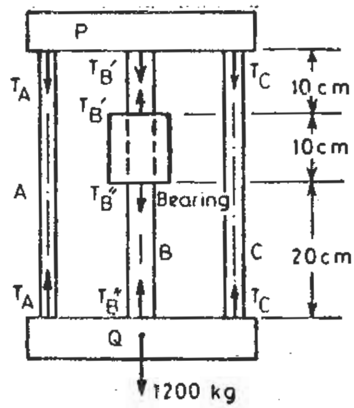


Fig. 1·46

Solution Say tensile forces developed in bars A, B and C are T_A, T_B', T_B'' (in two portions) and T_C as shown.

Considering the equilibrium for yoke P

$$T_A+T_B'+T_C=0 \quad \text{...(1)}$$

Considering the equilibrium of forces at yoke Q

$$T_A+T_B''+T_C=1200 \text{ kg} \quad \text{...(2)}$$

Say E=Young's modulus of the material of the rods. Since the yokes are rigid, elongation in each bar will be the same *i.e.*,

$$\delta l_A=\delta l_B=\delta l_C$$

$$\delta l_A=\frac{T_A\times 30}{aE} \quad \text{...(3)}$$

$$\delta l_C=\frac{T_C\times 30}{aE} \quad \text{...(4)}$$

$$\delta l_B=\frac{T_B'\times 10}{2aE}+\frac{T_B''\times 20}{2aE} \quad \text{...(5)}$$

Since the rods A and C are symmetrically placed about rod B and their area of cross section is the same.

$$T_A=T_C, \qquad \delta l_A=\delta l_C$$

So from equation (1) and (3)

$$T_B'=-2T_A$$

$$T_B''=1200-2T_A=1200+T_B'$$

or $$T_B''-T_B'=1200 \quad \text{.. (6)}$$

$$\frac{30\ T_A}{aE}=\frac{T_B'\times 10}{2aE}+\frac{T_B''\times 20}{2aE} \quad \text{...(7)}$$

$$30\ T_A=5T_B'+10T_B'' \quad \text{...(8)}$$

But $T_B''=1200+T_B'$ from equation (6)

So $30\ T_A=5T_B'+10\ (1200+T_B')$

$$30\ T_A=15T_B'+12000 \quad \text{...(9)}$$

From equation (1) $\quad 2T_A=-T_B'$

Substituting in equation (7) we get

$$30T_A = -30T_A + 12000$$

$$T_A = 200 \text{ kg} = T_C$$

$$T_B' = -2 \times 200 = -400 \text{ kg}$$

$$T_B'' = 1200 - 400 = 800 \text{ kg} \quad \text{from equation (6)}$$

Forces in bars are $T_A = T_C = 200$ kg

$T_B' = -400$ kg

$T_B'' = +800$ kg.

Problem 1·8. For the structure shown, member AC is a steel wire 3 mm in diameter and member AB is an aluminium rod 15 mm in diameter, supporting a vertical load $P=200$ N. Determine the horizontal and vertical displacements of the point A if

$$E \text{ for steel} = 210 \times 10^3 \text{ N/mm}^2$$

$$E \text{ for aluminium} = 70 \times 10^3 \text{ N/mm}^2.$$

Solution. Figure 1·47 shows an aluminium bar and steel wire carrying the load $P=200$ N. If the force polygon is drawn for the point A, as shown in the Fig. 1·47.

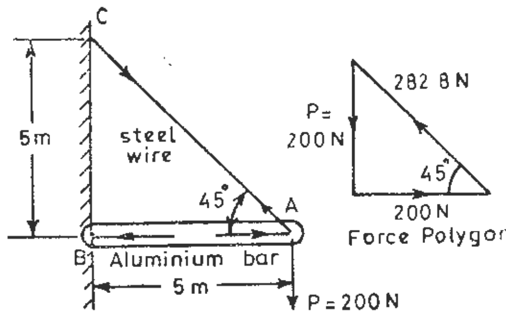


Fig. 1·47

Tension in steel wire $= 282{\cdot}8$ N.

Compressive force in aluminium rod $= 200{\cdot}8$ N.

Extension in steel wire

$$= \frac{282{\cdot}8 \times \text{length of steel wire}}{\text{Area of cross section} \times E_s}$$

Length of steel wire

$= 5\sqrt{2} = 5 \times 1{\cdot}414 = 7{\cdot}07$ m

$= 7070$ mm

Area of cross section $= \frac{\pi}{4}(3)^2 = 7{\cdot}0685 \text{ mm}^2$

Extension in steel wire $= \frac{282{\cdot}8 \times 7070}{7{\cdot}0685 \times 210 \times 1000} = 1{\cdot}347$ mm

Length of aluminium rod $= 5000$ mm

Area of cross section of aluminium rod

$$= \frac{\pi}{4}(15)^2 = 176{\cdot}7 \text{ mm}^2$$

Contraction in aluminium rod

$$= \frac{200 \times 5000}{176{\cdot}7 \times E_a} = \frac{200 \times 5000}{176{\cdot}7 \times 70 \times 100} = {\cdot}080 \text{ mm}.$$

Displacements at point A

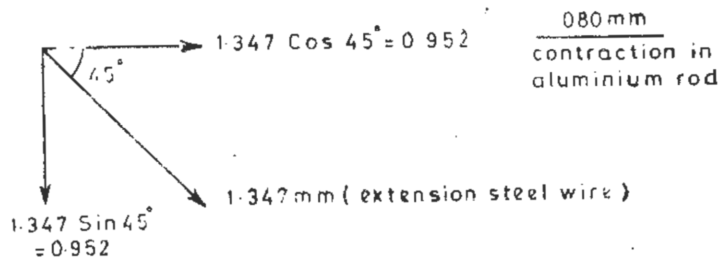


Fig. 1·48

Vertical displacement of point A

$$=0{\cdot}952 \text{ mm} \downarrow$$

Horizontal displacement of point A

$$=0{\cdot}952-0{\cdot}080=\overrightarrow{0{\cdot}872} \text{ mm.}$$

Probem 1·9. The cross section of a bar is given by $(169+0{\cdot}01x^2)$ mm² where x is the distance from one end in mm. If the length of the bar is 200 mm, find the change in length under a load of 5 kN.

$$E=2\times 10^5 \text{ N/mm}^2.$$

Solution. Considering a small length dx at a distance of x mm from one end.

Area of cross section $=(169+{\cdot}01x^2)$ mm²

Load $=5$ kN

Stress, $f_x=\dfrac{5000}{(169+0{\cdot}01x^2)}$

Strain, $\epsilon_x=\dfrac{5000}{E(169+0{\cdot}01x^2)}$

Change in length over $dx=\dfrac{5000\,dx}{E(169+0{\cdot}01x^2)}$

Total change in length, $\delta l=\displaystyle\int_0^{200}\frac{5000\,dx}{2\times 10^5\,(169+0{\cdot}01x^2)}$

$$=2{\cdot}5\times 10^{-2}\int_0^{200}\frac{dx}{(169+0{\cdot}01x^2)}$$

$$=2{\cdot}5\times 10^{-2}\int_0^{200}\frac{dx}{0{\cdot}01(16900+x^2)}$$

$$=2{\cdot}5\times\left|\frac{1}{130}\tan^{-1}\frac{x}{130}\right|_0^{200}$$

$$=2.5\times\frac{1}{130}\times\tan^{-1}\frac{200}{130}$$

$$=\frac{2.5}{130}\times 0.9931=0.019 \text{ mm.}$$

Problem 1·10. Determine the reduction in length in a circular tapered steel bar with a cylindrical hole under a compressive force $P=40$ kN as shown in the Fig. 1·49.

$$E=2100\times 100 \text{ N/mm}^2.$$

Solution. Consider the two portions *AB* and *BC* separately.

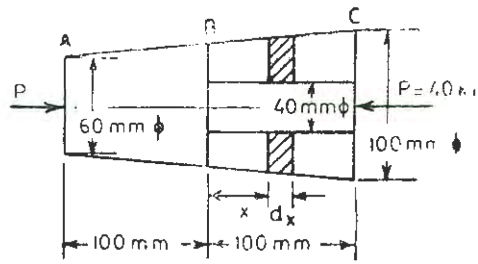


Fig. 1·49

Portion AB

Diameter at A $=60$ mm

Diameter at B $=\dfrac{100+60}{2}=80$ mm

Length AB $=100$ mm

Force $P=40$ kN

Contraction in the length AB,

$$\delta l=\frac{4PL}{\pi D_1 D_2 E}$$

$$=\frac{4\times 40\times 1000\times 100}{\pi\times 60\times 80\times 2100\times 100}=0.00505 \text{ mm.}$$

Portion BC

Consider an elementary ring of length dx at a distance of x from B.

Diameter at x, $\quad D_x=80+\dfrac{100-80}{100}\times x$

$$=(80+0.2x)$$

Area of cross section, $A_x=\dfrac{\pi}{4}(80+0.2x)^2-\dfrac{\pi}{4}(40)^2$

$$=\frac{\pi}{4}[80+0.2x+40][80+0.2x-40]$$

$$=\frac{\pi}{4}(120+0.2x)(40+0.2x)$$

$$=\frac{\pi}{4}(0.2)^2[600+x][200+x]$$

$$=\frac{\pi}{100}[600+x][200+x]$$

Stress, $\quad f_x=\dfrac{P}{A_x}=\dfrac{40\times 1000\times 100}{\pi[600+x][200+x]}$

Strain, $\quad \epsilon_x=\dfrac{f_x}{E}$

Change in length over $dx=\epsilon_x\,.\,dx$

Total change in length for BC,

$$\delta l_2=\int_0^{100} \epsilon_x \, dx$$

$$=\int_0^{100} \frac{4\times 10^6 \, dx}{\pi E\,[600+x][200+x]}$$

$$=\frac{4\times 10^6}{\pi\times 2100\times 100}\int_0^{100} \frac{1}{400}\left[\frac{1}{200+x}-\frac{1}{600+x}\right] dx$$

$$=\frac{1}{21\pi}\Big|\ \ln(200+x)-\ln(600+x)\ \Big|_0^{100}$$

$$=\frac{1}{21\pi}\left(\ln\frac{300}{200}-\ln\frac{700}{600}\right)$$

$$=\frac{1}{21\pi}\ \ln\ \frac{3}{2}\times\frac{6}{7}=\frac{1}{21\pi}\ \ln\ \frac{9}{7}$$

$$=\frac{1}{21\pi}\times 0{\cdot}2515=0{\cdot}00381 \text{ mm}$$

Total change in length $=0{\cdot}00505+0{\cdot}00381$

$=0{\cdot}00886$ mm (contraction).

Problem 1·11. A load W is suspended by ropes as shown in Fig. 1·50. In case (*a*) a rope of area of cross section 400 mm² passes over a pulley of diameter 200 mm and its ends are connected to the ceiling as shown. The pulley carries a load of 2000 N. In case (*b*), load

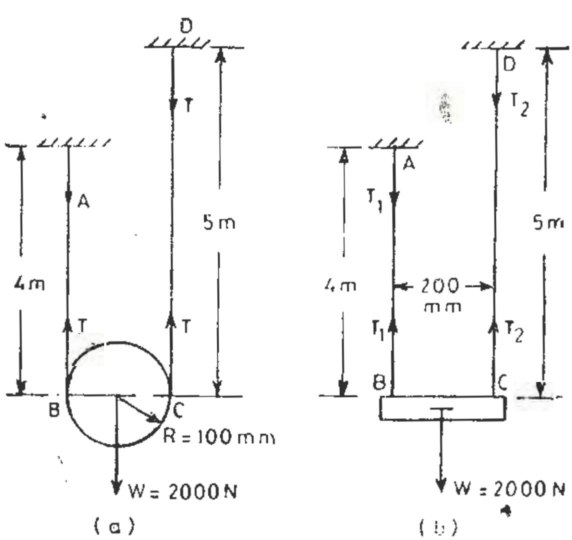


Fig. 1·50

of 2000 N is suspended at a bar attached to the ropes of area of cross section 400 mm^2 and lengths 4 m and 5 m respectively as shown. The load is suspended in such a manner that the bar remains horizontal in both cases determine the stress in the ropes and downward movement of the pulley and the bar. $E=210\times 10^3$ N/mm^2.

Solution. In case (*a*) there is a continuous rope from *ABC* to *D*. The load *W* produces tension $T=W/2$ in both the portions.

Length of the rope, $l=4000+5000+\pi\times R$

where *R* is the radius of the pulley.

$$=9000+\pi\times 100=9314\cdot16 \text{ mm}$$

Stress in the rope $f=\dfrac{W}{2 \text{ area}}=\dfrac{T}{\text{area}}=\dfrac{1000}{400}=2\cdot5$ N/mm^2

Change in length of the rope,

$$=\frac{f}{E}\times l=\frac{2\cdot5}{210\times 1000}\times 9314\cdot16$$

$$=0\cdot11088 \text{ mm}$$

Downward movement of pulley

$$=\frac{0\cdot11088}{2}=\cdot05544 \text{ mm.}$$

In case (*b*) the length of the ropes *AB* and *CD* are different, but the bar *BC* is to remain horizontal *i e.*, the change in length in both the ropes is the same

i.e., $\delta l_1=\delta l_2$

$$\epsilon_1\times l_1=\epsilon_2\times l_2$$

where ϵ_1 and ϵ_2 are strains in *AB* and *CD*

or $\epsilon_1\times 4000=\epsilon_2\times 5000$

$$\epsilon_1=1\cdot25\ \epsilon_2$$

But $\epsilon_1=\dfrac{f_1}{E},\quad \epsilon_2=\dfrac{f_2}{E}$

or $\dfrac{f_1}{E}=1\cdot25\dfrac{f_2}{E}$ Both the ropes of the same material.

or $f_1=1\cdot25 f_2$

Tension in rope 1+tension in rope 2=

$$W=2000 \text{ N}$$

or $T_1+T_2=2000$ N

$$f_1 A+f_2 A=2000 \text{ N}$$

$$f_1\times 400+f_2\times 400=2000 \text{ N}$$

$$f_1+f_2=5$$

or $1\cdot25 f_1+f_2=5$ N/mm^2

$$f_2=2\cdot222 \text{ N/mm}^2$$

$$f_1=2\cdot777 \text{ N/mm}^2$$

Stress in rope *AB* $=2\cdot777$ N/mm^2

Stress in rope *CD* $=2\cdot222$ N/mm^2

Downward displacement of bar $BC=\delta l_1 \;=\delta l_2$

$$=\frac{f_1}{E}\times l_1$$

$$=\frac{2{\cdot}777}{210\times 1000}\times 4000=0{\cdot}0529 \text{ mm.}$$

Problem 1·12. A steel wire 5 mm in diameter is used for hoisting purposes during construction of building. If 100 m length of the wire is hanging vertically and a load of 0·6 kN is being lifted at the lower end of the wire, determine the total elongation of the wire. Given specific weight of steel=0·008 kg/cm³.

$$E=210\times 1000 \text{ N/mm}^2.$$

Solution. Diameter of wire,

$$d=5 \text{ mm}$$

Area of cross section, $A=\frac{\pi}{4}(5)^2=19{\cdot}635 \text{ mm}^2$

Length of steel wire, $l=100 \text{ m}=100\times 1000 \text{ mm}$

Load, $W=0{\cdot}6 \text{ kN}=600 \text{ N}$

$$E=210\times 1000 \text{ N/mm}^2$$

Elongation in wire due to load W,

$$\delta l_1=\frac{W}{AE}\times l$$

$$=\frac{600\times 100\times 1000}{19{\cdot}635\times 210\times 1000}=14{\cdot}551 \text{ mm.}$$

Density of steel $=\frac{0{\cdot}008\times 9{\cdot}8}{10^3}=0{\cdot}0784\times 10^{-3} \text{ N/mm}^3$

Elongation due to the self weight,

$$\delta l_2=\frac{wl^2}{2E}=\frac{0{\cdot}0784\times 10^{-3}\times (10^5)^2}{2\times 210\times 1000}$$

$$=\frac{{\cdot}0784\times 10^3}{42}=1{\cdot}866 \text{ mm}$$

Total elongation $=\delta l_1+\delta l_2=14{\cdot}551+1{\cdot}866$

$$=16{\cdot}417 \text{ mm.}$$

Problem 1·13. A cone of base diameter D and height L is securely fixed with base at the top as shown in Fig. 1·51. If the weight density of the material is w, determine the extension in the length of the cone due to its own weight.

Solution. Consider an elementary disc of length dx at a distance of x from the apex.

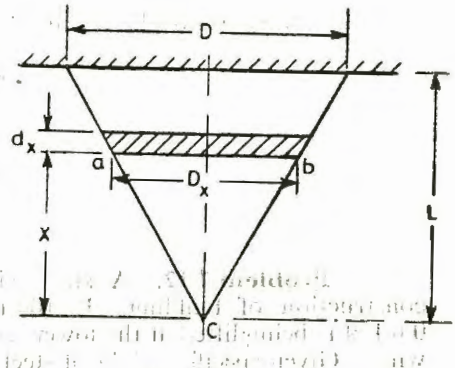


Fig. 1·51

Diameter $D_x = \frac{x}{L} \times D$

Weight of the cone abc

$$W_x = w\left[\frac{1}{3}\ \frac{(\pi D_x^2)}{4} \times x\right]$$

$$= \frac{w}{12} \cdot \pi \left(\frac{x}{L}\ D\right)^2 . x$$

$$= \frac{\pi w}{12} \cdot \frac{x^3 D^2}{L^2}$$

Stress, $f_x = \frac{4W_x}{\pi D_x^2} = \frac{\pi w}{12} \times \frac{x^3 D^2}{L^2} \times \frac{4L^2}{\pi x^2 D^2} = \frac{wx}{3}$

Strain $\epsilon_x = \frac{f_x}{E} = \frac{wx}{3E}$

Change in length over

$$dx = \frac{wx\ dx}{3E}$$

Total change in length,

$$\delta l = \int_0^L \frac{wx\ dx}{3E} = \frac{wL^2}{6E}.$$

Problem 1·14. A stepped bar 1·6 m long has area of cross section 4 cm² over a certain length and 8 cm² over remainder of its length. The strain energy of this stepped bar is 40% of that of a bar 8 cm² in area, 1·6 m long subjected to the same maximum stress. What is the length of the portion of 4 cm² in area ?

Solution. Say the bar is subjected to same axial load and

Stress in portion I $= f$

Then stress in portion II$= \frac{f \cdot 4}{8} = \frac{f}{2}$

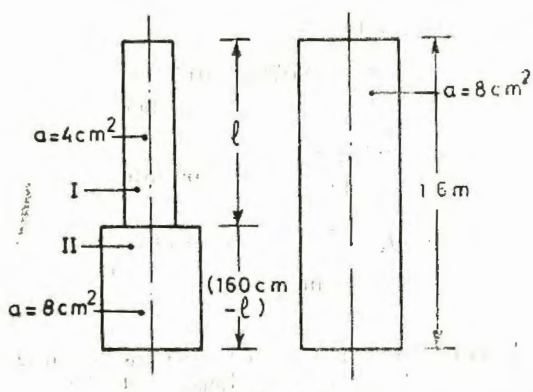


Fig. 1·52

Say E is the Young's modulus of the material.

Strain energy, $$U=\frac{f^2}{2E}(4\times l)+\left(\frac{f}{2}\right)^2\frac{1}{2E}(8)(160-l)$$

$$=\frac{f^2}{2E}\left[4l+2(160-l)\right]=\frac{f^2}{E}\left[l+160\right].$$

Now the uniform bar of area 8 cm² is also subjected to the stress f (maximum stress in stepped bar).

Strain energy, $$U'=\frac{f^2}{2E}(8)(160)$$

but $$U=0{\cdot}4\,U'$$

$$\frac{f^2}{E}(l+160)=0{\cdot}4\,\frac{f^2}{2E}\times 1280$$

$$l+160=256$$

$$l=96 \text{ cm.}$$

Problem 1·15. Compare the strain energy absorbed by the bars A and B as shown. Bar A of length L having diameter d at one end and uniformly increasing to the diameter D at

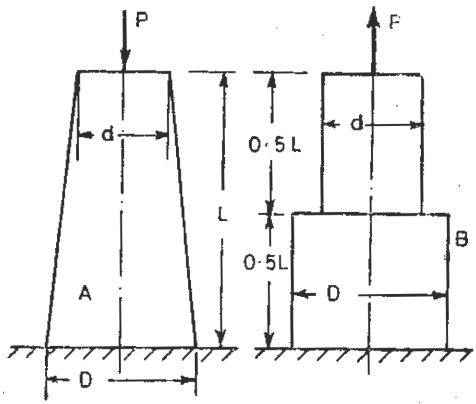


Fig. 1·53

the other end is subjected to compressive force P. Bar B of the same material, but a stepped bar of diameter d for half of its length and diameter D of the remaining half of its length is subjected to the same compressive force P. Given $d=0{\cdot}6\,D$.

Solution. Say E=Young's modulus of the material.

δL, contraction in bar $A=\dfrac{PL}{\pi DdE}$

Strain energy, $$U_A=\frac{1}{2}P\delta L=\frac{2PL^2}{\pi DdE}$$

but $$d=0{\cdot}6\,D$$

$$U_A=\frac{2PL^2}{\pi\times 0{\cdot}6D^2E}\quad\ldots(1)$$

$\delta L'$, contraction in bar $B = \dfrac{4P}{\pi d^2} \times \dfrac{L}{E} \times \dfrac{1}{2} + \dfrac{4P}{\pi D^2} \times \dfrac{L}{E} \times \dfrac{1}{2}$

$$= \frac{2PL}{\pi E}\left[\frac{1}{d^2} + \frac{1}{D^2}\right] = \frac{2PL}{\pi E}\left[\frac{1}{0{\cdot}36D^2} + \frac{1}{D^2}\right]$$

$$= \frac{2PL \times 1{\cdot}36}{0{\cdot}36\pi ED^2}$$

Strain energy, $U_B = \dfrac{1}{2}\ P\delta L' = \dfrac{1{\cdot}36PL^2}{0{\cdot}36\pi ED^2}$

or $\dfrac{U_A}{U_B} = \dfrac{2PL^2}{\pi ED^2 \times 0{\cdot}6} \times \dfrac{\pi ED^2 \times 0{\cdot}36}{1{\cdot}36\ PL^2} = \dfrac{2}{0{\cdot}6} \times \dfrac{0{\cdot}36}{1{\cdot}36} = \dfrac{0{\cdot}72}{{\cdot}816}$

$$\frac{\text{Strain energy, bar } A}{\text{Strain energy, bar } B} = 0{\cdot}882.$$

Example 1·16. A steel rod 80 cm long and 2 cm in diameter suspended vertically is secured at its upper end, and a weight of 20 kg is allowed to slide freely on the rod through a height $h=3$ cm on to a collar at the lower end. Determine :—

(*a*) The stress developed in the rod.

(*b*) The stress developed, if the extension of rod while computing the potential energy given up by the weight is neglected.

(*c*) The stress developed in the rod when $h=0$.

Take $E=2100$ tonnes/cm².

Solution. (*a*) $W=20$ kg

$h=3$ cm

$E=3100\times 1000$ kg/cm²

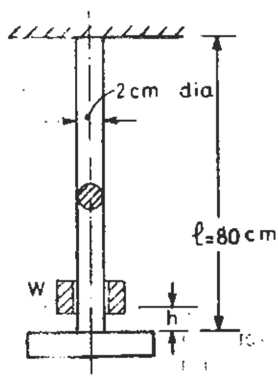


Fig. 1·54

Area of cross section,

$$A = \frac{\pi}{4} \times 4$$

$$= 3{\cdot}1416 \text{ cm}^2$$

Length of the bar,

$l=80$ cm

Stress, $f = \dfrac{W}{A}\left[1 + \sqrt{1 + \dfrac{2EAh}{Wl}}\right]$

$$\frac{2EAh}{Wl} = \frac{2\times 2100\times 1000\times 3{\cdot}1416\times 3}{20\times 80} = 24720$$

$$f = \frac{20}{3{\cdot}1416}\left[1+\sqrt{24721}\right]$$

$$= \frac{20\times 158{\cdot}2}{3{\cdot}1416} = 1007 \text{ kg/cm}^2.$$

(b) The extension of the rod when the potential energy given up by the weight is neglected.

Potential energy lost by the weight

$$=W\times h$$

Strain energy stored in the bar

$$=\frac{f^2}{2E}\times A\times l$$

$$W\times h=\frac{f^2}{2E}\times A\times l$$

$$f^2=\frac{2WhE}{Al}=\frac{2\times 20\times 3\times 2100\times 1000}{3{\cdot}1416\times 80}=1{\cdot}002\times 10^6$$

$f=$**1001 kg/cm²**.

(c) $h=0$, the load will suddenly act on the bar

$$f=\frac{W}{A}\times 2=\frac{20\times 2}{3{\cdot}1416}=\mathbf{12{\cdot}72\ kg/cm^2}.$$

Problem 1·17. A vertical steel rod 150 mm long is rigidly secured at its upper end and a weight of 15N is allowed to slide freely on the rod through a distance of 20 mm on to a stop at the lower end of the rod. What would be maximum stress developed if the upper 90 mm length of the rod has a diameter of 16 mm and the lower 60 mm length remains at 12 mm diameter.

$$E=210\times 10^3\ \text{N/mm}^2.$$

Solution. Say the instantaneous stress developed in portion I $=f_1$

Instantaneous stress developed in portion II $=f_2$

For equilibrium

$$f_1A_1=f_2A_1$$

$$f_1\times\frac{\pi}{4}(16)^2=f_2\times\frac{\pi}{4}(12)^2$$

$$f_1=0{\cdot}5625\,f_2.\qquad ...(1)$$

Volume, $V_1=\frac{\pi}{4}(16)^2\times 90$

$=18095{\cdot}6\ \text{mm}^3$

Volume, $V_2=\frac{\pi}{4}(12)^2\times 60$

$=6785{\cdot}8\ \text{mm}^3$

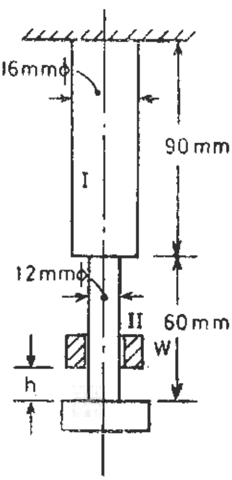


Fig. 1·55

Say the change in length under the instantaneous stress,

$$\delta l_1=\frac{f_1l_1}{E}=\frac{f_1\times 90}{E}$$

$$\delta l_2=\frac{f_2l_2}{E}=\frac{f_2\times 60}{E}$$

Loss of PE of the weight$=W(h+\delta l_1+\delta l_2)$

Strain energy absorbed by the bar

$$=\frac{f_1^2}{2E}V_1+\frac{f_2^2}{2E}\times V_2$$

Using the principle of conservation of energy

$$W(h+\delta l_1+\delta l_2) \quad =\frac{f_1^2}{2E}\times V_1+\frac{f_2^2}{2E}\times V_2$$

or $$W.h+\frac{W.f_1\times 90}{E}+\frac{W.f_2\times 60}{E}=\frac{f_1^2}{2E}\times 18095{\cdot}6+\frac{f_2^2}{2E}\times 6785{\cdot}8$$

$$E\times 15\times 20+15\times f_1\times 90+15f_2\times 60$$

$$=\frac{f_1^2}{2}\times 18095{\cdot}6+\frac{f_1^2}{2}\times 6785{\cdot}8$$

Putting $W=15$ N

$$2E+9f_1+6f_2=6{\cdot}032f_1^2+2{\cdot}26\,f_2^2$$

$$2\times 210\times 1000+9(0{\cdot}5625f_2)+6f_2$$

$$=60{\cdot}32(0{\cdot}5625f_2)^2+22{\cdot}6f_2^2$$

$$420\times 10^3+11{\cdot}0625f_2=41{\cdot}7\,f_2^2$$

$$f_2^2-{\cdot}02653f_2-10071{\cdot}94=0$$

$$f_2=\frac{0{\cdot}2653+\sqrt{(0{\cdot}2653)^2+4\times 10071{\cdot}94}}{2}$$

$$=\frac{{\cdot}2653+200{\cdot}72}{2}$$

Maximum stress developed in steel rod

$$=100{\cdot}49\ \text{N/mm}^2.$$

Problem 1·18. A load W suspended from a crane hook by a chain is being lowered at a speed of 1 metre/second. At a particular instant when the length of the unwound chain is 8 metres, the motion is suddenly arrested. The chain links are made of 10 mm round steel bar.

Determine the maximum load that the chain can carry under these conditions if the instantaneous stress produced in the chain is not to exceed 150 N/mm². Neglect weight of the chain.

Acceleration due to gravity $=9{\cdot}8$ m/sec²

E for steel $=210\times 10^3$ N/mm².

Solution. The chain link is shown in Fig. 1·65

Area of cross section of the chain

$$=2\times\frac{\pi}{4}(10)^2$$

$$=157{\cdot}58\ \text{mm}^2$$

Length of the unwound chain,

$$l=8000\ \text{mm}$$

Speed, $V=1$ m/sec

Say when the motion is stopped, the instantaneous stress developed in chain is f_t N/mm².

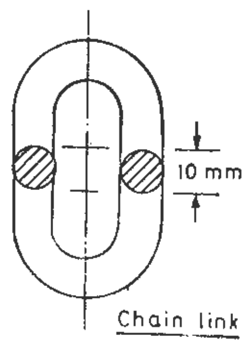


Fig. 1·56

Extension in chain, $\delta l = \frac{f_t}{E} \times l = \frac{f_t}{210 \times 1000} \times 8000$

$= 0{\cdot}038\, f_t$

Say the maximum load carried by the chain $= W$ N.

KE lost by the weight $= \frac{WV^2}{2g} = \frac{W \times 1000 \times 1000}{2 \times 9{\cdot}8 \times 1000}$

$= 51{\cdot}02\, W$ N mm ...(1)

P.E. lost by the weight $= W.\delta l$

$= 0{\cdot}038\, Wf_t$ ...(2)

Strain energy absorbed by the chain

$= \frac{f_t^2}{2E} \times \text{Volume}$

$= \frac{f_t^2 \times 157{\cdot}58 \times 8000}{2 \times 210 \times 1000} = 3{\cdot}00\, f_t^2$ ...(3)

Applying the principle of conservation of energy

$51{\cdot}02\, W + {\cdot}038\, W f_t = 3\, f_t^2$ ...(4)

But maximum stress developed is not to exceed 150 N/mm²

So $f_t = 150$ N/mm²

$51{\cdot}02 + 0{\cdot}038\, W \times 150 = 3 \times 150 \times 150$

$51{\cdot}02\, W + 5{\cdot}7\, W = 67500$

The maximum load which the chain can carry

$W = \frac{67500}{56{\cdot}72} = 1190{\cdot}0$ N

Problem 1·19. The load to be carried by a lift can be dropped on its floor through a height of 8 cm. The weight of the cage of the lift is 250 kg. The cage is supported by a wire rope 20 m in length weighing 0·80 kg/metre length. The wire rope consists of 49 wires of 1·5 mm dia. each. If the maximum stress in the wire rope is not to exceed 1000 kg/cm², determine the maximum load which can be carried by the lift. E for the rope material = 670 tonnes/cm²

Solution. Weight of the cage = 250 kg

Weight of the rope supporting the cage

$= 0{\cdot}8 \times 20 = 16$ kg.

The maximum stress in the wire rope will occur at tbe end of the rope coming out of the rope drum.

Area of cross section of wire rope

$= 49 \times \frac{\pi}{4}\, (0{\cdot}15)^2 = 0{\cdot}866$ cm²

Initial stress in wire rope $= \frac{250 + 16}{0{\cdot}866} = 307{\cdot}16$ kg/cm²

Allowable stress $= 1000$ kg/cm²

Stress due to the falling load,

$$f_i = 1000 - 307{\cdot}16$$
$$= 692{\cdot}84 \text{ kg/cm}^2$$

Extension in wire rope due to f_i

$$= \frac{f_i}{E} \times \text{length of the rope}$$
$$= \frac{692{\cdot}84 \times 20 \times 100}{670 \times 1000} = 2{\cdot}068 \text{ cm.}$$

Say the maximum load which can be dropped on the floor of the cage is W, then

$$W(h+\delta l) = \frac{f_i^2}{2E} \times \text{Volume of the rope}$$

where h=height through which the load falls

$$W(8+2{\cdot}068) = \frac{(692{\cdot}84)^2}{2 \times 670 \times 1000} \times 0{\cdot}866 \times 20 \times 100$$
$$= 620{\cdot}453$$
$$W = 61{\cdot}62 \text{ kg.}$$

Example 1·20. A vertical tie consisting a steel rod 1·6 m long and 25 mm diameter encased throughout in a brass tube 40 mm external diameter, is rigidly fixed at the top end. The rod and the tube are fixed together so as to form a compound bar. The compound bar is suddenly loaded in tension by a weight W falling freely through a height of 10 mm, before being arrested by a collar provided at the lower end of the tie. Determine the magnitude of W if the maximum stress in the tie is not to exceed 80 N/mm².

Given $E_{steel} = 2 \times 10^5$ N/mm²

$E_{brass} = 1 \times 10^5$ N/mm².

Solution. Since steel rod and brass tube are fixed together, there cannot be differential elongation for them. Both the rod and tube have to strain together under load *i.e.*, change in length in steel rod=change in length in brass tube

or $\epsilon_s = \epsilon_b$

(Strains are the same)

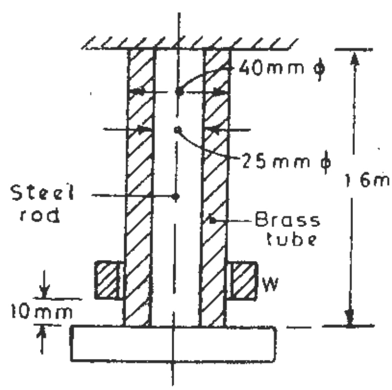


Fig. 1·57

Say the stress developed in steel rod=f_s

Stress developed in brass tube $=f_b$

then
$$\frac{f_s}{E_s} = \frac{f_b}{E_b}$$

or
$$\frac{f_s}{f_b} = \frac{E_s}{E_b} = \frac{2 \times 10^5}{1 \times 10^5} = 2$$

or
$$f_s = 2\, f_b \qquad \ldots(1)$$

Extension in compound bar

$$= \frac{f_s}{E_s} \times l \qquad \text{where } l = \text{length of the bar}$$

$$= \frac{f_s}{2 \times 10^5} \times 1600 = 0{\cdot}8 \times 10^{-2} f_s \qquad \ldots(2)$$

Height through which load falls,

$$h=10 \text{ mm}$$

Area of cross section of steel rod

$$=\frac{\pi}{4}(25)^2=490{\cdot}875 \text{ mm}^2$$

Area of cross section of brass tube

$$=\frac{\pi}{4}(40^2-25^2)=765{\cdot}765 \text{ mm}^2$$

Volume of steel rod, $V_s=490{\cdot}875\times 1600$ mm³

Volume of brass tube, $V_b=765{\cdot}765\times 1600$ mm³

Applying the principle of conservation of energy

$$W(h+\delta l)=\frac{f_s^2}{2E_s}\times V_s+\frac{f_b^2}{2E_b}\times V_b$$

$$W(10+{\cdot}008\, f_s)=\frac{f_s^2}{2E_s}\times V_s+\frac{f_b^2}{2E_b}\times V_b$$

Maximum stress in tie $=80$ N/mm²

Which means $f_s=80$ N/mm²

$f_b=40$ N/mm²

Substituting the values above

$$W(10+{\cdot}008\times 80)=\frac{80^2}{2\times 2\times 10^5}\times 490{\cdot}875\times 1600+\frac{40^2}{2\times 1\times 10^5}\times 765{\cdot}765\times 1600$$

$$W\,(10{\cdot}64)=12566{\cdot}4+9801{\cdot}8$$

Load, $$W=\frac{22368{\cdot}2}{10{\cdot}64}=2102{\cdot}27 \text{ Newtons.}$$

Problem 1·21. A steel bar 3 cm in diameter, 2 metres long is rigidly fixed to a bracket as shown in Fig.1·58. The other end with a collar rests on a support. A weight of 25 kg moving horizontally along the bar at the velocity of 300 cm/second is brought to rest by the collar at the other end. The bracket deflects horizontally by 0·04 cm/tonne of the load induced in the bar. Calculate the maximum tensile stress in the bar.

Given $E=2100$ tonnes/cm²

Acceleration due to gravity, $g=980$ cm/sec².

Solution.

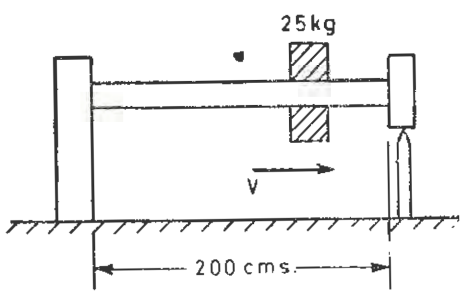


Fig. 1·58

K.E. lost by the travelling load

$$=\frac{25\times 300\times 300}{2\times 980}=1150 \text{ cm-kg}$$

Say the stress developed $=f$ kg/cm^2

Equivalent load induced, $W=f\,.\,A$ kg

$$=\frac{f\,.\,A}{1000} \text{ tonnes}$$

Cross sectional area, $A=\frac{\pi}{4}\times 9=7{\cdot}07$ cm^2

Length of the bar, $l=200$ cm.

Instantaneous horizontal deflection in the bracket,

$$\delta=0{\cdot}04\ \ W=\frac{{\cdot}04\,f\,.\,A}{1000} \text{ cm}$$

$$=\frac{f\,.\,A}{25000} \text{ cm}$$

Strain energy in the bracket

$=W\delta$ (since the load is sudden)

Strain energy in the bar $=\frac{f^2}{2E}\times Al$

$$\therefore \qquad 1150=\frac{f\,.\,A}{1000}\times\frac{f\,.\,A}{25000}+\frac{f^2\,A\times 200}{2\times 2100\times 1000}$$

$$1150\times 10^6=f^2\times\frac{7{\cdot}07\times 7{\cdot}07}{25}+\frac{f^2\times 200\times 7{\cdot}07}{4{\cdot}2}$$

$$=f^2\,[2{\cdot}003+337]$$

$$f^2=\frac{1150}{339{\cdot}003}\times 10^6=3{\cdot}39\times 10^6$$

$f=1840$ kg/cm^2. **Ans.**

SUMMARY

1. F_R is the resultant force on a plane, making an angle θ to the plane, then normal force $=F_R \sin\theta$, shear force$=F_R\cos\theta$ on the plane.
2. Normal stress=Normal force per unit area of the plane
   +ve normal stress—a tensile stress (pointing away from the plane)
   —ve normal stress—a compressive stress (pointing towards the plane).
3. Normal strain=change in length per unit length along the direction of the applied force
   +ve normal strain—elongation in the length of specimen
   —ve normal strain—contraction in the length of the specimen.

   Within the elastic limit, Young's modulus $=\frac{\text{Normal stress}}{\text{Normal strain}}$.

4. Lateral strain$=\dfrac{\text{Change in lateral dimension}}{\text{Original lateral dimension}}$ (lateral to the direction of force) lateral strain is always of the opposite sign to the normal (or linear) strain.

5. Poisson's ratio$=\dfrac{\text{Lateral strain}}{\text{Normal strain}}$ (within the elastic limit of the material).

6. Within the elastic limit of the material, if the load is removed from the specimen, the specimen returns to its original dimension and original shape. Or the strains produced on the material within the elastic limit are recovered after the removal of the load.

7. A tapered bar of length L, diameter at one end D_1 and diameter at the other end D_2 is subjected to an axial force P. If E=Young's modulus of the material, then change in the length of the bar is

$$\delta L=\frac{4P}{\pi D_1 D_2}\times\frac{L}{E}.$$

8. Bar of uniform strength, area of cross section at a distance y from the end where area is A, is given by

$$A=A_1\ e^{\frac{wy}{f}}$$

where w=weight density of the material

f=uniform stress in bar.

9. Extension in bar due to its own weight$=\dfrac{wh^2}{2E}$

where w=weight density

h=height of the bar.

10. Within the elastic limit, shear steess is proportional to shear strain,

Modulus of rigidity, $G=\dfrac{\text{Shear stress}}{\text{Shear strain}}$.

11. Within the elastic limit, volumetric stress is proportional to volumetric strain,

Bulk modulus, $K=\dfrac{\text{Volumetric stress}}{\text{Volumetric strain}}$.

12. Tensile test on mild steel.

Mild steel fails showing a cup and cone type fracture.

At the yield point there is considerable extension without increase in internal resistance.

Ultimate tensile strength is the maximum load withstood by the specimen divided by its original area of cross section.

Total change in length $\delta l=bl+c\sqrt{A}$ where b and c are

Barba's constants, l=length and A=area of cross section of the specimen.

13. Strain energy absorbed by a body within its elastic limit

$$=\frac{f^2}{2E}\times\text{Volume}$$

f=stress developed in the body.

14. Strain energy due to shear stress q,

$$U_s = \frac{q^2}{2G} \times \text{Volume}.$$

15. Volumetric strain energy, $U_v = \frac{p^2}{2K} \times \text{Volume}$

where p=volumetric stress.

16. Stress produced by a sudden load W on a bar of area of cross section A

$$_{sudden} = \frac{2W}{A}.$$

17. Instantaneous stress produced by an impact load

$$f_i = \frac{W}{A}\left[1+\sqrt{1+\frac{2EAh}{WL}}\right]$$

where A=area of cross section

h=height through which load W falls

L=length of the bar.

18. Stress concentration factor in a specimen with a discontinuity such as hole, fillet etc.

$$\text{SCF} = \frac{\text{Maximum stress}}{\text{Average stress at minimum section}}.$$

19. Factor of safety $= \frac{\text{Ultimate stress}}{\text{Working stress}}.$

MULTIPLE CHOICE QUESTIONS

1. On a plane resultant stress is inclined at an angle 30° to the plane. If the normal stress on the plane is 50 N/mm², the shear stress on the plane will be

(*a*) 43·3 N/mm² (*b*) 86·6 N/mm²
(*c*) 100 N/mm² (*d*) None of the above.

2. A bar of square cross section side a is subjected to a tensile load P. On a plane inclined at 45° to the axis of the bar, the normal stress will be

(*a*) $\frac{2P}{a^2}$ (*b*) $\frac{P}{a^2}$
(*c*) $\frac{P}{2a^2}$ (*d*) $\frac{P}{4a^2}$.

3. A steel bar 100 mm long is subjected to a tensile stress f. If the change in the length of the bar is 1/20 mm, what will be the value of f? E for steel$=200\times1000$ N/mm²

(*a*) 200 N/mm² (*b*) 100 N/mm²
(*c*) 50 N/mm² (*d*) 25 N/mm².

4. A steel bar of square section tapering from 25 mm $\times$ 25 mm to 20 mm $\times$ 20 mm over a length of 1 metre, is subjected to a tensile force of 1000 N. If E......, The change in the length of bar is given by (E=200 $\times$ 1000 N/mm²)

(*a*) ·2 mm (*b*) ·1 mm
(*c*) ·02 mm (*d*) ·01 mm.

5. Two tie rods are connected through a pin of 40 mm² area of cross section. If the tie rods carry a tensile load of 10 kN, the shear stress in pin will be

(*a*) 125 N/mm² (*b*) 250 N/mm²
(*c*) 375 N/mm² (*d*) 500 N/mm².

6. A rivet is connecting two plates through a lap joint subjected to a tensile force of 500 kg. If the maximum shear stress permissible in rivet is 625 kg/cm², what is the area of cross section of the rivet

(*a*) 125 mm² (*b*) 100 mm²
(*c*) 80 mm² (*d*) 40 mm².

7. A wire rope 10 metre long is suspended vertically from a pulley. The wire rope weighs 1·2 kg/metre length. The area of cross section of the wire rope is 20 mm². The maximum stress developed in wire rope is

(*a*) 12 kg/cm² (*b*) 24 kg/cm²
(*c*) 30 kg/cm² (*d*) 60 kg/cm².

8. A spherical ball of volume 10^6 mm³ is subjected to a hydrostatic pressure of 90 N/mm². If the bulk modulus for the material is 180 $\times$ 1000 N/mm². The change in the volume of the bar will be

(*a*) 50 mm³ (*b*) 100 mm³
(*c*) 250 mm³ (*d*) 500 mm³.

9. A bar 100 mm long and cross sectional area 64 mm² is tested under tension. The Barba's constants for the material are b=0·2, c=0·5. The percentage elongation of the bar is

(*a*) 29·6 (*b*) 24
(*c*) 20 (*d*) None of the above.

10. A load of 100 kg acts suddenly on a bar with 0·8 cm² area of cross section and length 10 cm. The maximum stress developed in the bar is

(*a*) 125 kg/cm² (*b*) 250 kg/cm²
(*c*) 500 kg/cm² (*d*) 1250 kg/cm².

11. A bar 1 m long and 4 cm² area of cross section is securely fixed at one end and a collar is provided on the bar at the other end. A weight of 10 kg falls through a height of 10 cm on to the collar so as to extend the bar instantaneously. Neglecting the effect of extension in bar, the maximum stress developed in the bar is— $E=2\times10^6$ kg/cm²

(*a*) 5 kg/cm² (*b*) 500 kg/cm²
(*c*) 1000 kg/cm² (*d*) 2000 kg/cm.

12. A round bar 'A' of length L and diameter D is subjected to an axial force producing stress f. Another round 'B' bar of the same material but diameter $2D$ and length 0·5 L is also subjected to the same stress f. The ratio of strain energy in A to the strain energy in B is given by

(*a*) 2 (*b*) 1·5
(*c*) 1·0 (*d*) 0·5.

13. A steel rod 1 m long of diameter 3 cm is completely encased in a brass tube of external diameter 5 cm. and internal diameter 3 cm. A shock load produces a stress of 900 N/mm² in steel rod. If $E_{steel} = 2E_{brass}$, the stress developed in brass tube is

(*a*) 1600 N/mm² (*b*) 1500 N/mm²
(*c*) 900 N/mm² (*d*) 450 N/mm².

14. A mild steel specimen is tested under tension and a continuous graph between load and extension is obtained. A load at which there is considerable extension without increase in resistance is called

(*a*) Ultimate load (*b*) Breaking load
(*c*) Upper yield load (*d*) Lower yield load.

15. The approximate value of Poisson's ratio for mild steel is

(*a*) 0·35 (*b*) 0·33
(*c*) 0·29 (*d*) 0·25.

ANSWERS

1. (*b*).	2. (*c*).	3. (*b*).	4. (*c*).	5. (*a*).	6. (*c*).
7. (*d*).	8. (*d*).	9. (*b*).	10. (*b*).	11. (*c*).	12. (*d*).
13. (*d*).	14. (*c*).	15. (*c*).			

EXERCISE

1·1. The parts of a certain machine component are joined by a rivet 25 mm in diameter. Determine the shear and normal stresses in the rivet if the axial force $P = 15$ kN and the angle of joint is 30° to the axis of the load. Fig. 1·59

[**Ans.** 15·28 N/mm², 26·46 N/mm²]

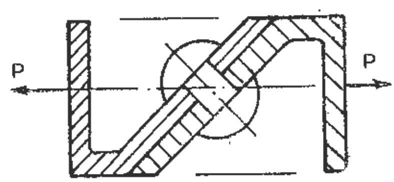


Fig. 1·59

1·2. A solid circular shaft and collar are forged in one piece. If the maximum permissible shearing stress is 300 kg/cm², what is the greatest compressive load W which may act on the shaft ? What will be the compressive stress in the shaft ? Fig. 1·60

[**Ans.** 22·62 tonnes, 288 kg/cm²]

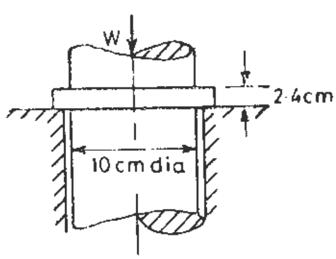


Fig. 1·60

1·3. A tie bar 2 cm in diameter carries a load which causes a stress of 1000 kg/cm². It is attached to a cast iron bracket by means of four bolts which can be stressed only upto 800 kg/cm². Determine the diameter of the bolts. [**Ans.** 1·12 cm]

1·4. The wooden pieces of square cross section 5 cm × 5 cm are glued together as shown in Fig. 1·61. The outer surface of the assembly are glued to the foundation. What will be the average shearing stress on the glued joints if the horizontal force = 5 kN.

[**Ans.** 50 N/mm²]

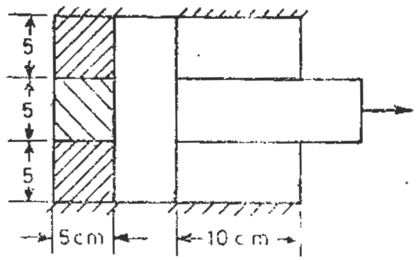


Fig. 1·61

1·5. The round bar as shown in Fig. 1·62 is subjected to a tensile load of 10 tonnes. What must be the diameter of the middle portion if the stress there is to be 1 tonne/cm².

What must be the length of the middle portion if the total extension of the bar under the given load is 0·010 cm

E = 2100 tonnes/cm².

[**Ans.** 3·568 cm. 9·633 cm]

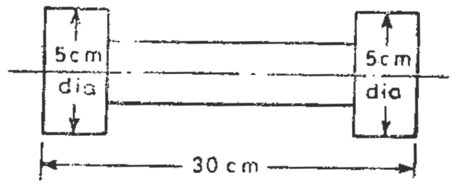


Fig. 1·62

1·6. A rigid plate 80 cm × 80 cm is supported by 4 elastic legs A, B, C and D as shown in Fig. 1·63. A load of 1200 N is applied on the plate at a distance of 30 cm from edge AD and at 20 cm from edge AB. Determine the required compressive force in each leg.

[**Ans.** 525, 375, 75, 225 N]

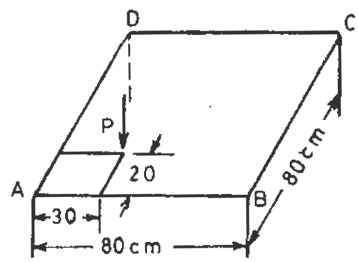


Fig. 1·63

1·7. Two rigid yokes P and Q are connected by three elastic rods A, B and C made of the same material as shown in Fig. 1·64. The area of cross section of bars A and C is a each, while the area of cross section of the bar B is $3a$. A load of 10 kN hangs from the lower yoke. Find the magnitude of the forces in the bars A and C and in two portions of the bar B. The frame is symmetrical about the central rod B, which is passing through a horizontal bearing.

[**Ans.** $T_A = T_C = 1086·96$ N,
$T_B' = -2173·91$ N, $T_B'' = +7826·09$ N]

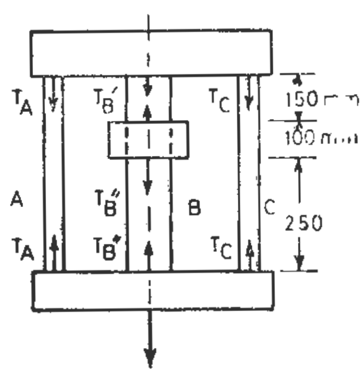


Fig. 1·64

1·8. For the simple structure shown in Fig. 1·65, member AC is a steel wire 4 mm in diameter and member AB is an aluminium rod 8 mm in diameter supporting a vertical load $P=100$ kg. Determine the horizontal and vertical displacements of the point A, if

$E_{\text{steel}}=2100$ tonnes/cm²

$E_{\text{aluminium}}=700$ tonnes/cm²

[**Ans.** 0·103 cm, 0·178 cm]

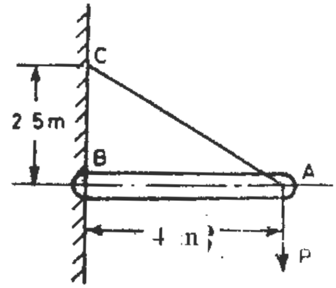


Fig 1·65

1·9. The cross section of a bar is given by $(2+{\cdot}02x^2)$ cm² where x is the distance from one end in cm. If the length of the bar is 15 cm, find the change in length under a load of 5 tonnes. $E=2\times10^6$ kg/cm². [**Ans.** 0·0123 cm]

1·10. Determine the extension in a rectangular steel bar 30 cm long with a triangular hole cut in it, under an axial force of $P=25$ tonnes. Given $E_{steel}=2100$ tonnes/cm².

[**Ans.** $12{\cdot}6\times10^{-3}$ cm]

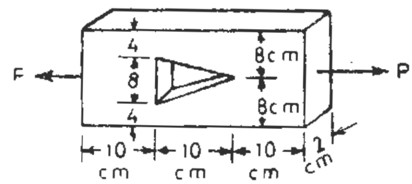


Fig 1·66

1·11. A gradually applied load $W=300$ kg is suspended by ropes as shown in Fig. 1·67. In both (a) and (b) the ropes have a cross sectional area 5 cm² and value of $E=2100$ tonnes/cm². In case (a), the rope ABC is continuous and the weight is suspended by a frictionless pulley of diameter 20 cm. In (b) AB and BC are separate ropes connected to a bar from which the ropes are suspended, such that the bar remains horizontal.

Find for both (a) and (b) the stresses in the ropes and find the downward movement of the pulley and the bar due to the load.

[**Ans.** (a) 30 kg/cm², 0·05 mm]
(b) 25·71 kg/cm², 34·285 kg/cm², 0·049 mm]

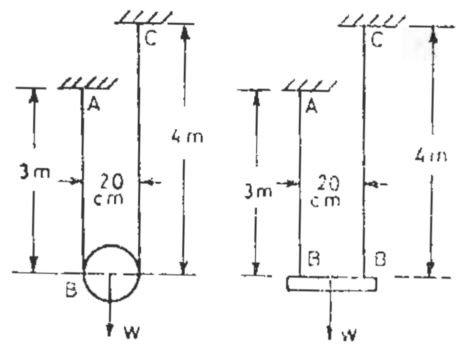


Fig. 1·67

1·12. A steel wire 6 mm in diameter is used for hoisting purposes in a building construction. If 150 metres of the wire is hanging vertically and a load of 100 kg is being lifted at the lower end of the wire, determine the total elongation of the wire. The specific weight of steel is 8 g/cm³, $E=2100$ tonnes/cm². [**Ans.** 2·954 cm]

1·13. A steel cone of base 200 mm and height 600 mm is securely fixed with base on the top. If the weight density of the material is 0·0078 kg/cm², determine the extension in the length of the cone due to its own weight. $E=2{\cdot}1\times10^6$ kg/cm². [**Ans**. $2{\cdot}228\times10^{-6}$ cm]

1·14. A brass rod 1 m long and 15 mm diameter suspended vertically is secured at its upper end and a weight of 40 N is allowed to slide freely on the rod through a height 50 mm on to a stop provided at the lower end. Determine

(*a*) Stress developed in the rod.

(*b*) The stress developed in the rod when $h=0$.

E for brass$=100\times1000$ N/mm². [**Ans**. (*a*) 47·96 N/mm², (*b*) 0·453 N/mm²]

1·15. A vertical steel rod 100 cm long is rigidly secured at its upper end and a weight of 10 kg is allowed to slide freely on the rod through a distanee of 5 cm on to a stop at the lower end of the rod, what would be the maximum stress developed if the upper 60 cm length of the rod has a diameter of 2 cm and lower 40 cm length remains at 1·6 cm diameter. $E=2{\cdot}1\times10^6$ kg/cm². [**Ans**. 1109·5 kg/cm²]

1·16. A load of 1 tonne suspended from a crane hook by a chain of cross sectional area 2 cm² is being lowered at a uniform speed of 50 cm/sec. At the instant when the length of the unwound chain is 10 metres, the machine stops working and the motion is suddenly arrested. Determine the instantaneous stress produced in the chain along with the elongation in its length due to sudden stoppage. Neglect weight of the chain.

g, acceleration due to gravity$=9{\cdot}8$ m/sec².

$E=2100$ tonnes/cm². [**Ans**. 2210 kg/cm², 1·05 cm]

1·17. A stepped bar L m long has area of cross section a cm² over a certain length and $2a$ cm² over the remainder of the length. The strain energy of the stepped bar is 30% of that of a bar $2a$ cm² in area, L m long subjected to the same maximum stress. What is the length of the portion of a cm² in area ? [**Ans**. 0·2 L]

1·18. Compare the strain energy absorbed by two bars A and B. Bar A 1·5 m long has diameter 8 cm at one end uniformly tapering to a diameter 4 cm at the other end. Bar B is a stepped bar of diameter 8 cm for 0·5 m length and 4 cm diameter for 1 m length. Both the bars are of the same material and are subjected to the same magnitude of axial load.

[**Ans**. 1·059]

1·19. The load to be carried by a lift can be dropped on its floor through a height of 15 cm. The weight of the cage of the lift is 200 kg, the cage is supported by a wire rope of 30 m length weighing 0·7 kg per metre length. The wire rope consists of 49 wires of 1·2 mm diameter each. If the maximum stress in the wire rope is not to exceed 90 N/mm². Determine the maximum load which can be carrried by the lift. $E=70{,}000$ N/mm². [**Ans**. 179·2 N]

1·20. A vertical tie consisting a steel rod 2 m long and 3 cm diameter encased throughout in an aluminium tube 4 cm external diameter and 3 cm internal diameter is rigidly fixed at the top end. The rod and the tube are fixed together so as to form a compound bar. The compound bar is suddenly loaded in tension by a weight W falling through a height of 2 cm, before being arrested by a collar provided at the lower end of the tie. Determine the magnitude of W if the maximum stress in the tie is not to exceed 1800 kg/cm².

$E_{steel}=2100$ tonnes/cm².

$E_{aluminium}=700$ tonnes/cm². [**Ans**. 632 kg]

1·21. A steel bar 20 mm in diameter 1·5 m long is rigidly fixed to a bracket as shown in Fig. 1·68 The other end with a collar rests on a support. A weight of 200 N moving horizontally along the bar at the velocity of 2m/sec is brought to rest by the collar at the other end. The bracket deflects horizontally by 2×10^{-5} mm/N of the load induced in the bar. Calculate the maximum tensile stress in the bar.

$E = 210 \times 10^3$ N/mm²

acceleration due to gravity, $g = 9·8$ m/sec².

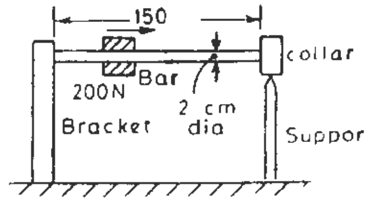


Fig. 1.68

[**Ans.** 190·2 N/mm²]

2

Composite Bars and Temperature Stresses

A bar made up of two or more than two different materials is called a composite or a compound bar. The bars of different materials in a composite bar are rigidly fixed together and there is no relative movement amongst these bars. Under the applied load, all the bars of different materials strain together. The most common example of a composite bar is an RCC column or a slab *i.e.*, concrete column or slab reinforced with steel bars to increase the strength of a concrete structure. Another common example is a bimetallic strip, made of two different materials, used in a refrigerator for temperature control.

1·2. STRESSES IN A COMPOSITE BAR

Fig. 2·1 shows a composite bar in which a solid circular rod of material 1 is completely

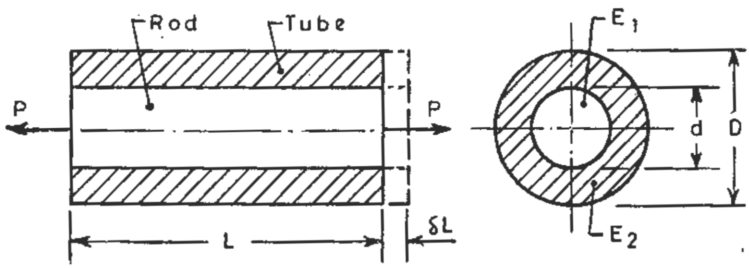


Fig. 2·1

encased in a tube of material 2. The outer tube can be force fitted or shrink fitted over the inner rod, so that both are perfectly fixed at the interface. The diameter of the rod is d and outer diameter of the tube is D. The composite bar is subjected to an axial tensile force P. Say the change in length in composite bar is δL change in length in rod, δL=change in length in tube, δL. The force P will be shared by rod and the tube in such a manner that change in length due to axial force for both rod and tube is the same.

Total load on composite bar $= P$

Say load shared by rod $= P_1$

Load shared by tube $= P_2$

Therefore $P_1 + P_2 = P$ ...(1)

A_1, Area of cross section of rod $= \frac{\pi}{4} \times d^2$

A_2. Area of cross section of tube $= \frac{\pi}{4}(D^2-d^2)$

Stress in rod, $f_1 = \frac{P_1}{A_1}$

Stress in tube, $f_2 = \frac{P_2}{A_2}$

Say Young's modulus of rod $= E_1$

Young's modulus of tube $= E_2$

Strain in rod, $\epsilon_1 = \frac{f_1}{E_1} = \frac{P_1}{A_1 E_1}$

Strain in tube, $\epsilon_2 = \frac{f_2}{E_2} = \frac{P_2}{A_2 E_2}$

Change in length of rod, $\delta l_1 = \epsilon_1 L = \frac{P_1 L}{A_1 E_1}$

Change in length of tube, $\delta l_2 = \epsilon_2 L = \frac{P_2 L}{A_2 E_2}$

But $\delta L_1 = \delta L_2$ (as per the condition for a composite bar)

or $$\frac{P_1 L_1}{A_1 E_1} = \frac{P_2 L}{A_2 E_2}$$

or $$P_1 = P_2 \frac{A_1}{A_2} \times \frac{E_1}{E_2} \qquad \ldots(2)$$

$$P_2 = P - P_1$$

Stress in the rod, $$f_1 = \frac{P_1}{A_1} = \epsilon_1 E_1 \qquad \ldots(3)$$

Stress in the tube, $$f_2 = \frac{P_2}{A_2} = \epsilon_2 E_2 \qquad \ldots(4)$$

But $\delta l_1 = \delta l_2$

$$\epsilon_1 L = \epsilon_2 L, \quad \text{or} \quad \epsilon_1 = \epsilon_2$$

So from equations (3) and (4).

$$\frac{f_1}{f_2} = \frac{E_1}{E_2} \qquad \ldots(5)$$

Now $$P = P_1 + P_2 = f_1 A_1 + f_2 P_2$$

$$P = f_2 \times \frac{E_1}{E_2} \times A_1 + f_2 A_2 \qquad \ldots(6)$$

From equation (6), stress f_2 can be worked out and then from equation (5) stress f_1 can be calculated.

Example 2·1-1. A steel bar 50 mm diameter is completely encased in a brass tube of 80 mm outside diameter. The length of the composite bar is 400 mm. If the assembly is subjected to a compressive force of 80 kN determine

(*i*) stresses in steel bar and brass tube (*ii*) change in the length of the assembly.

Given $E_{steel} = 208 \times 1000$ N/mm²

$E_{brass} = 104 \times 1000$ N/mm².

Solution. Area of cross section of steel bar,

$$A_s = \frac{\pi}{4}(50)^2 = 1963{\cdot}5 \text{ mm}^2$$

Area of cross section of brass tube,

$$A_b = \frac{\pi}{4}(80^2 - 50^2) = 3063{\cdot}06 \text{ mm}^2$$

Say the stress in steel bar $= f_s$

Stress in brass tube $= f_b$

But $\frac{f_s}{f_b} = \frac{E_s}{E_b}$

$$f_s = \frac{208 \times 1000}{104 \times 1000} \times f_b = 2 f_b \quad \ldots(1)$$

Total load $P = 80$ kN $= 80{,}000$ N $= P_s + P_b$ (compressive load)

where P_s = load shared by steel bar $= f_s A_s$

P_b = load shared by brass tube $= f_b A_b$

So $80{,}000 = f_s \times 1963{\cdot}5 + f_b \times 3063{\cdot}06$

$= 2f_b \times 1953{\cdot}5 + f_b \times 3063{\cdot}06$

or $$f_b = \frac{80000}{6990{\cdot}06} = \mathbf{11{\cdot}44} \text{ N/mm}^2 \text{ (compressive)}$$

$f_s = 2f_s = \mathbf{22{\cdot}88}$ N/mm² (compressive)

Change in length, $$\delta l_s = \frac{f_s}{E_s} = \frac{22{\cdot}88 \times 400}{208 \times 1000}$$

$= 0{\cdot}044$ mm = change in length in brass tube

Change in length of composite bar

$= \mathbf{0{\cdot}044}$ **mm.**

Exercise 2·1-2. A steel rod of 2 cm diameter is fully encased in an aluminium tube of outside diameter 4 cm, so as to make a composite bar. The assembly is subjected to a tensile force of 4 tonnes. Determine the stresses developed in steel rod and aluminium tube and change in the length of the assembly if its length is 1 metre.

Given $E_{steel} = 2100$ tonnes/cm²

$E_{aluminium} = 1/3\ E_{steel}$.

[**Ans.** 0·6366, 0·2122 tonne/cm² (tensile), 0·303 mm]

2·2. COMPOSITE BAR WITH MORE THAN 2 BARS OF DIFFERENT MATERIALS

A composite bar can be a combination of two or more than two bars of different materials. The Fig. 2·2 shows a composite bar of 3 different materials. Three bars of cross

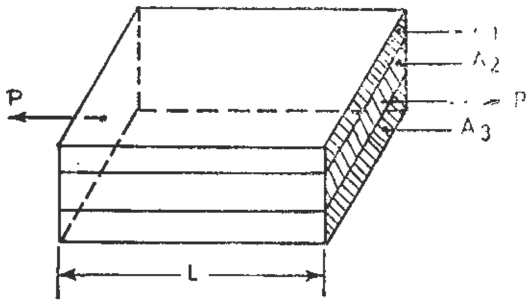


Fig. 2·2

sectional areas A_1, A_2 and A_3 respectively but of same length L and of different materials are perfectly joined together, say the Young's modulus of elasticity of these bars is respectively E_1, E_3 and E_3.

The composite bar is subjected to an axial tensile force P as shown. The load will bc shared by each rod *i.e.*,

$$P=P_1+P_2+P_3=f_1\,A_1+f_2\,A_2+f_3\,A_3$$

where f_1, f_2 and f_3 are the stresses developed in each bar.

But $$f_1=\epsilon_1\,E_1,\quad f_2=\epsilon_2\,E_2,\quad f_3=\epsilon_3\,E_3$$

But in a composite bar $\epsilon_1=\epsilon_2=\epsilon_3=\dfrac{\delta L}{L}=\dfrac{\text{change in length}}{\text{original length}}$

So $$P=\frac{\delta L}{L}\,[E_1\,A_1+E_2\,A_2+E_3\,A_3]$$

or change in length, $$\delta L=\frac{PL}{A_1E_1+A_2E_2+A_3E_3}$$

then $$\epsilon_1=\epsilon_2=\epsilon_3 \text{ or } \frac{\delta L}{L}=\frac{f_1}{E_1}=\frac{f_2}{E_2}=\frac{f_3}{E_3}$$

$$f_1=\frac{\delta L}{L}\times E_1 \text{ and } P_1=f_1A_1=\frac{\delta L}{L}\times E_1A_1$$

So the load shared by bar 1,

$$P_1=\frac{P\,E_1A_1}{E_1A_1+E_2A_2+E_3A_3}.$$

Similarly the load shared by other bars can be determined. Say in a composite bar there are n bars of same length but with areas of cross sections A_1, A_2, $A_3 \ldots A_n$ respectively.

Load shared by any ith bar ($i < n$)

$$P_i = \frac{P\,E_iA_i}{E_1A_1+E_2A_2+\ldots\ldots E_nA_n}$$

Stress in the ith bar $= \dfrac{P_i}{A_i}$.

Example 2·2-1. A flat bar of steel 2·5 cm wide and 5 mm thick is placed between two aluminium alloy flats, each 2·5 cm wide and 10 mm thick to form a composite bar of section 25 mm×25 mm. The three flats are fastened together at their ends. An axial tensile load of 2000 kg is applied to the composite bar. What are the stresses developed in steel and aluminium alloy.

$$E_{steel} = 2100 \text{ tonnes/cm}^2$$

$$E_{aluminium\ alloy} = 700 \text{ tonnes/cm}^2.$$

Solution. The figure 2·3 shows the cross section of the composite bar.

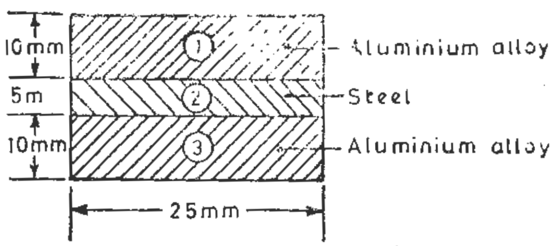


Fig. 2·3

Cross sectional areas, $A_1=10\times25=250$ mm^2, $A_2=125$ mm^2, $A_3=250$ mm^3

Modulus of elasticity, $E_1=700$ tonnes/cm^2, $E_2=2100$ tonnes/cm^2

$E_3=700$ tonnes/cm^2

$$E_1A_1=700\times2\text{·}5=1750 \text{ tonnes}\ ;\ E_2A_2=2100\times1\text{·}25=2625 \text{ tonnes}$$

$$E_3A_3=700\times2\text{·}5=1750 \text{ tonnes}$$

$$E_1A_1+E_2A_2+E_3A_3=6125$$

Load applied, $P=2000$ kg $=2$ tonnes

Load shared by each bar

$$P_1=P_3=\frac{2\times1750}{6125}=0\text{·}5715 \text{ tonne}$$

$$P_2=\frac{2\times2625}{6125}=0\text{·}857 \text{ tonne}$$

Stresses in each bar $\quad f_1=f_3=\dfrac{0\text{·}5715}{2\text{·}5}=0\text{·}2286$ tonne/cm^2

$$f_2=\frac{0\text{·}857}{1\text{·}25}=\text{·}6856 \text{ tonne/cm}^2$$

Stress in aluminium alloy $=0{\cdot}2286$ tonnes/cm^2

$=228{\cdot}6$ kg/cm^2 (tensile)

Stress in steel $=0{\cdot}6856$ tonne/cm$^2=685{\cdot}6$ kg/cm^2 (tensile).

Exercise 2·2-1. A steel bar of section 30 mm$\times$10 mm is placed between bars of aluminium and brass of section 30 mm$\times$10 mm each. The bars are fastened together at the ends. An axial compressive force of 60 kN is applied to the composite bar. Determine the stresses produced in steel, aluminium and brass bars. Take,

$$E_{steel}=210\times10^3 \text{ N/mm}^2$$
$$E_{aluminium}=70\times10^3 \text{ N/mm}^2$$
$$E_{brass}=105\times10^3 \text{ N/mm}^2.$$

Calculate also the change in length of the composite bar which is 2 m long.

[**Ans.** 36·36, 109·1, 54·53 N/mm^2 (compressive stresses) ; 1·09 mm]

2·3. COMPOSITE SYSTEMS

Two or more bars or wires of different materials may support a load. This type of system is called a composite system. Fig. 2·4 shows two bars of different materials and different areas of cross section but of the same length carrying a load through a horizontal bar.

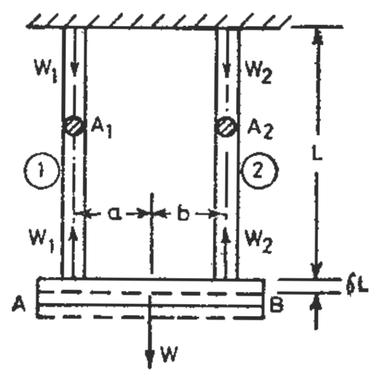


Fig. 2·4

Say the area of cross section of bar 1 is A_1 and that of bar 2 is A_2 ; Modulus of elasticity for bar 1 is E_1 and that for bar 2 is E_2. The load W is placed at such a position that the bar AB remains horizontal, which means

Elongation in bar 1 = Elongation in bar 2 $=\delta l$

or Strain in bar 1, $\epsilon_1=$strain in bar 2, ϵ_2

Load W will be shared by the two bars

$$W=W_1+W_2=f_1A_1+f_2A_2$$

where f_1 and f_2 are the stresses developed in bars 1 and 2 respectively.

But $$f_1=\epsilon_1E_1 \quad\text{and } f_2=\epsilon_2E_2$$

$$f_1=\frac{\delta L}{L}\times E_1 \quad\text{and } f_2=\frac{\delta L}{L}\times E_2$$

$$W=\frac{\delta L}{L}.\ E_1A_1+\frac{\delta L}{L}.\ E_2A_2$$

or $$\delta L=WL\left(\frac{1}{E_1A_1+E_2A_2}\right) \qquad ...(1)$$

Moreover $\epsilon_1=\epsilon_2=\dfrac{f_1}{E_1}=\dfrac{f_2}{E_2}$

or $$f_1=\frac{E_1}{E_2}\times f_2 \qquad ...(2)$$

$$W=f_1A_1+f_2A_2=\frac{E_1}{E_2}\times f_2A_1+f_2A_2 \qquad ...(3)$$

The stresses in bars can be determined by using equations (2) and (3),

Load shared by each bar

$$W_1=f_1A_1 \text{ and } W_2=f_2A_2 \qquad ...(4)$$

Position of the load W. Taking moments of the forces about the point A

$$W_1\times 0+Wa=W_2\,(a+b)$$

or $$W_2\,(a+b)=Wa$$

$$a=\frac{W_2(a+b)}{W}. \qquad ...(5)$$

Let us consider that the load is placed at the centre of the bar AB. then

load shared by bar 1=load shared by bar 2

$$=\frac{W}{2}$$

Elongation in the bar 1, δL_1

$$=\frac{W}{2A_1}\times\frac{L}{E_1}$$

Elongation in the bar 2, δL_2

$$=\frac{W}{2A_2}\times\frac{L}{E_2}$$

Depending upon the value of A_1E_1 and A_2E_2. δL_1 may not be equal to δL_2

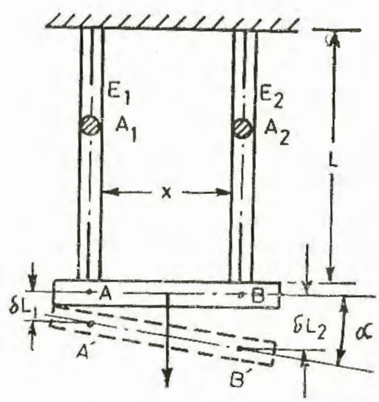


Fig. 2·5

Say $\delta L_2>\delta L_1$.

The bar AB will no longer remain horizontal but will now be inclined at an angle α

$$\tan\alpha=\frac{\delta L_2-\delta L_1}{x}$$

Example 2·3-1. A rigid bar is suspended from two wires of equal lengths 1·25 metres. One wire is 1 mm in diameter and made of steel, other wire of 2 mm diameter is made of brass. A load 10 kg is placed on the rigid bar such that the bar remains horizontal. If the horizontal distance between wires is 20 cm, determine.

(*i*) Stresses developed in steel and brass wires.

(*ii*) Elongation in the steel and brass wires.

(*iii*) Distance of the load from the steel wire.

Given $E_{steel}=2\,E_{brass}=210\times10^3$ N/mm².

Solution. Area of cross section of steel wire, $A_s = \frac{\pi}{4}(1)^2 = 0{\cdot}7854 \text{ mm}^2$

Area of cross section of brass wire,

$$A_b = \frac{\pi}{4}(2)^2 = 3{\cdot}1416 \text{ mm}^2$$

The bar AB (Fig. 2·6) is to remain horizontal

Strain in steel wire

= strain in brass wire

$\frac{f_s}{E_s} = \frac{f_b}{E_b}$ where f_s and f_b are the stresses developed in steel and brass wires

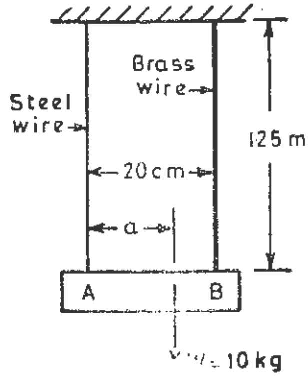


Fig. 2·6

$$f_s = \frac{E_s}{E_b} \times f_b = 2f_b. \qquad \ldots(1)$$

Load shared by steel wire, $W_s = f_s \times A_s = 0{\cdot}7854\, f_s$

Load shared by brass wire, $W_b = f_b \times A_b = 3{\cdot}1416\, f_b$.

But $\qquad W = W_s + W_b$ for equilibrium

$$9{\cdot}8 \times 10 = 0{\cdot}7854 f_s + 3{\cdot}1416 f_b$$

$$= 0{\cdot}7854 \times 2 \times f_b + 3{\cdot}1416 f_b$$

or Stress in brass wire, $\qquad f_b = \frac{98}{4{\cdot}7124} = 20{\cdot}8 \text{ N/mm}^2$

Stress in steel wire, $\qquad f_s = 2f_b = 41{\cdot}6 \text{ N/mm}^2$.

Load shared by brass wire, $W_b = 20{\cdot}8 \times 3{\cdot}1416 = 65{\cdot}34$ N

Elongation in steel wire, $\qquad \delta L_s = \frac{f_s}{E_s} \times L = \frac{41{\cdot}6}{210 \times 1000} \times 1250$

where L is the length of the wire

$= 0{\cdot}248$ mm = Elongation in brass wire, δL_b

Taking moments of the forces about the point A

$$W \times a = W_b \times 20$$

$$a = \frac{W_b \times 20}{W} = \frac{65.34 \times 20}{98}$$

Distance of load axis from steel wire $= 13{\cdot}33$ cm.

Example 2·3-2. A rigid bar is supported by wires of steel and aluminium alloy, each 1·5 m long. The diameter of each wire is 1·2 mm. A load of 20 kg is placed at the middle of the bar as shown in Fig. 2·7. After the load is applied, the bar is inclined to the horizontal. Determine this angle made by the rigid bar. The distance between the wires is 20 cm.

Given $E_{steel}=2{\cdot}1\times10^6$ kg/cm²

$E_{aluminium\ alloy}=0{\cdot}7\times10^6$ kg/cm².

Solution. Since the load is placed at the middle of the rigid bar AB. Load shared by steel wire

$=$Load shared by aluminium alloy wire

$=\frac{W}{2}=10$ kg.

Length of each wire, $L=150$ cm.

Diameter of each wire $=1{\cdot}2$ mm

$=0{\cdot}12$ cm

Change in length of steel wire,

$$\delta L_s=\frac{10\times4}{\pi(\cdot12)^2}\times\frac{150}{2{\cdot}1\times10^6}=\cdot063 \text{ cm}$$

Change in length of aluminium alloy wire

$$\delta L_a=\frac{10\times4}{\pi\times(0{\cdot}12)^2}\times\frac{150}{0{\cdot}7\times10^6}$$

$=0{\cdot}189$ cm

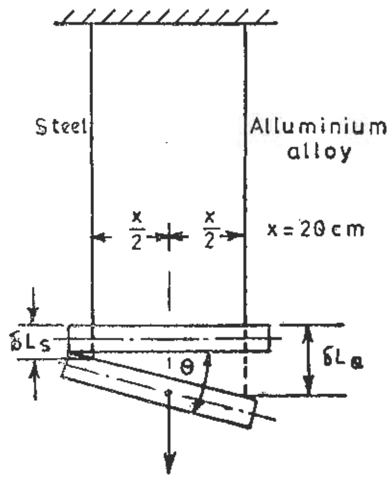


Fig. 2·7

Say the angle of inclination of the bar is θ

$$\tan\theta=\frac{\delta l_a-\delta l_s}{x}=\frac{0{\cdot}189-\cdot063}{20}=\frac{0{\cdot}126}{20}=\cdot0063$$

$\theta=0°\ 22'$.

Exercise 2·3-1. Two wires each of diameter 2 mm and length 2 metre are securely fixed at the top. At the bottom a rigid bar is suspended, keeping the distance between the wires equal to 0·25m. One wire is of brass and the other wire is of aluminium. A load of 30 kg is applied at the bar in a manner that bar remains horizontal. Determine the stress in each wire and change in their lengths

Given $E_{brass}=1000$ tonnes/cm²

$E_{aluminium}=700$ tonnes/cm². [**Ans.** 561·7 kg/cm², 393·2 kg/cm², 0·112 cm.]

Exercise 2·3-2. A rigid bar is suspended from two wires A and B. Wire A is of steel with diameter 2 mm, wire B is of brass with diameter 1 mm. Length of each wire is 2·5 metres. Horizontal distance between the wires is 240 mm. A load of 300 N is placed on the bar at a distance of 80 mm from the brass wire. The rigid bar is inclined with the horizontal, determine the angle of inclination of the bar.

$E_{steel}=2\ E_{brass}=210\times10^3$ N/mm². [**Ans.** 0° 16′]

2·4. BARS OF DIFFERENT LENGTHS SUBJECTED TO LOADS

Fig. 2·8 shows a rod of material 1 and diameter d placed co-axially in a tube of outside diameter D_2, inside diameter D_1 of material 2. The length of the tube L_2 is slightly less than the length of the rod L_1.

i.e., $L_1 = L_2 + C$

where C = clearance between rod and tube.

Area of cross section of rod,

$$A_1 = \frac{\pi}{4}(d^2)$$

Area of cross section of tube,

$$A_2 = \frac{\pi}{4}(D_2^2 - D_1^2)$$

Say, the Young's Modulus of elasticity of rod $= E_1$

Young's modulus of elasticity of tube $= E_2$.

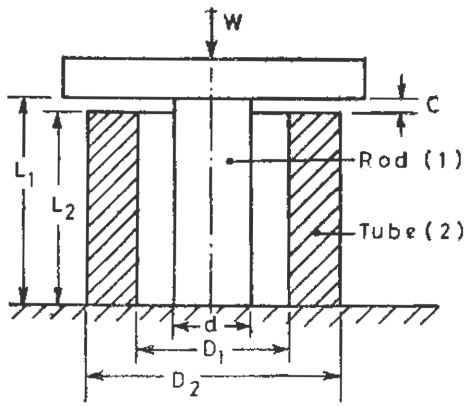


Fig. 2·8

The load W applied axially on the assembly would compress the rod and tube simultaneously and load will be distributed between rod and the tube.

Say the stress developed in rod $= f_1$

Stress developed in tube $= f_2$

Change in the length of rod, $\delta L_1 = \frac{f_1}{E_1} \times L_1$

Change in the length of the tube, $\delta L_2 = \frac{f_2}{E_2} \times L_2$

But $\delta L_1 = \delta L_2 + C$

So $\frac{f_1}{E_1} \times L_1 = \frac{f_2}{E_2} \times L_2 + C$

or
$$f_1 = f_2 \times \frac{E_1}{E_2} \times \frac{L_2}{L_1} + C\ \frac{E_1}{L_1}. \quad \ldots(1)$$

From this equation relationship between f_1 and f_2 is obtained.

Total load, $W = W_1 + W_2$

$$= f_1 A_1 + f_2 A_2$$

$$= \left(f_2 \times \frac{E_1}{E_2} \times \frac{L_2}{L_1} \times A_1 + C\ \frac{E_1}{L_1}\ A_1 \right) + f_2 A_2 \quad \ldots(2)$$

From equation (2) stress f_2 in tube is determined and then from equation (1) stress f_1 in rod is determined.

Example 2·4-1. A steel rod 20 mm diameter 1·5 m long is placed co-axially in an aluminium tube of outer diameter 60 mm and inner diameter 30 mm and 1500·4 mm length.

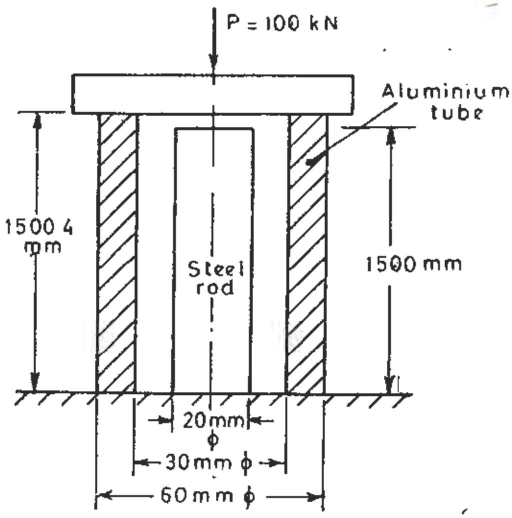


Fig. 2·9

A compressive force of 100 kN is applied on the assembly as shown in the Fig. 2·9. Determine the stresses developed in steel rod and aluminium tube.

Given $E_{steel}=210\times10^3$ N/mm²

$E_{aluminium}=70\times10^3$ N/mm²

Solution.

(*i*) Say the load is taken up by the aluminium tube only

Area of cross section of aluminium tube,

$$A_a=\frac{\pi}{4}(60^2-30^2)=2120{\cdot}58 \text{ mm}^2$$

Change in the length $=\dfrac{P}{A_a}\times\dfrac{L_a}{E_a}=\dfrac{100\times1000}{2120{\cdot}58}\times\dfrac{1500{\cdot}4}{70\times1000}=1{\cdot}01$ mm.

Note that 1·01 mm is greater than 1500·4−1500=0·4 mm *i.e.*, clearance between the rod and tube. So the load will be shared by both the rod and tube.

(*ii*) Say the stress developed in steel rod $=f_s$

Stress developed in aluminium tube $=f_a$

Area of cross section of steel rod, $A_s=\dfrac{\pi}{4}(20)^2=314{\cdot}16$ mm²

δL_s, change in length in steel rod $=\dfrac{f_s}{E_s}\times1500$

δL_a, change in length in aluminium tube $=\dfrac{f_a}{E_a}\times1500{\cdot}4$

but $\delta L_a = \delta L_s + 0{\cdot}4$

$$\frac{f_a}{E_a} \times 1500{\cdot}4 = \frac{f_s}{E_s} \times 1500 + 0{\cdot}4$$

Substituting the values of E_a and E_s $f_a = f_s \times \frac{70}{210} \times \frac{10^3}{10^3} \times \frac{1500}{1500{\cdot}4} + \frac{0{\cdot}4 \times 70 \times 10^3}{1500{\cdot}4}$

$$= 0{\cdot}333\, f_s + 18{\cdot}662.$$

But $P = P_a + P_s$ (load shared by the aluminium tube and steel rod)

$$100 \times 1000 = f_a \times 2120{\cdot}58 + f_s \times 314{\cdot}16$$

$$10{,}000 = (0{\cdot}333\, f_s + 18{\cdot}662)\ 2120{\cdot}58 + 314{\cdot}16\, f_s$$

$$100{,}000 - 39574{\cdot}26 = 1020{\cdot}313\, f_s$$

or stress in steel rod, $f_s = \frac{60425{\cdot}74}{1020{\cdot}313}$ N/mm² $= 59{\cdot}22$ N/mm²

Stress in aluminium tube, $f_a = 0{\cdot}333\, f_s + 18{\cdot}662$

$= 38{\cdot}382$ N/mm².

Exercise 2·4-1. A steel rod 40 mm diameter is placed co-axially inside a brass tube of inner diameter 42 mm and outer diameter 50 mm. The length of the brass tube is 2·5 metre

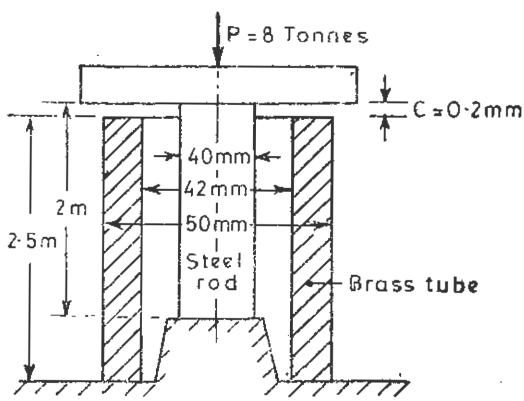


Fig. 2·10

and the length of steel rod is 2 metres. There is clearance at the top between rod and tube equal to 0·2 mm. The assembly is subjected to a compressive force of 5 tonnes. Determine the stresses developed in the rod and the tube.

Given that $E_{steel} = 2100$ tonnes/cm²

$E_{brass} = 1/2\ E_{steel}$. **[Ans.** 0·57 tonnes/cm², 0·144 tonnes/cm²]

2·5. BOLT AND TUBE ASSEMBLY TIGHTENED WITH A NUT

Fig. 2·11 shows an assembly of a bolt and a tube tightened with a nut. The figure shows a bolt passing through a tube co-axially. On both the ends of the tube there are washers. The nut is tightened on the threaded portion of the bolt, exerting pressure on the washers or consequently on the tube.

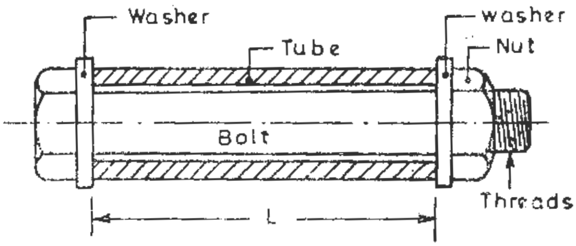


Fig. 2·11

When the nut is tightened on the bolt as shown, the bolt is extended and the tube is contracted, developing compressive stress in the tube and a tensile stress in the bolt.

Let the diameter of the bolt $= d$

Length of the bolt between the washers $= L$

Inner diameter of the tube $= D_1$

Outer diameter of the tube $= D_2$

Area of cross section of bolt $= A_1 = \frac{\pi}{4}\,(d^2)$

Area of cross section of tube $= A_2 = \frac{\pi}{4}\,(D_2^2 - D_1^2)$

Say Young's modulus of elasticity of bolt $= E_1$

Young's modulus of elasticity of tube $= E_2$

When the nut is being tightened, the bolt is extended and tube is reduced in length.

Extension in bolt + contraction in tube = axial movement of the nut. ...(1)

Compressive stress is developed in tube and tensile stress is developed in the bolt, for equilibrium.

Tensile force in bolt = Compressive force in tube

Say the stress developed in bolt $= f_1$

stress developed in tube $= f_2$

then
$$f_1 A_1 = f_2 A_2 \qquad ...(2)$$

Moreover $\frac{f_1}{E_1} \times L + \frac{f_2}{E_2} + L =$ axial movement of not ...(1)

note that f_1 is a positive tensile stress and f_2 is a negative compressive stress.

Example 2·5-1. A steel bolt of diameter 1·8 cm passes co-axially through a copper tube of inner diameter 2 cm and outer diameter 3 cm and length 50 cm. Washers are placed at both the ends of the tube. The bolt has threads at one end with a pitch of 2·4 mm as shown in Fig. 2·11 The nut is turned on the bolt through 45° so as to tighten the assembly. Determine the stresses developed in the bolt and the tube.

$$E_s = 2E_c = 200{,}000 \text{ N/mm}^2.$$

Solution. Pitch of the threads,

$$p = 2·4 \text{ mm}$$

Angle through which nut is tightened

$$= 45°$$

So axial movement of nut$=\frac{2\cdot4}{360}\times45°=0\cdot3$ mm

Area of cross section of bolt,

$$A_s=\frac{\pi}{4}(18)^2=254\cdot47 \text{ mm}^2$$

Area of cross section of tube,

$$A_c=\frac{\pi}{4}(30^2-20^2)=392\cdot70 \text{ mm}^2$$

Say the stress developed in bolt

$=+f_s$ (tensile)

Stress developed in tube $=-f_c$ (compressive)

Due to the tightening of the nut, bolt is extended and tube is contracted.

Now $\quad f_sA_s-f_cA_c=0$

$$f_s\times254\cdot47=f_c\times392\cdot70$$

$$f_s=1\cdot543\,f_c \qquad \ldots(1)$$

Extension in bolt $=\frac{f_s}{E_s}\times L=\frac{f_s\times500}{200{,}000}=\frac{f_s}{400}$

Contraction in tube $=\frac{f_c}{E_c}\times L=\frac{f_c\times500}{100{,}000}=\frac{f_c}{200}$

Now axial movement of nut=extension in bolt+contraction in tube

$$0\cdot3=\frac{f_s}{400}+\frac{f_c}{200}$$

$$120=f_s+2f_c \qquad \ldots(2)$$

or $\quad 1\cdot543\,f_c+2f_c=120$

$$f_c=\frac{120}{3\cdot543}=33\cdot87 \text{ N/mm}^2 \text{ compressive)}$$

$$f_s=1\cdot543\times33\cdot87=52\cdot26 \text{ N/mm}^2 \text{ (tensile).}$$

Exercise 2·5-1. A central steel rod 25 mm diameter passes through a brass sleeve 30 mm inside and 40 mm outside diameter and 60 cm long. It is provided with nuts and washers at each end. A nut is tightened so as to produce a compressive stress of 600 kg/cm² in tube. If the pitch of the threads on rod is 3 mm, determine

(*i*) Stress developed in steel rod

(*ii*) Angle through which the nut is turned during tightening.

Given $\quad E_s=2E_c=2100$ tonnes/cm².

[**Ans.** (*i*) 671·85 kg/cm² (tensile), (*ii*) 64·2°]

2·6. TEMPERATURE STRESSES IN A SINGLE BAR

A bar of diameter d and length L is fixed between two rigid walls as shown in the Fig. 2·12.

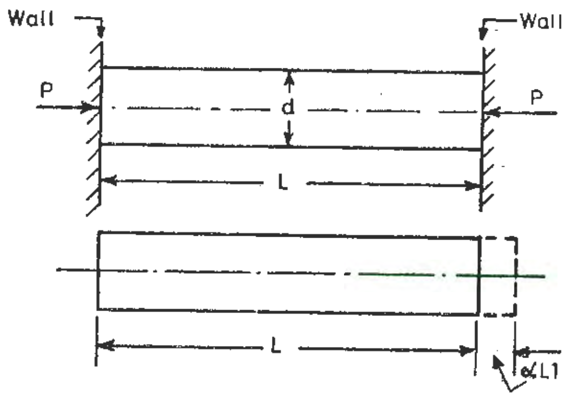


Fig. 2·12

The coefficient of linear expansion of the bar is α. The temperature of the bar is raised through $T°$. The bar is not free to expand. The bar tries to expand and exerts axial pressure on wall, and at the same time wall puts equal and opposite pressure on the bar.

If the bar is free to expand, then free expansion in the length of the bar$=\alpha LT$

Total length of the bar after expansion$=(L+\alpha LT)$.

But the initial and the final length of the bar after the temperature rise remains the same. In other words the wall exerts pressure on the bar and its length $(L+\alpha LT)$ is compressed to L.

So the change in length $=L+\alpha TL-L=\alpha LT$ (contraction)

Strain in the bar due to temperature rise,

$$\epsilon_T=\frac{\alpha LT}{L+\alpha LT}\simeq\frac{\alpha LT}{L}=\alpha T \text{ as } \quad \alpha LT<<L$$

f_T, stress in the bar due to temperature rise

$$=\epsilon_T\,E=\alpha TE \text{ (compressive)}$$

where E is the Young's modulus of the bar.

Similarly if there is a drop in the temperature of the bar, the bar will try to contract exerting pull on the wall and in turn the wall offers equal and opposite reaction exerting pull on the bar and developing tensile stress in the bar.

f_T, stress in bar due to fall in temperature

$$=\alpha TE \text{ (tensile)}$$

Example 2·6-1. A steel bar 2 cm in diameter, 2 metres long is rigidly held between two walls. The temperature of the bar is raised by 30°C. If the coefficient of linear expansion of steel is 11×10^{-6}/°C, determine

(*i*) Stress developed in bar

(*ii*) Force exerted by the wall on the bar.

Given $E_{steel} = 2080$ tonnes/cm².

Solution. Diameter of the steel bar,

$= 2$ cm

A, area of cross section of the bar

$$= \frac{\pi}{4}(2)^2 = 3{\cdot}1416 \text{ cm}^2$$

α, coefficient of linear expansion of steel

$= 11 \times 10^{-6}/°C$

Temperature rise, $T = 30°C$

$E_{steel} = 2080$ tonnes/cm²

Compressive strain in bar,

$$\epsilon_T = \alpha T = 11 \times 10^{-6} \times 30 = 330 \times 10^{-6}$$

Compressive stress in bar,

$$f_T = \alpha T E = 330 \times 10^{-6} \times 2080$$

$= 0{\cdot}6864$ tonnes/cm². **Ans.**

Force exerted by the wall,

$$P = f_T \times A = 0{\cdot}6864 \times 3{\cdot}1416$$

$= 2{\cdot}156$ tonnes. **Ans.**

Exercise 2·6-2. A copper bar of square section 3 cm × 3 cm and length 1 metre is held between rigid fixtures There is drop in the temperature of the bar by 20°F. Determine

(*i*) Stress developed in bar

(*ii*) Force exerted by the rigid fixture.

Given $E_{copper} = 100{,}000$ N/mm²

$\alpha_{copper} = 10 \times 10^{-6}/°F$.

[**Ans.** (*i*) 20 N/mm² (tensile), (*ii*) 18 kN]

2·7. TEMPERATURE STRESSES IN A COMPOSITE BAR

A composite bar made up of two bars *I* and *II* of different materials is shown in the Fig. 2·13. The length of the two bars is the same, say L. The area of cross section of bar

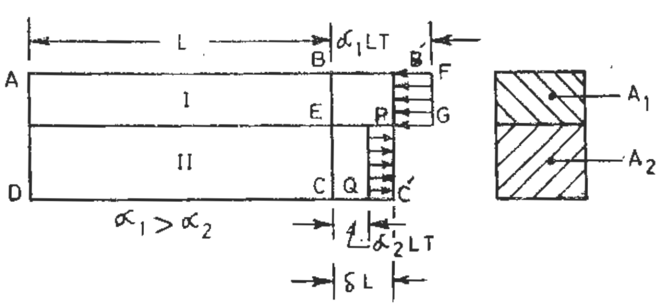


Fig. 2·13

1 is A_1 and area of cross section of bar 2 is A_2. The coefficient of linear expansion of bar 1 is α_1 while the coefficient of linear expansion of bar 2 is α_2. Say $\alpha_1 > \alpha_2$. Both the bars are permanently fixed together so as to form a composite bar. Now say the temperature of the composite bar is increased by T°. The length of the composite bar is increased by δL, *i.e.*, BB' or CC' as shown in the figure.

If the bar 1 is free to expand independently then it would expand or change in its length$=BF=EG=\alpha_1 LT$.

Similarly if the bar 2 is free to expand independently, then it would expand or change in its length$=EP=CQ=\alpha_2 LT$.

But in a composite bar, both the bars 1 and 2 expand unitedly, by the same amount *i.e.*, δL

$$\delta L < \alpha_1 LT, \quad \delta L > \alpha_2 LT$$

Contraction in the freely expanded length of bar 1

$$=\alpha_1 LT-\delta L$$

Extension in the freely expanded length of bar 2

$$=\delta L-\alpha_2 LT$$

Compressive strain in bar 1,

$$\epsilon_1=\frac{\alpha_1 LT-\delta L}{L+\alpha_1 LT}\simeq\frac{\alpha_1 LT-\delta L}{L}$$

Since $\qquad \alpha_1 LT << L$

Tensile strain in bar 2, $\epsilon_2=\dfrac{\delta L-\alpha_2 LT}{L+\alpha_2 LT}\simeq\dfrac{\delta L-\alpha_2 LT}{L}$

Contraction in length of bar 1,

$$\delta L_1=\frac{(\alpha_1 LT-\delta L)}{L}\times L=\alpha_1 LT-\delta L$$

Extension in length of bar 2,

$$\delta L_2=\left(\frac{(\delta L-\alpha_2 LT)}{L}\right) L=\delta L-\alpha_2 LT$$

Contration δL_1+extension, $\delta L_2=\alpha_1 LT-\alpha_2 LT=LT\,(\alpha_1-\alpha_2)$

Say the stress developed in bar 1,

$$=f_{1T} \text{ (compressive)}$$

Stress developed in bar 2$=f_{2T}$ (tensile)

For equilibrium compressive force in bar 1=Tensile force in bar 2

i.e., $\qquad f_{1T}\times A_1=f_{2T}\times A_2 \qquad \ldots(2)$

Contraction, $\qquad \delta L_1=\dfrac{f_{1T}}{E_1}\times L$

Extension, $\qquad \delta L_2=\dfrac{f_{2T}}{E_2}\times L$

where E_1 and E_2 are the Young's modulus of elasticity of bars 1 and 2 respectively.

From equation (1)

$$\frac{f_{1T}}{E_1}+\frac{f_{2T}}{E_2}=T\ (\alpha_1-\alpha_2) \qquad ...(1)$$

From equations (1) and (2) temperature stresses in bars 1 and 2 can be determined.

Example 2·7-1. A flat steel bar 20 mm×8 mm is placed between two aluminium alloy bars 20 mm×6 mm each, so as to form a composite bar of section 20 mm×20 mm. The three bars are fastened together at their ends when the temperature of each is 75°F. Find the stress in each when the temperature of the whole assembly is raised to 125°F. Determine the temperature stresses developed in the steel and aluminium alloy bars.

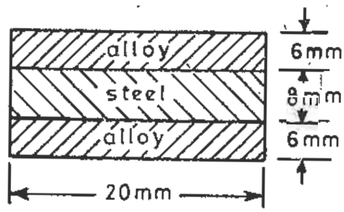


Fig. 2·14

$$E_s=210{,}000\ \text{N/mm}^2 \quad E_a=70{,}000\ \text{N/mm}^2$$

$$\alpha_s=6{\cdot}4\times10^{-6}/°\text{F} \qquad \alpha_a=12{\cdot}8\times10^{-6}/°\text{F}.$$

Say the stress developed in steel due to temperature rise $=f_{ST}$

Stress developed in aluminium alloy due to temperature rise $=f_{aT}$

$$E_s=210\times10^3\ \text{N/mm}^2,\ E_a=70\times10^3\ \text{N/mm}^2$$

$$\alpha_s=6{\cdot}4\times10^{-6}/°\text{F}, \qquad \alpha_a=12{\cdot}8\times10^{-6}/°\text{F}$$

Temperature rise, $T=125-75=50°\text{F}$

Area of cross section, $A_s=20\times8=160\ \text{mm}^2$

Area of cross section, $A_a=2\times20\times9=240\ \text{mm}^2$

Now $\quad f_{ST}\times A_s=f_{aT}\times A_a$

$$f_{ST}\times160=f_{aT}\times240$$

or $\quad f_{ST}=1{\cdot}5\,f_{aT} \qquad ...(1)$

$$\frac{f_{ST}}{E_s}+\frac{f_{aT}}{E_a}=T\ (\alpha_a-\alpha_s)$$

$$=50\ (12{\cdot}8-6{\cdot}4)\times10^{-6}=320\times10^{-6}$$

or $\quad \dfrac{f_{ST}}{210{,}000}+\dfrac{f_{aT}}{70{,}000}=320\times10^{-6} \qquad ...(2)$

$$\frac{1{\cdot}5\,f_{aT}}{210{,}000}+\frac{f_{aT}}{70{,}000}=320\times10^{-6} \qquad ...(2)$$

or $\quad 1{\cdot}5\,f_{aT}=22{\cdot}4$

$$f_{aT}=14{\cdot}93\ \text{N/mm}^2$$

$$f_{ST}=1{\cdot}5\,f_{aT}=22{\cdot}40\ \text{N/mm}^2$$

Since $\quad \alpha_a>\alpha_s$

$$f_{aT}=14{\cdot}93\ \text{N/mm}^2\ \text{(compressive)}$$

$$f_{ST}=22{\cdot}40\ \text{N/mm}^2\ \text{(tensile)}.$$

Exercise 2·7-2. A compound bar 1·2 m long is made up of two pieces of metal ; one of steel and the other of copper. The area of cross section of steel is 40 cm² and that of copper is 25 cm². Both pieces are 1·2 m long and are rigidly connected together at both the ends, the temperature of the bar is now raised by 100°C. The bar is restrained against bending. Determine the temperature stress in both the material.

$$\alpha_s = 12 \times 10^{-6}/°C, \; E_s = 2100 \text{ tonnes/cm}^2$$

$$\alpha_c = 17{\cdot}5 \times 10^{-6}/°C, \; E_c = 1000 \text{ tonnes/cm}^2.$$

[**Ans.** 0·265 tonne/cm² (tensile) in steel
0·424 tonne/cm² (compressive) in copper]

Problem 2·1. A weight of 25 tonnes is supported by a short concrete column 25 cm × 25 cm in section strengthened by 4 steel bars in the corners of the cross section. The diameter of each steel bar is 3 cm. Find the stresses in steel and in concrete.

$$E_s = 15E_c = 2100 \text{ tonnes/cm}^2.$$

If the stress in the concrete must not exceed 20 kg/cm², what area of steel is required in order that the column may support a load of 40 tonnes.

Solution. Load, $P = 25$ tonnes

Say the stresses developed

in steel $= f_s$

in concrete $= f_c$

Area of cross section of steel,

$$A_s = \frac{\pi}{4}(3)^2 \times 4 = 28{\cdot}27 \text{ cm}^2$$

Area of cross section of concrete,

$$A_c = 25 \times 25 - 28{\cdot}27 = 596{\cdot}73 \text{ cm}^2$$

Concrete column reinforced with steel bar is a composite bar

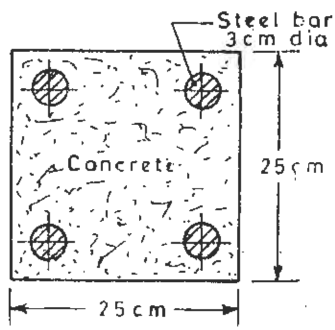


Fig. 2·15

strain in steel = strain in concrete

$$\frac{f_s}{E_s} = \frac{f_c}{E_c}$$

or $$f_s = f_c \times \frac{E_s}{E_c} = 15 f_c. \quad \ldots(1)$$

Now $$P = f_s A_s + f_c A_c = 15 f_c \cdot A_s + f_c A_c$$

$$25 = 15 \times f_c \times 28{\cdot}27 + f_c \times 596{\cdot}73$$

or $$f_c = \frac{25}{596{\cdot}73 + 424{\cdot}05} = \frac{25}{1020{\cdot}78} = 0{\cdot}0245 \text{ tonnes/cm}^2$$

$$f_s = 15 f_c = 0{\cdot}3675 \text{ tonne/cm}^2.$$

(*b*) Now the allowable stress in concrete = 20 kg/cm²

So maximum stress in steel $= 15 \times 20 = 300$ kg/cm²

Say the area of steel section $= A_s'$; $A_c' = (225 - A_s')$

Load $P=40$ tonnes $=40{,}000$ kg $=f_s A_s'+f_c A_c'$

$$40{,}000=300\times A_s'+20\times(625-A_s')$$

$$40{,}000=12{,}500=280\ A_s'$$

$$A_s'=\frac{27500}{280}$$

Area of the steel required $=$ **98·2 cm²**.

Problem 2·2. A short, hollow cast iron column 20 cm external diameter and 16 cm internal diameter is filled with concrete as shown in Fig. 2·16. The column carries a total load of 30 tonnes. If $E_{CI}=6E_{concrete}$ calculate the stresses in the cast iron and the concrete.

What must be the internal diameter of the cast iron column if a load of 40 tonnes is to be carried, the stresses and the external diameter being unchanged.

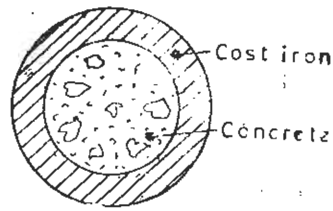


Fig. 2·16

Solution. (*a*) Load on column, $P=30$ tonnes

Area of cross section of cast iron, $A_1=\frac{\pi}{4}(20^2-16^2)=113{\cdot}098\text{ cm}^2$

Area of cross section of concrete, $A_2=\frac{\pi}{4}\times16^2=201{\cdot}062\text{ cm}^2$.

Say the stress developed in cast iron $=f_1$ (compressive)

Stress developed in concrete $=f_2$ (compressive)

(As the load on the column is a compressive force)

$$\frac{f_1}{E_1}=\frac{f_2}{E_2}\text{ (in a composite bar)}$$

$$f_1=\frac{E_1}{E_2}f_2=6\,f_2\text{ (as given in the problem)}$$

So

$$f_1A_1+f_2A_2=30{,}000\text{ kg}$$

$$6f_2A_1+f_2A_2=30{,}000$$

$$f_2[6\times113{\cdot}098+201{\cdot}062]=30{,}000$$

Stress in concrete, $f_2=\frac{30{,}000}{879{\cdot}65}=34{\cdot}10\text{ kg/cm}^2$

Stress in cast iron, $f_1=6f_2=204{\cdot}60\text{ kg/cm}^2$

(*b*) Load $=40{,}000$ tonnes

Stress in cast iron, $f_1=204{\cdot}60\text{ kg/cm}^2$

Stress in concrete, $f_2=34{\cdot}10\text{ kg/cm}^2$

Outside diameter of cast iron column $=20$ cm

Say inside diameter of cast iron column

$$=d \text{ cm}$$

Area of cross section of cast iron, $A_1' = \frac{\pi}{4}(20^2 - d^2) = \frac{\pi}{4}(400 - d^2)$ cm²

Area of cross section of concrete, $A_2' = \frac{\pi}{4} d^2$ cm²

Now $\qquad f_1A_1' + f_2A_2' = 40{,}000$

$$204.6 \times \frac{\pi}{4}(400 - d^2) + 34.1 \times \frac{\pi}{4} \times d^2 = 40{,}000$$

$$64277.13 - 160.69\, d^2 + 26.78\, d^2 = 40{,}000$$

$$24277.13 = 133.91\, d^2$$

$$d^2 = \frac{24277.13}{133.91} = 181.29$$

Internal dia. of column, $\qquad d = 13.464$ cm.

Problem 2·3. A circular ring is suspended by 3 vertical bars A, B and C, of different lengths. The upper ends of the bars are held at different levels. Bar A is 2 metres long, 15 mm diameter. Bar B is 1·6 metres long, 12 mm diameter and Bar C is 1 metre long and 18 mm diameter. Bar A is of steel, B of copper and C of aluminium. A load of 30 kN is hung on the ring. Calculate how much of this load is carried by each bar, if the circular ring remains horizontal after the application of the load.

$$E_s = 2E_c = 3E_a = 210 \times 1000 \text{ N/mm}^2$$

s stand for steel, c for copper and a for aluminium.

Solution. Area of cross sections of bars

$$A_A = \frac{\pi}{4}(15)^2 = 176.71 \text{ mm}^2$$

$$A_B = \frac{\pi}{4}(12)^2 = 113.098 \text{ mm}^2$$

$$A_C = \frac{\pi}{4}(18)^2 = 254.47 \text{ mm}^2.$$

Length of the bars, $L_A = 2000$ mm

$L_B = 1600$ mm

$L_C = 1000$ mm.

Modulus of Elasticity, $E_A = 210 \times 1000$ N/mm²

$E_B = 105 \times 1000$ N/mm²

$E_C = 70 \times 1000$ N/mm².

Say the change in length in each bar $= \delta L$

$$\delta L = \frac{f_A}{E_A} \times L_A = \frac{f_B}{E_B} \times L_B = \frac{f_C}{E_C} \times L_C$$

or

$$f_A = \delta L \cdot \frac{E_A}{L_A} = \delta L \times \frac{210 \times 1000}{2000} = 105\, \delta L$$

$$f_B = \delta L \cdot \frac{E_B}{L_B} = \delta L \times \frac{105 \times 1000}{1600} = 65{\cdot}62\ \delta L$$

$$f_C = \delta L \cdot \frac{E_C}{L_C} = \delta L \times \frac{70 \times 1000}{1000} = 70\ \delta L.$$

Now $f_A A_A + f_B A_B + f_C A_C = 30{,}000$ N

$$(105 \times 176{\cdot}71 + 65{\cdot}62 \times 113{\cdot}098 + 70 \times 254{\cdot}47)\ \delta L = 30{,}000. \quad \ldots(1)$$

$$\delta L = \frac{30{,}000}{18554{\cdot}55 + 7421{\cdot}49 + 17812{\cdot}9}$$

$$= \frac{30000}{43788{\cdot}94} = 0{\cdot}6851 \text{ mm.}$$

Load shared by bars, $P_A = f_A A_A = 18554{\cdot}55 \times 0{\cdot}6851 = 12711{\cdot}8$ N

$P_B = f_B A_B = 7421{\cdot}49 \times 0{\cdot}6851 = 5084{\cdot}5$ N

$P_C = f_C A_C = 17812{\cdot}9 \times 0{\cdot}6851 = 12203{\cdot}7$ N.

[Note that values of $f_A A_A$, $f_B A_B$ and $f_C A_C$ are taken from equation (1) above].

Problem 2·4. Prestressed concrete beam is fabricated as follows :

(*a*) A rod is loaded between the plates under tension, f_s.

(*b*) Then the concrete is poured to form a beam of the section shown.

(*c*) After the concrete is properly set the external force P is removed, and the beam is left in a prestressed condition.

If $\frac{E_s}{E_c} = 15$ and their cross sectional areas are in the ratio $\frac{A_s}{A_c} = \frac{1}{15}$, what will be the final residual stresses in the two materials.

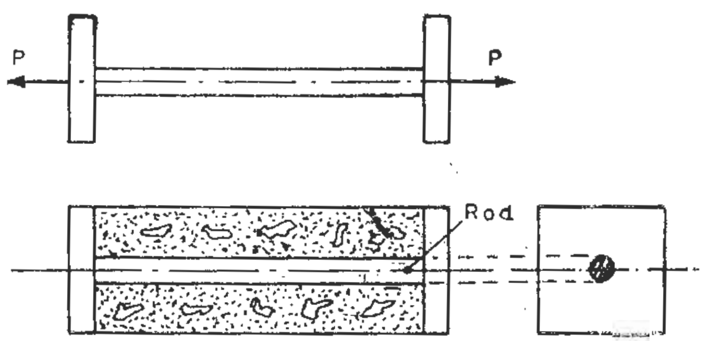


Fig. 2·17

Solution.

(*a*) Area of steel reinforcement $= A_s$

External force $= P$

Initial tensile stress in steel, $f_s=\frac{P}{A_s}$

After the concrete is properly set the external force P is removed.

The steel will try to contract but its contraction will be checked by the concrete portion because steel and concrete are bound together and released load will act as a compressive force on the composite beam.

Strain in steel bar=strain in concrete beam

$$\frac{f_s'}{E_s}=\frac{f_c'}{E_c}$$

where f_s'=stress in steel (compressive)

f_c'=stress in concrete (compressive)

$$\frac{f_s'}{f_c'}=\frac{E_s}{E_c}=15$$

Moreover $f_s' \cdot A_s+f_c' \cdot A_c=P$

$$f_s' \cdot A_s+\frac{f_s'}{15}\,15 \cdot A_s=P$$

$$2 \cdot f_s' \cdot A_s=P$$

$$f_s'=\frac{P}{2A_s}$$

$$f_c'=\frac{f_s'}{15}=\frac{P}{30\,A_s}$$

Final stresses, in steel, $f_s''=\frac{P}{A_s}-\frac{P}{2A_s}=\frac{P}{2A_s}$ (tensile)

In concrete, $f_c'=\frac{P}{30\,A_s}$ (compressive)

$$=\frac{P}{2A_c}$$

Ratio, $\frac{f_s''}{f_c'}=\frac{P}{2A_s}\times\frac{2A_c}{P}=\frac{A_c}{A_s}\;\frac{\text{(tensile stress)}}{\text{(compressive stress)}}=-15$

Problem 2·5. A steel bar 2 m long with area 500 mm² for 80 cm length and 1000 mm² for 120 cm length. The bar is fitted between two rigid supports at top and bottom as shown in Fig. 2·18. A uniformly distributed load of 20 kN is applied on the shoulder. Determine the stresses developed in the upper and lower portions of the bar.

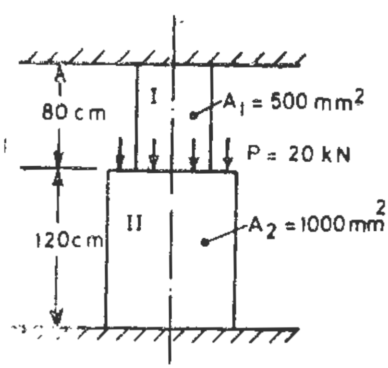


Fig. 2·18

Solution. When the load is applied, upper portion will come under tension and lower portion will come under compression.

Extension in upper portion = Contraction in lower portion, Say the stress in upper portion $= f_1$

Stress in lower portion $= f_2$

Young's modulus of elasticity of the material $= E$

Extension in upper portion $= \frac{f_1}{E} \times 80$

Contraction in lower portion $= \frac{f_2}{E} \times 120$

So $$\frac{f_1 \times 80}{E} = \frac{f_2 \times 120}{E}$$

$$f_1 = 1{\cdot}5\, f_2 \qquad ...(1)$$

Total force will be shared by the upper and lower portion

$$P = P_1 + P_2$$

$$20 \times 1000 = f_1 A_1 + f_2 A_2$$

$$= 1{\cdot}5\, f_2 \times 500 + f_2 \times 1000$$

Compressive stress in lower portion,

$$f_2 = \frac{20{,}000}{1750} = 11{\cdot}428 \text{ N/mm}^2$$

Tensile stress in upper portion,

$$f_1 = 1{\cdot}5\, f_2 = 1{\cdot}5 \times 11{\cdot}428 = 17{\cdot}142 \text{ N/mm}^2.$$

Problem 2·6. A steel rod 20 mm in diameter passes co-axially inside a brass tube of inner diameter 24 mm and outer diameter 40 mm It is provided with nuts and washers at each end and nuts are tightened until a stress of 150 kg/cm² is set up in steel.

The whole assembly is now placed in a lathe and a cut is taken along half the length of the tube reducing the outer diameter to 36 mm.

(*a*) Calculate the stress now existing in the steel.

(*b*) If an end thrust of 500 kg is applied at the ends of steel bar, calculate the final stress in steel.

$$E_{steel} = 2E_{brass} = 2100 \text{ tonnes/cm}^2$$

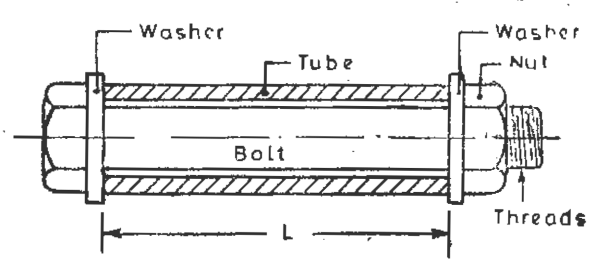


Fig. 2·19

Solution.

(1) Area of cross-section of steel rod, $A_s = \frac{\pi}{4}(2)^2 = 3{\cdot}1416$ cm²

Area of cross-section of brass tube, $A_B = \frac{\pi}{4}(4^2 - 2{\cdot}4^2) = 8{\cdot}0425$ cm²

Stress in steel, $f_s = 150$ kg/cm² (tensile)

Now for equilibrium Pull in steel rod = Push in brass tube

$$150\times 3\text{·}1416 = f_B \times 8\text{·}0425$$

or $$f_B = \frac{150\times 3\text{·}1416}{8\text{·}0425} = 58\text{·}59 \text{ kg/cm}^2 \text{ (compressive)}$$

(*ii*) When the tube is turned for half of its length on a lathe contraction of the tube will increase and extension of the rod is decreased.

Change in length of steel rod = change in length of brass tube

Say f_{B_1} = stress in the reduced section of the tube

f_{B_2} = stress in the uncut section of the tube

f_{S_1} = stress in the steel rod

Load on the tube = load on the rod

Reduced section of the tube

$$A_B' = \frac{\pi}{4}\ (3\text{·}6^2 - 2\text{·}4^2) = 5\text{·}655 \text{ cm}^2$$

$$f_{B_1}\times A_B' = f_{B_2}\times A_B = f_{S_1}\times A_S$$

$$f_{B_1}\times 5\text{·}655 = f_{B_2}\times 8\text{·}0425 = f_{S_1}\times 3\text{·}1416$$

or $$f_{B_1} = 0\text{·}555\, f_{S_1} \qquad \ldots(1)$$

$$f_{B_2} = 0\text{·}390\, f_{S_1} \qquad \ldots(2)$$

Now reduction in the length of the rod = Reduction in the length of the tube

$$\frac{f_S - f_{S_1}}{E_s}\times l = \frac{f_{B_1} - f_B}{E_B}\times\frac{l}{2} + \frac{f_{B_2} - f_B}{E_B}\times\frac{l}{2} \qquad \ldots(3)$$

or $$2(f_S - f_{S_1}) = \frac{E_S}{E_B}\ [(f_{B_1} - f_B) + (f_{B_2} - f_B)]$$

$$= 2(f_{B_1} + f_{B_2} - 2f_B)$$

$$f_S - f_{S_1} = f_{B_1} + f_{B_2} - 2f_B$$

Substituting the values of f_S and f_B

$$150 - f_{S_1} = 0\text{·}555\, f_{S_1} + 0\text{·}390\, f_{S_1} - 2\times 58\text{·}59$$

$$f_{S_1}\ (1 + 0\text{·}555 + 0\text{·}390) = 150 + 2\times 58\text{·}59 = 267\text{·}18$$

$$f_{S_1} = \frac{267\text{·}18}{1\text{·}945} = 137\text{·}37 \text{ kg/cm}^2.$$

Stress now existing in the steel rod = 137·37 kg/cm³.

(*iii*) When the end thrust is applied on the steel rod, there will be further reduction in its tension f_{S_1} and the compressive stresses in the two portions of tube will increase.

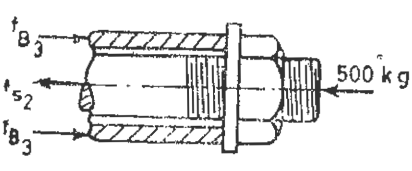


Fig. 2·20

Say f_{B_1} increases to f_{B_3} (in the reduced section)

f_{B_2} increases to f_{B_4} (in the uncut section).

So $\quad 500$=compressive force in tube—tensile force in rod (for equilibrium)

$$500=f_{B_3}\times 5{\cdot}655-f_{S_2}\times 3{\cdot}1416$$
$$=f_{B_4}\times 8{\cdot}0425-f_{S_2}\times 3{\cdot}1416$$

(Because the compressive force along the tube will be constant throughout its length).

$$f_{B_3}=88{\cdot}417+0{\cdot}555\,f_{S_2}$$
$$f_{B_4}=62{\cdot}170+0{\cdot}390\,f_{S_2}.$$

Reduction in length of the tube=Reduction in length of the rod

$$\frac{f_S-f_{S_2}}{E_S}\times l=\frac{f_{B_3}-f_B}{E_B}\times\frac{l}{2}+\frac{f_{B_4}-f_B}{E}\times\frac{l}{2}$$

But $\quad E_S=2E_B$

$$f_S-f_{S_2}=f_{B_3}-f_B+f_{B_4}-f_B$$
$$150-f_{S_2}=88{\cdot}417+0{\cdot}555\,f_{S_2}+62{\cdot}170+0{\cdot}390\,f_{S_2}-2\times 58{\cdot}59$$
$$f_{S_2}\,(1+0{\cdot}555+0{\cdot}390)=150+1\times 58{\cdot}59-88{\cdot}417-62{\cdot}170$$
$$f_{S_2}\,(1{\cdot}945)=116{\cdot}593$$

Final stress in steel rod,

$$f_{S_2}=\frac{116{\cdot}593}{1{\cdot}945}=59{\cdot}945 \text{ kg/cm}^2.$$

Problem 2·7. A rigid bar EF 3 m long is supported by two wires AB and CD af shown in the Fig. 2·21. Wire AB is of steel, 2 m long and 6 mm in diameter. Wire CD is of

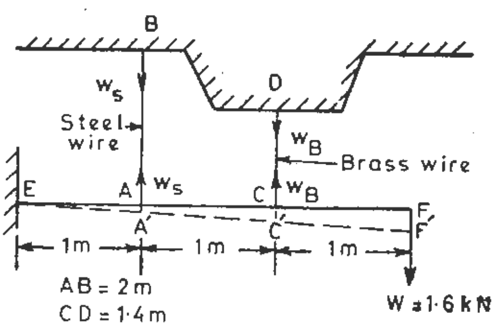


Fig. 2·21

brass, 1·4 m long and 5 mm in diameter. The bar carries a vertical load of 1·6 kN, at the end F and end E is hinged. Determine the stresses in steel and brass wires.

$$E_S=2E_B=210\times 1000 \text{ N/mm}^2.$$

Solution. Under the applied load at the end F, the rigid bar EF will be inclined as shown in Fig. 2·21.

Extension in steel wire, $\quad \delta l_S \propto EA$

Extension in brass wire, $\quad \delta l_B \propto EC$

but $$EC=2EA$$

So $$\delta l_B=2\delta l_S. \quad \ldots(1)$$

Say the stress developed in steel wire is f_S and in brass wire it is f_B.

Then $$\frac{f_B}{E_B}\times l_B=2\times\frac{f_S}{E_S}\times l_S$$

$$\frac{f_B}{E_B}\times 1400=2\times\frac{f_S}{2E_B}\times 2000$$

or $$\frac{f_B}{f_S}=\frac{2000}{1400}=\frac{10}{7}. \quad \ldots(2)$$

Load on steel wire, $$W_S=f_SA_S=f_S\times\frac{\pi}{4}(6)^2=28{\cdot}27\,f_S$$

Load on brass wire, $$W_B=f_BA_B=f_B\times\frac{\pi}{4}(5)^2=19{\cdot}63\,f_B$$

Taking moments of the forces about the point E

$$1{\cdot}6\text{ kN}\times 3=W_S\times 1+W_B\times 2$$

or $$W_S+2W_B=4{\cdot}8\text{ kN}=4800\text{ N}. \quad \ldots(3)$$

or $$28{\cdot}27\,f_S+2\times 19{\cdot}63\,f_B=4800\text{ N}$$

But $$f_B=\frac{10}{7}f_S$$

So $$28{\cdot}27f_S+2\times 19{\cdot}63\times\frac{10}{7}f_S=4800$$

$$f_S[28{\cdot}27+56{\cdot}086]=4800$$

Stress in steel wire, $$f_S=\frac{4800}{84{\cdot}356}=56{\cdot}90\text{ N/mm}^2$$

Stress in brass wire, $$f_B=81{\cdot}29\text{ N/mm}^2.$$

Problem 2·8. A rigid steel plate is supported by three vertical concrete posts of 2 m height each, but accidentally the height of the middle post is 0·5 mm less as shown in Fig. 2·22.

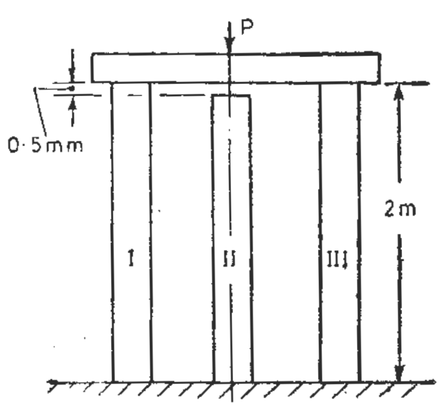


Fig. 2·22

Area of cross section of each post is 200 mm$\times$200 mm. Determine the safe value of the load P if the allowable stress for concrete in compression is 16 N/mm^2.

$$E \text{ for concrete} = 12 \times 1000 \text{ N/mm}^2.$$

Solution. Say the stress developed in outer posts $= f_1$

Stress developed in middle post $= f_2$

Area of cross section of each post, A $= 200 \times 200$

or $A = 4 \times 10^4$ mm

Now contraction in outer post $=$ Contraction in middle post $+ 0{\cdot}5$ mm

$$\frac{f_1}{E} \times 2000 = \frac{f_2}{E} \times (2000 - 0{\cdot}5) + 0{\cdot}5$$

$$2000\, f_1 = 1999{\cdot}5\, f_2 + 0{\cdot}5 \times 12 \times 1000 = 1999{\cdot}5\, f_2 + 6000$$

$$f_1 = 0{\cdot}99975\, f_2 + 3. \qquad \ldots(1)$$

i.e., $f_1 > f_2$

So $f_1 = 16$ N/mm^2 allowable stress

$$16 = 0{\cdot}99975\, f_2 + 3$$

or $f_2 = 13{\cdot}00325$ N/mm^2. ...(2)

Safe load, $P = 2f_1 \times A + f_2 \times A$

$$= 2 \times 16 \times 4 \times 10^4 + 13{\cdot}00325 \times 4 \times 10^4$$

$$= 128 \times 10^4 + 52{\cdot}013 \times 10^4 = \mathbf{1800{\cdot}13\ kN}.$$

Problem 2·9. A combination of stepped brass bar 599·97 mm long and aluminium tube 600 mm long is subjected to an axial compressive force of 20 kN. Both the tube and the

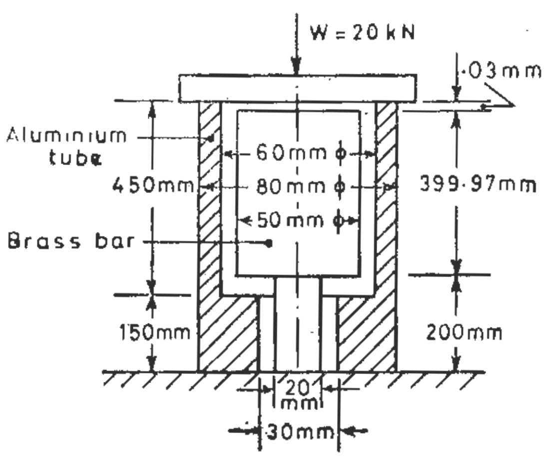


Fig. 2·23

rod are co-axial. Determine the maximum stresses in brass bar and aluminium tube.

$$E_{brass} = 105 \times 10^3 \text{ N/mm}^2$$

$$E_{aluminium} = 70 \times 10^3 \text{ N/mm}^2.$$

Solution. Say the load shared

by brass bar $= W_1$

by aluminium tube $= W_2$.

Contraction in brass bar

$$\delta l_b = \frac{4W_1}{\pi} \times \frac{200}{(20)^2 E_b} + \frac{4W_1}{\pi(50)^2} \times \frac{399{\cdot}97}{E_b}$$

$$= \frac{W_1}{E_b}\left[\frac{2}{\pi} + \frac{0{\cdot}640}{\pi}\right] = \frac{0{\cdot}84}{E_b} \times W_1$$

Contraction in aluminium tube

$$\delta l_a = \frac{4W_2}{\pi} \times \frac{150}{(80^2-30^2)\,E_a} + \frac{4W_2 \times 450}{\pi \times (80^2-60^2)\,E_a}$$

$$= \frac{W_2}{E_a}\left[\frac{600}{\pi \times 5500} + \frac{1800}{\pi \times 2800}\right]$$

$$= \frac{W_2}{E_a}\,[0{\cdot}0347 + 0{\cdot}2046] = \frac{0{\cdot}2393\,W_2}{E_a}.$$

But contraction in tube $=$ Contraction in bar $+ 0{\cdot}03$

$$\frac{0{\cdot}2393\,W_2}{E_a} = \frac{0{\cdot}84\,W_1}{E_b} + 0{\cdot}03$$

$$\frac{0{\cdot}2393\,W_2}{70 \times 10^3} = \frac{0{\cdot}84 \times W_1}{105 \times 10^3} + 0{\cdot}03$$

$$0{\cdot}342\,W_2 = 0{\cdot}8\,W_1 + 3000$$

or $$W_1 = 0{\cdot}4275\,W_2 - 3750$$

But $$W_1 + W_2 = 20{,}000 \text{ N}$$

$$W_1 = 2000 - W_2$$

or $$0{\cdot}4275\,W_2 - 3750 = 20{,}000 - W_2$$

$$1{\cdot}4275\,W_2 = 23750$$

$$W_2 = 16637{\cdot}5 \text{ N}$$

$$W_1 = 3362{\cdot}5 \text{ N}$$

Maximum stress in brass bar $= \dfrac{4 \times 3362{\cdot}5}{\pi \times (20)^2} = \dfrac{4 \times 3362{\cdot}5}{400 \times \pi} = 10{\cdot}7 \text{ N/mm}^2$.

Maximum stress in aluminium tube

$$= \frac{4 \times 16637{\cdot}5}{\pi \times (80^2 - 60^2)} = \frac{4 \times 16637{\cdot}5}{\pi \times 2800} = 7{\cdot}565 \text{ N/mm}^2.$$

Problem 2·10. A railway is laid so that there is no stress in the rails at 110°F. Determine the stress in the rail at 50 °F if all contraction is prevented. $E = 2100$ tonnes/cm², $\alpha = 6{\cdot}5 \times 10^{-6}$/°F.

The rails are 30 m long. If however, there is 5 mm allowance for contraction per rail, what is the stress at 50 °F ?

Solution. (*a*) T, Fall in temperature

$$=110-50=60°\text{F}$$

Coefficient of linear expansion, $\alpha=6{\cdot}5\times10^{-6}/°\text{F}$

Say the stress developed in rail $=f_{ST}$

$$E=\text{Young's modulus of elasticity}$$
$$=2100\times1000\ \text{kg/cm}^2.$$

Due to fall in temperature, rail will try to contract, but its contraction is prevented, so tensile stress is developed in rail.

$$\frac{f_{ST}}{E}=\alpha\times T$$

$$f_{ST}=2100\times1000\times6{\cdot}5\times10^{-6}\times60$$
$$=2{\cdot}1\times6{\cdot}5\times60=819{\cdot}0\ \text{kg/cm}^2.$$

(*b*) Length of the rail, $L=30\text{ m}=3000\text{ cm}$

Allowance for contraction $=0{\cdot}5$ cm

Say stress developed $=f_{ST}'$

$$\frac{f_{ST}'}{E}\times L=\alpha LT-\text{contraction allowance}$$

$$=6{\cdot}5\times10^{-6}\times3000\times60-0{\cdot}5$$
$$=1{\cdot}17-0{\cdot}5=0{\cdot}67\text{ cm}$$

$$f_{ST}'=\frac{0{\cdot}670\times2100\times1000}{3000}=469\ \text{kg/cm}^2.$$

Problem 2 11. A steel wire 2·4 mm in diameter is stretched tightly between two rigid supports 1 metre apart under an initial tensile force of 1 kN. If the temperature of the wire drops by 20°C, determine the maximum tensile stress in the wire.

$$E_S=210\times10^3\ \text{N/mm}^2$$
$$\alpha_S=11\times10^{-6}/°\text{C}.$$

Solution. Wire diameter, $d=2{\cdot}4$ mm

Area of cross section, $A_S=\frac{\pi}{4}(2{\cdot}4)^2=4{\cdot}524\ \text{mm}^2$

Tensile force $=1000$ N

Initial tensile stress, $f_S=\frac{1000}{4{\cdot}524}=221{\cdot}04\ \text{N/mm}^2.$

Now due to the temperature drop, the wire will try to contract but the rigid supports will prevent this contraction and thus further increasing the tensile stress in the wire.

Say the tensile stress in wire due to drop in temperature $=f_{ST}$

Drop in temperature, $T=20°\text{C}$

$$\frac{f_{ST}}{E_S}=\alpha\times T$$

$$f_{sT}=210\times 10^3\times 11\times 10^{-6}\times 20=4{\cdot}2\times 11$$
$$=46{\cdot}2\ \text{N/mm}^2.$$

Maximum tensile stress in wire,

$$f_{s_{max}}=f_s+f_{sT}=221{\cdot}04+46{\cdot}2$$
$$=267{\cdot}24\ \text{N/mm}^2.$$

Problem 2·12. Two steel rods, each 5 cm in diameter are joined end to end by means of a turnbuckle, as shown in Fig. 2·24. The other end of each rod is rigidly fixed and there is initially a small tension in the rods. The effective length of each rod is 4 metres. Calculate the increase in this tension, when the turnbuckle is tightened one quater of a turn.

There are threads on each rod with a pitch of 5 mm.

$$E=2080000\ \text{kg/cm}^2.$$

If $\alpha=6{\cdot}5\times 10^{-6}/°\text{F}$, what rise in temperature would nullify this increase in tension.

Solution.

Diameter of steel rod $=5$ cm

Area of cross section, $A_s=\frac{\pi}{4}(5)^2=19{\cdot}635\ \text{cm}^2$

Length of each rod, $L=400$ cm

Pitch of threads, $p=5$ mm

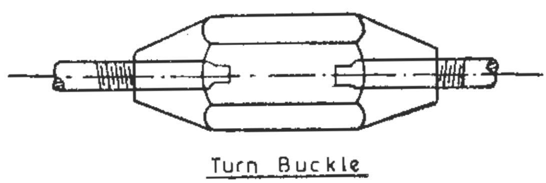


Fig. 2·24

Elongation in each rod,

$$\delta L=\frac{p}{4}=\frac{5}{4}=1{\cdot}25\ \text{mm}=0{\cdot}125\ \text{cm}$$

(The turn buckle is tightened one quarter of a turn)

Young's modulus, $E=2080{,}000\ \text{kg/cm}^2$

Strain in each rod, $\epsilon=\frac{\delta L}{L}=\frac{0{\cdot}125}{400}$

f, stress in each rod, $\epsilon E=\frac{0{\cdot}125}{400}\times 2080{,}000=650\ \text{kg/cm}^2$

Increase in tension, $F=f\,.\,A_s=650\times 19{\cdot}635$

$=12762{\cdot}75\ \text{kg}=12{\cdot}76$ tonnes

Now if the temperature of each bar is increased, it would expand nullifying the tensile stress developed, Say the temperature rise $=T°F$

$$\frac{f}{E}=\alpha T$$

$$\frac{650}{2080\times1000}=6{\cdot}5\times10^{-6}\times T$$

Increase in temperature required,

$$T=\frac{650\times10^6}{6{\cdot}5\times2080\times1000}=48{\cdot}08\ °F$$

Problem 2·13. Three vertical wires one of steel and two of copper are suspended in the same vertical plane from a horizontal support. They are all of the same length and same area of cross section and carry a load by means of a rigid cross bar at their lower ends. The load is increased and temperature is changed in such a way that stress in each wire is increased by 100 kg/cm². Find the change in temperature.

$$E_S=2000 \text{ tonnes/cm}^2,\ E_c=1000 \text{ tonnes/cm}^2$$

$$\alpha_S=11\times10^{-6}/°C,\qquad \alpha_c=18\times10^{-6}/°C.$$

Solution. Say the stresses developed due to load in steel and copper wires are f_s and f_c respectively. The stresses developed due to temperature change by $T°C$ are f_{ST} and f_{CT} respectively,

then
$$f_s+f_{ST}=100 \text{ kg/cm}^2$$
$$f_c+f_{CT}=100 \text{ kg/cm}^2.$$

Say the area of cross section of each wire$=A$ cm².

Stresses due to direct load

$$f_s\times A+2f_c\times A=100\times3\ A$$

or
$$f_s+2f_c=300 \qquad ...(1)$$

(The load due to temperature change will be compressive in one and tensile in the other so that total load change due to temperature is balanced).

Moreover strain in steel wire=strain in copper wire due to direct load.

So
$$\frac{f_s}{E_s}=\frac{f_c}{E_c}$$

$$f_s=\frac{E_s}{E_c}\times f_c=\frac{2000}{1000}\times f_c=2f_c \qquad ...(2)$$

So
$$2f_c+2f_c=300$$
$$f_c=75 \text{ kg/cm}^2$$
$$f_s=150 \text{ kg/cm}^2.$$

(Stress due to the direct load)

Resultant stress in steel wire

$$=f_s+f_{ST}=100 \text{ kg/cm}^2$$

So
$$f_{ST}=-50 \text{ kg/cm}^2 \text{ (compressive)}$$
$$f_C+f_{ST}=100$$
$$f_{CT}=100-75=25 \text{ kg/cm}^2 \text{ (tensile)}$$

Since $\alpha_s < \alpha_c$, and the temperature stress in steel wire is compressive, there will be fall in temperature say by T°C.

$$\frac{f_{ST}}{E_s}+\frac{f_c}{E_c}=(\alpha_c-\alpha_s)\times 10^{-6}\times T$$

$$\frac{50}{2000\times 1000}+\frac{25}{1000\times 1000}=(18-11)\times 10^{-6}\times T, \quad 50=7\,T$$

$$T=\frac{50}{7}=7{\cdot}14°\text{C (fall in temperature)}$$

Let us check for equilibrium

Compressive force in steel wire $=50\,A$

Tensile force in copper wires $=2\times 25\times A=50\,A.$

Problem 2·14. Three vertical rods carry a tensile load of 10 tonnes. The area of cross section of each bar is 5 cm². Their temperature is raised by 60°C and the load of 10 tonnes is now so adjusted that they extend equally. Determine the load shared by each. The two outer rods are of steel and the middle one is of brass.

$$E_S=2100 \text{ tonnes/cm}^2,\ E_B=1050 \text{ tonnes/cm}^2$$

$$\alpha_S=11\times 10^{-6}/°\text{C}, \qquad \alpha_B=18\times 10^{-6}/°\text{C}.$$

Solution. Each bar extends due to the rise in its temperature by 60°C and due to the load shared by it.

Say the total extension in each rod $=\delta$

Expansion in each steel rod due to rise in temperature $=\alpha_S\,TL$

Expansion in brass rod due to rise in temperature $=\alpha_B\,TL$

Extension in each steel rod due to load $=\delta-\alpha_S\,TL$

Extension in brass rod due to load $=\delta-\alpha_B\,TL$

So strain in each steel rod

$$=\frac{\delta-\alpha_S\,TL}{L}=\left(\frac{\delta}{L}-\alpha_S\,T\right)$$

Strain in brass rod $=\dfrac{\delta-\alpha_B\,TL}{L}=\left(\dfrac{\delta}{L}-\alpha_B T\right)$

Total load $=10 \text{ tonnes}=2\left(\dfrac{\delta}{L}-\alpha_S\,T\right)E_S\,.\,A+\left(\dfrac{\delta}{L}-\alpha_B T\right)E_B\,.\,A$

$$10=2\times 2100\times 5\left(\frac{\delta}{L}-\alpha_S\,T\right)+1050\times 5\left(\frac{\delta}{L}-\alpha_B\,T\right)$$

$$\frac{10}{5250}=\frac{4\delta}{L}-4\,\alpha_S\,T+\frac{\delta}{L}-\alpha_B\,T$$

$$=\frac{5\delta}{L}-4\times 60\,(11\times 10^{-6})-60(18\times 10^{-6})$$

$$\frac{1}{525}=\frac{5\delta}{L}-60\times 10^{-6}\,(62)$$

$$\frac{1}{5\times 525}=\frac{\delta}{L}-12\times 62\times 10^{-6}$$

$$\frac{\delta}{L}=\frac{1}{2625}+12\times 62\times 10^{-6}$$

$$=(0{\cdot}381+0{\cdot}744)\times 10^{-6}=1{\cdot}125\times 10^{-3}$$

Load on each steel rod $=\left(\frac{\delta}{L}-\alpha_s\,T\right)E_s\,A$

$$=(1{\cdot}125\times 10^{-3}-11\times 10^{6}\times 60)\;2100\times 5$$

$$=11{\cdot}8125-6{\cdot}930=4{\cdot}88 \text{ tonnes}$$

Load on brass rod $=\left(\frac{\delta}{L}-\alpha_B\,T\right)\times E_B\times A$

$$=(1{\cdot}125\times 10^{-3}-18\times 10^{-6}\times 60)\;1050\times 5$$

$$=5{\cdot}91-5{\cdot}67=0{\cdot}24 \text{ tonne.}$$

Problem 2·15. A steel tie rod 20 mm diameter is encased in a copper tube of external diameter 36 mm and internal diameter 24 mm with the help of washers and nuts The nut on the tie rod is tightened so as to produce a tensile stress of 400 kg/cm² in steel rod. This combination is subjected to a tensile load of 2 tonnes. Determine the resultant stresses in steel rod and the copper tube, if

$$E_s=2\,E_c=2100 \text{ tonnes/cm}^2.$$

Now if the temperature of the assembly is raised by 80°C, determine the resultant stresses developed in the rod and the tube.

Given $\alpha_{steel}=11\times 10^{-6}/°C$

$\alpha_{copper}=18\times 10^{-6}/°C.$

Solution. (*i*) **Stresses due to tightening the nut.**

Area of cross section of steel rod,

$$A_s=\frac{\pi}{4}(2)^2=3{\cdot}1416 \text{ cm}^2$$

Area of cross section of copper tube,

$$A_c=\frac{\pi}{4}(3{\cdot}6^2-2{\cdot}4^2)=5{\cdot}655 \text{ cm}^2$$

Tensile stress developed in steel rod,

$$f_s=400 \text{ kg/cm}^2$$

Tensile force in steel rod,

$$F_s=400\times 3{\cdot}1416 \text{ kg}$$

$=$ compressive force in copper tube, F_c

$=f_c\times 5{\cdot}655$ (for equilibrium)

So compressive stress in copper tube

$$=\frac{400\times 3{\cdot}1416}{5{\cdot}655}=222{\cdot}2 \text{ kg/cm}^2$$

(*ii*) **Stresses due to tensile force**

Say the stresses developed due to tensile force of 4 tonnes in steel rod is f_s' and in copper tube it is f_c'.

$$2000 = f_s' \times A_s + f_c' \times A_c \qquad \ldots(1)$$

But $$\frac{f_s'}{E_s} = \frac{f_c'}{E_c} \text{ (in a composite bar under direct force)}$$

$$f_s' = 2f_c' \qquad \ldots(2)$$

Substituting in equation (1)

So $$2f_c' \times 3{\cdot}1416 + f_c' \times 5{\cdot}655 = 2000$$

$$f_c' = \frac{2000}{11{\cdot}9382} = 167{\cdot}53 \text{ kg/cm}^2 \text{ (tensile)}$$

$$f_s' = 2f_c' = 335{\cdot}06 \text{ kg/cm}^2$$

Resultant stresses in steel rod

$$= 400 + 335{\cdot}06 = 735{\cdot}06 \text{ kg/cm}^2 \text{ (tensile)}$$

In copper tube $$= 222{\cdot}20 - 167{\cdot}53 = 54{\cdot}67 \text{ kg/cm}^2 \text{ (compressive).}$$

(*iii*) **Temperature Stresses**

$$\alpha_c > \alpha_s$$

Due to temperature rise compressive stress will be developed in copper tube and tensile stress will be developed in steel rod. Say the stresses developed are f_{ST} and f_{CT} respectively.

Then for equilibrium

$$f_{ST} \times A_s = f_{CT} \times A_c$$

$$f_{ST} = f_{CT} \times \frac{5{\cdot}655}{3{\cdot}1416} = 1{\cdot}8\, f_{CT} \qquad \ldots(1)$$

Rise in temperature, $T = 80°C$

then $$\frac{f_{ST}}{E_s} + \frac{f_{CT}}{E_c} = (\alpha_c - \alpha_s) \times T$$

$$\frac{f_{ST}}{2100 \times 1000} + \frac{f_{CT}}{1050 \times 1000} = (18 - 11) \times 10^{-6} \times 80$$

$$\frac{1{\cdot}8\, f_{CT}}{2{\cdot}1 \times 10^{6}} + \frac{f_{CT}}{1{\cdot}05 \times 10^{-6}} = 500 \times 10^{-6}$$

$$f_{CT}\,(0{\cdot}857 + 0{\cdot}952) = 569$$

$$f_{CT} = \frac{560}{1{\cdot}809} = 309{\cdot}56 \text{ kg/cm}^2 \text{ (compressive)}$$

$$f_{ST} = 1{\cdot}8\, f_{CT} = 557{\cdot}28 \text{ kg/cm}^2 \text{ (tensile)}$$

Resultant stress in steel rod

$$= 735{\cdot}06 + 557{\cdot}20 = 1292{\cdot}26 \text{ kg/cm}^2 \text{ (tensile)}$$

Resultant stress in copper tube

$$= 54{\cdot}67 + 309{\cdot}56 = 364{\cdot}23 \text{ kg/cm}^2 \text{ (compressive)}$$

Problem 2·16. Fig. 2·25 shows a steel bolt 20 mm diameter and 200 mm long passing centrally through a copper tube 150 mm long, outside diameter 40 mm and inside diameter 28 mm. The screw on bolt has a pitch of 2 mm and initially the nut is just tight.

Find (*i*) Changes in the stresses in the bolt and the tube due to tightening the nut by rotating it through 45°.

(*ii*) Changes in the stresses in the bolt and tube due to temperature rise of 40°F.

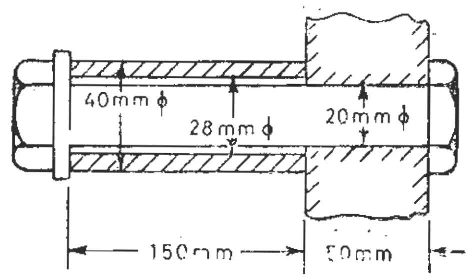


Fig. 2·25

$E_s=210\times10^3$ N/mm², $E_c=100\times10^3$ N/mm²
$\alpha_s=6{\cdot}4\times10^{-6}/°F$, $\alpha_c=10\times10^{-6}/°F$.

Solution. (*i*) **Stresses due to tightening of the nut**

Pitch of the threads $=2$ mm
Rotation of the nut $=45°$

Axial movement of the nut

$$=2\times\frac{45}{360}=0{\cdot}25 \text{ mm}$$

= contraction in tube + extension in bolt

Area of cross section of the bolt,

$$A_s=\frac{\pi}{4}(20)^2=314{\cdot}16 \text{ mm}^2$$

Area of cross section of the tube,

$$A_c=\frac{\pi}{4}(40^2-28^2)=640{\cdot}88 \text{ mm}^2$$

Say the stresses developed
In bolt $=f_s$
In tube $=f_c$
Pull in bolt = Push in tube

$$f_s\,.\,A_s=f_c\,.\,A_c$$

$$f_s=f_c\,.\,\frac{640{\cdot}88}{314{\cdot}16}=2{\cdot}04\,f_c \quad \ldots(1)$$

Extension in bolt $=\dfrac{f_s}{E_s}\times l_s=\dfrac{f_s}{210\times10^3}\times200$

Contraction in tube $=\dfrac{f_c}{E_c}\times l_c=\dfrac{f_c}{100\times10^3}\times150$

So $$0{\cdot}25=\frac{f_s}{1050}+\frac{1{\cdot}5\,f_c}{1000}=\frac{2{\cdot}04\,f_c}{1050}+\frac{1{\cdot}5\,f_c}{1000} \quad \ldots(2)$$

$$3{\cdot}443\,f_s=250$$

$$f_s=72{\cdot}61 \text{ N/mm}^2$$

$$f_c=35{\cdot}59 \text{ N/mm}^2.$$

(*ii*) **Stresses due to temperature rise**

$\alpha_c > \alpha_s$, so compressive stress will be developed in copper tube and tensile stress will be developed in steel bolt due to temperature rise.

Say the stresses developed are f_{CT} and f_{ST} respectively.

$$f_{CT} \times A_c = f_{ST} \times A_s$$

$$f_{CT} = f_{ST} \times \frac{314 \cdot 16}{640 \cdot 88} = 0 \cdot 49\, f_{ST} \qquad \ldots(3)$$

Moreover $\frac{f_{ST}}{E_s} + \frac{f_{CT}}{E_c} = (\alpha_c - \alpha_s) \times \text{temperature rise}$

$$= (10 - 6 \cdot 4) \times 10^{-6} \times 40$$

$$\frac{f_{ST}}{210 \times 10^3} + \frac{0 \cdot 49\, f_{ST}}{100 \times 1000} = 144 \times 10^{-6}$$

$$0 \cdot 966\, f_{ST} = 14 \cdot 4$$

$$f_{ST} = 14 \cdot 9 \text{ N/mm}^2, \quad f_{CT} = 7 \cdot 30 \text{ N/mm}^2$$

Final stresses in bolt $= 72 \cdot 61 + 14 \cdot 90 = 87 \cdot 51$ N/mm² (tensile)

In tube $= 35 \cdot 59 + 7 \cdot 30 = 42 \cdot 89$ N/mm² (compressive)

Problem 2·17. Four steel bars of length L and area A each support a square rigid plate. The bars are symmetrically arranged. A load P is then applied at the middle of the square plate.

A steel rod of length $L - \delta$ and area a is now attached to the rigid support where the four bars are secured and its temperature is raised by $T°$ above the normal so that it can be connected at the middle of the square plate. When the central bar returned to normal temperature, it was found that the load in each of the four bars has been reduced by 20%. Show that

$$\delta = \frac{PL}{5E}\left(\frac{1}{a} - \frac{1}{A}\right)$$

$$T° = \frac{P}{5E\alpha}\left(\frac{1}{a} + \frac{1}{4A}\right)$$

where α = coefficient of linear expansion of steel

and E = Young's modulus of elasticity of steel.

Solution. Initially the four bars of area A and length L carry the load P,

then load on each bar $= \frac{P}{4}$

Initial extension in each bar

$$= \frac{PL}{4AE} = \delta' \text{ (as shown)}$$

When the middle bar is heated by $T°$, it has to expand by $(\delta + \delta')$.

So that its connection can be made with the rigid square plate.

When the central bar returns to normal temperature, Load shared by each bar of area A

$$=0{\cdot}8\ (0{\cdot}25\ P)=0{\cdot}2\ P$$

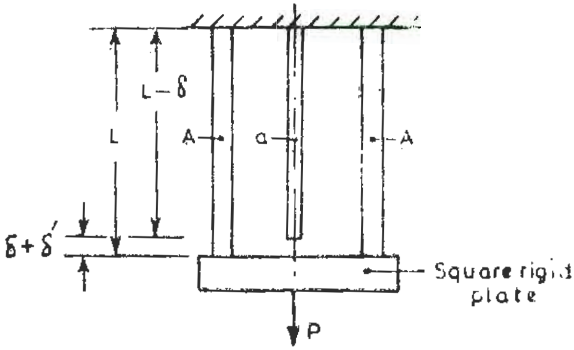


Fig. 2·26

So the load on middle bar

$$=P-4\times0{\cdot}2\ P=0{\cdot}2\ P$$

When the central bar tries to contract, tensile stress will be developed in this bar and compressive stress will be developed in outer four bars.

Tensile force in central bar

$$=0{\cdot}2\ P$$

Compressive force in outer bars (showing equilibrium)

$$=4\ (0{\cdot}25\ P-0{\cdot}20\ P)=0{\cdot}2\ P$$

f_{ST}, tensile stress in central bar

$$=\frac{0{\cdot}2\ P}{a}$$

f_{CT}, compressive stress in each of outer bars

$$=\frac{0{\cdot}2\ P}{4A}=\frac{0{\cdot}05\ P}{A}$$

Now $$\frac{f_{ST}}{E}+\frac{f_{CT}}{E}=\alpha(T)$$

$$\frac{0{\cdot}2P}{aE}+\frac{0{\cdot}05P}{AE}=\alpha T$$

or $$T=\frac{P}{5E\alpha}\left[\frac{1}{a}+\frac{1}{4A}\right]$$

Now total expansion in central bar,

$$\delta+\delta'=\alpha T\ (L-\delta)\simeq\alpha LT$$

Since $$\delta<<L$$

or $$\delta=\alpha LT-\delta'$$

$$=\alpha L\left[\frac{1}{a}+\frac{1}{4A}\right]\frac{P}{5Ea}-\frac{PL}{4AE}$$

$$=\frac{PL}{5Ea}+\frac{PL}{20EA}-\frac{PL}{4AE}=\frac{PL}{5Ea}-\frac{PL}{5AE}$$

$$\delta=\frac{PL}{5E}\left[\frac{1}{a}-\frac{1}{A}\right].$$

Problem 2·18. A 10 mm diameter steel rod passes centrally through a copper tube 25 mm external diameter and 15 mm internal diameter and 2·5 m long. The tube is closed at each end by thick steel plates secured by nuts. The nuts are tightened until the copper tube is reduced in length by 0·6 mm. The whole assembly is then raised in temperature by 20°C. Calculate the stresses in the steel rod and copper tube before and after the rise in temperature. The thickness of the end plates remains unchanged.

$$E_{steel}=208000 \text{ N/mm}^2$$

$$E_{copper}=104000 \text{ N/mm}^2$$

$$\alpha_s=12\times10^{-6}/^\circ\text{C},\ \alpha_c=17{\cdot}5\times10^{-6}/^\circ\text{C}.$$

Solution. Copper tube

External diameter $=25$ mm

Internal diameter $=15$ mm

Area of cross section, $A_c=\frac{\pi}{4}(25^2-15^2)=314{\cdot}16 \text{ mm}^2$

Length of the copper tube $=2500$ mm

Contraction in length $=0{\cdot}6$ mm

Strain, $\epsilon_c=\frac{0{\cdot}6}{2500}$

Young's modulus, $E_c=208000 \text{ N/mm}^2$

Stress in copper tube, $f_c=\epsilon_c\times E_c$

$$=\frac{0{\cdot}6}{2500}\times208000=49{\cdot}92 \text{ N/mm}^2 \text{ (compressive)}$$

Compressive force in copper tube,

$$P_c=49{\cdot}92\times314{\cdot}16=15682{\cdot}87 \text{ N}$$

For equilibrium compressive force in tube,

$$P_c=\text{tensile force in rod, } P_s$$

A_s, Area of cross section of steel rod

$$=\frac{\pi}{4}(10)^2=78{\cdot}54 \text{ mm}^2$$

Stress in steel rod, $f_s=\frac{15682{\cdot}87}{78{\cdot}54}=199{\cdot}68 \text{ N/mm}^2$ (tensile)

Temperature Stresses. Say the stresses developed in steel rod and copper tube due to temperature rise are f_{ST} and f_{CT}. As $\alpha_s < \alpha_c$; f_{sT} will be tensile and f_{cT} will be compressive.

$$f_{ST} \cdot A_s = f_{CT} \cdot A_c \text{ (for equilibrium)} \quad ...(1)$$

$$\frac{f_{ST}}{E_s} + \frac{f_{CT}}{E_c} = (\alpha_c - \alpha_s)\, T \quad ...(2)$$

Temperature rise, $T = 20°C$

So $f_{ST} \times 78{\cdot}54 = f_{CT} \times 314{\cdot}16$

$$f_{sT} = 4\, f_{cT} \quad ...(1)$$

Substituting the values in equation (2)

$$\frac{4f_{ST}}{208{,}000} + \frac{f_{CT}}{104{,}000} = (17{\cdot}5 - 12) \times 10^{-6} \times 20$$

$$3\, f_{CT} = 5{\cdot}5 \times 20 \times 10^{-6} \times 104{,}000 = 11{\cdot}44$$

$$f_{CT} = 3{\cdot}813 \text{ N/mm}^2$$

$$f_{ST} = 4\, f_{CT} = 15{\cdot}253 \text{ N/mm}^2$$

Stresses after the rise in temperature in steel rod

$$= f_s + f_{ST} = 199{\cdot}68 + 15{\cdot}253 = 214{\cdot}933 \text{ N/mm}^2 \text{ (tensile)}$$

In copper tube $= f_c + f_{CT} = 49{\cdot}92 + 3{\cdot}813 = 53{\cdot}733 \text{ N/mm}^2$ (compressive)

Problem 2·19. A circular aluminium rod of area 300 mm² is fitted in a square steel frame of area of cross section 400 mm² as shown in the Fig. 2·27. At a temperature of 25°C there is a clearance of 0·05 mm between the upper end of the rod and the top of the frame. Determine the compressive force in the aluminium if the temperature of the system is raised to 50°C, neglecting the bending of the frame and the bar.

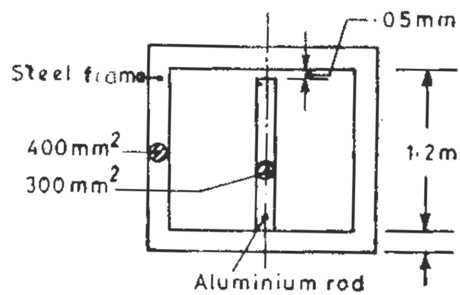


Fig. 2·27

Given $\alpha_s = 11 \times 10^{-6}/°C$

$\alpha_a = 22 \times 10^{-6}/°C$

$E_s = 3\, E_a = 2100 \times 1000 \text{ N/mm}^2$.

Solution. At 25°C, clearance between steel frame and aluminium rod $= 0{\cdot}05$ mm

At 50°C, say extension in steel frame $= \delta$ mm

Then extension in aluminium rod $= \delta + 0{\cdot}05$ mm

Since $\alpha_a > \alpha_s$, aluminium rod will tend to extend more than steel but steel frame wil prevent free expansion of aluminium rod and in turn aluminium rod will exert pull on tw vertical steel bars and steel bars will extend beyond their free expansion limit.

ϵ_a, compressive strain in aluminium

$$= \frac{\alpha_a(50-25)(1200-0{\cdot}05)-(\delta+0{\cdot}05)}{1200-0{\cdot}05} \quad \text{taking}(1200-0{\cdot}05) \simeq 12$$

$$\epsilon_a = \frac{22 \times 10^{-6} \times 25 \times 1200 - (\delta + 0{\cdot}05)}{1200} = \frac{0{\cdot}66 - \delta - 0{\cdot}05}{1200} = \frac{0{\cdot}61 -}{1200}$$

Area of cross section of aluminium rod,

$$A_a = 300 \text{ mm}^2$$

Compressive force in aluminium rod

$$F_a = \epsilon_a \,.\, E_a \times A_a = \epsilon_a \times E_a \times 300$$

Tensile strain in steel bar,

$$\epsilon_s = \frac{\delta - \alpha_s\,(50-25) \times 1200}{1200}$$

Tensile force in steel bars,

$$F_s = \epsilon_s\, E_s\, A_s = \epsilon_s \times 3\, E_a \times 800$$

But for equilibrium $F_a = F_s$

$$\epsilon_a \,.\, E_a \,.\, 300 = \epsilon_s \,.\, 3\, E_a\, 800$$

$$\epsilon_a = 8\, \epsilon_s \qquad \ldots(1)$$

$$\epsilon_s = \frac{\delta - 11 \times 10^{-6} \times 25 \times 1200}{1200} = \frac{\delta - 0{\cdot}33}{1200}$$

So $\epsilon_a = 8\, \epsilon_s$

$$\frac{0{\cdot}61 - \delta}{1200} = 8\, \frac{(\delta - 0{\cdot}33)}{1200}$$

or $0{\cdot}61 + 8 \times 0{\cdot}33 = 9\,\delta$

$$\frac{3{\cdot}25}{9} = \delta, \quad \delta = 0{\cdot}361 \text{ mm}$$

Now $\epsilon_a = \dfrac{0{\cdot}61 - \delta}{1200} = \dfrac{0{\cdot}61 - 0{\cdot}361}{1200} = \dfrac{0{\cdot}249}{1200}$

Stress in aluminium rod,

$$f_a = \frac{0{\cdot}249}{1200} \times E_a = \frac{0{\cdot}249}{1200} \times 70 \times 1000 = 14{\cdot}525 \text{ N/mm}^2$$

(compressive)

Compressive force in aluminium rod,

$$F_a = 14{\cdot}525 \times A_a = 14{\cdot}525 \times 300 = 4357{\cdot}5 \text{ N}$$

$$= 4{\cdot}3575 \text{ kN} \quad \textbf{Ans.}$$

SUMMARY

1. In a composite bar of two materials with cross sectional areas A_1, A_2 and length L, the stresses developed under a load W are

$$f_1 A_1 + f_2 A_2 = W. \qquad \ldots(1)$$

$$\frac{f_1}{E_1} = \frac{f_2}{E_2}. \qquad \ldots(2)$$

where E_1 and E_2 are the Modulus of elasticity for both the materials

$$W_1 + W_2 = W. \qquad \ldots(3)$$

Change in length, $\delta L = \dfrac{f_1}{E_1} \times L = \dfrac{f_2}{E_2}\, L.$

2. In a composite bar with more than 2 materials, with areas of cross sections A_1, A_2, ..., A_n etc.

$$W_1+W_2+W_3......W_n=W \text{ (Total load)}$$

$$\frac{f_1}{E_1}=\frac{f_2}{E_2}=\frac{f_3}{E_3}......=\frac{f_n}{E_n}$$

Stress in any bar i, $\quad f_i=\dfrac{W_i}{A_i}$.

3. Bars of different lengths L_1 and L_2 placed co-axially subjected to load.

$$L_1-L_2=C \text{ (a small clearance)}$$

$$\delta L_1-C=\delta L_2. \quad ...(1)$$

$$\delta L_1=\frac{f_1}{E_1}\times L_1\,; \quad \delta L_2=\frac{f_2}{E_2}\times L. \quad ...(2)$$

$$P=f_1A_1+f_2A_2. \quad ...(3)$$

4. A bolt and a tube assembly tightened with a nut

$$f_1A_1=f_2A_2 \quad ...(1)$$

(f_1 and f_2 are tensile and compressive stresses in bolt and tube)

A_1 and A_2—area of cross section of bolt and tube respectively.

Axial movement of the nut $=\dfrac{f_1}{E_1}L+\dfrac{f_2}{E_2}L$

where L is the length of bolt and tube.

5. A single bar fixed between rigid supports at both the ends, subjected to temperature change by T°.

Tensile stress in bar $=\alpha T$ (for decrease in temperature)

Compressive stress in bar $=\alpha T$ (for increase in temperature)

$\alpha=$ coefficient of linear expansion of bar.

6. A composite bar, of two materials, cross sections A_1 and A_2 subjected to temperature change T, $\alpha_1>\alpha_2$

$$f_1A_1=f_2A_2$$

$$\frac{f_1}{E_1}+\frac{f_2}{E_2}=(\alpha_1-\alpha_2)\,T$$

f_1 is compressive and f_2 is tensile stress for increase in temperature.

f_1 is tensile and f_2 is compressive stress for decrease in temperature.

MULTIPLE CHOICE QUESTIONS

1. A composite bar is made by encasing a brass rod in steel tube. If $E_{steel}=2\,E_{brass}$ and there is a change of length of 0·1 mm in brass rod in a length of 1 metre of the composite bar, due to an applied force, then change in length of steel tube is :—

(a) 0·2 mm (b) 0·1 mm

(c) 0·05 mm (d) 0·025 mm.

2. A copper bar 2 cm in diameter is completely encased in a steel tube of 3 cm external diameter so as to make a composite bar. The bar is subjected to a compressive force of P tonnes so as to cause a stress of 150 kg/cm² in copper bar, the stress developed in steel tube will be (if $E_s=2E_c$).

   (*a*) 300 kg/cm² (*b*) 150 kg/cm²
   (*c*) 75 kg/cm² (*d*) None of the above.

3. A composite bar made of steel and aluminium strips each having 2 cm² area of cross section. The composite bar is subjected to load P. If the stress in aluminium is 100 kg/cm² and $E_{steel}=3\ E_{aluminium}$ the value of load P is

   (*a*) 400 kg (*b*) 600 kg
   (*c*) 800 kg (*d*) 100 kg.

4. A composite bar is made of strips of material 1 and material 2, having area 100 mm² each. The stress in material 1 is 20 N/mm² due to an applied load of 6000 N. If the value of E for material 1 is 100×10^3, the value of E for material 2 will be

   (*a*) 25×10^3 N/mm² (*b*) 50×10^3 N/mm²
   (*c*) 100×10^3 N/mm² (*d*) 200×10^3 N/mm².

5. A steel bolt passes centrally through a brass tube. At the ends washers and nuts are provided. Nuts are tightened so as to produce a compressive stress of 100 N/mm² in brass tube. The area of cross section of brass tube is 1000 mm² and that of steel bolt is 500 mm². The value of E for steel is 2 times the value of E for brass. The stress developed in steel rod is

   (*a*) 50 N/mm$_2$ (*b*) 100 N/mm²
   (*c*) 200 N/mm² (*d*) None of the above.

6. A steel bolt passes centrally through a brass tube. At the ends washers and nuts are provided. The whole assembly is raised in temperature by 50°C The area of the cross section of steel bolt is 2000 mm² and that of brass tube is 1000 mm². $E_{steel}=2\ E_{brass}$. If the stress due to temperature rise is 40 N/mm² (tensile) in steel bolt, the stress in brass tube will be

   (*a*) 80 N/mm² (compressive) (*b*) 60 N/mm² (tensile)
   (*c*) 40 N/mm² (tensile) (*d*) 20 N/mm² (compression).

7. A wire of a material, 1 mm in diameter, 1 m long is stretched between two rigid supports. the temperature of the wire drops by 10°C. If $\alpha=10\times10^{-6}$/°C and $E=100\times10^3$ N/mm² for the wire, the stress developed in wire will be

   (*a*) 1 N/mm² (*b*) 10 N/mm²
   (*c*) 100 N/mm² (*d*) 1000 N/mm².

8. A steel rail track is laid by joining 30 m long rails end to end. A 30°C there is no stress in rails. At 50°C what will be the stress in rail if $\alpha=11\times10^{-6}$/°C and $E=200\times10^3$ N/mm.

   (*a*) 88 N/mm² (compressive) (*b*) 88 N/mm² (tensile)
   (*c*) 44 N/mm² (compressive) (*d*) 44 N/mm² (tensile).

9. Three wires of equal cross section and equal length but of different materials fixed at the top support a ring, $E_1=2E_2=3E_3$. A load of 3 kN is applied on the ring in such a manner that the ring remains horizontal. The load shared by wire of material 1 is

   (*a*) 0·5 kN (*b*) 1 kN
   (*c*) 1·5 kN (*d*) 2 kN.

10. A steel bolt passes centrally through a copper tube. At the ends nuts and washers are provided. The area of cross section of both bolt and tube is the same. $E_{steel}=2E_{brass}$. If the assembly is tightened by rotating a nut through 60° on the thread of pitch 3·6 mm. The contraction in the length of the tube is

(a) 0·6 mm (b) 0·4 mm

(c) 0·2 mm (d) 0·1 mm.

ANSWERS

1. (b)	2. (a)	3. (c)	4. (d)	5. (c)	6. (a)
7. (b)	8. (c)	9. (c)	10. (b).		

EXERCISE

2·1. A weight of 200 kN is supported by a short concrete column of 30 cm diameter strengthened by 6 steel bars of 2·5 cm diameter symmetrically placed in the section of concrete. Find the stresses in steel and concrete.

$$E_{steel}=15\ E_{concrete}=210\times10^5\ \text{N/mm}^2.$$

If the stress in the concrete is not to exceed 3 N/mm², what area of steel is required in order that the column may support a load of 400 kN ?

[**Ans.** 26·81 N/mm², 1·787 N/mm² ; 4474·8 mm²]

2·2. A short hollow cast iron column, 250 mm external diameter and 200 mm internal diameter is filled with concrete. The column carries a total load of 500 kN. If $E_{cast\ iron}=6\ E_{concrete}$, calculate stresses in cast iron and concrete.

What must be the internal diameter of the cast iron column if a load of 650 kN is to be carried. The stresses in cast iron and concrete and external diameter of the column being unchanged. [**Ans.** 21.828 N/mm², 3·628 N/mm², 171·76 mm]

2·3. A circular ring is suspended by three vertical bars A, B and C of different lengths. The upper end of the bars are held at different levels. Bar A is 1·5 m long with 2 cm² cross sectional area, bar B is 1 m long with 1·5 cm² cross sectional area and bar C is 70 cm long with cross sectional area equal to 2·5 cm². Bar A is of steel, B of copper and C of aluminium. A load of 2 tonnes is hung on the ring. Calculate how much of this load is carried by each bar, if the circular ring remains horizontal after the application of the load.

$E_{steel}=2100$ tonnes/cm², $E_{coppper}=1100$ tonnes/cm², $E_{aluminium}=700$ tonnes/cm².

[**Ans.** 0·941, 0·555, 0·504 tonne]

2·4. Pre-stressed concrete beam is fabricated as follows :

(i) A steel rod is loaded in tension between two plates.

(ii) Then the concrete is poured to form a beam of square cross section with steel rod in centre.

(iii) After the concrete is properly set, the external force on the rod is removed and the beam is left in a pre-stressed condition.

If $E_s=15\ E_c$, beam section 15 cm×15 cm and the steel rod is of 3·5 cm diameter, what will be the ratio of final residual stresses in the two materials. [**Ans.** $-22·39$]

2.5. A copper rod of length $2l$, with cross sectional area $A_1=4$ cm² over the upper half and $A_2=6$ cm² over the lower half is supported and loaded as shown in Fig. 2·28. Calculate the stresses in the upper and lower portions if $P=1$ tonne.

Ans. [100 kg/cm² (tensile) in the upper portion, 100 kg/cm² (compression) in the lower portion].

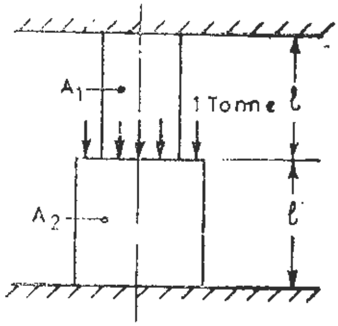


Fig. 2·28

2·6. A steel rod 25 mm in diameter passes co-axially inside an aluminium tube of inner diameter 30 mm and outer diameter 40 mm. It is provided with washers at each end and nuts are tightened until a stress of 20 N/mm² is set up in the aluminium tube.

The whole assembly is now placed in a lathe and a cut is taken along half the length of the tube reducing the outer diameter to 37 mm.

(*a*) Calculate stress now existing in the steel.

(*b*) If an additional end thrust of 4000 N is applied at the ends of the steel bar, calculate the final stress in steel.

$$E_{steel}=3\ E_{aluminium}=210\times10^3 \text{ N/mm}^2.$$

[**Ans.** (*a*) 18·997 N/mm², (*b*) 12·73 N/mm²]

2·7. A rigid bar EF 3·5 m long is supported by two wires AB and CD as shown in Fig. 2·29. Wire AB is 140 cm long, 5 mm diameter and made of copper. While wire CD is

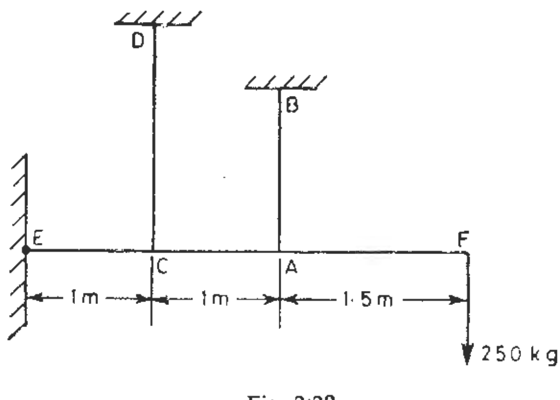


Fig. 2·29

180 cm long, 5 mm diameter and made of steel. The bar carries a vertical load of 250 kg at the end F and end E is hinged, determine the stresses in steel and copper wire.

$E_{steel}=2100$ tonnes/cm², $E_{copper}=1080$ tonnes/cm².

[**Ans.** 1220 kg/cm² (steel), 1615 kg cm² (copper)]

2·8. A rigid steel plate is supported by three vertical concrete posts of 80 cm height each, but accidentally the height of the middle post is 0·04 cm less as shown in Fig. 2·30. Area of cross section of each post is 12 cm×12 cm. Determine the safe value of the load P if the allowable stress for concrete in compression is 150 kg/cm².

$E_{concrete}=120$ tonnes/cm². [**Ans.** 56·16 tonnes]

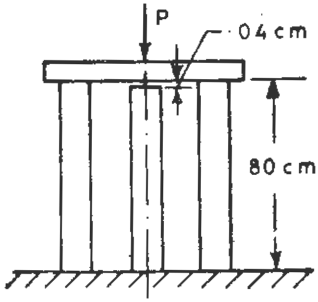


Fig. 2·30

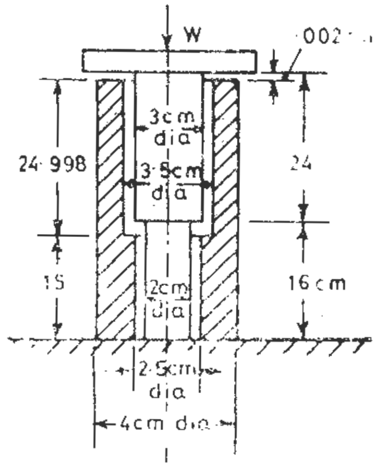


Fig. 2·31

2·9. A combination of stepped steel rod 40 cm long and brass tube 39·998 cm long is subjected to an axial compressive load $W=3$ tonnes. Both the rod, and the tube are co-axial as shown in Fig. 2·31. Determine the maximum stresses in steel and brass.

$E_{steel}=2\ E_{brass}=2000$ tonnes/cm², [**Ans.** 232 kg/cm², 772 kg/cm²

2·10. A railway is laid so that there is no stress in rails at 80°F. Determine the stress in the rail at 130 F, if its expansion is prevented.

$$E=210\times10^3 \text{ N/mm}^2$$
$$\alpha=6{\cdot}4\times10^{-6}/°\text{F}.$$

The rails are 30 m long. If, however, there is 6 mm allowance for expansion for rail, what is the stress at 130°F. [**Ans.** 67·2 N/mm², 25·2 N/mm²]

2·11. A copper wire 2 mm in diameter is stretched tightly between two supports 1 m apart under an initial tension of 20 kg. If the temperature drops by 10°C, determine the maximum tensile stress in the wire material.

$$E=1050 \text{ tonnes/cm}^2,\ \alpha=18\times10^{-6}/°\text{C}.$$

[**Ans.** 825 kg/cm²]

2·12. Two steel rods 25 mm in diameter are joined end to end by means of a turn buckle. The other end of each rod is rigidly fixed and there is initially a small tension in each rod. The effective length of each rod is 5 m. Calculate the increase in tension of each rod, when the turn buckle is tightened through one half of a turn. There are threads on each rod with a pitch of 3·18 mm.

$E=208\times10^3$ N/mm². If $\alpha=11\times10^{-6}/°\text{C}$,

what rise in temperature would nullify the increase in tension. [**Ans.** 32·408 kN, 28·9°C]

2·13. Three vertical wires, two of steel and one of aluminium are suspended in a vertical plane from a horizontal support. They are all of the same length and same area of cross section and carry a load by means of a rigid bar at their lower ends. The load is now increased and temperature is changed in such a way that the stress in each wire is increased by 28 N/mm². Find the change in temperature.

$$E_s=3E_a=210\times10^3 \text{ N/mm}^2$$
$$\alpha_s=6{\cdot}5\times10^{-6}/°\text{F},\quad \alpha_a=12{\cdot}5\times10^{-6}/°\text{F}.$$

[**Ans.** $-44{\cdot}44$°F]

2·14. Three vertical rods carry a tensile load of 120 kN. The area of cross section of each bar is 600 mm². Their temperature is raised by 100°C and the load is so adjusted that they extend equally. Determine the load shared by each. The outer two rods are of aluminium and the middle one is of brass.

$$E_a=700\times10^2\ \text{N/mm}^2,\quad E_b=1050\times10^2\ \text{N/mm}^2$$
$$\alpha_a=23\times10^{-6}/^\circ\text{C},\quad \alpha_b=18\times10^{-6}/^\circ\text{C}.$$

[**Ans.** 25·285 kN in each aluminium bar
69·43 kN in brass bar]

2·15. A steel tie rod of 25 mm diameter is enclosed in a brass tube of external diameter 40 mm and internal diameter 30 mm with the help of washers and nuts. The nut on the tie rod is tightened so as to produce a tensile stress of 30 N/mm² in steel rod. This combination is now subjected to a tensile load of 30 kN. Determine the resultant stress in steel tie rod and brass tube if

$$E_s=2\times10^5\ \text{N/mm}^2,\quad E_b=0{\cdot}8\times10^5\ \text{N/mm}^2.$$

Now if the temperature of the assembly is raised by 50°C, determine the final stresses in tie rod and tube.

$$\alpha_s=11\times10^{-6}/^\circ\text{C},\quad \alpha_b=19\times10^{-6}/^\circ\text{C}.$$

[**Ans.** (*i*) 72·19 N/mm² (tensile), 10·10 N/mm² (compressive)
(*ii*) 98·94 N/mm² (tensile), 32·21 N/mm² (compressive)]

2·16. Fig. 2·32 shows a steel bolt 2·5 cm diameter and 250 mm long passing centrally through an aluminium tube 180 mm long, outside diameter 4 cm and inside diameter 3 cm. The thread on bolt has a pitch of 3·18 mm. Find (*i*) changes in the stresses in bolt and tube due to the tightening of the nut through 30°. (*ii*) changes in the stresses in bolt and tube due to increase in temperature by 30°C.

$$E_s=3\,E_a=2100\ \text{tonnes/cm}^2$$
$$\alpha_s=11\times10^{-6}/^\circ\text{C},\quad \alpha_a=22\times10^{-6}/^\circ\text{C}.$$

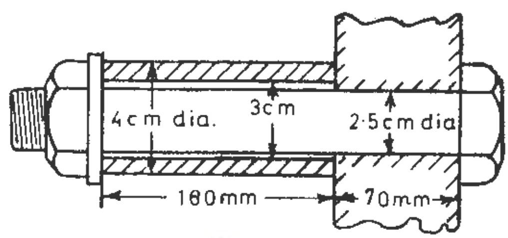


Fig. 2·32

2·17. Three brass wires of length L, area A support an equilateral triangular rigid plate. The bars are arranged at the corners of the triangle. A load P is then applied at the C.G. of the triangular plate.

A brass rod of length $(L-\delta)$ and area a is now attached to the rigid support, where the 3 bars are fixed, and its temperature is raised by $T°$ above normal so that it can be connected to the C.G. of the triangular plate. When the middle bar returned to normal temperature, it was found that the load in each of the 3 bars has been reduced by 25%. Show that

$$\delta=\frac{P}{4E\alpha}\left[\frac{1}{a}+\frac{1}{3A}\right]$$

$$T = \frac{PL}{4E}\left[\frac{1}{a} - \frac{1}{A}\right]$$

where α=coefficient of linear expansion

E=Young's modulus.

2·18. A 2 cm diameter steel rod passes centrally through on aluminium tube 2·4 cm internal diameter and 4 cm external diameter. The tube is closed at each end by thick steel plates secured by nuts. The nuts are tightened until the aluminium tube is reduced in length by 0·4 mm. The whole assembly is then raised in temperature by 50°C. Calculate the stresses in steel rod and aluminium tube before and after the rise in temperature. The thickness of the steel plates remain unchanged.

$$E_s = 3\ E_a = 2100 \text{ tonnes/cm}^2$$

$$\alpha_a = 2\ \alpha_s = 21 \times 10^{-6}/°\text{C}.$$

[**Ans.** (*i*) 448 kg/cm² (tensile), 175 kg/cm² (compressive)
(*ii*) 979·92 kg/cm² (tensile), 382·78 kg/cm² (compressive)]

2·19. A circular copper rod is fitted in a square steel frame of circular section. The diameter of the copper rod is 2 cm and the diameter of the rod of the steel frame is 4 cm. At a temperature of 15°C, there is a clearance of 0·02 mm, between the copper rod and the frame as shown in Fig. 2·33. Determine the compressive force in the copper rod if the temperature of the system is raised to 25°C, neglecting the bending of the frame.

Given $\alpha_c = 18 \times 10^{-6}/°\text{C}$

$\alpha_s = 11{\cdot}2 \times 10^{-6}/°\text{C}$

$E_{steel} = 2\ E_{copper} = 2080 \text{ tonnes/cm}^2$

[**Ans.** 132·32 kg]

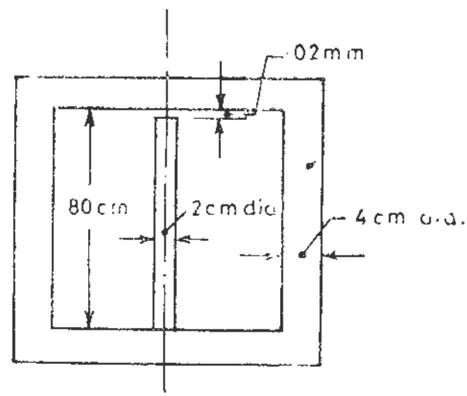


Fig. 2·33

3

Principal Stresses and Strains

While designing a machine member or a component of any structure, a designer has to determine in the critical region of member, where the stress developed due to loads is maximum, the nature and magnitude of the maximum stress. However, complex may be the state of stress at a point, there always exists a set of three orthogonal planes, perpendicular to each other on which the stresses are wholly normal and the shear stress does not accompany these direct stresses on any of the three orthogonal planes. The normal stresses on these three planes are called principal stresses and the planes are ealled principal planes. Out of these three principal stresses, one is maximum, other is minimum and third one is of some intermediate value. The designer has to consider this maximum principal stress while deciding about the dimensions of the machine member under consideration.

3·1. STRESSES ON AN INCLINED PLANE

Fig. 3·1 shows a small triangular element of a body with thickness t. Horizontial plane BC and vertical plane AC make a right angle at the point C. On plane BC, f_1 is the direct tensile stress and q is the shear stress. On plane AC, f_2 is the direct tensile stress and q is the shear stress. Shear stress q on plane AC is complementary to the shear stress q on plane BC.

Let us determine the nature and magnitude of the direct and shear stresses on the inclined plane AB. Now,

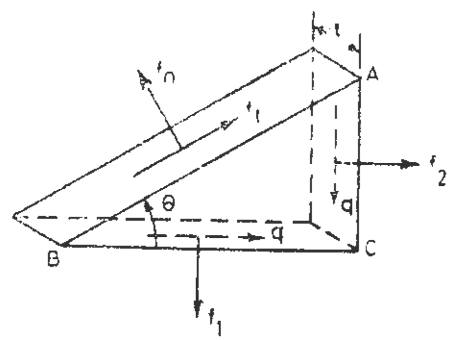


Fig. 3·1

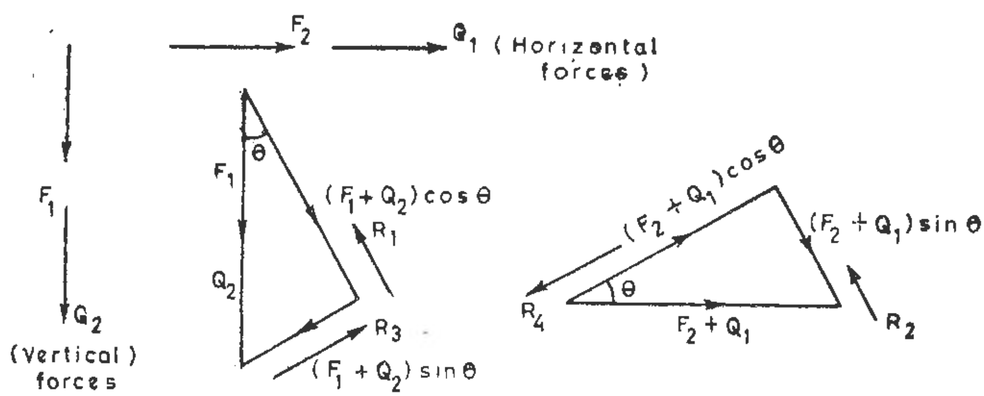


Fig. 3·2

Normal force on plane BC, $F_1=f_1\times BC\times t$
Shear force on plane BC, $Q_1=q\times BC\times t$
Normal force on plane AC, $F_2=f_2\times AC\times t$
Shear force on plane AC, $Q_2=q\times AC\times t$.

Forces F_1 and Q_2 are vertical while the forces Q_1 and F_2 are horizontal as shown in Fig. 3·2. Take the components of vertical and horizontal forces, along and perpendicular to the inclined plane AB. The triangular element is in static equilibrium. So the reactions to the applied forces are

$$R_1=(F_1+Q_2)\cos\theta, \qquad R_2=(F_2+Q_1)\sin\theta$$
$$R_3=(F_1+Q_2)\sin\theta, \qquad R_4=(F_2+Q_1)\cos\theta.$$

From these reactions,

Normal force on the inclined plane AB,

$$F_n=R_1+R_2 \qquad \ldots(1)$$
$$=(F_1+Q_2)\cos\theta+(F_2+Q_1)\sin\theta \text{ (a tensile force)}$$

(Since R_1 and R_2 are pointing away from the plane AB)

Shear force on the inclined plane AB,

$$F_t=R_3-R_4 \qquad \ldots(2)$$

(Since R_3 is producing clockwise moment and R_4 is producing anti-clockwise moment on the body)

$$F_n=F_1\cos\theta+Q_2\cos\theta+F_2\sin\theta+Q_1\sin\theta$$

Substituting the values of the forces

$$F_n=f_1\times BC\times t\cos\theta+q\times AC\times t\times\cos\theta$$
$$+f_2\times AC\times t\sin\theta+q\times BC\times t\sin\theta$$
$$=f_n\times AB\times t$$

where f_n is the direct tensile stress developed on the plane AB.

From equation (1)

$$f_n=f_1\times\frac{BC}{AB}\cos\theta+q\times\frac{AC}{AB}\cos\theta+f_2\times\frac{AC}{AB}\sin\theta+q+\frac{BC}{AB}\sin\theta$$

But $$\frac{BC}{AB}=\cos\theta \quad\text{and}\quad \frac{AC}{AB}=\sin\theta$$

So $$f_n=f_1\cos^2\theta+q\sin\theta\cos\theta+f_2\sin^2\theta+q\cos\theta\sin\theta \qquad \ldots(1)$$

where $$\cos^2\theta=\frac{1+\cos 2\theta}{2} \quad\text{and}\quad \sin^2\theta=\frac{1-\cos 2\theta}{2}$$

$$2\sin\theta\cos\theta=\sin 2\theta$$

Substituting these values above in equation (1)

$$f_n=f_1\left(\frac{1+\cos 2\theta}{2}\right)+f_2\left(\frac{1+\cos 2\theta}{2}\right)+q\sin 2\theta$$

$$f_n=\frac{f_1+f_2}{2}+\left(\frac{f_1-f_2}{2}\right)\cos 2\theta+q\sin 2\theta$$

Similarly from equation (2)

$$F_t = f_t \times AB \times t = (F_1 + Q_2)\sin\theta - (F_2 + Q_1)\cos\theta$$
$$= F_1 \sin\theta + Q_2 \sin\theta - F_2 \cos\theta - Q_1 \cos\theta$$
$$= f_1 \times BC \times t \sin\theta + q \times AC \times t \sin\theta - f_2 \times AC \times t \cos\theta - q \times BC \times t \cos\theta$$

or
$$f_t = f_1 \times \frac{BC}{AB}\sin\theta + q \times \frac{AC}{AB}\sin\theta - f_2 \times \frac{AC}{AB}\cos\theta - q \times \frac{BC}{AB}\cos\theta$$
$$= f_1 \cos\theta \sin\theta + q\sin^2\theta - f_2 \sin\theta\cos\theta - q\cos^2\theta$$
$$= \left(\frac{f_1 - f_2}{2}\right)\sin 2\theta - q(\cos^2\theta - \sin^2\theta)$$
$$= \left(\frac{f_1 - f_2}{2}\right)\sin 2\theta - q\cos 2\theta \qquad \ldots(2)$$

Note that if BC is taken as a reference plane with which angle θ is measured, then shear stress on this reference plane is negative (as is obvious from the figure).

Example 3·1-1. The stresses at a point on two perpendicular planes BC and AC are shown in the Fig. 3·3. Determine the normal and shear stresses on the inclined plane AB.

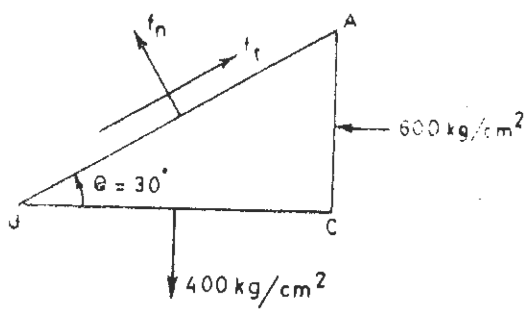


Fig. 3·3

Solution. Normal stress on plane BC,

$$f_1 = +400 \text{ kg/cm}^2 \text{ (tensile)}$$

Shear stress on plane BC,

$$q = 0$$

Normal stress on plane AC,

$$f_2 = -600 \text{ kg/cm}^2 \text{ (compressive)}$$

Angle of inclined plane AB with reference to plane BC,

$$\theta = 30°$$

Normal stress on inclined plane,

$$f_n = \frac{f_1 + f_2}{2} + \frac{f_1 - f_2}{2}\cos 2\theta$$
$$= \frac{400 - 600}{2} + \frac{400 + 600}{2}\cos 60°$$
$$= -100 + 250 = +150 \text{ kg/cm}^2 \text{ (tensile)}$$

Shear stress on inclined plane,

$$f_t = \frac{f_1 - f_2}{2} \sin 2\theta \quad \text{as } q=0$$

$$= \frac{400+600}{2} \cos 60^\circ = 433 \text{ kg/cm}^2 \text{ (+ve).}$$

Example 3·1-2. The stress at a point on two planes perpendicular to each other are shown in the Fig. 3·4. Determine the position of the plane AB such that shear stress on this plane is zero. What will be the normal stress on such a plane.

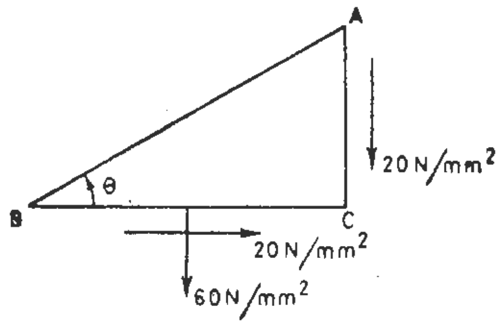


Fig. 3·4

Solution. Normal stress on plane BC, $f_1 = 60$ N/mm²

Shear stress on plane BC, $q = -20$ N/mm²

Normal stress on plane AC, $f_2 = 0$

Shear stress on the inclined plane AB,

$$f_t = \frac{f_1}{2} \sin 2\theta - q \cos 2\theta$$

$$= \frac{60}{2} \sin 2\theta - 20 \cos 2\theta = 0 \text{ (as given)}$$

So $$\tan 2\theta = \frac{20}{30} = 0{\cdot}666$$

$$2\theta = 33^\circ\ 42'$$

or $$\theta_1 = 16^\circ\ 51'$$

$$\theta_2 = \theta_1 + 90^\circ = 106^\circ\ 51'.$$

There are two planes inclined at angles θ_1 and θ_2 on the plane BC, on which the shear stress is zero.

Normal stress as the inclined plane AB,

(taking $\theta = 16^\circ\ 51'$) $$f_n = \frac{f_1}{2} + \frac{f_1}{2} \cos 2\theta + q \sin 2\theta$$

$$= \frac{60}{2} + \frac{60}{2} \cos 33^\circ\ 42' + 20 \sin 33^\circ\ 42'$$

$$= 30 + 30 \times 0{\cdot}8325 + 20 \times 0{\cdot}555$$

$$= \mathbf{66{\cdot}075\ N/mm^2}.$$

Normal stress on inclined plane

(taking $\theta=106° 51'$) AB $=30+30\cos(213° 42')+20\sin/213° 42')$

$=30-30\times0{\cdot}8325-20\times0{\cdot}555$

$=-6{\cdot}075$ N/mm².

Example 3·1-3. At a point in a strained body, planes BC and AC are perpendicular to each other. On plane BC the normal stress is 80 N/mm² tensile, shear stress is 15 N/mm². On plane AC, the normal stress is 40 N/mm² tensile and a shear stress 15 N/mm². Plane AB is inclined at an angle of 25° to the plane BC. Determine normal and shear stresses on the plane AB.

Solution. Let us show the stress-system graphically as in Fig. 3·5.

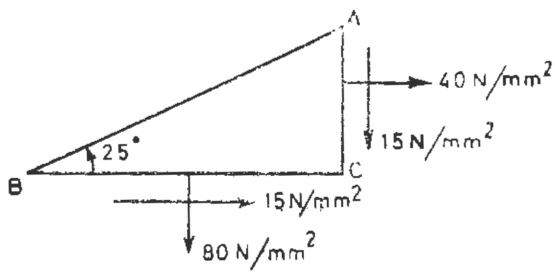


Fig. 3·5

Normal stress BC, $f_1=+80$ N/mm²

Shear stress on BC, $q=-15$ N/mm²

Normal stress on AC, $f_2=40$ N/mm³

Shear stress on AC, $q=+15$ N/mm³

Normal stress as inclined plane,

$$f_n=\frac{f_1+f_2}{2}+\frac{f_1-f_2}{2}\cos 2\theta+q\sin 2\theta$$

$$=\frac{80+40}{2}+\frac{80-40}{2}\cos 50°+15\sin 50°$$

$$=60+20\times0{\cdot}6428+15\times0{\cdot}7660=84{\cdot}346 \text{ N/mm}^2 \text{ (tensile)}$$

Shear stress on inclined plane, $f_t=\dfrac{f_1-f_2}{2}\sin 2\theta-q\cos 2\theta$

$$=\frac{80-40}{2}\sin 50°-15\times\cos 50°$$

$$=20\times0{\cdot}7660-15\times0{\cdot}6428=5{\cdot}678 \text{ N/mm}^2.$$

Example 3·1-4. The stresses at a point on two perpendicular planes AC and CB are as shown in Fig. 3·6. Determine the normal and shear stresses on plane AC, inclined at 35° to the plane AB.

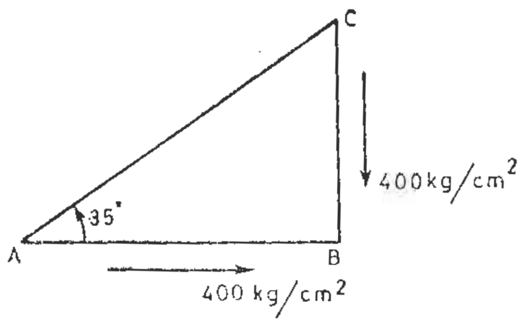


Fig. 3·6

Solution. In this case no normal stress is acting on plane AB and CB, therefore,

$$f_1 = f_2 = 0.$$

Normal stress on inclined plane,

$$f_n = q \sin 2\theta = 400 \times \sin 70° = 400 \times 0{\cdot}9397$$
$$= 375{\cdot}88 \text{ kg/cm}^2 \text{ (tensile)}$$

Shear stress on inclined plane, $f_t = -q \cos 2\theta$

$$= -400 \times \cos 70° = -400 \times 0{\cdot}342 = -133{\cdot}8 \text{ kg/cm}^2.$$

Exercise 3·1-1. The stress at a point on two perpendicular planes BC and AC are shown in the Fig. 3·7. Determine the normal and shear stresses on the inclined plane AB.
[**Ans.** 40 N/mm², 10 N/mm²].

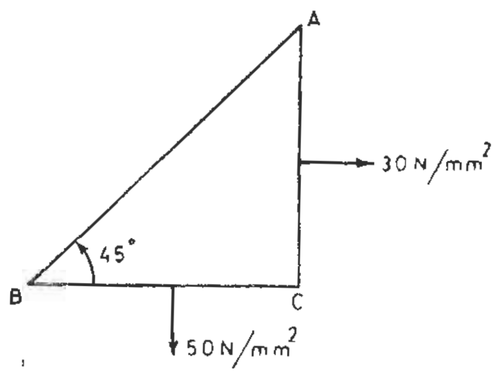


Fig. 3·7

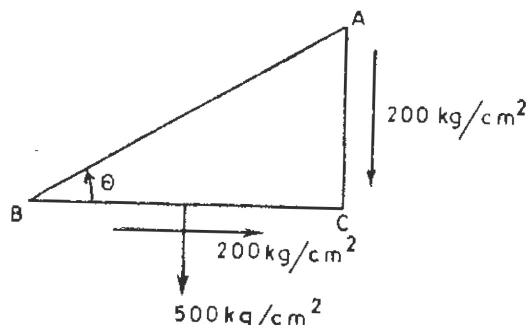


Fig. 3·8

Exercise 3·1-2. The stresses at a point on two planes perpendicular to each other are shown in Fig 3·8. Determine the position of the plane AB such that shear stress on this plane is zero. What will be the normal stress on such a plane.
[**Ans.** $\theta = 19° \, 19'$, $109° \, 19'$, 570·155, $-70{\cdot}155$ kg/cm²].

Exercise 3·1-3. At a point in a strained body, planes BC and AC are perpendicular to each other. On plane BC, the normal stress is 400 kg/cm² (tensile) and shear stress is 200 kg/cm². On plane AC the normal stress is 200 kg/cm² (compression) and the shear stress is 200 kg/cm². Plane AB is inclined at an angle of 30° to the plane BC. Determine the normal and shear stress on the AB. [**Ans.** 423·2 kg/cm², 159·8 kg/cm²]

Exercise 3·1·4. At a point in a strained body, planes BC and AC are perpendicular to each other. Shear stress 50 N/mm² is acting on both these planes. Determine the normal and shear stress on a plane AB inclined at an angle of 45° to the plane BC.

[**Ans.** 50 N/mm², 0·0]

3·2. PRINCIPAL STRESSES

Considering a three dimensional case is not within the scope of this book. We will consider only a plane stress problem *i.e.* the stress on the third plane is zero or say on the plane of the paper, the stress is zero. Therefore we will determine two principal stresses only and the third principal stress will be zero.

Now firstly on the principal planes shear stresses is zero and secondly the principal stresses are maximum and minimum normal stresses at a point.

Considering shear stress to be zero, from equation (2)

$$f_t=\frac{f_1-f_2}{2}\sin 2\theta-q\cos 2\theta=0.$$

$$\tan 2\theta=\frac{2q}{f_1-f_2}. \qquad \ldots(3)$$

Since $\tan 2\theta=\tan(180+2\theta)$

The principal angles with reference to plane BC (Fig. 3·1) are

$$\theta_1=\frac{1}{2}\tan^{-1}\frac{q}{(f_1-f_2)/2}$$

$$\theta_2=\theta_1+90°.$$

From equation (1), normal stress on any plane is

$$f_n=\frac{f_1+f_2}{2}+\frac{f_1-f_2}{2}\cos 2\theta+q\sin 2\theta. \qquad \ldots(1)$$

For maximum and minimum normal stresses

$$\frac{df_n}{d\theta}=0=\frac{f_1-f_2}{2}(-2\sin 2\theta)+q\,(2\cos\theta)$$

or

$$\tan 2\theta=\frac{2q}{f_1-f_2}=\frac{q}{(f_1-f_2)/2}. \qquad \ldots(4)$$

From equations (3) and (4), we learn that principal plane is inclined at an angle

$$\theta=\frac{1}{2}\tan^{-1}\frac{2q}{f_1-f_2}$$

to the reference plane (refer Fig. 3.1).

From equation (4),

$$\sin 2\theta=+\frac{q}{\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}}\;;\;-\frac{q}{\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}}$$

$$\cos 2\theta=+\frac{\left(\frac{f_1-f_3}{2}\right)}{\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}}\;;\;-\frac{\left(\frac{f_1-f_2}{2}\right)}{\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^3}}$$

Substituting the first set of values of sin 2θ and cos 2θ, we get principal stress

$$p_1=\frac{f_1+f_2}{2}+\left(\frac{f_1-f_2}{2}\times\left(\frac{\left(\frac{f_1-f_2}{2}\right)}{\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}}+q\times\frac{q}{\sqrt{\left(\frac{f_1-f_2}{2}\right)+q^2}}\right.\right.$$

$$p_1=\frac{f_1+f_2}{2}+\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}. \qquad \text{...(5)}$$

Similarly substituting second set of values of cos 2θ and sin 2θ we get other principal stress

$$p_2=\frac{f_1+f_2}{2}-\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2} \qquad \text{...(6)}$$

The third principal stress at the point is

$$p_3=0.$$

Maximum Shear Stress. The shear stress on any inclined plane given by equation (2) is

$$f_t=\frac{f_1-f_2}{2}\sin 2\theta-q\cos 2\theta. \qquad \text{...(2)}$$

For maximum value of shear stress $\frac{df_t}{d\theta}=0$

or $$\left(\frac{f_1-f_2}{2}\right)(2\cos 2\theta)+q\times 2\sin 2\theta=0$$

$$\tan 2\theta=-\frac{(f_1-f_2)/2}{q} \qquad \text{...(7)}$$

$$=\frac{f_2-f_1}{2q}.$$

From equation (7) $$\sin 2\theta=\frac{(f_1-f_2)/2}{\pm\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}}$$

$$\cos 2\theta=-\frac{q}{\pm\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}}.$$

Substituting the values in equation (2)

$$(f_t)_{max}=\left(\frac{f_1-f_2}{2}\right)\frac{\left(\frac{f_1-f_2}{2}\right)}{\pm\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}}+\frac{q\times q}{\pm\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}}$$

$$=\pm\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}. \qquad \text{...(8)}$$

Example 3·2-1. At a point in a strained material, on plane *BC* there are normal and shear stresses of 560 N/mm² and 14 N/mm² respectively. On plane *AC*, perpendicular to plane *BC*; there are normal and shear stresses of 28 N/mm² and 14 N/mm³ respectively as shown in the Fig. 3·9. Determine

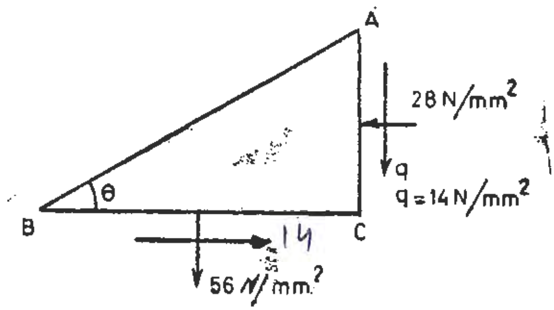


Fig. 3·9

(*i*) Principal stresses and principal angles.

(*ii*) Maximum shear stress and the plane on which it acts.

Solution. Taking plane *BC* to be the reference plane

$$f_1 = +56 \text{ N/mm}^2$$
$$q = -14 \text{ N/mm}^2$$
$$f_2 = -28 \text{ N/mm}^2.$$

Principal stresses

$$p_1 = \frac{f_1+f_2}{2} + \sqrt{\left(\frac{f_1-f_2}{2}\right)^2 + q^2}$$

$$= \frac{56-28}{4} + \sqrt{\left(\frac{56+28}{2}\right) + (14)^2}$$

$$= 14 + 44{\cdot}27 = 58{\cdot}27 \text{ N/mm}^2$$

$$p_2 = \frac{f_1+f_2}{2} - \sqrt{\left(\frac{f_1-f_2}{2}\right)^2 + q^2}$$

$$= 14 - 44{\cdot}27 = -30{\cdot}27 \text{ N/mm}^2$$

$$p_3 = 0$$

Principal Angles

$$\theta_1 = \frac{1}{2}\tan^{-1}\frac{2q}{f_1-f_2} = \frac{1}{2}\tan^{-1}\frac{2\times 14}{56+28} = \frac{1}{2}[18^\circ\ 30']$$

$$= 9^\circ\ 15'$$

$$\theta_2 = 90^\circ + \theta_1 = 99^\circ\ 15'$$

Maximum shear stress, $(f_t)_{max} = \pm\sqrt{\left(\frac{f_1-f_2}{2}\right)^2 + q^2}$

$$= \pm 44{\cdot}27 \text{ N/mm}^2.$$

Angles of planes for maximum shear stress

$$\theta_3 = \frac{1}{2}\tan^{-1}\frac{f_2-f_1}{2q} = \frac{1}{2}\tan^{-1}\left(\frac{-28-56}{2\times 24}\right)$$

$$= \frac{1}{2}[-71^\circ\ 30'] = -35^\circ\ 45'$$

$$\theta_4 = \theta_3 + 90^\circ = 54^\circ\ 15'.$$

One can learn from the observations as above, that the plane carrying maximum shear stress is at an angle of 45° to the principal plane.

Example 3·2-2. At a point in a strained material, on two planes BC and AC only the shear stress of intensity 35 N/mm² acts. Determine the magnitude of principal stresses and direction of principal planes.

Solution. Let us represent the stresses on the element as shown in Fig. 3·10.

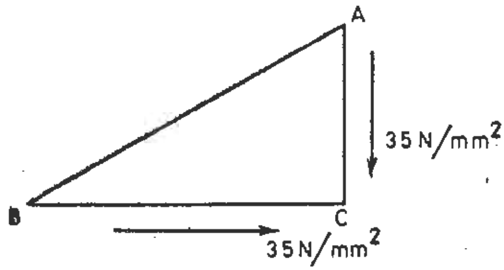


Fig. 3·10

Since in this case $f_1 = f_2 = 0$

$q = 35$ N/mm²

Principal stresses $p_1, p_2 = \pm\sqrt{q^2} = \pm 35$ N/mm².

Principal angles. $\theta_1 = \frac{1}{2}\tan^{-1}\frac{2q}{f_1 - f_2} = \frac{1}{2}\tan^{-1}\left(\frac{2\times 35}{0-0}\right) = \frac{1}{2}(90°)$

$= 45°$

$\theta_2 = 45° + 90° = 135°$.

In this example, planes BC and AC are planes of maximum shear stress. Principal planes are at angles of 45° and 135° the plane BC.

Exercise 3·2-1. At a point in a strained material, on plane BC there are normal and shear stresses of values -600 kg/cm² and 200 kg/cm² respectively. On plane AC, perpendicular to plane BC, there are normal and shear stresses of values $+300$ kg/cm² and 200 kg/cm² respectively. Determine

(*i*) Magnitude of principal stresses.

(*ii*) Directions of principal planes.

(*iii*) Magnitude of maximum shear stress.

(*iv*) Directions of planes carrying maximum shear stress with respect to the plane BC.
[**Ans.** 310·98, $-610·98$ kg/cm² ; $-11°$ 59°, $+78°$ 1′, $\pm 460·98$ kg/cm² ; 33° 1′, 123° 1′]

Exercise 3·2-2. At a point in a strained material, two plane BC and AC perpendicular to each other carry only the shear stress of intensity 200 kg/cm². Determine the magnitude of principal stresses and directions of principal planes.
[**Ans.** 200 kg/cm², -200 kg/cm² ; 45°, 135°]

3·3. GRAPHICAL SOLUTION

The stresses on any plane inclined to a reference plane or principal stresses and directions of principal planes can be easily obtained through a graphical solution. Following sign conventions can be taken for stresses :—

(*i*) Direct tensile stress (+ve).

(*ii*) Direct compressive stress (—ve).

(*iii*) Shear stress tending to rotate the body in the clockwise direction (+ve).

(*iv*) Shear stress tending to rotate the body in the anti-clockwise direction (—ve)

Direct stress is perpendicular to the shear stress on a plane. Therefore direct stress can be represented along the abscissae and shear stress can be represented along the ordinate of an *x-y* co-ordinate system, consider an element *ABC* subjected to the stresses as hown in Fig. 3·11.

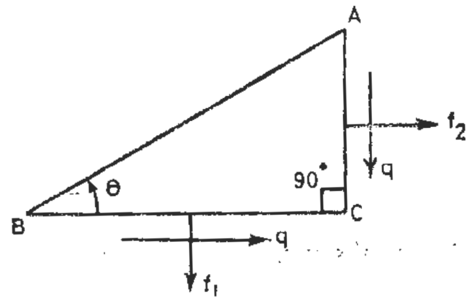


Fig. 3·11

On plane BC

(1) f_1 is +ve tensile stress.

(2) q is —ve shear stress.

On place AC

(1) f_2 is +ve tensile stress.

(2) q is +ve shear stress.

Fig 3·12 shows a co-ordinate system representing direct stresses along the abscissa and shear stresses along the ordinate.

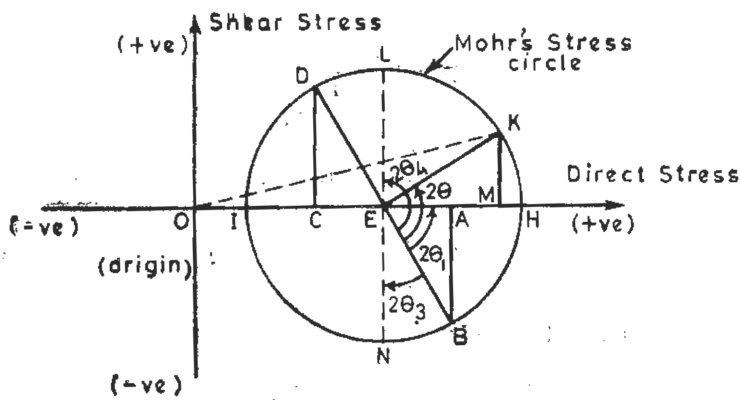


Fig. 3·12

To some suitable scale, take

$OA = f_1$ (co-ordinates of point *B* give the state of stress on plane *BC*)
$AB = -q$

$OC = f_2$ (co-ordinates of point *D* give the state of stress on plane *AC*).
$CD = +q$

Join BD intersecting the abscissa at E. Since $AB=CD$

$CE=EA=\frac{f_1-f_2}{2}$, with E as centre and radius EB or ED draw a circle. This is called a Mohr's stress circle.

Radius of the circle, $$R=\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}.$$

This circle intersects the abscissa at points H and I where the shear stress is zero. Therefore,

Principal stress, $$p_1=OH=OE+EH=OE+R$$
$$=OC+CE+R$$
$$=f_2+\frac{f_1-f_2}{2}+\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}$$
$$=\frac{f_1+f_2}{2}+\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}$$

Principal stress, $$p_2=OI=OE-EI=OC+CE-R$$
$$=\frac{f_1+f_2}{2}-\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}$$

Angles $$\angle BEH=2\theta_1$$

and $$\angle BEI=2\theta_2 \text{ are principal angles}$$

$$2\theta_1=\tan^{-1}\frac{AB}{EA}=\tan^{-1}\frac{2q}{f_1-f_2}$$

or $$\theta_1=\frac{1}{2}\tan^{-1}\frac{2q}{f_1-f_2}$$
$$\theta_2=\theta_1+90° \text{ as } \angle BEI=\angle BEH+180°.$$

To determine the stresses on a plane inclined at an angle θ to the reference plane, let us take $\angle BEK=2\theta$.

The co-ordinates of the point K on the Mohr's stress circle *i.e.* OM and KM give the normal and shear stresses respectively on the inclined plane.

Now $$OM=OF+EM$$
$$=\frac{f_1+f_2}{2}+R\cos(2\theta-2\theta_1)$$
$$=\frac{f_1+f_2}{2}+R\cos 2\theta\cos 2\theta_1+R\sin 2\theta\sin 2\theta_1$$

where $$\cos 2\theta_1=\frac{EA}{EB}=\frac{f_1-f_2}{2R}$$
$$\sin 2\theta_1=\frac{AB}{EB}=\frac{q}{R}.$$

So normal stresses on inclined plane,

$$f_n = \frac{f_1+f_2}{2} + R\cos 2\theta \cdot \frac{f_1-f_2}{2R} + R\sin 2\theta \cdot \frac{q}{R}$$

$$= \frac{f_1+f_2}{2} + \frac{f_1-f_2}{2}\cos 2\theta + q\sin 2\theta$$

(as proved analytically in article 3·1)

Shear stress on inclined plane,

$$f_t = KM = EK\sin(2\theta - 2\theta_1)$$

$$= R\sin 2\theta\cos 2\theta_1 - R\cos 2\theta\sin 2\theta_1$$

$$= R\sin 2\theta \cdot \frac{f_1-f_2}{2R} - R\cos 2\theta \cdot \frac{q}{R}$$

$$= \frac{f_1-f_2}{2}\sin 2\theta - q\cos 2\theta$$

(as proved analytically in article 3·1)

The resultant stress on the inclined plane,

$$f_r = OK = \sqrt{OM^2 + KM^2} = \sqrt{f_n^2 + f_t^2}$$

Points L and N on the Mohr's stress circle represent the maximum shear stress +ve and −ve at the point.

$$(f_t)_{max} = \pm \text{Radius of the circle}$$

$$= \pm\sqrt{\left(\frac{f_1-f_2}{2}\right)^2 + q^2}$$

To determine the direction of planes carrying maximum shear stress.

Let us take $\angle BEN = -2\theta_3$, (in the opposite direction to the normal convention of +ve angle)

$$\tan 2\theta_3 = \frac{EA}{AB} = \frac{f_1-f_2}{2q}$$

or

$$\theta_3 = -\frac{1}{2}\tan^{-1}\frac{f_1-f_2}{2q}$$

$$\theta_4 = \theta_3 + 90^\circ$$

Example 3·3-1. At a point in a strained material, stresses on two planes BC and AC, perpendicular to each other are as shown in the Fig. 3·13. Draw the Mohr's stress circle and determine

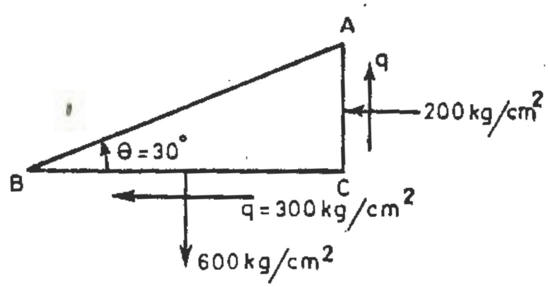


Fig. 3·13

(*i*) Stresses on the inclined plane AB
(*ii*) Magnitude of principal stresses
(*iii*) Direction of principal planes with respect to the plane BC
(*iv*) Magnitude of maximum shear stress
(*v*) Direction of planes carrying maximum shear stress.

Solution. Choose a co-ordinate system and to some scale take

$OA=+600$ kg/cm²
$AB=+300$ (shear stress on plane BC is +ve)
$OC=-200$ (normal stress on plane AC is −ve)
$CD=-300$ (shear stress on plane AC is −ve).

Join BD, intersecting abscissa at E. From E as centre draw a circle with radius ED. This is the Mohr's stress circle shown in Fig. 3·14.

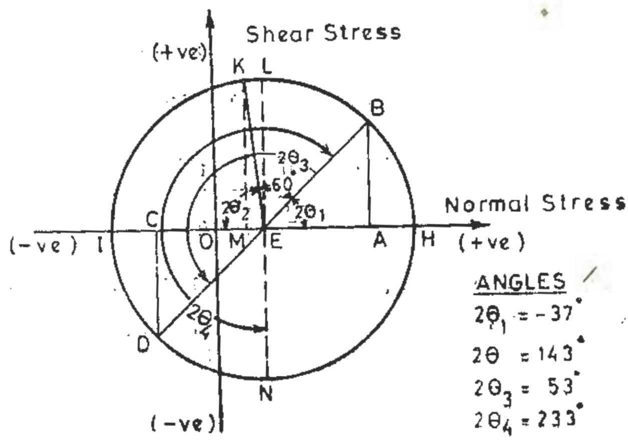


Fig. 3·14

(*i*) Draw an angle

$$\angle BEK=2\times 30°=60°$$
$$OM=f_n=+140 \text{ kg/cm}^2$$
$$KM=f_t=+495 \text{ kg/cm}^2$$

(*ii*) *Principal stresses* $p_1=OH=700$ kg/cm²
$p_2=OI=-300$ kg/cm²

(*iii*) Directions of principal planes

$$\frac{1}{2}\angle BEH=\theta_1=-18°\ 30'$$

$$\frac{1}{2}\angle BEI=\theta_2=+71°\ 30'$$

(*iv*) Maximum shear stress,

$$(f_t)_{max}=EL=EN=\pm 500 \text{ kg/cm}^2$$

(v) Directions of planes carrying maximum shear stress

$$\frac{1}{2}\ \angle BEL=\theta_3=26^\circ\ 30^{\circ\prime}$$

$$\frac{1}{2}\ \angle BEN=\theta_4=116^\circ\ 30'.$$

3·4. ELLIPSE OF STRESSES

If we know the principal stresses at a point, then stresses on any plane inclined to principal planes can be determined graphically with the help of an ellipse, with major axis $2p_1$ and minor axis $2p_2$ as shown in Fig. 3·15. YY is the plane of major principal stress p_1 and XX is the plane of minor principal stress p_2. Now make $\angle YOY'=\theta$, *i e.*, the inclined plane. The normal stress on the inclined plane will be in the direction perpendicular to $Y'Y'$ *i.e.*, along

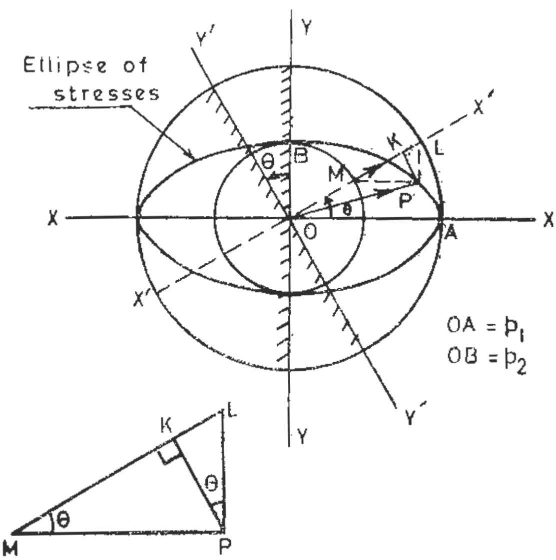


Fig. 3·15

$X'X'$ as shown. With O as centre draw two concentric circles one with radius p_1 and the other with radius p_2. The straight line $X'X'$ intersects the concentric circles at the points L and M respectively. From M draw a line MP parallel to OX and from L draw a line PL parallel to OB, intersecting the line MP at the point P. This point P lies on the ellipse with major axis $2p_1$ and minor axis $2p_2$. The resultant stress on the inclined plane is given by OP.

The components of OP, perpendicular to $Y'Y'$ and parallel to $Y'Y'$ represent in magnitude and direction the normal and shear stresses on the inclined plane,

$$f_n=OK,\quad f_t=KP$$

Resultant stress, $\quad OP=\sqrt{OK^2+KP^2}$

Geometrical proof

$$\angle PMK=\angle KPL=\theta$$

Normal stress, $\quad f_n=OK+KM=p_2+MP\cos\theta$

$$=p_2+(ML\cos\theta)\cos\theta=p_2+ML\cos^2\theta$$

$$=p_1+(p_1-p_2)\cos^2\theta$$

$$=p_2+(p_1-p_2)\left(\frac{1+\cos 2\theta}{2}\right) \quad \text{where} \quad ML=p_1-p_2$$

$$=\frac{p_1+p_2}{2}+\left(\frac{p_1-p_2}{2}\right)\cos 2\theta$$

Shear stress, $f_t=KP=MP \sin\theta=ML\cos\theta \,.\, \sin\theta$

$$=(p_1-p_2)\sin\theta\cos\theta$$

$$=\left(\frac{p_1-p_2}{2}\right)\sin 2\theta \quad \text{(already derived analytically in article 3·1)}$$

Example 3·4-1. The major and minor principal stresses at a point are $+70$ N/mm² and -30 N/mm². With the help of ellipse of stresses determine normal and shear stresses on a plane inclined at angle of 30° to the plane of major principal stress.

Solution. With O as centre and radii equal to $+70$ N/mm² and -30 N/mm² to some suitable scale, two concentric circles are drawn. $OA=+70$ N/mm², YY is the plane for major stress, $OB=-30$ N/mm², XX is the plane for minor principal stress. Draw plane $Y'Y'$ at an angle θ to YY as shown in Fig. 3·16.

Draw a line $X'X'$ perpendicular to $Y'Y'$ intersecting the bigger circle at C in the 1st quadrant in which major principal stress is positive and intersecting the smaller circle at D in the 3rd quadrant in which the minor principal stress is negative. From C draw a line parallel to YY and from D draw a line parallel to XX both meeting at P. Then OP is the resultant stress or plane $Y'Y'$. Component OK is the shear stress and component PK is the normal stress

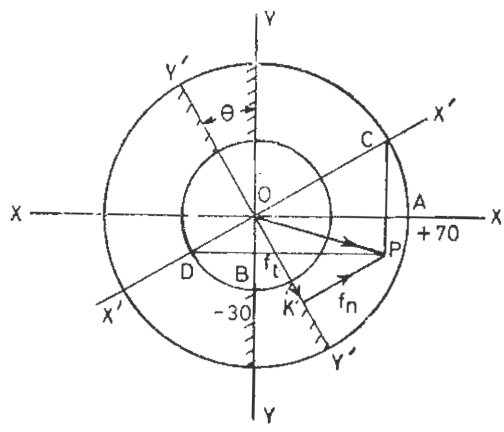


Fig. 3·16

$$f_n=PK=44·4 \text{ N/mm}^2$$

$$f_t=OK=43·8 \text{ N/mm}^2$$

These values can be verified analytically (see article 3·1)

$$f_1=+70/\text{mm}^2,\; f_2=-30 \text{ N/mm}^2,\; \theta=30°$$

$$f_n=\frac{f_1+f_2}{2}+\frac{f_1-f_2}{2}\cos 2\theta=\frac{70-30}{2}+\frac{70+30}{2}\cos 60°$$

$$=20+25=45 \text{ N/mm}^2$$

$$f_t=\frac{f_1-f_2}{2}\sin 2\theta=\frac{70+30}{2}\sin 60°=50\times 0·866$$

$$=43·3 \text{ N/mm}^2.$$

The solution through ellipse of stresses may show some graphical error.

Exercise 3·4-1. The major and minor principal stresses at a point in a strained material are -600 kg/cm^2 and -200 kg/cm^2. With the help of an ellipse of stresses, determine the normal and shear stresses on a plane inclined at an angle of 60° to the plane of major principal stress. [**Ans.** -300 kg/cm^2, -173 kg/cm^2]

3·5. STRAIN COMPONENTS

Let us consider only the plane strain of a body *i.e.*, we consider a body whose particles all lie in the same plane and which deform only in this plane. In chapter 1 we have studied the uniform strain, *i.e.*, all elements of a bar have been deformed by the same amount. But n a general case, the deformation can be non-uniform *i.e.*, a straight line is rotated, distorted, lisplaced from its original state in the undeformed geometry and the deformation is non-uniform, strain becomes more complicated. However if we examine a sufficiently small area, the deformation can be approximated uniform. In the limit as the small area centered on O shrinks to zero, this uniform deformation becomes the deformation at point O, (Fig. 3·17).

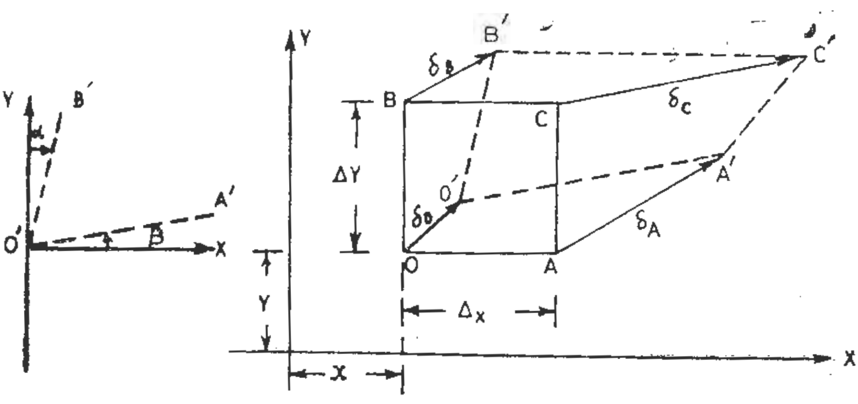


Fig. 3·17

Consider a thin continuous body $OACB$, lying entirely in the xy plane and undergoing a nall geometric deformation in the x-y plane. Let us express the deformation in the vicinity point O quantitatively by giving the changes in the length of two lines OA and OB. The *rmal strain* component is defined as the fractional change in the original length of a line and designated by the symbol ϵ with a subscript to indicate the original direction of the line for hich strain is measured.

Strains
$$\epsilon_x = \lim_{\Delta x \to 0} \frac{O'A' - OA}{OA}$$
$$\epsilon_y = \lim_{\Delta y \to 0} \frac{O'B' - OB}{OB}$$

This normal strain is positive when the line elongates and is negative when the line ntracts.

The shear strain component is specified with respect to two axes which are perpendi-lar in the undeformed geometry of the body and is designated by the symbol γ, with two bscripts to indicate these two axes. Shear strain is defined as the tangent of the change angle between these two originally perpendicular axes. These shear strains of engineering interest are very small such a 0·001 radian. It is adequate to define shear strain in terms of change in angle itself.

So $$\gamma_{xy}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}(\angle AOB-\angle A'O'B')=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left(\frac{\pi}{2}-\angle A'O'B'\right)$$

Figure 3·17 again shows an angle

$$\alpha=\angle YO'B' \text{ and angle } \beta=\angle XO'A'.$$

Total change in angle $\angle BOA=\alpha-\beta$

i e., shear strain is positive when α is clockwise and is negative when β is anticlockwise.

Total shear strain $=\alpha-\beta=\gamma_{xy}$.

Shear strain γ_{xy} is assumed to be equally divided about *OB* and *OA* axes. *i.e.*, shear strain $\gamma_{xy}/2$ about *OB* axis is positive while the shear strain $\gamma_{xy}/2$ about *OA* axis is negative.

3·6. STRAIN COMPONENTS ON INCLINED PLANE

Consider an element shown iu Fig. 3·18 subjected to strain components ϵ_x, ϵ_y and $\gamma_{xy}/2$. Say the block *ABCD* is deformed to $AB_1C_3D_2$ as shown. In other words *B* is displaced to

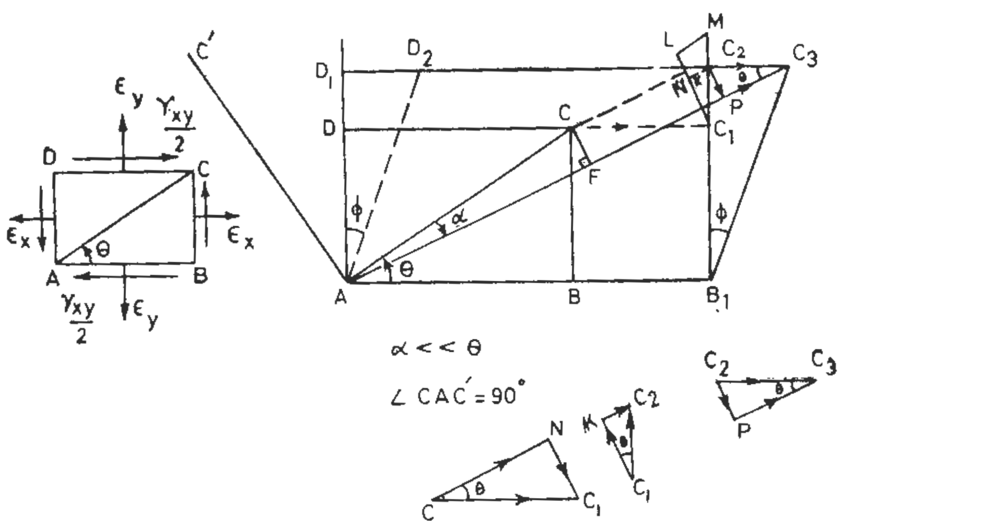


Fig. 3·18

B_1 and *C* is displaced to C_1. Then *D* is displaced to D_1 and C_1 is displaced to C_2. Then *C* is displaced to C_3 and D_1 is displaced to D_2.

Displacements BB_1, C_1C_2, C_2C_3 etc. are very very small in comparison to the dimension *AB* and *BC*. We have to determine the normal and shear strain components along th diagonal *AC*, inclined at an angle θ to the direction of strain component ϵ_x.

As per the definition of strain

$$\epsilon_x=\frac{BB_1}{AB}=\frac{CC_1}{AB}$$

$$\epsilon_y = \frac{C_1C_2}{B_1C_1} = \frac{DD_1}{AD}$$

$$\gamma_{xy} = \frac{C_2C_3}{B_1C_2} = \frac{C_2C_3}{B_1C_1+C_2C_1} \qquad \text{but} \qquad C_2C_1 << B_1C_1$$

$$= \frac{C_2C_3}{BC}.$$

Fig. 3·18 shows the components of displacements CC_1, C_1C_2, C_2C_3 along the diagonal AC and perpendicular to the diagonal AC.

Displacement components along $AC = CN + KC_2 + PC_3$

$$FC_3 = CC_1 \cos\theta + C_1C_2 \sin\theta + C_2C_3 \cos\theta$$

or
$$\frac{FC_3}{AC} = \frac{CC_1}{AC}\cos\theta + \frac{C_1C_2}{AC}\sin\theta + \frac{C_2C_3}{AC}\cos\theta$$

Strain component,
$$\epsilon_\theta = \frac{CC_1}{AB} \times \frac{AB}{AC}\cos\theta + \frac{C_1C_2}{BC} \times \frac{BC}{AC}\sin\theta + \frac{C_2C_3}{BC} \times \frac{BC}{AC}\cos\theta$$

$$= \epsilon_x \cos\theta \cos\theta + \epsilon_y \sin\theta \sin\theta + \gamma_{xy} \sin\theta \cos\theta$$

$$= \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

Since α angle is very small, CF is perpendicular to $AF. AC \simeq AF$.

$$\tan\alpha = \alpha = \frac{CF}{AC} \text{ as } \alpha \text{ is very small.}$$

Displacement component CF perpendicular to the diagonal AC,

$$CF = NC_1 + C_2P - KC_1$$

$$= CC_1 \sin\theta + C_2C_3 \sin\theta - C_1C_2 \cos\theta$$

or angle α,
$$\frac{CF}{AC} = \frac{CC_1}{AC}\sin\theta - \frac{C_1C_2}{AC}\cos\theta + \frac{C_2C_3}{AC}\sin\theta$$

$$\alpha = \frac{CC_1}{AB} \times \frac{AB}{AC}\sin\theta - \frac{C_1C_2}{BC} \cdot \frac{BC}{AC}\cos\theta + \frac{C_2C_3}{BC} \times \frac{BC}{AC}\sin\theta$$

$$= \epsilon_x \cos\theta \sin\theta - \epsilon_y \sin\theta \cos\theta + \gamma_{xy} \sin^2\theta$$

$$= (\epsilon_x - \epsilon_y)\sin\theta\cos\theta + \gamma_{xy}\sin^2\theta.$$

Say β is the change in angle in the clockwise direction for the direction AC' perpendicular to AC, *i.e.*, $\left(\theta + \frac{\pi}{2}\right)$ with respect to AB

$$\beta = (\epsilon_x - \epsilon_y)\sin\left(\theta + \frac{\pi}{2}\right)\cos\left(\theta + \frac{\pi}{2}\right) + \gamma_{xy}\sin^2\left(\theta + \frac{\pi}{2}\right)$$

$$= -(\epsilon_x - \epsilon_y)\cos\theta\sin\theta + \gamma_{xy}\cos^2\theta$$

Total shear strain, $= \alpha - \beta$

$$\alpha - \beta = 2(\epsilon_x - \epsilon_y)\sin\theta\cos\theta + \gamma_{xy}(\sin^2\theta - \cos^2\theta)$$

or
$$2\gamma_\theta = (\epsilon_x - \epsilon_y)\sin 2\theta - \gamma_{xy}\cos 2\theta$$

$$\gamma_\theta = \left(\frac{\epsilon_x - \epsilon_y}{2}\right)\sin 2\theta - \frac{\gamma_{xy}}{2}\cos 2\theta.$$

Example 3·6-1. A sheet of metal is deformed uniformly in its own plane such that the strain compenents related to xy axis are

$$\epsilon_x=-200\times10^{-6},\quad \epsilon_y=500\times10^{-6}$$
$$\gamma_{xy}=450\times10^{-6}.$$

Determine the normal and shear strain components on a plane inclined at an angle of 35° to the plane of ϵ_x.

Solution. The normal strain on inclined plane

$$\epsilon_\theta=\frac{\epsilon_x+\epsilon_y}{2}+\frac{\epsilon_x-\epsilon_y}{2}\cos 2\theta+\frac{\gamma_{xy}}{2}\sin 2\theta$$
$$=\left[\left(\frac{-200+500}{2}\right)+\left(\frac{-200-500}{2}\right)\cos 70+\frac{450}{2}\times\sin 70°\right]\times10^{-6}$$
$$=[150-350\cos 70°+225\sin 70°]\times10^{-6}$$
$$=[150-350\times0{\cdot}342+225\times0{\cdot}9397]\times10^{-6}=-58{\cdot}27\times10^{-6}$$

The shear strain on inclined plane

$$\gamma_\theta=\frac{\epsilon_x-\epsilon_y}{2}\sin 2\theta-\frac{\gamma_{xy}}{2}\cos 2\theta$$
$$=\left[\left(\frac{-200-500}{2}\right)\sin 70°-\frac{450}{2}\cos 70°\right]\times10^{-6}$$
$$=[-350\times0{\cdot}9397-225\times0{\cdot}342]\times10^{-6}=[-328{\cdot}89-76{\cdot}95]\times10^{-6}$$
$$=-405{\cdot}84\times10^{-6}.$$

Exercise 3·6-1. A sheet of metal is deformed in its own plane such that the strain components related to xy axes are

$$\epsilon_x=400\times10^{-6};\ \epsilon_y=-200\times10^{-6};\ \gamma_{xy}=500\times10^{-6}.$$

Determine the normal and shear strain components on a plane inclined at an angle of 45° to the plane of ϵ_x. [**Ans.** 350×10^{-6}, 300×10^{-6}]

3·7. MOHR'S STRAIN CIRCLE

Fig. 3·19 shows plane strains on an element *i.e.*, on plane AC, normal strain is ϵ_x, shear strain is $\gamma_{xy}/2$, on plane BC, normal strain is ϵ and shear strain is $\frac{\gamma_{xy}}{2}$. Choose the x—y co-ordinate system. Normal stresses are represented along the abscissa and shear strains are represented along the ordinate. Take to same suitable scale $OA=\epsilon_x$, $AB=-\gamma_{xy}/2$ (because the shear strain on plane AC tends to rotate the body in the anticlockwise direction. Then take $OC=\epsilon_y$, $CD=+\frac{\gamma_{xy}}{2}$

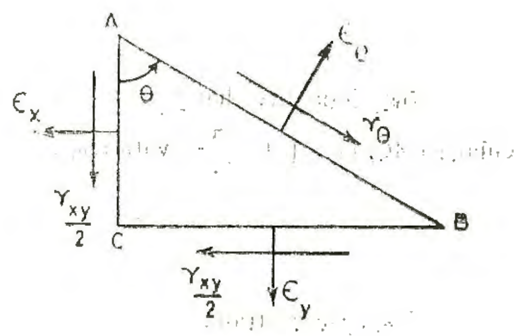


Fig. 3·19

(because the shear strain on plane BC tends to rotate the body in the clockwise direction). Join BD, intersecting the abscissa at E. With E as centre and radius equal to EB or

ED draw a circle as shown in Fig. 3·20. This is called the Mohr's strain circle. The circle intersects the abscissa at point *H* and *I*, where shear strain is zero. i.e., these points represent the principal strains.

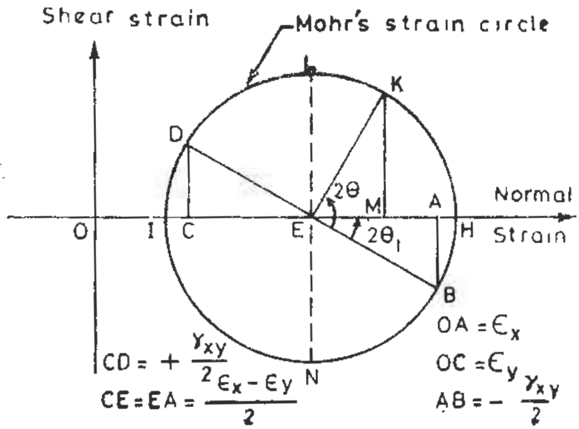


Fig. 3·20

Principal strains

$$\epsilon_1 = OH = OC + CE + EH = OC + CE + \text{Radius of the circle}$$

Radius of the circle, $R = \sqrt{EA^2 + AB^2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$

$$\epsilon_1 = \epsilon_y + \frac{\epsilon_x - \epsilon_y}{2} + \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= \frac{\epsilon_x + \epsilon_y}{2} + \frac{1}{2}\sqrt{(\epsilon_x - \epsilon_y)^2 + (\gamma_{xy})^2}$$

$$\epsilon_2 = OI = OE - EI = OE - \text{Radius of the circle } (R)$$

$$= \frac{\epsilon_x + \epsilon_y}{2} - \frac{1}{2}\sqrt{(\epsilon_x - \epsilon_y)^2 + (\gamma_{xy})^2}$$

Points *L* and *N* on the circle represent the maximum shear strain on the element of the body.

$$\gamma_{max} = \pm\frac{1}{2}\sqrt{(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2}.$$

Principal angle. To determine the directions of principal planes carrying principal strains with respect to the reference plane *AC* Consider angle *BEH*.

$$\angle BEH = 2\theta_1 = \tan^{-1}\frac{AB}{EA} = \tan^{-1}\frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

$$\therefore \quad \theta_1 = \frac{1}{2}\tan^{-1}\frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

$$\angle BEI = 2\theta_2 = 2\theta_1 + 180°$$

$$\theta_2 = \theta_1 + 90°.$$

Strains on the inclined plane. Plane AB is inclined at an angle θ to the reference plane AC. Draw an angle $BEK=2\theta$, intersecting the Mohr's strain circle at K. Then co-ordinates of the point K determine the normal and shear strains on the inclined plane.

Normal strain on inclined plane,

$$\epsilon_\theta=OM=OE+EM$$

Shear strain on inclined plane,

$$\gamma_\theta=KM$$

$$\epsilon_\theta=OE+EM=\frac{\epsilon_x+\epsilon_y}{2}+R\cos(2\theta-2\theta_1)$$

$$=\frac{\epsilon_x+\epsilon_y}{2}+R\cos 2\theta\cos 2\theta_1+R\sin 2\theta\sin 2\theta_1$$

$$=\frac{\epsilon_x+\epsilon_y}{2}+R\cos 2\theta\,.\,\frac{EA}{R}+R\sin 2\theta\,.\,\frac{AB}{R}$$

$$=\frac{\epsilon_x+\epsilon_y}{2}+\frac{\epsilon_x-\epsilon_y}{2}\cos 2\theta+\frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\gamma_\theta=KM=R\sin(2\theta-2\theta_1)$$

$$=R\sin 2\theta\cos 2\theta_1-R\cos 2\theta\sin 2\theta_1$$

$$=R\sin 2\theta\,.\,\frac{EA}{R}-R\cos 2\theta\,.\,\frac{AB}{R}$$

$$=\frac{\epsilon_x-\epsilon_y}{2}\sin 2\theta-\frac{\gamma_{xy}}{2}\cos 2\theta.$$

Example 3·7-1. The normal and shear strains acting at a point are

$$\epsilon_x=+500\mu\text{ cm/cm},\ \epsilon_y=-200\mu\text{ cm/cm},$$

$\gamma_{xy}/2=\pm150\mu$ cm/cm. Determine the (*i*) principal strains (*ii*) principal angles (*iii*) normal and shear strain on a plane inclined at an angle of 25° to the plane of ϵ_x.

Solution. $1\mu\text{ cm/cm}=1\times10^{-6}\dfrac{\text{cm}}{\text{cm}}=10^{-6}$ strain

$$\epsilon_x=500\times10^{-6},\ \epsilon_y=-100\times10^{-6}$$

$$\frac{\gamma_{xy}}{2}=150\times10^{-6},\ \frac{\epsilon_x+\epsilon_y}{2}=150\times10^{-6}$$

$$\frac{\epsilon_x-\epsilon_y}{2}=350\times10^{-6}$$

Principal strains

$$\epsilon_1=\frac{\epsilon_x+\epsilon_y}{2}+\sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$=[150+\sqrt{(350)^2+(150)^2}]\times10^{-6}$$

$$=[150+380{\cdot}79]\times10^{-6}=530{\cdot}79\times10^{-6}$$

$$\epsilon_2=\frac{\epsilon_x+\epsilon_y}{2}-\sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$=-230{\cdot}79\times10^{-6},$$

Principal angles

$$\theta_1=\frac{1}{2}\tan^{-1}\frac{\gamma_{xy}}{\epsilon_x-\epsilon_y}=\frac{1}{2}\tan^{-1}\frac{300}{700}=\frac{1}{2}(23^\circ\ 12')$$
$$=11^\circ\ 36',$$
$$\theta_2=\theta_1+90^\circ=101^\circ\ 36'.$$

Strains on inclined plane

$$\theta=25^\circ$$
$$\epsilon_\theta=\frac{\epsilon_x+\epsilon_y}{2}+\frac{\epsilon_x-\epsilon_y}{2}\cos 50^\circ+\frac{\gamma_{xy}}{2}\sin 50^\circ$$
$$=(150+350\times0{\cdot}6428+150\times0{\cdot}7660)\times10^{-6}$$
$$=(150+224{\cdot}98+114{\cdot}90)\times10^{-6}=489{\cdot}88\times10^{-6}$$
$$\gamma_\theta=\frac{\epsilon_x-\epsilon_y}{2}\sin 50^\circ-\frac{\gamma_{xy}}{2}\cos 50^\circ$$
$$=(350\times0{\cdot}7660-150\times0{\cdot}6428)\times10^{-6}=(268{\cdot}10-96{\cdot}42)\times10^{-6}$$
$$=171{\cdot}68\times10^{-6}.$$

Exercise 3·7-1. The normal and shear strains acting at a point are $\epsilon_x=450\times10^{-6}$, $\epsilon_y=250\times10^{-6}$, $\gamma_{xy}/2=\pm300\times10^{-6}$. Determine (*i*) principal strains (*ii*) principal angles (*iii*) strains on a plane inclined at an angle of 60° to the plane of ϵ_x.

[**Ans.** (*i*) $666{\cdot}22\times10^{-6}$, $33{\cdot}78\times10^{-6}$; (*ii*) 35° 48′, 125° 48′ ; (*iii*) $559{\cdot}8\times10^{-6}$, $236{\cdot}6\times10^{-6}$]

3·8. PRINCIPAL STRAINS IN TERMS OF PRINCIPAL STRESSES

As we already know that at any point, there always exists a set of 3 orthogonal planes on which the stresses are only the normal stresses. The normal stresses on these planes are called principal stresses and the strains in the direction of principal stresses are called *principal strains.*

Consider at a point, three principal stresses p_1, p_2, p_3 on three principal planes *OAFE*, *OEDC* and *OABC* respectively as shown in the Fig. 3·21.

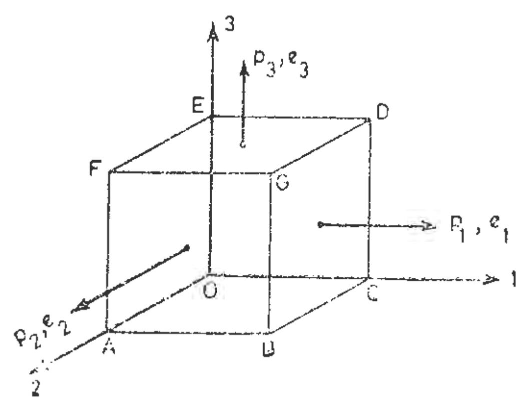


Fig. 3·21

Say the Young's modulus of the material is E and its Poisson's ratio is $1/m$.

Linear strain due to p_1 in direction 1

$$=+\frac{p_1}{E}$$

Lateral strain due to p_1 in direction 2

$$=-\frac{p_1}{mE}$$

Lateral strain due to p_1 in direction 3

$$=-\frac{p_1}{mE}$$

Similarly the linear and lateral strains due to p_2 and p_3 in directions 1, 2 and 3 can be determined.

Total strain in the direction 1,

$$\epsilon_1 = \frac{p_1}{E} - \frac{p_2}{mE} - \frac{p_3}{mE}$$

Total strain in the direction 2,

$$\epsilon_2 = \frac{p_2}{E} - \frac{p_1}{mE} - \frac{p_3}{mE}$$

Total strain in the direction 3,

$$\epsilon_3 = \frac{p_3}{E} - \frac{p_1}{mE} - \frac{p_2}{mE}.$$

ϵ_1, ϵ_2 and ϵ_3 are the principal strains in the directions of principal stresses.

In a two dimensional case where $p_3=0$

$$\epsilon_1 = \frac{p_1}{E} - \frac{p_2}{mE}\ ; \qquad \epsilon_2 = \frac{p_2}{E} - \frac{p_1}{mE}\ ; \qquad \epsilon_3 = \frac{-p_1}{mE} - \frac{p_2}{mE}.$$

Example 3·8-1. The principal strains at a point in a strained material subjected to principal stresses p_1 and p_2 are 720×10^{-6}, -560×10^{-6}. Determine the magnitude of the principal stresses if

$$E=200\ \text{GN/m}^2,\ \frac{1}{m}=0{\cdot}29.$$

Solution. Principal strain, $\epsilon_1 = \frac{p_1}{E} - \frac{p_2}{mE} = 720\times10^{-6}$

$$\epsilon_2 = \frac{p_2}{E} - \frac{p_1}{mE} = -560\times10^{-6}$$

or
$$p_1 - 0{\cdot}29\ p_2 = 720\times10^{-6}\times200\times10^{9} = 1420\times10^{5}. \qquad \ldots(1)$$
$$p_2 - 0{\cdot}29\ p_1 = -560\times10^{-6}\times200\times10^{9} = -1120\times10^{5}. \qquad \ldots(2)$$

Simultaneously solving these equations we get

$$p_1 = 1662{\cdot}4\times10^5\ \text{N/m}^2 = 166{\cdot}24\ \text{MN/m}^2$$
$$p_2 = -766{\cdot}9\times10^5\ \text{N/m}^2 = -76{\cdot}69\ \text{MN/m}^2.$$

Exercise 3·8-1. The principal stresses at a point in a strained material are $+1200$ kg/cm², $+800$ kg/cm², -400 kg/cm². Determine the values of principal strains if $E=2\times10^{-6}$ kg/cm², $1/m=0{\cdot}3$. [**Ans.** 540×10^{-6}, 280×10^{-6}, -500×10^{-6}]

3·9. MODIFIED MODULUS OF ELASTICITY

While determining the Young's modulus of elasticity of any material, tensile test is performed on a specimen of standard dimensions (as in Fig. 3·22) the tensile force P is applied

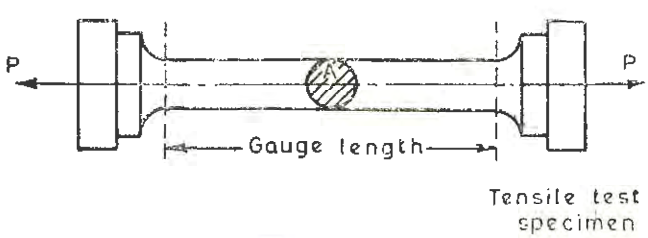


Fig. 3·22

along the axis of the cylindrical specimen. The Young's modulus of elasticity is determined as the ratio of axial tensile stress $p(=P/A)$ and the axial strain ϵ. In this case there is only one principal stress $p_1=P/A$ in the direction of load, while principal stresses p_2 and p_3 are zero. In actual practice a machine member or a structure may be subjected to principal stresses p_1, p_2 and p_3. In that case the ratio of actual stress and actual strain in a direction is called as modified modulus of elasticity in that direction.

So modified modulus of elasticity in direction 1 (see Fig. 3·21)

$$Em_1=\frac{p_1}{\epsilon_1}=\frac{Ep_1}{\left(p_1-\frac{p_2}{m}-\frac{p_3}{m}\right)}$$

Similarly $$Em_2=\frac{p_2}{\epsilon_2},\quad Em_3=\frac{p_3}{\epsilon_3}.$$

Example 3·9-1. The principal stresses at a point in a strained material are 100 N/mm² and −70 N/mm². Determine the modified modulus of elasticity of the material in the directions of principal stresses. Given $E=2\times10^5$ N/mm², $1/m=0·3$.

Solution. Principal strains, $$\epsilon_1=\frac{100}{E}+\frac{70}{E}\times0·3=\frac{121}{E}$$

$$\epsilon_2=\frac{-70}{E}-\frac{100\times0·3}{E}=\frac{-100}{E}$$

modified modulus of elasticity $$Em_1=\frac{p_1}{\epsilon_1}=\frac{100}{121}\times E$$

$$=\frac{100}{121}\times2\times10^5=1·653\times10^5 \text{ N/mm}^2$$

$$Em_2=\frac{p_2}{\epsilon_2}=\frac{-70}{-100}\times E$$

$$=0·7\times2\times10^5=1·4\times10^5 \text{ N/mm}^2.$$

Exercise 3·9-1. The principal stresses at a point in a strained material are 900 kg/cm² and 600 kg/cm². Determine the modified modulus of elasticity of the material along principal stress directions if $E=2·1\times10^6$ kg/cm², $1/m=0·28$. [**Ans.** $2·58\times10^6$ kg/cm², $3·62\times10^6$ kg/cm²]

Problem 3·1. In a piece of material, a tensile stress f_1 and a shearing stress q act on a given plane, while a tensile stress f_2 and a shearing stress q act on other plane perpendicular to the first plane, and all the stresses are coplanar. Find the conditions for which both the principal stresses will be of the same sign.

Solution. Fig. 3·23 shows a stress system, in which plane BC carries the tensile stress f_1 and shear stress q. While the plane AC carries the tensile stress f_2 and a shearing stress q.

Principal stresses are $$p_1=\frac{f_1+f_2}{2}+\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}$$

$$p_2=\frac{f_1+f_2}{2}-\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}.$$

Now if the principal stresses are both tensile then

$$\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2} < \frac{f_1+f_2}{2}$$

Squaring both the sides

$$\left(\frac{f_1-f_2}{2}\right)^2+q^2 < \left(\frac{f_1+f_2}{2}\right)^2$$

$$\frac{f_1^2}{4}+\frac{f_2^2}{4}-\frac{f_1 f_2}{2}+q^2 < \frac{f_1^2}{4}+\frac{f_1^2}{4}+\frac{f_1 f_2}{2}$$

or $q^2 < f_1 f_2$ condition for which both the principal stresses are of the same sign.

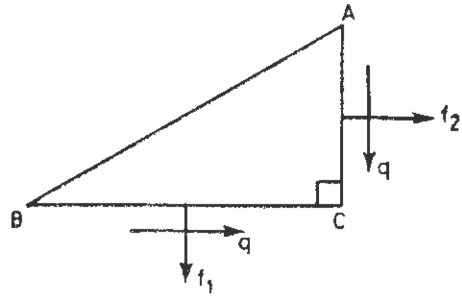


Fig. 3·23

Problem 3·2. Prove that the sum of the normal stresses on two perpendicular planes is constant.

Solution. Let us consider a general case.

On plane BC

Normal stress $=f_1$

Shear stress $=q$

On plane $AC \perp BC$

Normal stress $=f_2$

Shear stress $=q$

Sum of the normal stresses on two perpendicular planes BC and AC

$$=f_1+f_2$$

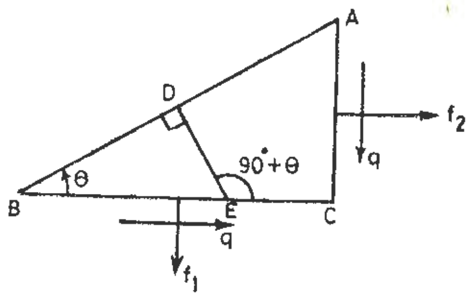


Fig. 3·24

Again consider a plane AB at an angle θ to the plane BC and plane DE perpendicular to the plane AB or inclined at an angle $90°+\theta$ to the plane BC, as shown in Fig. 3·24.

Normal stress on AB, $$f_n=\frac{f_1+f_2}{2}+\frac{f_1-f_2}{2}\cos 2\theta+q\sin 2\theta. \quad ...(1)$$

Normal stress on DE, $$f_n'=\frac{f_1+f_2}{2}+\frac{f_1-f_3}{2}\cos 2\,(90°+\theta)+q\sin 2\,(90°+\theta)$$

$$=\frac{f_1+f_2}{2}-\frac{f_1-f_2}{2}\cos 2\theta-q\sin 2\theta. \quad ...(2)$$

or $$f_n+f_n'=\frac{f_1+f_2}{2}+\frac{f_1+f_2}{2}=f_1+f_2. \quad ...(3)$$

This proves that the sum of the normal stresses on any two perpendicular planes is constant

Problem 3·3. On two perpendicular planes of a body, direct stresses of 15 N/mm² tensile and 90 N/mm² compressive are applied. The major principal stress at the point is not to exceed 200 N/mm², what shearing stress can be applied to the given planes ? What will be the minor principal stress and the maximum shearing stress at the point ?

Solution. On two planes BC and AC, perpendicular to each other, the direct stresses are as shown

$$f_1=150 \text{ N/mm}^2,\ f_2=-90 \text{ N/mm}^2.$$

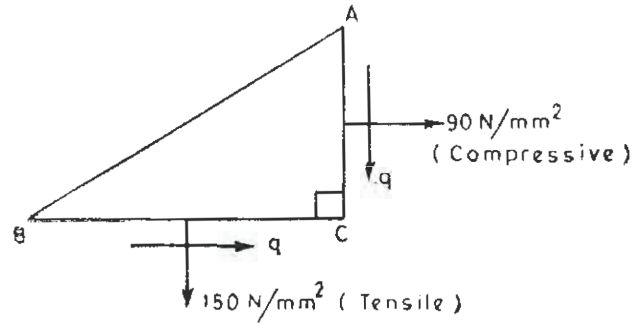


Fig. 3·25

Shear stress, q=unknown

Major principal stress $=\frac{f_1+f_2}{2}+\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}$

$$=\frac{150-90}{2}+\sqrt{\left(\frac{150+90}{2}\right)^2+q^2}$$

$$200=30+\sqrt{(120)^2+q^2}$$

or $\sqrt{(120)^2+q^2}=170, \quad (120)^2+q^2=170^2$

$$q^2=170^2-120^2=14500$$

Shear stress on given planes

$$q=\pm 120{\cdot}416 \text{ N/mm}^2$$

Minor principal stress $=\frac{f_1+f_2}{2}-\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}$

$$=30-170=-140 \text{ N/mm}^2$$

Maximum shearing stress $=\pm\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}$

$$=\pm 170 \text{ N/mm}^2$$

Problem 3·4. A piece of material is subjected to two tensile stresses at right angles, of values 120 N/mm² and 50 N/mm². Find the position of plane on which the resultant stress is most inclined to the normal. Find the value of this resultant stress.

Solution. To draw Mohr's stress circle, take

$$OA=120 \text{ N/mm}^2$$

$$OB=50 \text{ N/mm}^2 \text{ (to some suitable scale)}$$

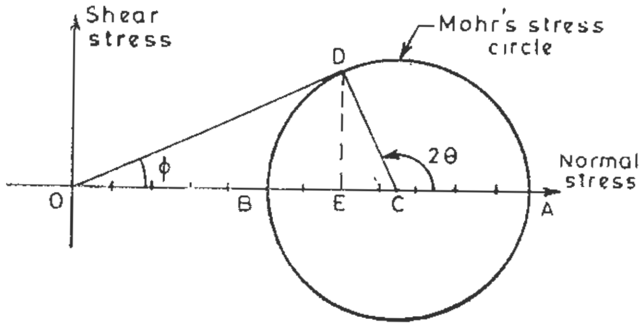


Fig. 3·26

C is the centre of AB, it is the centre of Mohr's stress circle..

Radius of the Mohr's stress circle

$$=BC=CA=\frac{120-50}{2}=35 \text{ N/mm}^2$$

The maximum angle ϕ is obtained when OD is tangent to the stress circle

$$OC=50+35=85 \text{ N/mm}^2$$
$$CD=35 \text{ N/mm}^2$$
$$\sin\phi=\frac{35}{85}=0{\cdot}41176$$
$$\phi=24^\circ\ 18'$$

angle of inclination of resultant stress to normal stress

$$OD=OC\cos\phi=85\times0{\cdot}9114$$
$$=77{\cdot}469 \text{ N/mm}^2.$$

The plane on which the resultant stress is most inclined to the normal stress is inclined at an angle θ to the plane of normal stress 120 N/mm²

$$2\theta=90+24^\circ\ 18'=114^\circ\ 18'$$
$$\theta=57^\circ\ 9'.$$

Problem. 3·5. In a stressed body, on a plane AB, the resultant stress is 67 N/mm² inclined at an angle of 15° to the normal stress and an another plane CD, the resultant stress

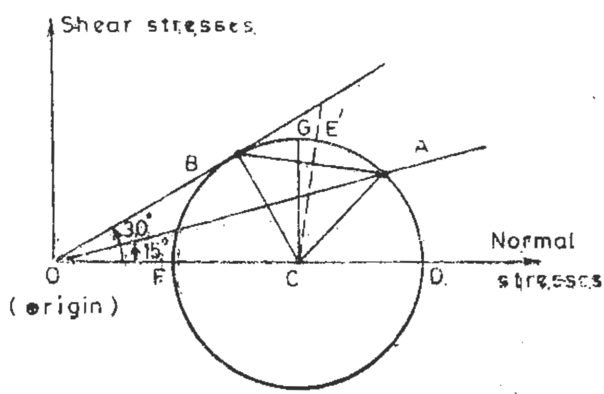


Fig. 3·27

is 45 N/mm² inclined at an angle of 30° to the normal stress. Determine the angle between the planes *AB* and *CD*. Find the magnitude of principal stresses and maximum shear stress on the body.

Solution. Take the co-ordinate axis with oxigin *O* as shown. Take to some scale $OA=67$ N/mm² at an angle of 15° with the abscissa and $OB=45$ N/mm² at an angle of 30°, with the abscissa. The points *A* and *B* lie on the Mohr's stress circle. Therefore draw the perpendicular bisector *EC* of the line *AB*, which meets the abscissa at the point *C*, which is the centre of the Mohr's stress circle.

With *C* as centre and radius *CB* or *CA* draw the circle *i.e.*, Mohr's stress circle.

Angle between the planes *AB* and *CD*

$$\theta=1/2\ \angle ACB=33^\circ$$

Principal stresses $p_1=OD=72$ N/mm² (tensile)

$p_2=OF=23$ N/mm² (tensile)

Maximum shear stress, $q_{max}=CG=$ Radius of the Mohr's stress circle

$=24{\cdot}5$ N/mm².

Problem 3·6. At a point in a stressed material, the normal stress on plane *AB* is 470 kg/cm² and the resultant stress on plane *BC* is 850 kg/cm² as shown in the Fig. 3·28. Determine the magnitude of principal stresses and maximum shearing stress at the point and directions of planes carrying these stresses.

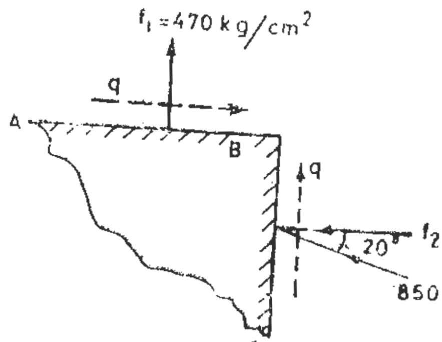


Fig. 3·28

Solution. Normal stress on plane *BC*,

$$f_2=850\cos 20^\circ=85\times 0{\cdot}9397$$
$$=798{\cdot}745 \text{ (compressive)}$$
$$=-798{\cdot}745 \text{ kg/cm}^2$$

Shear stress on plane *BC*,

$$q=850\sin 20^\circ=850\times 0{\cdot}342$$
$$=290{\cdot}7 \text{ kg/cm}^2$$

So the shear stress on plane *AB*,

$$q=290{\cdot}7 \text{ kg/cm}^2 \text{ (complementary shear stress)}$$

f_1, Normal stress on the plane $AB=470$ kg/cm²

Say the plane *AB* is the reference plane (on which the shear stress q is +ve).

Principal stresses $$p_1=\frac{f_1+f_2}{2}+\sqrt{\left(\frac{f_1-f_2}{2}\right)^2+q^2}$$

$$=\frac{470-798{\cdot}745}{2}+\sqrt{\left(\frac{470+798{\cdot}745}{2}\right)^2+(290{\cdot}7)^2}$$

$$=-164{\cdot}37+\sqrt{402428{\cdot}46+84506{\cdot}49}$$

$$=-164{\cdot}37+697{\cdot}60=533{\cdot}43 \text{ kg/cm}^2 \text{ (tensile)}$$

$$p_2=-164{\cdot}37-\sqrt{402428{\cdot}46+84506{\cdot}49}$$

$$=-164{\cdot}37-697{\cdot}80=-862{\cdot}17 \text{ kg/cm}^2 \text{ (compressive)}$$

Maximum shear stress

$$q_{max} = \pm\sqrt{\left(\frac{f_1 - f_2}{2}\right)^2 + q^2}$$

$$= \pm 697{\cdot}80 \text{ kg/cm}^2$$

Angles of principal planes

$$\tan 2\theta_1 = \frac{-2q}{(f_1 - f_2)} \text{ since the shear stress on reference plane is +ve}$$

$$= \frac{-2 \times 290{\cdot}7}{470 - 798{\cdot}745} = 1{\cdot}7685$$

$$\theta_1 = 30^\circ\ 15' \text{ w.r.t. plane } AB$$

$$\theta_2 = \theta_1 + 90^\circ = 120^\circ\ 15'$$

Angle for the plane of maximum shear,

$$\theta_3 = \theta_1 + 45^\circ = 75^\circ\ 15'.$$

Problem 3·7. Passing through a point in a material, there are two planes XY and YZ. Plane YZ is inclined at 45° clockwise to XY. The direct and shear stresses on plane XY are 80 MN/m² tensile and 40 MN/m² respectively. On the plane YZ there is a tensile stress of magnitude 150 MN/m² and a shearing stress. Determine (*i*) the magnitude of shearing stress on plane YZ (*ii*) magnitude of principal stresses (*iii*) maximum shearing stress (*iv*) directions of principal planes with the respect to the plane XY.

Solution. Let us take XY as reference plane. Consider a plane YM at right angles to plane XY, then shear stress on this plane will be 40 MN/m²

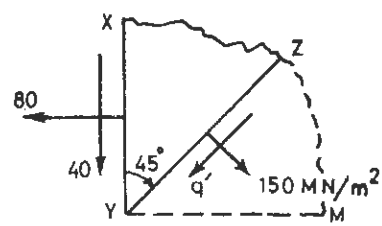


Fig. 3·29

Say the direct stress on plane YM

$$= f_2$$

Direct stress on plane YZ

$$= \frac{80 + f_2}{2} + \frac{80 - f_2}{2} \cos(-90^\circ) + 40 \sin(-90^\circ)$$

Since the angle of plane YZ w.r.t. XY is -45°

Therefore $$150 = \frac{80 + f_2}{2} + \frac{80 - f_2}{2}(0) - 40$$

or $$300 = 80 + f_2 - 80$$

$$f_2 = 300 \text{ MN/m}^2 \qquad \ldots(1)$$

Shear stress on plane YZ,

$$q' = \frac{80 - f_2}{2} \sin(-90^\circ) - 40 \cos(-90^\circ)$$

$$= \left(\frac{80 - 300}{2}\right)(-1) = 110 \text{ MN/m}^2,$$

Principal stresses

$$p_1=\frac{80+300}{2}+\sqrt{\left(\frac{80-300}{2}\right)^2+(40)^2}$$

$$=190+\sqrt{(110)^2+(40)^2}$$

$$=190+117{\cdot}05=307{\cdot}05 \text{ MN/m}^2$$

$$p_2=\frac{80+300}{2}-\sqrt{\left(\frac{80-300}{2}\right)^2+(40)^2}$$

$$=190-117{\cdot}05=72{\cdot}95 \text{ MN/m}^2.$$

Maximum shear stress

$$q_{max}=\pm\sqrt{\left(\frac{80-300}{2}\right)^2+(40)^2}$$

$$=\pm 117{\cdot}05 \text{ MN/m}^2.$$

Directions of principal planes

$$\tan 2\theta_1=\frac{2q}{f_1-f_2}$$

where $q=40$ MN/m²

$f_1=80$ MN/m², $f_2=300$ MN/m²

$$\tan 2\theta_1=\frac{2\times 40}{80-300}=\frac{-80}{220}=-0{\cdot}3636$$

$$2\theta_1=-90^\circ\ 54'$$

or $\theta_1=-9^\circ\ 57'$

$$\theta_2=-9^\circ\ 57'+90^\circ=80^\circ-3'.$$

One principal plane is inclined at an angle of 9° 57′ in clockwise direction and other principal plane is inclined at an angle of 80° 3′ in anticlockwise direction to the plane XY.

Problem 3·8. The minor principal stress at a point in the cross section of a beam is 30 MN/m² compressive and the magnitude of maximum shearing stress is 100 MN/m². Determine :

(*a*) the major principal stress if it is compressive and the direct and shear stresses on the plane making an angle of 60° in the clockwise direction with the plane of minor principal stress and

(*b*) The major principal stress if it is tensile and the direct stress on the planes of maximum shearing stress.

Solution. (*a*) When the major principal stress is also compressive

Say p_1=major principal stress

p_2=minor principal stress

$p_1-p_2=2\times$ maximum shearing stress

$$p_1=p_2+2q_{max}=30+2\times 100$$

$$=230 \text{ MN/m}^2 \text{ (compressive)}$$

Taking the plane of minor principal stress as the reference plane,

$\theta=-60^\circ$ (angle for the plane on which direct and shear stresses are to de determined.)

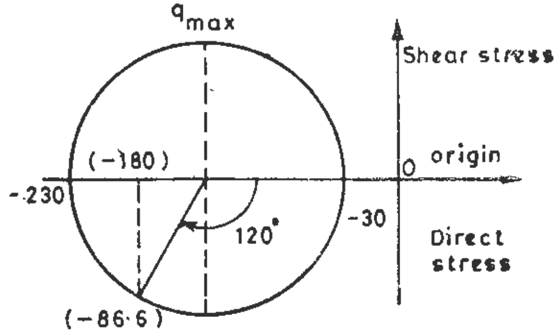


Fig. 3·30

Direct stress, $f_n = \frac{p_2+p_1}{2} + \frac{p_2-p_1}{2} \cos 2\theta$

$$= \frac{-30-230}{2} + \frac{-30+230}{2} \cos(-120°)$$

$$= -130+100 \cos(120°) = -130-50$$

$$= -180 \text{ MN/mm}^2.$$

Shear stress, $f_t = \frac{p_2-p_1}{2} \sin 2\theta$

$$= \frac{-30+230}{2} \sin(-120°) = +100\,(-0{\cdot}866)$$

$$= -86{\cdot}6 \text{ MN/m}^2.$$

(*b*) When the major principal stress is tensile

$$p_1 - p_2 = 2 \times q_{max}$$

$$p_1 = p_2 + 2q_{max} = -30 + 2 \times 100$$

Major principal stress $= 170$ MN/mm² (tensile).

Direct stress at the centre of the Mohr's stress circle

$$= p_2 + q_{max} = -30 + 100 = +70 \text{ MN/mm}^2.$$

This is the direct stress on the planes which carry the maximum shear stress.

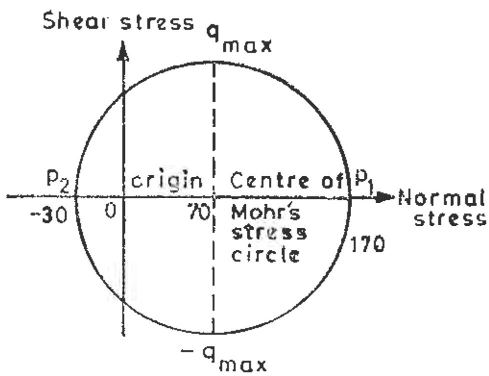


Fig. 3·31

Problem 3·9. At a point in a strained material, the stresses on the planes XY and XZ at an angle θ as shown in Fig. 3·32 are $+1{\cdot}2$ tonnes/cm² normal stress, 0·8 tonne/cm² shear stress on XY plane and $-2{\cdot}1$ tonnes/cm² normal stress, 0·6 tonne/cm² shear stress on XZ plane. Determine

(a) The angle between the planes XY and XZ.

(b) Magnitude of maximum and minimum principal stresses.

(c) The directions of principal planes with respect to the plane XY.

Solution. Let us consider a plane XZ' perpendicular to the plane XY.

Normal stress on the plane $XZ'=f_2$ (unknown)

Shear stress on the plane XZ',

$q=-0{\cdot}8$ tonne/cm²

(complementary shear stress)

Normal stress on the plane XZ,

$f_n=-2{\cdot}1$ tonnes/cm²

Shear stress on the plane XZ,

$f_t=-0{\cdot}6$ tonne/cm²

(producing anticlockwise moment on the body)

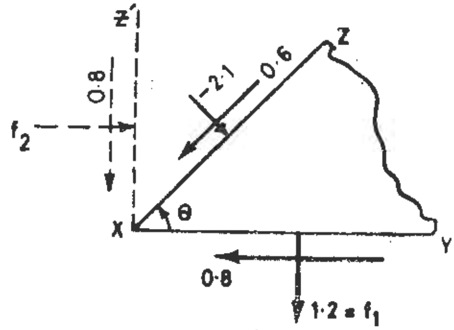


Fig. 3·32

Now,
$$f_n=\frac{f_1+f_2}{2}+\frac{f_1-f_2}{2}\cos 2\theta-q\sin 2\theta$$

$$-2{\cdot}1=\frac{1{\cdot}2+f_2}{2}+\frac{1{\cdot}2-f_2}{2}\cos 2\theta-0{\cdot}8\sin 2\theta$$

$$-2{\cdot}1\times 2=1{\cdot}2+f_2+1{\cdot}2\cos 2\theta-f_2\cos 2\theta-1{\cdot}6\sin 2\theta. \qquad ...(1)$$

$$f_t=\frac{f_1-f_2}{2}\sin 2\theta+q\cos 2\theta$$

$$-0{\cdot}6=\frac{1{\cdot}2-f_2}{2}\sin 2\theta+0{\cdot}8\cos 2\theta$$

$$-1{\cdot}2=1{\cdot}2\sin 2\theta-f_2\sin 2\theta+1{\cdot}6\cos 2\theta$$

$$f_2\sin 2\theta=1{\cdot}2\sin 2\theta+1{\cdot}6\cos 2\theta+1{\cdot}2$$

or
$$f_2=1{\cdot}2+1{\cdot}6\cot 2\theta+1{\cdot}2\operatorname{cosec} 2\theta. \qquad ...(2)$$

Substituting the value of f_2 in (1) we get

$$-4{\cdot}2=1{\cdot}2+1{\cdot}2+1{\cdot}6\cot 2\theta+1{\cdot}2\operatorname{cosec} 2\theta+1{\cdot}2\cos 2\theta$$
$$-\cos 2\theta[1{\cdot}2+1{\cdot}6\cot 2\theta+1{\cdot}2\operatorname{cosec} 2\theta]-1{\cdot}6\sin 2\theta$$

$$-6{\cdot}6=1{\cdot}6\cot 2\theta+1{\cdot}2\operatorname{cosec} 2\theta+1{\cdot}2\cos 2\theta$$
$$-1{\cdot}2\cos 2\theta-1{\cdot}6\frac{\cos^2 2\theta}{\sin 2\theta}-1{\cdot}2\cot 2\theta-1{\cdot}6\sin 2\theta$$

$$-6{\cdot}6=0{\cdot}4\cot 2\theta+\frac{1{\cdot}2}{\sin 2\theta}-1{\cdot}6\frac{\cos^2 2\theta}{\sin 2\theta}-1{\cdot}6\sin 2\theta$$

$$=0{\cdot}4\frac{\cos 2\theta}{\sin 2\theta}+\frac{1{\cdot}2}{\sin 2\theta}-\frac{1{\cdot}6}{\sin 2\theta}[\cos^2 2\theta+\sin^2 2\theta]$$

$$-6{\cdot}6=0{\cdot}4\frac{\cos 2\theta}{\sin 2\theta}-\frac{0{\cdot}4}{\sin 2\theta}$$

$$\frac{6{\cdot}6}{0{\cdot}4}=\frac{1}{\sin 2\theta}-\frac{\cos 2\theta}{\sin 2\theta}=\frac{1-\cos 2\theta}{\sin 2\theta}$$

$$16{\cdot}5=\frac{1-\cos 2\theta}{\sin 2\theta}=\frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$16{\cdot}5=\tan\theta$$

$$\theta=86^\circ-33'$$

$$\sin\theta=\frac{16{\cdot}5}{\sqrt{273{\cdot}25}},\quad \cos\theta=\frac{1}{\sqrt{273{\cdot}25}}$$

$$\sin 2\theta=\frac{33}{273{\cdot}25},\ \tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=-\frac{33}{271{\cdot}25}.$$

Substituting these values in the expression for f_2

$$f_2=1{\cdot}2-\frac{1{\cdot}6\times 271{\cdot}25}{33}+\frac{1{\cdot}2\times 273{\cdot}25}{33}$$

$$=-2{\cdot}01 \text{ tonnes/cm}^2.$$

Minimum principal stress, $p_{min}=\dfrac{1{\cdot}2-2{\cdot}01}{2}+\sqrt{\left(\dfrac{1{\cdot}2+2{\cdot}01}{2}\right)^2+(0{\cdot}8)^2}$

$$=-0{\cdot}405+1{\cdot}790=1{\cdot}385 \text{ tonnes/cm}^2 \text{ (tensile)}$$

$$\tan 2\theta_1=-\frac{2q}{(f_1-f_2)}=-\frac{2\times 0{\cdot}8}{1{\cdot}2+2{\cdot}01}=-\frac{1{\cdot}6}{3{\cdot}21}$$

$$=-0{\cdot}498 \qquad 2\theta_1=-26{\cdot}5^\circ,\ \theta_1=-13^\circ\ 15'.$$

Maximum principal stress, $p_{max}=\dfrac{1{\cdot}2-2{\cdot}01}{2}-\sqrt{\left(\dfrac{1{\cdot}2+2{\cdot}01}{2}\right)^2+(0{\cdot}8)^2}$

$$=-2{\cdot}195 \text{ tonnes/cm}^2 \text{ (compressive)}$$

$$\theta_2=90+\theta_1=76^\circ\ 45'.$$

Graphical Method

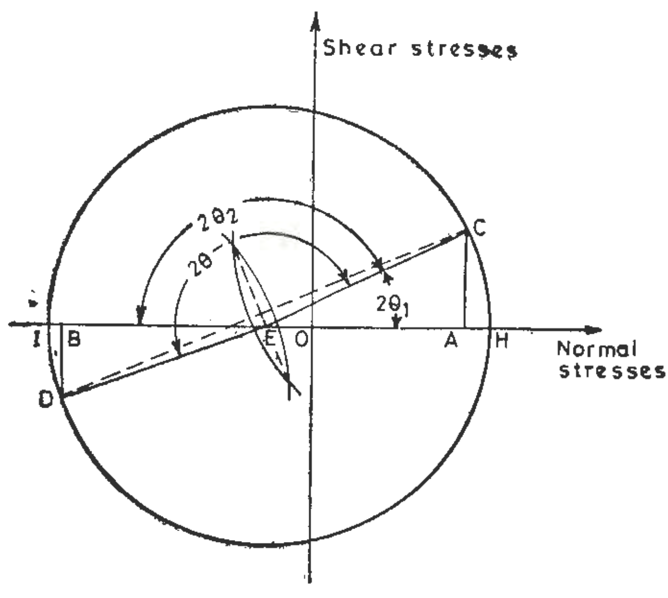


Fig. 3·33

To some suitable scale take

$$OA = f_1 = +1{\cdot}2 \quad \text{normal stress}$$
$$AC = +0{\cdot}8 \quad \text{shear stress}$$
$$OB = -2{\cdot}1 \quad \text{normal stress}$$
$$BD = -0{\cdot}6 \quad \text{shear stress.}$$

Join CD and draw the right bisector of CD and produce so as to cut the abscissa at the point E. Then E is the centre of the Mohr's stress circle.

With E as centre and radius EC or ED draw a circle.

$$\angle CED = 2\theta = 173° - 6'$$

or $$\theta = 86° \ 33'.$$

Maximum principal stress, $p_{max} = GI = -2{\cdot}195$ tonnes/cm²

$$2\theta_2 = 153° \ 30', \ \theta_2 = 76° \ 45'.$$

Minimum principal stress, $p_{min} = OH = -1{\cdot}385$ tonnes/cm²

$$2\theta_1 = -26° \ 30', \ \theta_1 = -13° \ 15'.$$

Problem 3·10. A circle of 100 mm diameter is inscribed on a steel plate before it is stressed. Then the plate is loaded so as to produce stresses as shown in Fig. 3·34 and the circle is deformed into an ellipse. Determine the major and minor axes of the ellipse and their directions.

Given $$E = 2100 \text{ tonnes/cm}^2, \ 1/m = 0{\cdot}28.$$

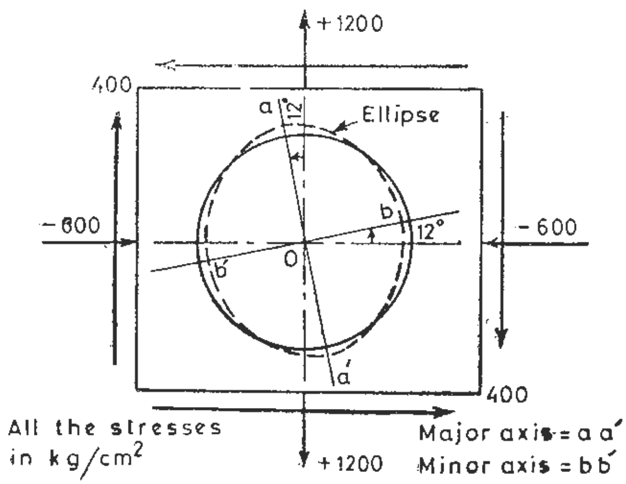


Fig. 3·34

Solution. Stresses

$$f_1 = +1200 \text{ kg/cm}^2$$
$$f_2 = -600 \text{ kg/cm}^2$$
$$q = 400 \text{ kg/cm}^2.$$

The circle will be deformed into an ellipse, due to the applied stresses. The major axis of the ellipse will be along the major principal stress p_1 and minor axis of the ellipse will be along the minor principal stress p_2.

Principal stresses, $p_1 = \frac{f_1+f_2}{2} + \sqrt{\left(\frac{f_1-f_2}{2}\right)^2 + q^2}$

$$= \frac{1200-600}{2} + \sqrt{\left(\frac{1200+600}{2}\right)^2 + 400^2}$$

$$= 300 + 985 = 1285 \text{ kg/cm}^2.$$

$$p_2 = 300 - 985 = -685 \text{ kg/cm}^2$$

$$\tan 2\theta_1 = \frac{2q}{f_1-f_2} = \frac{2\times 400}{1200+600} = 0{\cdot}44$$

$$2\theta_1 = 24°, \quad \theta_1 = 12°$$

and $\theta_2 = 90° + \theta_1 = 102°.$

Principal strains

$$\epsilon_1 = \frac{1285}{E} + \frac{1}{m}\cdot\frac{685}{E} = \frac{1285+0{\cdot}28\times 685}{2100\times 1000} = 0{\cdot}702\times 10^{-3}$$

Change in diameter $= \epsilon_1 \times 100 = 0{\cdot}702 \times 10^{-3} \times 100 = 0{\cdot}0702$ mm

Major axis of ellipse $= 100{\cdot}0702$ mm

$$\epsilon_2 = \frac{-685}{E} - \frac{1}{m} \times \frac{1285}{E} = \frac{-685 - 0{\cdot}28\times 1285}{2100\times 1000}$$

$$= -0{\cdot}498 \times 10^{-3}$$

Change in diameter $= \epsilon_2 \times 100 = -0{\cdot}498 \times 10^{-3} \times 100 = -0{\cdot}0498$ mm

Minor axis of the ellipse $= 100 - 0{\cdot}0498 = 99{\cdot}9502$ mm

Problem 3·11. Strains at a point on a specimen are recorded in direction 0°, 45° and 90° with the help of strain gauges. The readings are given below :

$$\epsilon_{0°} = 400 \ \mu \text{ cm/cm}$$

$$\epsilon_{45°} = +175 \ \mu \text{ cm/cm}$$

$$\epsilon_{90°} = -300 \ \mu \text{ cm/cm}$$

Determine the magnitude of principal strains and principal stresses and principal angles.

Given $E = 2 \times 10^5$ N/mm²

$$\frac{1}{m} = 0{\cdot}3.$$

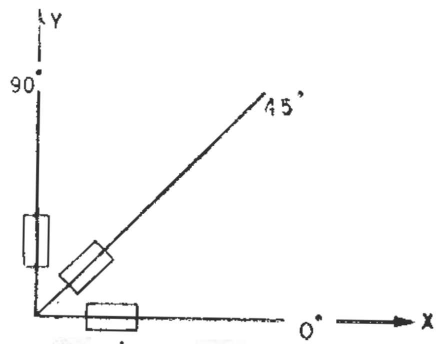


Fig. 3·35

Solution. Electrical resistance strain gauges are used to record the normal strains in any direction. The principal of operation of a strain gauge is that strain

$$\epsilon = \frac{\Delta R}{R \times GF}$$

where ΔR = Change in resistance R of the gauge due to strain ϵ

GF = Gauge factor specified for the gauge.

Let us take x axis along 0° as shown in Fig. 3·35.

$$\epsilon_{0°} = \epsilon_x = 400 \times 10^{-6} \quad ...(1)$$

Since $1\mu \text{ cm/cm} = 1\times 10^{-6} \frac{\text{cm}}{\text{cm}} = 1\times 10^{-6}$

$$\epsilon_{45^\circ} = \frac{\epsilon_x+\epsilon_y}{2} + \frac{\epsilon_x-\epsilon_y}{2} \cos 90^\circ + \frac{\gamma_{xy}}{2} \sin 90^\circ$$

$$175\times 10^{-6} = \frac{\epsilon_x+\epsilon_y}{2} + \frac{\gamma_{xy}}{2} \qquad ...(2)$$

$$\epsilon_{90^\circ} = \epsilon_y = -300\times 10^{-6} \qquad ...(3)$$

From equation (2)

$$\frac{\gamma_{xy}}{2} = 175\times 10^{-6} - \frac{\epsilon_x}{2} - \frac{\epsilon_y}{2} = (175-200+150)\times 10^{-6}$$

$$= 125\times 10^{-6}.$$

Principal strains

$$\epsilon_1 = \frac{\epsilon_x+\epsilon_y}{2} + \sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= \frac{400-300}{2} + \sqrt{\left(\frac{400+300}{2}\right)^2 + (125)^2}\ \mu \text{ cm/cm}$$

$$= 50 + 371{\cdot}65\ \mu \text{ cm/cm} = 421{\cdot}65\times 10^{-6}$$

$$\epsilon_2 = 50 - 371{\cdot}65\ \mu \text{ cm/cm} = -321{\cdot}65\times 10^{-6}.$$

Principal stresses

$$p_1 = \frac{E}{\left(1-\frac{1}{m^2}\right)} \left(\epsilon_1 + \frac{1}{m}\ \epsilon_2\right)$$

$$= \frac{2\times 10^5}{1-0{\cdot}3^2} (421{\cdot}65 - 0{\cdot}3\times 321{\cdot}65)\times 10^{-6}$$

$$= \frac{2\times 10^5}{0{\cdot}91} (325{\cdot}16)\times 10^{-6} = +71{\cdot}46 \text{ N/mm}^2$$

$$p_2 = \frac{E}{\left(1-\frac{1}{m^2}\right)} \left(\epsilon_2 + \frac{1}{m}\ \epsilon_1\right)$$

$$= \frac{2\times 10^5}{1-0{\cdot}3^2} (-321{\cdot}65 + 0{\cdot}3\times 421{\cdot}68)\times 10^{-6}$$

$$= \frac{2\times 10^5}{0{\cdot}91} (-195{\cdot}16)\times 10^{-6} = -42{\cdot}89 \text{ N/mm}^2$$

Principal angles

$$\tan 2\theta_1 = \frac{\gamma_{xy}}{\epsilon_x-\epsilon_y} = \frac{250}{400+300} = 0{\cdot}357$$

$$2\theta_1 = 19^\circ\ 38',\ \theta_1 = 9^\circ\ 49',\ \theta_2 = 99^\circ\ 49'.$$

Problem 3·12. For a delta rosette, the following observations are made with gauges mounted on an aluminium specimen

$$\epsilon_{0^\circ} = -100\ \mu\ \text{cm/cm}$$
$$\epsilon_{60^\circ} = +700\ \mu\ \text{cm/cm}$$
$$\epsilon_{120^\circ} = -600\ \mu\ \text{cm/cm}$$

Determine the principal strains, the principal stresses and the principal angles θ_1 and θ_2.

Given

E for aluminium $= 0{\cdot}7 \times 10^5$ N/mm²

$\frac{1}{m}$ for aluminium $= 0{\cdot}33$.

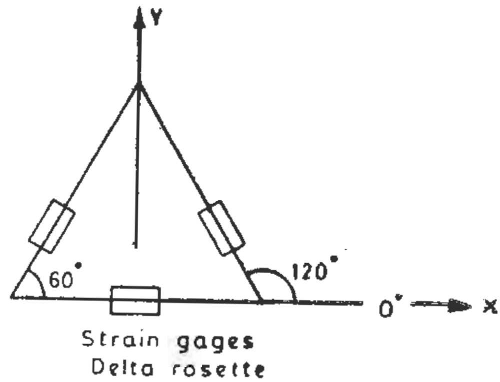


Fig. 3·36

Solution. In the case of delta-rosette, 3 strain gauges are available in a combination along the sides of an equilateral triangle as shown in Fig. 3·36 and these are mounted on the specimen.

Let us choose the x-axis along 0° direction, then

$$\epsilon_{0^\circ} = \epsilon_x = -100 \times 10^{-6} \qquad \ldots(1)$$

$$\epsilon_{60^\circ} = \frac{\epsilon_x+\epsilon_y}{2} + \frac{\epsilon_x-\epsilon_y}{2}\cos 120^\circ + \frac{\gamma_{xy}}{2}\sin 120^\circ$$

or

$$700 \times 10^{-6} = \frac{\epsilon_x+\epsilon_y}{2} + \frac{\epsilon_x-\epsilon_y}{2}(-0{\cdot}5) + \frac{\gamma_{xy}}{2}(0{\cdot}866) \qquad \ldots(2)$$

$$\epsilon_{120^\circ} = \frac{\epsilon_x+\epsilon_y}{2} + \frac{\epsilon_x-\epsilon_y}{2}\cos 240^\circ + \frac{\gamma_{xy}}{2}\sin 240^\circ$$

$$-600 \times 10^{-6} = \frac{\epsilon_x+\epsilon}{2} + \frac{\epsilon_x-\epsilon_y}{2}(-0{\cdot}5) + \frac{\gamma_{xy}}{2}(-0{\cdot}866) \qquad \ldots(3)$$

Substracting equation (3) from equation (2) we get

$$1300 \times 10^{-6} = \frac{\gamma_{xy}}{2}(1{\cdot}732)$$

$$\frac{\gamma_{xy}}{2} = 750{\cdot}58 \times 10^{-6} \qquad \ldots(4)$$

Putting the values of ϵ_x, $\frac{\gamma_{xy}}{2}$ in equation (2)

$$700 \times 10^{-6} = -\frac{100 \times 10^{-6}}{2} + \frac{\epsilon_y}{2} + \frac{100 \times 10^{-6}}{4} + \frac{\epsilon_y}{4} + 750{\cdot}58 \times 10^{-6} \times (0{\cdot}866)$$

$$\frac{3\epsilon_y}{4} = (700+50-25-650) \times 10^{-6}$$

$$\epsilon_y = 100 \times 10^{-6}.$$

Principal strains

$$\epsilon_1=\frac{\epsilon_x+\epsilon_y}{2}+\sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$=\left[-\frac{100+100}{2}+\sqrt{\left(\frac{-100-100}{2}\right)^2+(750{\cdot}58)^2}\right]\times10^{-6}$$

$$=+757{\cdot}21\times10^{-6}$$

$$\epsilon_2=\frac{\epsilon_y+\epsilon_y}{2}-\sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$=-757{\cdot}21\times10^{-6}.$$

Principal stresses

$$p_1=\frac{E}{1-\frac{1}{m^2}}\left[\epsilon_1+\frac{1}{m}\;\epsilon_2\right]$$

$$=\frac{0{\cdot}7\times10^5}{1-0{\cdot}33^2}\,[757{\cdot}21+0{\cdot}33\,(-757{\cdot}21)]\times10^{-6}$$

$$=\frac{0{\cdot}7\times10^5}{0{\cdot}891}\,[507{\cdot}33]\times10^{-6}=36{\cdot}2\text{ N/mm}^2$$

$$p_2=\frac{E}{1-\frac{1}{m^2}}\left[\epsilon_2+\frac{1}{m}\;\epsilon_1\right]$$

$$=\frac{0{\cdot}7\times10^5}{1-0{\cdot}33^2}\,[-757{\cdot}21+0{\cdot}33\times757{\cdot}21]\times10^{-6}$$

$$=-36{\cdot}2\text{ N/mm}^2.$$

Principal angles

$$\tan 2\theta_1=\frac{\gamma_{xy}}{\epsilon_x-\epsilon_y}=\frac{750{\cdot}58\times2}{-100-100}=-7{\cdot}5058$$

$$2\theta_1=-82^\circ\ 24'$$

$$\theta_1=-41^\circ\ 12'$$

$$\theta_2=\theta_1+90^\circ=+48^\circ\ 58'.$$

Problem 3·13. A rectangular bar of a material is subjected to an axial compressive stress p_1. In addition to the axial pressure, the lateral pressures act on the bar in other two

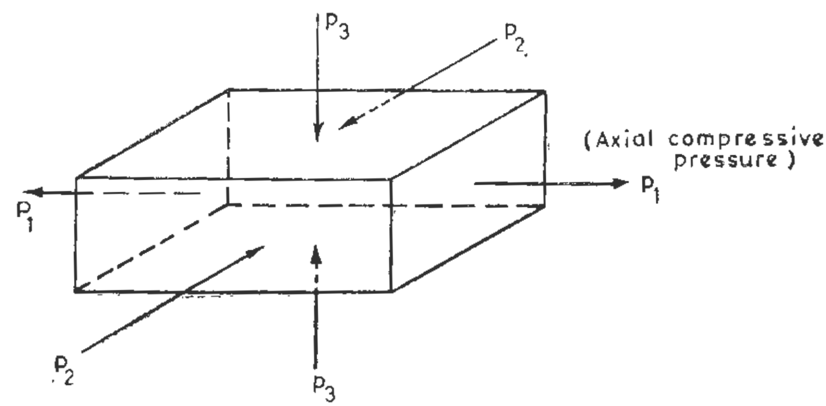


Fig. 3·37

directions such that the lateral strain in direction 2 is reduced to 50 per cent and in direction 3 it is reduced to 40 per cent of the strain if the bar is free to contract or expand laterally under the axial stress. Determine the modified modulus of elasticity in the direction of the stress p_1.

Solution. The free lateral expansion of the bar is prevented and reduced to half in direction 2 and 40 per cent in direction 3. Obviously with the help of compressive stresses p_1 and p_3 in directions 2 and 3 respectively as shown in Fig. 3·37.

(*a*) When the bar is subjected to p_1 only, strains in 3 directions will be

$$\epsilon_1=-\frac{p_1}{E},\quad \epsilon_2=+\frac{p_1}{mE},\quad \epsilon_3=+\frac{p_1}{mE}$$

(*b*) When bar is subjected to p_1, p_2, p_3 stresses, strain in 3 directions will be

$$\epsilon_1'=-\left[\frac{p_1}{E}-\frac{p_2+p_3}{mE}\right]$$

$$\epsilon_2'=-\left[\frac{p_2}{E}-\frac{p_1+p_3}{mE}\right]$$

$$\epsilon_3'=-\left[\frac{p_3}{E}-\frac{p_1+p_2}{mE}\right]$$

(*c*) Now $\epsilon_2'=0{\cdot}5\ \epsilon_2,\ \epsilon_3'=0{\cdot}4\ \epsilon_3$

Therefore $$\frac{p_1}{2mE}=-\left[\frac{p_2}{E}-\frac{p_1+p_3}{mE}\right]$$

or $$\frac{p_1}{2m}=-p_2+\frac{p_1}{m}+\frac{p_3}{m}$$

or $$-\frac{p_1}{2m}=-p_2+\frac{p_3}{m} \qquad \ldots(1)$$

$$\frac{0{\cdot}4\,p_1}{mE}=-\left[\frac{p_3}{E}-\frac{p_1+p_2}{mE}\right]$$

or $$\frac{0{\cdot}4\,p_1}{m}=-p_3+\frac{p_1}{m}+\frac{p_2}{m}$$

or $$-\frac{0{\cdot}6\,p_1}{m}=-p_3+\frac{p_2}{m} \qquad \ldots(2)$$

Multiplying equation (2) by $\frac{1}{m}$ and adding to equation (1) we get

$$-\frac{p_1}{2m}-\frac{0{\cdot}6\,p_1}{m^2}=\frac{p_2}{m^2}-p_2$$

$$-p_1\left(\frac{0{\cdot}5m+0{\cdot}6}{m^2}\right)=p_2\left(\frac{1-m^2}{m^2}\right)$$

$$p_2=\frac{0{\cdot}5m+0{\cdot}6}{m^2-1}\ p_1 \qquad \ldots(3)$$

Substituting the value of p_2 in equation (1) we get

$$-\frac{0{\cdot}5\,p_1}{m}=-\frac{0{\cdot}5m+0{\cdot}6}{m^2-1}\ p_1+\frac{p_3}{m}$$

or $$p_3=\frac{0{\cdot}6m+0{\cdot}5}{m^2-1}\times p_1 \qquad \ldots(4)$$

Now strain $\epsilon_1' = -\frac{p_1}{E} + \frac{p_2}{mE} + \frac{p_3}{mE}$

$$= -\frac{p_1}{m} + \frac{0{\cdot}5m+0{\cdot}6}{m(m^2-1)\,E} \times p_1 + \frac{0{\cdot}6m+0{\cdot}5}{m(m^2-1)\,E}\,p_1$$

$$\epsilon_1' = +\frac{p_1}{E}\left[\frac{-m(m^2-1)+0{\cdot}5m+0{\cdot}6+0{\cdot}6m+0{\cdot}5}{m(m^2-1)}\right]$$

$$\epsilon_1' = \frac{p_1}{E}\left[\frac{-m^3+m+1{\cdot}1m+1{\cdot}1}{m(m^2-1)}\right]$$

or Modified modulus of elasticity,

$$E_{m_1} = \frac{p_1}{\epsilon_1'} = \frac{Em\,(m^2-1)}{(1{\cdot}1m+1{\cdot}1-m^3)}.$$

SUMMARY

1. If f_1 and f_2 are the direct stresses and q is the shear stress on two planes perpendicular to each other at a point in a strained material then (*i*) normal stress and shear stress on a plane inclined at an angle θ to a reference plane (say plane of direct stress f_1) are

Normal stress, $f_n = \frac{f_1+f_2}{2} + \frac{f_1-f_2}{2}\cos 2\theta + q \sin 2\theta$

Shear stress, $f_t = \left(\frac{f_1-f_2}{2}\right)\sin 2\theta - q\cos 2\theta$

and (*ii*) principal stresses at the point are

$$p_1, p_2 = \frac{f_1+f_2}{2} \pm \sqrt{\left(\frac{f_1-f_2}{2}\right)^2 + q^2}$$

$$p_3 = 0$$

(*iii*) Principal angles with respect to principal planes are

$$\theta_1 = \frac{1}{2}\tan^{-1}\frac{2q}{(f_1-f_2)}$$

$$\theta_2 = \theta_1 + 90^\circ$$

(*iv*) Maximum shear stress at the point is

$$f_{t\,max} = \pm\sqrt{\left(\frac{f_1-f_2}{2}\right)^2 + q^2}.$$

2. To draw the Mohr's stress circle, direct stresses are taken along the abscissa and shear stresses are taken along the ordinate of a co-ordinate system. Two points representing the the state of stress on two perpendicular planes are located. Distance between these two points is the diameter of the Mohr's stress circle. The two points on the circle along the abscissa give the principal stresses. The maximum shear stress is equal to the radius of the circle.

3. If two principal stresses at a point are known then an ellipse with major and minor axes equal to two times the major and minor principal stresses at a point, is drawn. From this ellipse of stresses, normal and shear stresses on a plane inclined to a given principal plane can be determined.

4. If ϵ_x, ϵ_y and γ_{xy} are the normal and shear strain components on two perpendicular planes at a point, then principal strains at the point are

$$\epsilon_1, \epsilon_2 = \frac{\epsilon_x+\epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2}$$

and the normal strain on a plane inclined at an angle θ to a reference plane (say the plane of normal strain ϵ_x) is

$$\epsilon_\theta = \frac{\epsilon_x+\epsilon_y}{2} + \frac{\epsilon_x-\epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta.$$

5. If p_1, p_2 and p_3 are the principal srresses at a point, then principal strains are

$$\epsilon_1 = \frac{p_1}{E} - \frac{p_2}{mE} - \frac{p_3}{mE} \; ; \; \epsilon_2 = \frac{p_2}{E} - \frac{p_1}{mE} - \frac{p_3}{mE}$$

$$\epsilon_3 = \frac{p_3}{E} - \frac{p_1}{mE} - \frac{p_2}{mE}$$

where E=Young's modulus and $\frac{1}{m}$=Poisson's ratio.

6. Modified modulus of elasticity is the ratio of principal stress and principal strain in a particular direction.

MULTIPLE CHOICE QUESTIONS

1. At a point in a strained material, planes AB and BC perpendicular to each other pass through the point. The normal stress on plane AB is 80 N/mm² and on plane BC the normal stress is 40 N/mm². On both the planes there is a shear stress 15 N/mm². The normal stress on a plane inclined at an angle of 45° to AB is

   (*a*) 95 N/mm² (*b*) 80 N/mm²
   (*c*) 75 N/mm² (*d*) 55 N/mm².

2. Planes AB and BC perpendicular to each, pass through a point in a strained material. The normal and shear stresses on planes AB are 600 kg/cm² and −200 kg/cm². The normal and shear stresses on plane BC are 200 kg/cm² and +200 kg/cm² respectively. The plane on which shear stress is zero is inclined to the plane AB at an angle

   (*a*) 22° 30′ (*b*) 45°
   (*c*) 67° 30′ (*d*) 90°.

3. On two perpendicular planes passing through a point the normal and shear stresses are 80 MN/m², −60 MN/m² ; −80 MN/m², 60 MN/m² respectively. The maximum principal stress at the point is

   (*a*) 160 MN/m² (*b*) 100 MN/m²
   (*c*) 80 MN/m² (*d*) 60 MN/m².

4. The major and minor principal stresses at a point are 120 N/mm² and 40 N/mm² respectively. If a Mohr's stress circle is drawn for the stresses, the radius of the Mohr's stress circle will be

   (*a*) 120 N/mm² (*b*) 80 N/mm²
   (*c*) 40 N/mm² (*d*) 20 N/mm².

5. Two planes XY and YZ are passing through a point in a strained material. The normal and shear stresses on plane XY are $+60$ MN/m^2 and -30 MN/m^2 respectively, while the normal and shear stresses on plane YZ are -60 MN/m^2 and $+30$ MN/m^2 respectively. The angle between the planes XY and YZ is

(a) 30° (b) 60°
(c) 90° (d) 135°.

6. The major and minor principal stresses at a point are $+120$ N/mm^2 and -40 N/mm^2. A plane XY is passing through a point on which the normal stress is 80 N/mm^2 and the shear stress is q. Another plane YZ perpendicular to the plane XY is also passing through the same point. The normal and shear stresses on plane YZ are f and q respectively. The magnitude of the stress f is

(a) 60 N/mm^2 (b) 40 N/mm^2
(c) 20 N/mm^2 (d) 0 N/mm^2.

7. The major and minor principal stresses at a point are 1200 kg/cm^2 and 700 kg/cm^2 respectively. On a plane passing through a point, the normal stress is 1150 kg/cm^2. The shear stress on this plane will be

(a) 250 kg/cm^2 (b) 200 kg/cm^2
(c) 150 kg/cm^2 (d) 100 kg/cm^2.

8. On two perpendicular planes passing through a point there are complementary shear stresses $\pm$ 150 N/mm^2. The normal stress on these planes is zero. The maximum principal stress at the point is

(a) 300 N/mm^2 (b) 150 N/mm^2
(c) 750 N/mm^2 (d) None of the above.

9. In a strained material, at a point the strains are $\epsilon_x = 600\ \mu$ cm/cm, $\epsilon_y = 200\ \mu$ cm/cm, $\gamma_{xy}/2 = 150\ \mu$ cm/cm.

The maximum principal strain at the point is

(a) 100 μ cm/cm (b) 800 μ cm/cm
(c) 650 μ cm/cm (d) 500 μ cm/cm.

10. In a rectangular strain gauge rosette, the readings recorded are $\epsilon_{0^\circ} = 400\ \mu$ cm/cm, $\epsilon_{45^\circ} = 375\ \mu$ cm/cm, $\epsilon_{90^\circ} = 200\ \mu$ cm/cm ;

The maximum principal strain at the point is

(a) 775 μ cm/cm (b) 600 μ cm/cm
(c) 525 μ cm/cm (d) 425 μ cm/cm.

ANSWERS

1. (c)	2. (a)	3. (b)	4. (c)	5. (c)	6. (d)
7. (c)	8. (b)	9. (c)	10. (d).		

EXERCISES

3·1. On two perpendicular planes of a body, direct stresses of 1200 kg/cm^2 tensile and 700 kg/cm^2 tensile are applied. The major principal stress at the point is not to exceed 1350 kg/cm^2 ; What shearing stress can be applied to the given planes ? What will be the minor principal stress and the maximum shearing stress at the point ?

[**Ans.** 312·25/cm^2 ; 550 kg/cm^2, 400 kg/cm^2]

3·2. A piece of material is subjected to two tensile stresses at right angles of magnitude 800 kg/cm² and 400 kg/cm². Find the position of the plane on which the resultant stress is most inclined to the normal. Find the magnitude of this resultant stress.

[**Ans.** 54° 44′ ; 565 68 kg/cm²]

3·3. In a stressed body on a plane AB, the resultant stress is 80 MN/m² inclined at an angle of 20° to the normal stress and on another plane CD, the resultant stress is 50 MN/m² inclined at an angle of 35° to the normal stress. Determine the angle between the plane AB and CD. Find the magnitude of the principal stresses and the maximum shear stress on the body. [**Ans.** 32°, 91 MN/m², 27 MN/m² ; 32 MN/m²]

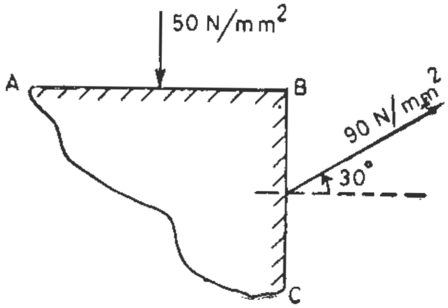


Fig. 3·38

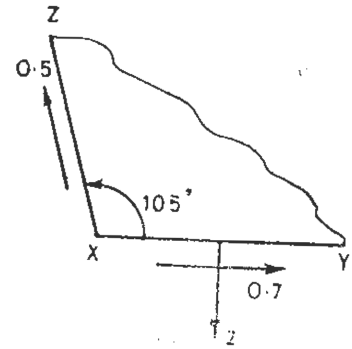


Fig. 3·39

3·4. At a point in a stressed material the normal stress on a plane AB is −50 N/mm² and the resultant stress on plane BC is 90 N/mm² as shown in the Fig. 3·38. Determine the magnitude of principal stresses and maximum shearing stress at the point and direction of planes carrying these stresses.

[**Ans.** 92·18 N/mm². 17° 33′ ; 64·24 N/mm², 107° 33′ ; 78·21 N/mm², 62° 33′]

3·5. At a point in a stressed material, the stresses on the plane XY and XZ at an angle 105° are as shown in Fig. 3·39, *i.e.* 2 tonnes/cm² normal tensile stress, 0·7 tonnes/cm² shear stress on XY plane and 0·5 tonne/cm² shear stress on XZ plane. Determine :

(*a*) The normal stress on plane XZ.

(*b*) The magnitude of the maximum and minimum principal stresses.

(*c*) The magnitude of the maximum shear stress and the direction of the planes carrying the maximum shear stress. [**Ans.** (*a*) 2·12 tonnes/cm² (*b*) 2·52 tonnes/cm², 1·056 tonnes/cm²

(*c*) ±0·731 tonne/cm², 81° 33′ and 126° 33′ with respect to the plane XY]

3·6. The minor principal stress at a point in the cross section of a beam is 50 N/mm² tensile and the magnitude of the maximum shearing stress is 80 N/mm². Determine :

(*a*) The major principal stress if it is tensile and the direct and shear stresses on a plane making an angle of 45° with a plane of minor principal stress.

(*b*) The major principal stress if it is compressive and the direct stress on the planes of maximum shearing stress. [**Ans.** (*a*) 210 N/mm². 130 N/mm², −80 N/mm²

(*b*) −110 N/mm² ; −30 N/mm²]

3·7. A circle of 15 cm diameter is inscribed on an aluminium plate before it is stressed. The plate is then loaded so as to produce stresses as shown in Fig. 3·40 and the circle is deformed to an ellipse. Determine the major and minor axis of the ellipse and their directions. Given $E=70\times10^3$ N/mm², $1/m=0·33$.

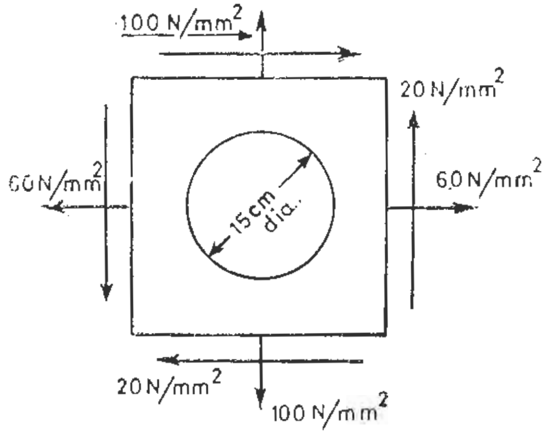


Fig. 3·40

[**Ans.** Major axis 15·0195 cm
Minor axis 15·0034 cm
Major axis is at an angle of 22° 30′ to the direction of 100 N/mm² stress]

3·8. At a point in a strained material, the stresses on the planes XY and XZ at an angle θ as shown in Fig. 3·41 are +75 MN/m² normal stress, 50 MN/m² shear stress on XY plane and −100 MN/m² normal stress and 40 MN/m² shear stress on XZ plane. Determine :

(*a*) The angle between the planes XY and XZ.

(*b*) Magnitude of maximum and minimum principal stresses.

(*c*) The directions of principal planes with respect to plane XY.

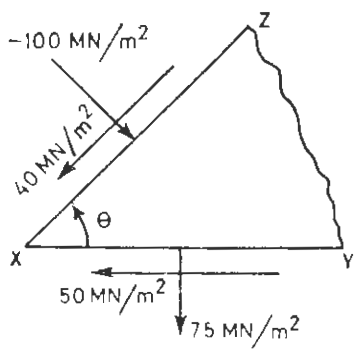


Fig. 3·41

[**Ans.** (*a*) 86° (*b*) −108 MN/m², +89 MN/m² (*c*) +74° 30′, −15° 30′].

3·9 Strains on an aluminium specimen in 3 directions 0°, 45°, 90° are recorded as follows :

$$\epsilon_0 = +400\ \mu\ \text{cm/cm}$$
$$\epsilon_{45^\circ} = -200\ \mu\ \text{cm/cm}$$
$$\epsilon_{90^\circ} = +200\ \mu\ \text{cm/cm.}$$

Determine the magnitude of principal strains, principal stresses and principal angles.

Given $E = 0·7 \times 10^5$ N/mm², $1/m = 0·33$ for aluminium.

[**Ans.** 810 μ cm/cm, −210 μ cm/cm ; 58·2 N/mm², 4·5 N/mm², −38° 21′, +51° 39′]

3·10. For a delta rosette, the following observations are made with the gauges mounted on a steel specimen.

$$\epsilon_0 = +600\ \mu\ \text{cm/cm}$$
$$\epsilon_{60^\circ} = -200\ \mu\ \text{cm/cm}$$
$$\epsilon_{120^\circ} = +200\ \mu\ \text{cm/cm}.$$

Determine the principal strains, principal stresses and principal angles. Given

$E = 2 \times 10^5$ N/mm², $1/m = 0{\cdot}3$ for steel.

[**Ans.** 661·88 μ cm/cm, $-261{\cdot}88$ μ cm/cm, 128·2 N/mm², $-13{\cdot}9$ N/mm², 15°, 105°]

3·11. A brass rod of 20 mm diameter encased in a sheath is subjected to an axial thrust of 16 kN. The sheath reduces the lateral expansion to one-third of its value if free. Determine :

(*a*) The pressure exerted by the sheath and

(*b*) The longitudinal strain in the bar.

Given $E = 102{,}000$ N/mm², $1/m = 0{\cdot}35$ for brass.

[**Ans.** 18·28 N/mm², $-0{\cdot}0003738$]

4

Relations Between Elastic Constants

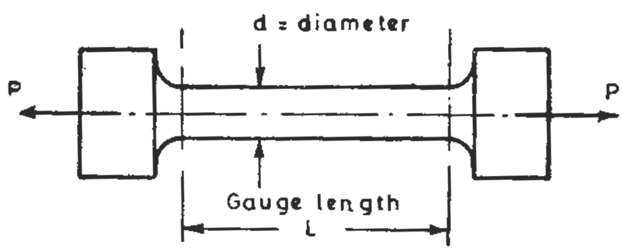
The deformations of a stressed body such as elongation, contraction, distortion etc., depend upon the values of its elastic constants. A bar having high Young's modulus of elasticity E will elongate much less under a tensile force in comparison to the elongation of a bar having low value of E. A block having high Modulus of rigidity G will distort much less under a shear force in comparison to the distortion of a block having low value of G. Similarly a sphere having high value of bulk modulus K will have much less change in its volume under hydrostatic pressure in comparison to the change in volume suffered by a sphere of low value of K. All these elastic constants including the Poisson's ratio $1/m$ for a material remain constant if the material is stressed with in the elastic limit.

These elastic constants can be determined experimentally and there is definite relationship between them.

4·1. YOUNG'S MODULUS OF ELASTICITY AND POISSON'S RATIO

To determine the values of E and $1/m$, a test piece of the material as shown in Fig. 4·1 is tested under tension. To record the change in length δL and change in diameter δD, extensometers of high precision are used or the strain gauges (as discussed in last chapter) are fixed on the test piece so as to find the axial strain (linear strain) and diameter strain (lateral strain). Tensile load P, vs. change in length δL or in other words f, stress (P/A) vs. strain, ϵ $(\delta L/L)$ is plotted as shown in Fig. 4·1 (*b*). The slop of this curve *i.e.*, the ratio f/ϵ is called Young's modulus of elasticity.

Another graph between change in diameter δD and change in length δL is plotted or in other words a graph between ϵ' lateral strain $(\delta D/D)$ and ϵ, linear strain $(\delta L/L)$ is plotted as shown in Fig. 4·1 (*c*). The slope of this curve *i.e.*, ϵ'/ϵ is called the Poisson's ratio of the



(a)

Fig. 4·1

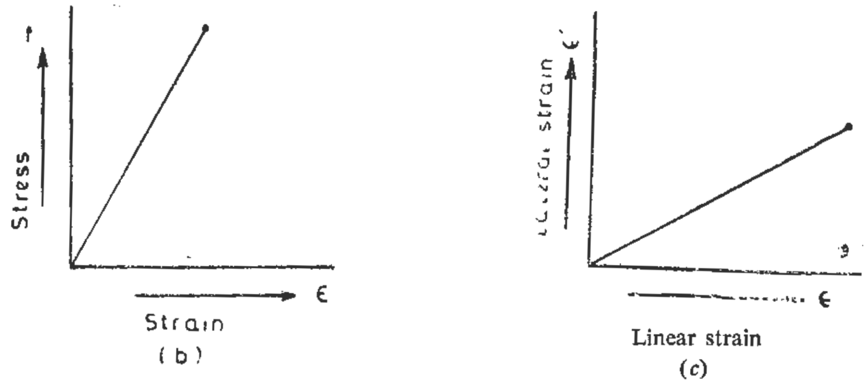


Fig. 4·1

material. Please note that $\delta L/L$ is positive and $\delta D/D$ is negative because as the length gradually increases, diameter gradually decreases. But Poisson's ratio is expressed only as a ratio and no sign is attached with this.

Example 4·1-1. A mild steel bar of 10 mm diameter and 100 mm gauge length is tested under tension. A tensile force of 10 kN produces an extension of 0·060 mm while its diameter is reduced by 0·0018 mm. Determine

$$E \text{ and } \frac{1}{m} \text{ for mild steel.}$$

Solution.

Tensile force, $P = 10\text{ kN}$

Area of cross section, $A = \frac{\pi}{4}(10)^2 = 78{\cdot}54\text{ mm}^2$

Stress, $f = \frac{10\text{ kN}}{78{\cdot}54} = 127{\cdot}32\text{ N/mm}^2$

Change in length, $\delta L = 0{\cdot}060\text{ mm}$

Original length, $L = 100\text{ mm}$

Linear strain, $\epsilon = \frac{0{\cdot}060}{100} = 0{\cdot}0006$

Young's modulus of elasticity,

$$E = \frac{f}{\epsilon} = \frac{127{\cdot}32}{0{\cdot}0006} = 212{\cdot}2 \times 10^3\text{ kN/mm}^2$$

Change in diameter, $\delta D = 0{\cdot}0018\text{ mm}$ (reduction)

Original diameter D, $= 10\text{ mm}$

Lateral strain, $\epsilon' = \frac{0{\cdot}0018}{10} = \cancel{0{\cdot}0018}$ 0·00018

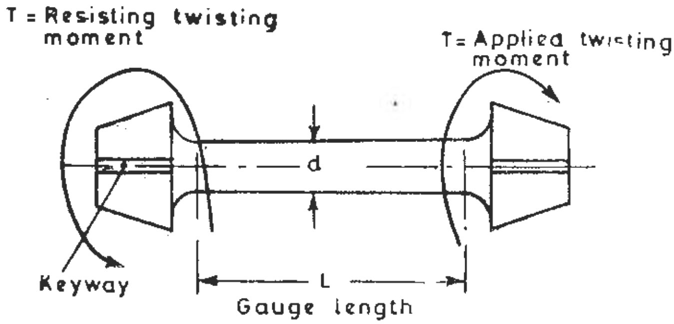
Poisson's ratio, $\frac{1}{m} = \frac{0{\cdot}00018}{0{\cdot}0006} = 0{\cdot}3$

Exercise 4·1-1. A brass bar specimen of guage length 150 mm and diameter 12 mm is tested under tension. A tensile forces of 14 kN produces an extension of 0·18 mm and its diameter is reduced by 0·0046 mm Determine the values of E and $1/m$ for brass.

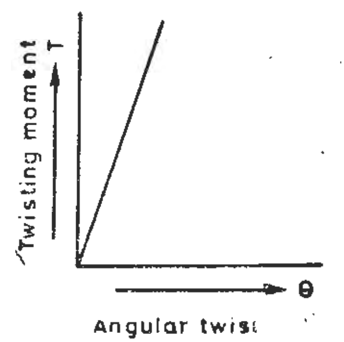
[**Ans.** $103·15 \times 10^3$ N/mm², 0·32]

4·2. MODULUS OF RIGIDITY

A round bar specimen as shown in Fig. 4.2 is fixed at one end and a twisting moment T is applied at the other end through a Torsion Testing Machine. Keyways are provided on the ends of the specimen so that specimen is firmly fixed in the fixtures of the machine. The angle of the rotation of one end with respect to the fixed end *i.e.*, angle of twist θ is continuously recorded as the twisting moment T is gradually increased.



(a)



(b)

Fig. 4·2

Within the elastic limit angular twist θ is directly proportional to the applied twisting moment T. The relationships between shear stress q and T, between shear strain ϕ and θ are discussed in chapter 13 on Torsion. However, the expression for relations are given as below :

Shear stress, $q=\dfrac{16\ T}{\pi d^3}$

where d is the diameter along the guage length of the specimen

Shear strain, $\phi=\dfrac{d\theta}{2L}$

where L=gauge length of the specimen.

Modulus of rigidity is obtained by the ratio of $\dfrac{q}{\phi}$ at any point within the elastic limit.

Modulus of rigidity, $G=\dfrac{q}{\phi}=\dfrac{32\ TL}{\pi d^4\theta}$

Example 4·2-1. An aluminium specimen of gauge length 200 mm and diameter 25 mm is tested under torsion. A torque of $16·5 \times 10^3$ Nmm produces an angular twist of 0·2 degree in the specimen. Determine the Modulus of rigidity of aluminium.

Solution. Torque, $T=16·5 \times 10^3$ N/mm

Gauge length, L=200 mm

Specimen diameter, d=25 mm

Angular twist, $\theta = 0{\cdot}2°$

$$= \frac{0{\cdot}2 \times \pi}{180} = \frac{\pi}{900} \text{ radian}$$

Modulus of rigidity, $G = \dfrac{32\,TL}{\pi d^4 \theta} = \dfrac{32 \times 16{\cdot}5 \times 10^3 \times 200 \times 900}{\pi \times 25^4 \times \pi}$

$$= 24{\cdot}65 \times 10^3 \text{ N/mm}^2.$$

Exercise 4·2-1. A mild steel specimen for torsion tests has guage length 250 mm and diameter 25 mm. A torque of 52 Nm produces an angular twist of 0·25 degree. Determine the modulus of rigidity for mild steel. [**Ans.** $77{\cdot}69 \times 10^3$ N/mm²]

4·3. RELATION BETWEEN E AND G

Consider a cube *ABCD* subjected to shear stress q at the top, while the bottom face is fixed. The cube is deformed as shown in the Fig. 4·3 (*a*) and at the same time a complementary shear q, at an angle of 90° to the applied shear stress is induced. The angle of shear ϕ, within the elastic limit is very much less (much less than even 1 degree for most of the metals) and not so large as shown in Fig. 4·3 (*a*). Therefore, angle *AB′D* is taken as 45°. Due to the shear stress applied as shown in the figure, the diagonal *DB* is increased in length to *DB′* and the diagonal *AC* is reduced in length to *A′C*.

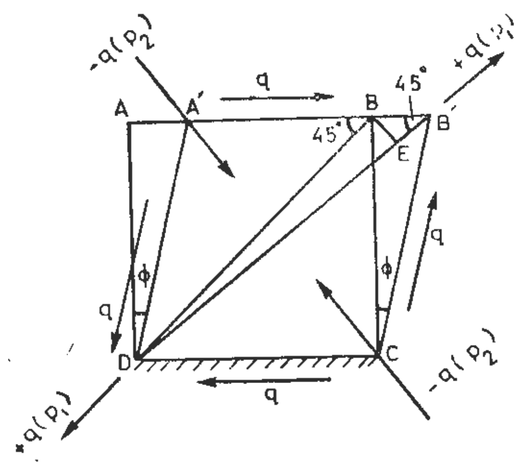


Fig. 4·3 (*a*)

Taking the sides of the cube as $\Delta_x = \Delta_y = \Delta_z$ limiting to zero, we can consider that the stresses are acting at a point in a strained material. Mohr's stress circle can be drawn for this point, taking $OP = +q$ and $OQ = -q$, the shear stresses acting on the planes *AB* and *BC*, right angle to each other. With O as the centre of Mohr's stress circle, the circle is drawn. From the diagram $OR = +q =$ principal stress p_1 and $OS = -q =$ principal stress p_2. In other words diagonal *DB* is extended due to principal stress p_1 which is equal to $+q$, a tensile stress.

Change in the length, $DB = DB' - DB$

Since angle ϕ is very small $DB \cong DE$ where BE is perpendicular to the line DB'.

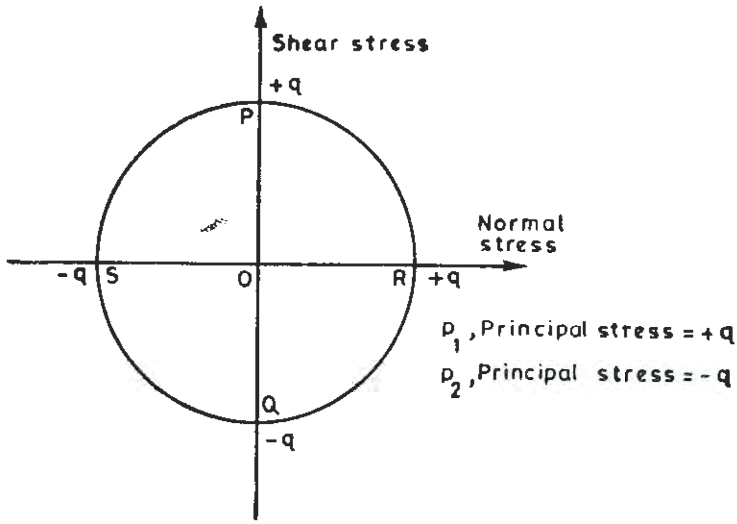


Fig. 4·3 (*b*)

or change in length $\qquad = EB' = BB' \times \cos 45° = \dfrac{BB'}{\sqrt{2}}$

ϵ, strain along $\qquad DB = \dfrac{\text{Change in length}}{\text{Original length}}$

$$= \frac{BB'}{\sqrt{2}\,DB} = \frac{BB'}{\sqrt{2}\sqrt{2}\,BC} = \frac{BB'}{2BC} \quad \text{where } DB = \sqrt{2}\,BC$$

as DB is the diagonal of the square $ABCD$.

Strain along DB in terms of principal stresses

$$\epsilon = \frac{p_1}{E} - \frac{p_2}{mE} = \frac{q}{E} + \frac{q}{mE}$$

$$\epsilon = \frac{q}{E}\left(1 + \frac{1}{m}\right) = \frac{BB'}{2BC}$$

or $$\frac{q(1+1/m)}{E} = \frac{BB'}{2BC}$$

Shear strain, $$\phi = \frac{BB'}{BC}$$

or $$\frac{q}{E}\left(1 + \frac{1}{m}\right) = \frac{\phi}{2}$$

or $$\frac{2q}{\phi}\left(1 + \frac{1}{m}\right) = E$$

but $$\frac{q}{\phi} = \frac{\text{Shear stress}}{\text{Shear strain}} = G, \text{ modulus of rigidity}$$

or $$E = 2G\left(1 + \frac{1}{m}\right).$$

Example 4·3-1. On a steel bar specimen of 15 mm diameter and 150 mm gauge length, when tested as a tensile test specimen, a force of 15 kN produce an extension of 0·063 mm. When the specimen is tested under torsion, a twisting moment of 6 94 Nm produces an angular twist of 0·15 degree. Determine the Poisson's ratio for the material of the bar.

Solution. Diameter of the bar,

$$d=15 \text{ mm}$$

Area of cross section, $A=\frac{\pi}{4}(15)^2=176{\cdot}7 \text{ mm}^2$

Tensile load, $P=15\times 1000$ N

Stress, $f=\frac{15000}{176{\cdot}7}=84{\cdot}89 \text{ N/mm}^2$

Extension, $\delta L=0{\cdot}063$ mm

Strain, $\epsilon=\frac{\delta L}{L}=\frac{0{\cdot}063}{150}=0{\cdot}00042$

Young's modulus of elasticity,

$$E=\frac{f}{\epsilon}=\frac{84{\cdot}89}{0{\cdot}00042}=202{\cdot}1\times 10^3 \text{ N/mm}^2$$

Twisting moment, $T=6{\cdot}94 \text{ Nm}=6940 \text{ Nmm}$

Angular twist, $\theta=0{\cdot}15^\circ=\frac{0{\cdot}15\times\pi}{180}$ radian

Gauge length, $L=150$ mm

Modulus of rigidity, $G=\frac{32\,TL}{\pi d^4\theta}$

$$=\frac{32\times 6940\times 150\times 180}{\pi\times 15^4\times 0{\cdot}15\times\pi}=80{\cdot}004\times 10^3 \text{ N/mm}^2$$

As per the relationship between G and E

$$\left(1+\frac{1}{m}\right)=\frac{E}{2G}=\frac{202{\cdot}1\times 10^3}{2\times 80{\cdot}004\times 10^3}=1{\cdot}263$$

or Poisson' ratio $\frac{1}{m}=1{\cdot}263-1=0{\cdot}263.$

Exercise 4·3-1. A brass bar of gauge length 100 mm and diameter 10 mm, when subjected to a load of 400 kg extends by 0·024 mm. What will be angular twist produced in this bar by a twisting moment of 0·5 kg-m. The Poisson's ratio for brass is 0·32.

[**Ans.** 0·363°]

4·4. RELATION BETWEEN MODULUS OF ELASTICITY AND BULK MODULUS K

Consider a rectangular block of dimensions : length L, breadth B and depth D subjected to the three principal stresses p each in the directions 1, 2 and 3 as shown in the Fig. 4·4.

Volume of the block,

$$V = LBD$$

Taking the partial derivatives of V

$$\partial V = LD\,\partial B + LB\,\partial D + BD\,\partial L$$

or

$$\frac{\partial V}{V} = \frac{LD\partial B}{V} + \frac{LB\partial D}{V} + \frac{BD\partial L}{V}$$

$$= \frac{\partial B}{B} + \frac{\partial D}{D} + \frac{\partial L}{L}$$

Volumetric strain,

$$\epsilon_v = \epsilon_1 + \epsilon_3 + \epsilon_2$$

= sum of the strains along 3 directions of co-ordinate axes.

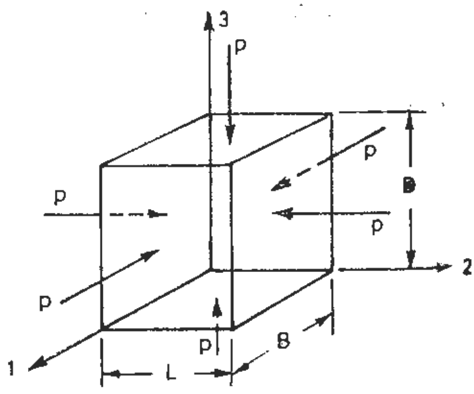


Fig 4·4

When the rectangular block is subjected to volumetric stress p in each direction, this stress acts as principal stresses in 3 directions.

Principal strains. $\epsilon_1 = \dfrac{p}{E} - \dfrac{p}{mE} - \dfrac{p}{mE}$ contraction

$\epsilon_2 = \dfrac{p}{E} - \dfrac{p}{mE} - \dfrac{p}{mE}$ contraction

$\epsilon_3 = \dfrac{p}{E} - \dfrac{p}{mE} - \dfrac{p}{mE}$ contraction

Volumetric strain, $\epsilon_v = \dfrac{3p}{E}\left[1 - \dfrac{2}{m}\right]$ reduction in volume

Bulk modulus, $K = \dfrac{p}{\epsilon_v} = \dfrac{pE}{3p\left[1 - \dfrac{2}{m}\right]}$

or Modulus of elasticity. $E = 3K\left[1 - \dfrac{2}{m}\right]$.

Example 4·4-1. What change in volume would a 20 cm cube of steel suffer at a depth of 4 km in sea water ?

Given E for steel $= 2{\cdot}05 \times 10^6$ kg/cm²

$\dfrac{1}{m}$ for steel $= 0{\cdot}29$.

Solution. For steel, $E = 2{\cdot}05 \times 10^6$ kg/cm²

Poisson's ratio, $\dfrac{1}{m} = 0{\cdot}29$

Bulk modulus, $K = \dfrac{E}{3\left(1 - \dfrac{2}{m}\right)} = \dfrac{2{\cdot}05 \times 10^6}{3(1 - 2 \times 0{\cdot}29)} = 1{\cdot}627 \times 10^6$ kg/cm²

Depth of the cube in sea water,

$$h = 4 \text{ km} = 4 \times 10^5 \text{ cm}$$

Density of sea water, $w = 1{\cdot}02 \text{ g/cm}^3 = 0{\cdot}00102 \text{ kg/cm}^3$

Hydrostatic pressure on cube,

$$p = wh = 0{\cdot}00102 \times 4 \times 10^5 = 408 \text{ kg/cm}^2$$

Volumetric strain, $\epsilon_v = \dfrac{p}{K} = \dfrac{408}{1{\cdot}627 \times 10^6} = 2{\cdot}508 \times 10^{-4}$

Original volume of cube,

$$V = 20^3 = 8000 \text{ cm}^3$$

Change in volume, $\delta V = \epsilon_v \,.\, V = 2{\cdot}508 \times 10^{-4} \times 8000 = 2{\cdot}0064 \text{ cm}^3$.

Exercise 4·4-1. What change in volume a brass sphere of 10 cm diameter would suffer at a depth of 2 kilometer in sea water.

E for brass $= 1000$ tonnes/cm^2

$\dfrac{1}{m}$ for brass $= 0{\cdot}32$

Density of sea water $= 1{\cdot}02 \times 10^3$ kg/m^3 **[Ans.** 0·115 c.c.]

Problem 4·1. Derive the relationship between modulus of elasticity E, modulus of rigidity G and Bulk modulus K

Solution. We know that

$$E = 2\,G\left(1 + \frac{1}{m}\right)$$

$$E = 3\,K\left(1 - \frac{2}{m}\right), \quad \text{where } \frac{1}{m} \text{ is the Poisson's ratio}$$

or

$$\frac{E}{G} = 2 + \frac{2}{m} \qquad \ldots(1)$$

$$\frac{E}{3K} = 1 - \frac{2}{m} \qquad \ldots(2)$$

Adding the equations (1) and (2), we get

$$\frac{E}{G} + \frac{E}{3K} = 3$$

or

$$\frac{3EK + GE}{3GK} = 3$$

or

$$E = \frac{9GK}{3K + G}.$$

Problem 4·2. At a point in a strained material the principal stresses are p_1, p_2 and p_3 ; and the principal strains are ϵ_1, ϵ_2 and ϵ_3. Show that the principal stress p_1 is given by

$$\alpha\, \epsilon_v + 2G\, \epsilon_1$$

where $\alpha=\dfrac{mE}{(m+1)(m-2)}$

ϵ_v=Volumetric strain

G=Modulus of rigidity

$\dfrac{1}{m}$=Poisson's ratio.

In a certain test, the principal strains observed are 700, 1400 and −1800 microstrain. Determine the three principal stresses.

Given $E=205\times10^3$ N/mm²

and Poisson's ratio $\dfrac{1}{m}=0{\cdot}28.$

Solution. The principal strains in terms of principal stresses are

$$\epsilon_1=\frac{p_1}{E}-\frac{p_2}{mE}-\frac{p_3}{mE} \qquad ...(1)$$

$$\epsilon_2=\frac{p_2}{E}-\frac{p_1}{mE}-\frac{p_3}{mE} \qquad ...(2)$$

$$\epsilon_3=\frac{p_3}{E}-\frac{p_1}{mE}-\frac{p_2}{mE} \qquad ...(3)$$

Volumetric strain, $\epsilon_v=\epsilon_1+\epsilon_2+\epsilon_3=\left(\dfrac{p_1+p_2+p_3}{E}\right)\left(1-\dfrac{2}{m}\right)$

or
$$\epsilon_v\,\frac{mE}{(m-2)}=p_1+p_2+p_3 \qquad ...(4)$$

From equation (1)

$$(p_2+p_3)=\left(\frac{p_1}{E}-\epsilon_1\right)mE=p_1\,m-\epsilon_1\,mE$$

Substituting the value in equation (4)

$$\epsilon_v\,.\,\frac{mE}{(m-2)}=p_1+p_1\,m-\epsilon_1\,mE$$

or
$$\epsilon_v\,\frac{mE}{m-2}+\epsilon_1\,mE=p_1\,(1+m)$$

or
$$\frac{\epsilon_v}{m-2}+\epsilon_1=\frac{p_1}{E}\left(\frac{1+m}{m}\right)$$

But
$$E=2G\left(1+\frac{1}{m}\right)=2G\left(\frac{m+1}{m}\right)$$

So
$$\frac{\epsilon_v}{m-2}+\epsilon_1=p_1\left(\frac{m+1}{m}\right)\times\frac{m}{2G(m+1)}=\frac{p_1}{2G}$$

or
$$p_1=\frac{2G\,\epsilon_v}{m-2}+2G\,\epsilon_1,\quad \text{again } 2G=\frac{E}{\left(1+\dfrac{1}{m}\right)}=\frac{Em}{m+1}$$

$$p_1=\epsilon_v\,.\,\frac{mE}{(m+1)(m-2)}+2G\,\epsilon_1$$
$$=\alpha\,\epsilon_v+2G\,\epsilon_1$$

where $$\alpha=\frac{mE}{(m+1)(m-2)}.$$

(b) $$E=205\times10^3\ \text{N/mm}^2$$

$$\frac{1}{m}=0.28$$

$$G=\frac{205\times10^3}{2\left(1+\frac{1}{m}\right)}=\frac{205\times10^3}{2(1.28)}=80.0\times10^3\ \text{N/mm}^2$$

$$\epsilon_1=700\ \mu\ \text{mm/mm},\ \ \epsilon_2=1400\ \mu\ \text{mm/mm}$$

$$\epsilon_3=-1800\ \mu\ \text{mm/mm}$$

$$\epsilon_v=\epsilon_1+\epsilon_2+\epsilon_3=+300\ \mu\ \text{mm/mm}$$

$$\frac{Em}{(m+1)(m-2)}=\frac{E}{\left(1+\frac{1}{m}\right)(m-2)}=\frac{205\times10^3}{(1+0.28)\left(\frac{1}{0.28}-2\right)}$$

$$\alpha=101.94\times10^3\ \text{N/mm}^2$$

Principal stresses

$$\begin{aligned}p_1&=\alpha\ \epsilon_v+2G\ \epsilon_1\\&=101.94\times10^3\times300\times10^{-6}+80\times10^3\times2\times700\times10^{-6}\\&=30.58+112\ \text{N/mm}^2=142.58\ \text{N/mm}^2\\p_2&=\alpha\ \epsilon_v+2G\ \epsilon_2\\&=30.58+80\times10^3\times2\times1400\times10^{-6}=30.58+224\\&=254.58\ \text{N/mm}^2\\p_3&=\alpha\ \epsilon_v+2G\ \epsilon_3\\&=30.58-80\times10^3\times2\times1800\times10^{-6}=30.58-288\\&=-257.42\ \text{N/mm}^2.\end{aligned}$$

Problem 4.3. The materials A and B have the same bulk modulus, but the value of E for A is 2% greater than that for B. Find the value of G for the material A in terms of E and G for the material B.

Solution. Bulk modulus, $K_A=K_B$

Young's modulus, $E_A=1.02\ E_B$

Now $$E=\frac{9GK}{G+3K}$$

or $$E_A=\frac{9\ G_A\ K_A}{G_A+3\ K_A},\quad E_B=\frac{9\ G_B\ K_B}{G_B+3K_B}$$

or $$E_A\ (G_A+3K_A)=9\ G_A\ K_A$$

or $$K_A=\frac{E_AG_A}{9\ G_A-3\ E_A}$$

Similarly $$K_B=\frac{E_BG_B}{9\ G_B-3\ E_B}$$

But $K_A = K_B$

$$\frac{E_A G_A}{9G_A - 3E_A} = \frac{E_B G_B}{9G_B - 3E_B}$$

But $E_A = 1{\cdot}02\ E_B$

So
$$\frac{1{\cdot}02\ E_B\ G_A}{9\ G_A - 3{\cdot}06\ E_B} = \frac{E_B \times G_B}{9\ G_B - 3\ E_B}$$

or
$$\frac{1{\cdot}02\ G_A}{3\ G_A - 1{\cdot}02\ E_B} = \frac{G_B}{3\ G_B - E_B}$$

or
$$G_A = \frac{306\ G_B\ E_B}{102\ E_B - 6G_B}.$$

Problem 4·4. The determination of E and G for a particular material gives the values as 208000 N/mm² and 80,000 N/mm² respectively. Calculate Poisson's ratio and bulk modulus.

If both the moduli are liable to an error of $\pm 1\%$, find the maximum percentage error in the derived value of Poisson's ratio.

Solution.
$$E = 208000\ \text{N/mm}^2$$
$$G = 80{,}000\ \text{N/mm}^2$$

Poisson's ratio,
$$\frac{1}{m} = \frac{E}{2G} - 1$$
$$= \frac{208000}{2 \times 80000} - 1 = 0{\cdot}30$$

Bulk modulus,
$$K = \frac{E}{3\left(1 - \dfrac{2}{m}\right)} = \frac{208000}{3(1 - 0{\cdot}3 \times 2)}$$
$$= 173{,}333\ \text{N/mm}^2$$

Now
$$\frac{1}{m} = \frac{E}{2G} - 1$$

Error in the determination of E and G is $\pm 1\%$

So the maximum value of $\dfrac{1}{m} = \dfrac{208000 \times 1{\cdot}01}{2 \times 80000 \times 0{\cdot}99} - 1 = 0{\cdot}326$

The minimum value of $\dfrac{1}{m} = \dfrac{208000 \times 0{\cdot}99}{2 \times 8000 \times 1{\cdot}01} - 1 = 0{\cdot}274$

So Error in the calculation of $\dfrac{1}{m} = +0{\cdot}026$ to $-0{\cdot}026$, $\quad \%\text{Error} = \dfrac{0{\cdot}026}{0{\cdot}3} \times 100$

$$\%\ \text{Error} = \pm\ 8{\cdot}66.$$

Problem 4·5. A steel bar 5 cm diameter, 1 metre long is subjected to an axial compressive load of 10 kN What will be the percentage change in its volume ? What change in volume would a 10 cm cube of steel suffer at a depth of 5 kilometres in sea water

$$E = 208{,}000\ \text{N/mm}^2$$
$$G = 83{,}000\ \text{N/mm}^2$$

Wt. Density of sea water $= 1030\ \text{kg/m}^3$.

Solution. $E=208000$ N/mm²

$G=83000$ N/mm²

Poisson's ratio, $\frac{1}{m}=\frac{E}{2G}-1$

$=\frac{208000}{2\times 83000}-1=1{\cdot}253-1=0{\cdot}253$

Diameter of steel bar, $d=5$ cm$=50$ mm

Area of cross section, $A=\frac{\pi}{4}\times 50^2=1963{\cdot}5$ mm²

Compressive load, $P=10$ kN

Axial compressive stress, $f=\frac{10\times 1000}{1963{\cdot}5}=5{\cdot}093$ N/mm² (compressive)

Axial strain, $\epsilon_a=-\frac{-5{\cdot}093}{208{,}000}$

Lateral strain, $\epsilon_D=+\frac{0{\cdot}253\times 5{\cdot}093}{208{,}000}$

Volumetric strain, $\epsilon_v=\epsilon_a+2\epsilon_D$

$=-\frac{5{\cdot}093\times 0{\cdot}494}{208{,}000}=1{\cdot}2\times 10^{-5}$

% change in volume $=\epsilon_v\times 100$

$=0{\cdot}0012\%$ (reduction)

Bulk modulus, $K=\frac{E}{3\left(1-\frac{2}{m}\right)}$

$=\frac{208000}{3(1-2\times 0{\cdot}253)}=140350{\cdot}87$ N/mm²

Pressure at a depth of 5 km, $p=wh=1030\times 5000$ kg/m²

$=5{\cdot}15$ kg/mm²$=50{\cdot}47$ N/mm²

Volumetric strain, $\epsilon_v=\frac{p}{K}=\frac{50{\cdot}47}{140350{\cdot}87}=35{\cdot}96\times 10^{-5}$

Original volume, $V=10^3=1000$ cm³

Change in volume, $dV=\epsilon_v\times V$

$=35{\cdot}90\times 10^{-5}\times 10^3=0{\cdot}3596$ cm².

Problem 4·6. A small piston of area 150 mm² compresses oil in a rigid container of 20,000 cm³. When a weight of 90 N is gradually applied to the piston its movement is observed to be 27 mm. If a weight of 50 N falls from a height of 100 mm on to the 90 N load, determine the maximum pressure developed in the oil container. Neglect the effects of friction and loss of energy.

Solution. Piston area, $A=150$ mm²

Original volume of oil, $V=20{,}000$ cm³$=20\times 10^6$ mm³

Weight on piston, $W=90$ N

Pressure on piston or on oil, $p=\dfrac{90}{150}=0{\cdot}6$ N/mm²

Change in volume of the oil, $\delta V=\dfrac{p}{K}\times V$

$$\delta V=\frac{0{\cdot}6\times 20\times 10^6}{K}$$

Change in volume of the oil, $\delta V=A\times S$

$=$piston area$\times$piston displacement

$=150\times 27=4050$ mm³

So Bulk modulus, $K=\dfrac{0{\cdot}6\times 20\times 10^6}{4050}=2{\cdot}963\times 10^3$ N/mm²

Say the additional pressure developed on oil due to the falling load is p' N/mm²

$\delta V'$, change in volume of the oil$=\dfrac{p'}{K}\times V=\dfrac{p'}{2{\cdot}963\times 10^3}\times 20\times 10^6$

S', displacement of the piston $=\dfrac{\delta V'}{A}=\dfrac{p'\times 20\times 10^6}{2{\cdot}963\times 10^3\times 150}=45\,p'$ mm

Strain energy absorbed by the oil

$$=\frac{p'^2}{2K}\times V=\frac{p'^2\times 20\times 10^6}{2\times 2{\cdot}963\times 10^3}$$

Loss of Potential energy by the weight

$=W'(h+S')=50(100+45\,p')$

$=$Strain energy absorbed by the oil.

$$\frac{p'^2\times 20\times 10^6}{2\times 2{\cdot}963\times 10^3}=5000+2250\,p'$$

$$p'^2-\frac{2\times 2250\times 2{\cdot}963}{20{,}000}\,p'-\frac{2\times 5000\times 2{\cdot}963\times 10^3}{20\times 10^6}=0$$

$$p'^2-0{\cdot}666\,p'-1{\cdot}481=0$$

$$p'=\frac{0{\cdot}666+\sqrt{(0{\cdot}666)^2+4\times 1{\cdot}481}}{2}$$

$$=\frac{0{\cdot}666+2{\cdot}5234}{2}=1{\cdot}595 \text{ N/mm}^2$$

Maximum pressure developed in the oil container

$=0{\cdot}60+1{\cdot}595=2{\cdot}195$ N/mm².

Problem 4·7. A round bar 10 mm in diameter and 100 mm long is tested in tension. It is observed that the longitudinal strain is 4 times the lateral strain. Calculate the modulus of rigidity and the bulk modulus if its elastic modulus is 200 $G\,P_a$. Find the change in volume when the bar is subjected to a hydrostatic pressure of 100 $M\,P_a$.

Solution. Longitudinal strain $=4\times$ lateral strain

Poisson's ratio, $\frac{1}{m}=\frac{\text{Lateral strain}}{\text{Longitudinal strain}}=0\cdot25$

Young's modulus, $E=200\ G\ P_a$

Modulus of rigidity, $G=\frac{E}{2\left(1+\frac{1}{m}\right)}=\frac{200\times10^9\ P_a}{2\ (1\cdot25)}=80\ G\ P_a$

Bulk modulus, $K=\frac{E}{3\left(1-\frac{2}{m}\right)}=\frac{200\ G\ P_a}{3\ (1-0\cdot5)}=133\cdot33\ G\ P_a$

$1\ P_a=1$ Pascal $=1\ \text{N/m}^2$

$M\ P_a=10^6\ N/\text{m}^2=1\ \text{N/mm}^2$

p, hydrostatic pressure $=100\ M\ P_a=100\times10^6\ \text{N/m}^2=100\ \text{N/mm}^2$

Bulk modulus, $K=133\cdot33\ G\ P_a$

$=133.33\times10^9\ \text{N/m}^2=133\ 33\times10^3\ \text{N/mm}^2$

Volumetric strain, $\epsilon_v=\frac{p}{K}=\frac{100}{133\cdot33\times10^3}=0\cdot75\times10^{-3}$

Change in volume, $\delta V=e_v\ .\ V=0\cdot75\times10^{-3}\times\frac{\pi}{4}\ (10)^2\times100$

$=5\cdot89\ \text{mm}^3$

Problem 4·8. The modulus of rigidity of a material is 380 tonnes/cm³. A 10 mm diameter rod of the material is subjected to an axial tensile force of 500 kg and the change in its diameter is observed to be 0·0002 cm. Calculate the Poisson's ratio and modulus of elasticity of the material.

Solution. Modulus of rigidity,

$G=380\times1000\ \text{kg/cm}^2$

Diameter, $d=1$ cm

Tensile force $=500$ kg

Tensile stress, $f=\frac{500\times4}{\pi\times(1)^2}=\frac{2000}{\pi}\ \text{kg/cm}^2$

Change in diameter, $\delta d=0\cdot0002$ cm

Lateral strain, $e'=\frac{0\cdot0002}{1}=0\cdot0002$

Say (axial) linear strain $=\epsilon$

Young's modulus $=E$

Axial strain, $\epsilon=\frac{2000}{\pi\times E}$

Lateral strain, $\epsilon'=\frac{1}{m},\ \epsilon=\frac{2000}{\pi E\ m}=0\cdot0002$ ...(1)

$G=\frac{E}{2+\frac{2}{m}}=380\times1000$ ...(2)

From equation (1), $E = \frac{2000}{\pi \times m \times 0{\cdot}0002}$

From equation (2), $E = 380 \times 1000\left(2 + \frac{2}{m}\right)$

So $\frac{2000}{\pi m \times 0{\cdot}0002} = 380 \times 1000 \left(2 + \frac{2}{m}\right)$

$$\frac{1}{m}\left[\frac{1000}{\pi}\right] = \left[1 + \frac{1}{m}\right]\left[760\right]$$

$$\frac{1}{m}[3183{\cdot}1 - 760] = 760$$

$$\frac{1}{m} = \frac{760}{2423{\cdot}1}$$

or Poisson's ratio, $\frac{1}{m} = 0{\cdot}3136$

Young's modulus, $E = \frac{1}{m} \times \frac{2000}{\pi \times 0{\cdot}0002} = 998{\cdot}2 \times 1000 \text{ kg/cm}^2.$

SUMMARY

1. In a tensile test performed on a bar of diameter d and gauge length L, a force P producing, change in length δL and change in diameter $-\delta d$.

Young's modulus of Elasticity, $E = \frac{4P}{\pi d^2} \times \frac{L}{\delta L}$

Poisson's ratio, $\frac{1}{m} = \frac{\delta d}{d} \times \frac{L}{\delta L}$.

2. In a Torsion test performed on a bar of diameter d and gauge length L subjected to twisting moment T producing the angular twist θ

Modulus of rigidity, $G = \frac{32\, TL}{\pi d^4 \theta}$.

3. $E = 2G\left(1 + \frac{1}{m}\right)$.

4. $E = 3K\left(1 - \frac{2}{m}\right)$ where K is the Bulk Modulus.

MULTIPLE CHOICE QUESTIONS

1. The modulus of elasticity for a material 208×10^3 N/mm^2 and its Poisson's ratio is 0·3. The modulus of rigidity for the material is
(a) 160×10^3 N/mm^2 (b) 104×10^3 N/mm^2
(c) 80×10^3 N/mm^2 (d) None of the above.

2. For a material, the value of Bulk modulus is 170×10^3 N/mm^2 and the Poisson's ratio is 0·3. The Young's modulus of elasticity for the material is
(a) 200×10^3 N/mm^2 (b) 204×10^3 N/mm^2
(c) 208×10^3 N/mm^2 (d) 212×10^3 N/mm^2.

3. For a material $E=1000\times10^3$ N/mm² $G=375\times10^3$ N/mm². The value of its Poisson's ratio is
   (a) 0·25 (b) 0·30
   (c) 0·33 (d) 0·35.

4. For a material when tested within the elastic limit, the value of Poisson's ratio is 0·35. When the same material is tested in the plastic stage, its Poisson's ratio in the plastic stage will be
   (a) 0·35 (b) 0·4
   (c) 0·5 (d) 0·6.

5. The ratio of Young's modulus E and modulus of rigidity G is given by—
   (a) $1+\frac{1}{m}$ (b) $2\left(1-\frac{1}{m}\right)$
   (c) $3\left(1-\frac{2}{m}\right)$ (d) $2\left(1+\frac{1}{m}\right)$

   where $\frac{1}{m}$ is the Poisson's ratio of the material.

6. The ratio of Young's modulus of Elasticity E and Bulk modulus K for a material is given by
   (a) $2\left(1+\frac{2}{m}\right)$ (b) $3\left(1-\frac{2}{m}\right)$
   (c) $3\left(1-\frac{1}{m}\right)$ (d) None of the above.

7. The value of Young's modulus E in terms of modulus of rigidity G and Bulk modulus K is
   (a) $\frac{9KG}{K+G}$ (b) $\frac{9\,GK}{K+3G}$
   (c) $\frac{9\,GK}{G+3K}$ (d) None of the above.

8. The principal strains at a point are $+800\ \mu$ cm/cm, $+400\ \mu$ cm/cm and $-1200\ \mu$ cm/cm. The volumetric strain will be —
   (a) 2400 μ cm/cm (b) $-2400\ \mu$ cm/cm
   (c) 1600 μ cm/cm (d) 0·0 μ cm/cm.

9. A body is subjected to a hydrostatic pressure of 100 N/mm². Young's modulus for the material is 100×1000 N/mm² and Poisson's ratio is 0·33. The volumetric strain on the body is
   (a) 340 μ cm/cm (b) 680 μ cm/cm
   (c) 1020 μ cm/cm (d) 0·0 cm/cm.

10. Which of the following materials has the value of Poisson's ratio approximately equal to 0·33—
   (a) Cast iron (b) Aluminium
   (c) High carbon steel (d) Glass.

ANSWERS

1. (c)	2. (b)	3. (c)	4. (c)	5. (d)
6. (b)	7. (c)	8. (d)	9. (c)	10. (b).

EXERCISES

4·1. Express the value of Poisson's ratio $1/m$ in terms of the modulus of rigidity. G and Bulk modulus, K.

$$\left[\textbf{Ans.}\quad \frac{1}{m}=\frac{3K-2G}{6K+2G}\right]$$

4·2. At a point in a strained material, the principal stresses are p_1, p_2 and p_3, while the principal strains are ϵ_1, ϵ_2 and ϵ_3 respectively. Show that

Principal stress, $p_1=2G\,[\beta\,.\,\epsilon_v+\epsilon_1]$

where G=modulus of rigidity

$$\beta=\frac{1}{m-2}\ ,\quad \frac{1}{m}=\text{Poisson's ratio}$$

The principal strains observed are

$$\epsilon_1=+800\ \mu\ \text{cm/cm},\ \epsilon_2=+600\ \mu\ \text{cm/cm},$$

$$\epsilon_3=-500\ \mu\ \text{cm/cm}.$$

Determine the principal stresses at the point.

Given $G=380$ tonnes/cm^2 and $1/m=0{\cdot}32$.

[**Ans.** 1216, 1064. 228 kg/cm^2]

4·3. The two materials A and B have the same modulus of rigidity but the value of E (Young's modulus) for A is 5% greater than that for B. Find the value of Bulk modulus K for the material A in terms of E and K for the material B.

$$\left[\textbf{Ans.}\quad \frac{7\,E_B\,K_B}{7\,E_B-3\,K_B}\right]$$

4·4. The determination of modulus of elasticity and modulus of rigidity for a particular material gives the values as 1000 tonnes/cm^2 and 380 tonnes/cm^2 respectively. Calculate Poisson's ratio and Bulk modulus for the material.

If both the moduli are liable to have an error of $\pm$ 0·5 %, find the maximum percentage error in the derived value of Poisson's ratio.

[**Ans.** 0·316, 679·35 tonnes/cm^2 ; 4·1 %]

4·5. A bar of steel 100 cm long, 2 cm diameter is subjected to an axial compressive force of 2 tonnes. Determine the percentage change in volume.

What will be the change in volume of a spherical steel ball 10 cm in diameter when submerged in sea water to a depth of 5 kilometers ?

Given E for steel=2100 tonnes/cm^2

G for steel=820 tonnes/cm^2

Weight density of sea water =0·00105 kg/cm^3. [**Ans.** $1{\cdot}33\times10^{-3}$ %, 0·173 cm^3]

4·6. A small piston of area 1·2 cm² compresses oil in a rigid container of 15000 cm³, when a weight of 12 kg is gradually applied to the piston.

Now a weight of 5 kg falls from a height of 15 cm on to the 12 kg load. Determine the maximum pressure developed in the oil containter. Neglect the effects of friction and loss of energy.

K for oil=35000 kg/cm² [**Ans.** 33·33 kg/cm²]

4·7. A round bar 16 mm in diameter and 150 mm long is tested in tension. It is observed that the ratio of lateral strain to longitudinal strain is 0·32. Calculate the modulus of rigidity and Bulk modulus if its Young's modulus of elasticity is 100 *GPa*. Find the change in volume when the bar is subjected to a hydrostatic pressure of 80 *MPa*.

[**Ans.** 37·88 *GPa*, 92·59 *GPa* ; 26·06 mm³]

4·8. The modulus of rigidity of a material is 78×1000 N/mm². *A* 15 mm diameter rod of the material is subjected to an axial tensile force of 10 kN and the change in its diameter is observed to be 0·00126 mm. Calculate the Poissons's ratio and modulus of elasticity of the material. [**Ans.** 0·3, $202{\cdot}1 \times 10^3$ N/mm²]

5

Thin Cylindrical and Spherical Shells

The thin cylindrical shell when subjected to internal fluid pressure or gas pressure, circumferential and axial stresses are developed in its wall. If the ratio of thickness t and diameter D *i.e.*, t/D is less than 0·05, it can be assumed with sufficient accuracy that the hoop stress and the axial stress are constant throughout the thickness of the cylinder wall and such a cylinder is classified as a thin cylinder. Similarly in the case of thin spherical shells, hoop stress or circumferential stress is developed in its wall and this stress is assumed to be constant throughout the thickness.

5·1. STRESSES IN THIN CYLINDERS

When the pressure inside the cylinder is developed, the volume of the liquid or gas pumped inside the cylinder is more than the initial volume of the cylinder. This additional volume of the liquid or gas will exert pressure on the cylinder wall which increases the volume of the cylinder and in turn cylinder wall offers equal resistance and compresses the liquid or gas inside the cylinder as shown in Fig. 5·1.

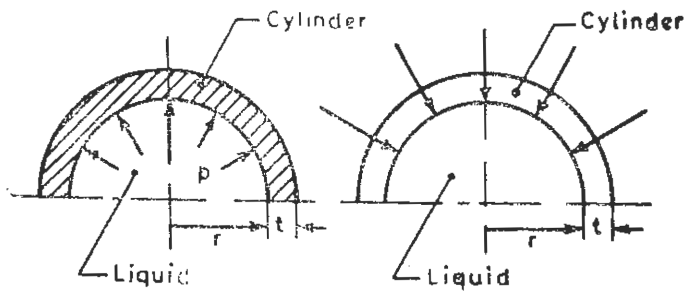


Fig. 5·1

Mathematically it can be written

$$\delta V = \delta V_1 + \delta V_2$$

where δV = additional volume of liquid pumped inside the cylinder

δV_1 = increase in the volume of the cylinder

δV_2 = decrease in the volume of liquid or gas.

On any element of the cylinder, three stresses orthogonal to each other are acting as shown in Fig. 5·2.

f_c = circumferential stress

f_l = axial stress

p = radial pressure

p_a = atmospheric pressure on outer surface.

The Fig. 5·3 shows a thin cylinder subjected to internal pressure p. The internal diameter is D and length is l.

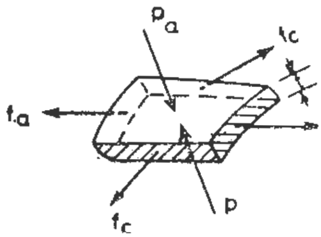


Fig. 5·2

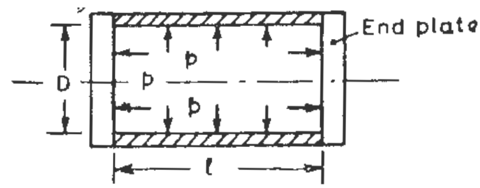


Fig. 5·3

Considering the axial bursting force tending to break the cylinder along the circumference as shown in Fig. 5·4.

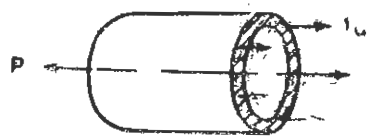


Fig. 5·4

Axial bursting force, $P_a = p \times \frac{\pi}{4} D^2$

Area of cross section resisting the axial bursting force

$= \pi D t$

Say axial stress developed $= f_a$

Then for equilibrium

$$f_a \,.\, \pi D t = p \times \frac{\pi}{4} D^2$$

$$f_a = \frac{pD}{4t} \qquad \ldots(1)$$

To determine the bursting force along a diameter, tending to break the cylinder along its length, consider a small length dl of the cylinder and an elementary area dA as shown in the Fig. 5·5.

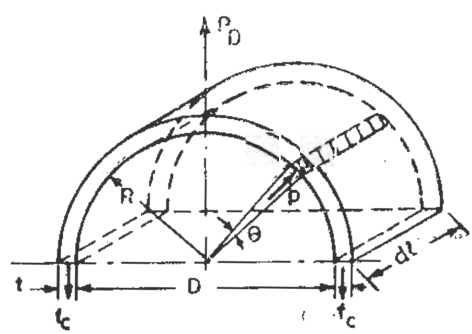


Fig. 5·5

Pressure on the inner surface

$= p$

δF, Force on elementary area

$= p dA = p\, R d\theta\, dl$

$= p \frac{D}{2} d\theta\, dl$

Vertical component of $dF = p \frac{D}{2} dl \,.\, \sin\theta \,.\, d\theta$

Horizontal component of dF

$$= p \frac{D}{2} dl \,.\, \cos\theta \,.\, d\theta$$

The horizontal component of the force is cancelled out when the force is integrated over the semi circular portion.

Therefore total diametral bursting force,

$$P_D = \int_0^{\pi} p \frac{D}{2} dl \, . \sin \theta \, d\theta$$

$$= \frac{pD}{2} dl \Big| -\cos \theta \Big|_0^{\pi}$$

$$= pD \, dl = p \times \text{projected area of the curved surface}$$

Area of cross section resisting the diametral busting force

$$= 2 \times dl \times t$$

Say circumferential stress developed

$$= f_c$$

For equilibrium $f_c \times 2t \times dl = pD \, dl$

$$f_c = \frac{pD}{2t} \qquad \ldots(2)$$

The circumferential stress f_c and axial stress f_a are quite large in comparison to the radial stress p, therefore, in the calculation of strains the radial stress p is not considered.

Circumferential strain, $\epsilon_c = \dfrac{f_c}{E} - \dfrac{1}{m} \dfrac{f_a}{E}$

where E=Young's modulus of elasticity of the material

and $\dfrac{1}{m}$=Poisson's ratio

$$\epsilon_c = \frac{pD}{2tE} - \frac{pD}{4mtE}$$

$$= \frac{pD}{4tE}\left(2 - \frac{1}{m}\right) \qquad \ldots(3)$$

=diameteral strain

Change in diameter, $\delta D = e_c \times D$

$$= \frac{pD^2}{4tE}\left(2 - \frac{1}{m}\right) \qquad \ldots(4)$$

Axial strain, $\epsilon_a = \dfrac{f_a}{E} - \dfrac{f_c}{mE}$

$$= \frac{pD}{4tE} - \frac{pD}{2mtE} = \frac{pD}{4tE}\left(1 - \frac{2}{m}\right) \qquad \ldots(5)$$

Change in length, $\delta l = e_a \times l$

$$= \frac{pDl}{4tE}\left(1 - \frac{2}{m}\right) \qquad \ldots(6)$$

Volumetric strain $=\dfrac{\text{Final volume}-\text{Initial volume}}{\text{Initial volume}}$

$$=\frac{\pi/4\,(D+\delta D)^2\,(l+\delta l)-\pi/4\,D^2 l}{\pi/4\,D^2 l}$$

$$=\frac{2\delta D}{D}+\frac{\delta l}{l} \quad \text{(neglecting higher order terms of } \delta D \text{ and } \delta l)$$

$$=2\,\epsilon_c+\epsilon_a$$

$$=\frac{2pD}{4tE}\left(2-\frac{1}{m}\right)+\frac{pD}{4tE}\left(1-\frac{2}{m}\right)$$

$$=\frac{pD}{4tE}\left(5-\frac{4}{m}\right) \qquad \ldots(7)$$

Change in volume of the cylinder,

$$\delta V_1=\frac{pDV}{4tE}\left(5-\frac{4}{m}\right)$$

where V is the initial volume of cylinder

$$=\frac{\pi p D^3 l}{16tE}\left(5-\frac{4}{m}\right) \qquad \ldots(8)$$

Change in volume of liquid,

$$\delta V_2=\frac{p}{K}\times V=\frac{\pi}{4K}\times pD^2 l$$

where K is the bulk modulus of the liquid.

Example 5·1-1. A closed cylindrical vessel made of steel plate 4 mm thick with plane ends carries fluid under a pressure of 30 kg/cm². The diameter of the cylinder is 25 cm and length is 75 cm, calculate the longitudinal and hoop stresses in the cylinder wall and determine the change in diameter, length and volume of the cylinder.

$$E=2100 \text{ tonnes/cm}^2$$

$$\frac{1}{m}=0·286.$$

Solution. Internal pressure

$p=30$ kg/cm²

Diameter, $D=25$ cm

Wall thickness, $t=0·4$ cm

Length $l=75$ cm

Longitudinal or axial stress,

$$f_a=\frac{pD}{4t}=\frac{30\times 5}{4\times 0·4}=468·75 \text{ kg/cm}^2$$

Hoop or circumferential stress,

$$f_c=\frac{pD}{2t}=\frac{30\times 25}{2\times 0·4}=937·5 \text{ kg/cm}^2$$

Change in diameter, $\delta D=\frac{pD^2}{4tE}\left(2-\frac{1}{m}\right)$

$$=\frac{30\times25\times25}{4\times0{\cdot}4\times2100\times1000}\ (2-0{\cdot}286)=0{\cdot}01\ \text{cm}$$

Change in length, $\delta l=\frac{pDl}{4tE}\left(1-\frac{2}{m}\right)$

$$=\frac{30\times25\times75}{4\times0{\cdot}4\times2100\times1000}\ (1-2\times0{\cdot}286)$$

$$=0{\cdot}007165\ \text{cm}$$

Change in volume, $\delta V=\frac{\pi pD^3l}{16tE}\left(5-\frac{4}{m}\right)$

$$=\frac{\pi\times30\times25^3\times75}{16\times0{\cdot}4\times2100\times1000}\ (5-4\times0{\cdot}286)=31{\cdot}68\ \text{cm}^3.$$

Example 5·1-2. A thin cylindrical shell made of copper plate 5 mm thick is filled with water under a pressure of 4 N/mm². The internal diameter of the cylinder is 200 mm and its length is 0·8 m. Determine the additional volume of the water pumped inside the cylinder to develop the required pressure.

$$E_{copper}=104{,}000\ \text{N/mm}^2$$

$$\frac{1}{m}\ \text{for copper}=0{\cdot}32$$

$$K\ \text{for water}=2100\ \text{N/mm}^2.$$

Solution. Diameter of the cylinder,

$$D=200\ \text{mm}$$

Length of the cylinder, $l=800$ mm

Wall thickness $t=5$ mm

Pressure $p=4$ N/mm²

Initial volume of the cylinder,

$$V=\frac{\pi}{4}\ D^2l=\frac{\pi}{4}\times(200)^2\times800$$

$$=25{\cdot}1328\times10^6\ \text{mm}^3$$

Volumetric strain, $\epsilon_v=\frac{pD}{4tE}\left(5-\frac{4}{m}\right)$

$$=\frac{4\times200}{4\times5\times104{,}000}\ (5-4\times0{\cdot}32)=1{\cdot}430\times10^{-3}$$

Change in volume of the cylinder,

$$\delta V_1=\epsilon_v\times V=1{\cdot}430\times10^{-3}\times25{\cdot}1328\times10^6$$

$$=35{\cdot}94\times10^3\ \text{mm}^3$$

Say the compression in the volume of water,

$$=\delta V_2$$

Then total volume of water,

$$V' = V + \delta V_1 + \delta V_2.$$

This has been compressed to $V + \delta V_1$

So $$\delta V_2 = \frac{p}{K} \times V'$$

$$= \frac{4}{2100} \times (25{\cdot}1328 \times 10^6 + 35{\cdot}94 \times 10^3 + \delta V_2)$$

$$= \frac{4}{2100} (25{\cdot}16874 \times 10^6 + \delta V_2)$$

$$\delta V_2 = 48{\cdot}03 \times 10^3 \text{ mm}^3$$

Additional volume of water pumped in,

$$\delta V = \delta V_1 + \delta V_2 = (35{\cdot}94 + 48{\cdot}03) \times 10^3$$

$$= 83{\cdot}97 \times 10^3 \text{ mm}^3.$$

Exercise 5·1-1. A thin cylindrical shell made of copper plate is subjected to an internal fluid pressure of 3 N/mm². The wall thickness is 2·5 mm, diameter of the cylinder is 150 mm and length 0·8 m. Determine (*i*) axial stress (*ii*) hoop stress (*iii*) change in diameter, length and volume.

E for copper = 104,000 N/mm²

$1/m$ for copper = 0·32.

[**Ans.** 45 N/mm² (*ii*) 90 N/mm³ (*iii*) 0·109 mm, 0·1246 mm, 22755·5 mm³]

Exercise 5·1·2. A thin cylindrical shell made of steel plate is 3·5 mm thick and is filled with oil under a pressure of 50 kg/cm². The internal diameter of cylinder is 17·5 cm and its length is 60 cm. Determine the modulus of compressibility of oil if the total volume of oil filled in the cylinder is 14482 cm³.

E for steel = 2100 tonnes/cm²

$1/m$ for steel = 0·28. [**Ans.** 25000 kg/cm²]

5·2. THIN SPHERICAL SHELLS

A thin spherical shell of internal diameter D and wall thickness t subjected to an internal fluid pressure p is shown in Fig. 5·6. Due to the bursting force, the spherical shell is

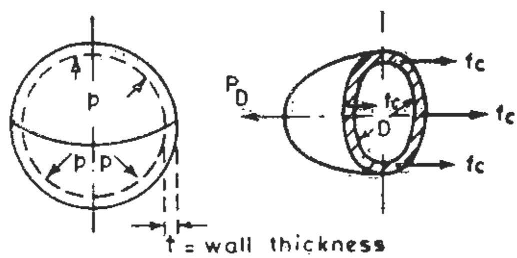


Fig. 5·6

going to fail along the circumferential area as shown in the figure-breaking into two hemispheres. Diametral bursting force,

$$P_D = p \times \text{ projected area of the hemisphere}$$

$$= p \times \frac{\pi}{4} D^2$$

Area of cross section resisting the bursting force

$$=\pi Dt$$

Say the circumferential stress developed

$$=f_c$$

For equilibrium, $f_c \pi Dt = p \times \frac{\pi}{4} D^2$

$$f_c = \frac{pD}{4t}.$$

The stresses acting on an element of the spherical shell are shown in Fig. 5·7. Where p_a is the atmospheric pressure on the outer surface and p is the radial pressure on the inner surface. Since the value of p is very small in comparison to the value of f_c, its effect on the calculation of diametral strain is not considered.

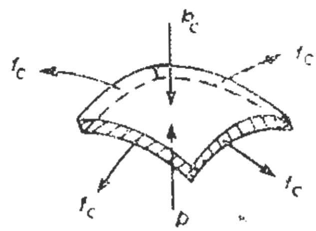


Fig. 5·7

Circumferential strain,

$$\epsilon_c = \frac{f_c}{E} - \frac{f_a}{mE} = \frac{pD}{4tE}\left(1-\frac{1}{m}\right) \qquad ...(9)$$

$$= \text{Diametral strain}$$

V, Initial volume of shell $= \frac{\pi D^3}{6}$

If δD is the change in diameter due to p then final volume of shell

$$= \frac{\pi}{6}(D+\delta D)^3$$

Volumetric strain, $\epsilon_v = \dfrac{\frac{\pi}{6}(D+\delta D)^3 - \frac{\pi}{6}D^3}{\frac{\pi D^3}{6}}$

$$\simeq \frac{3\delta D}{D} \text{ (neglecting higher order terms of } \delta D).$$

$$= 3 \times \text{diametral strain}$$

$$= \frac{4pD}{4tE}\left(1-\frac{1}{m}\right). \qquad ...(10)$$

Change in volume, $\delta V_1 = \frac{3pD}{4tE} \times V \,.\, \left(1-\frac{1}{m}\right)$

$$= \frac{\pi p D^4}{8tE}\left(1-\frac{1}{m}\right). \qquad ...(11)$$

Example 5·2-1. A thin spherical shell of wall thickness 4 mm and diameter 30 cm is subjected to an internal pressure p. Determine the magnitude of p if the diametral strain is $\frac{1}{2000}$.

$$E = 205{,}000 \text{ N/mm}^2$$

$$\frac{1}{m} = 0{\cdot}3.$$

Solution. Internal diameter,

$$D = 300 \text{ mm}$$

Wall thickness, $t = 4$ mm

Diametral strain, $\epsilon_c = \dfrac{1}{2000} = \dfrac{pD}{4tE}\left(1 - \dfrac{1}{m}\right)$

or
$$\frac{1}{2000} = \frac{p \times 300}{4 \times 4 \times 205{,}000}\,(1 - 0{\cdot}3)$$

$$p = 7{\cdot}81 \text{ N/mm}^2.$$

Exercise 5·2-1. A thin spherical shell of wall thickness 5 mm, and diameter 30 cm is subjected to an internal pressure of 50 kg/cm². Determine (*a*) hoop stress (*b*) diametral strain (*c*) volumetric strain.

Given $E = 1080$ tonnes/cm²

$\dfrac{1}{m} = 0{\cdot}32$. [**Ans.** (*a*) 750 kg/cm² (*b*) $0{\cdot}472 \times 10^{-3}$ (*c*) $1{\cdot}416 \times 10^{-3}$]

5·3. CYLINDRICAL SHELL WITH HEMISPHERICAL ENDS

A thin cylindrical shell with hemispherical ends as shown in Fig. 5·8, is subjected to internal fluid pressure *p*.

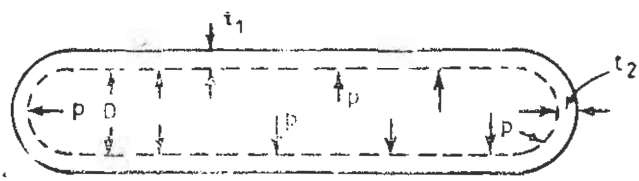


Fig. 5·8

The internal diameter of the cylinder, $= D$

Wall thickness of cylindrical portion, $= t_1$

Wall thickness of hemispherical portion, $= t_2$

Circumferential stress developed in

Cylindrical portion, $f_{c1} = \dfrac{pD}{2t_1}$

Axial stress in cylindrical portion, $f_{a1} = \dfrac{pD}{4t_1}$

Circumferential strain in cylindrical portion, $= \dfrac{f_{c1}}{E} - \dfrac{f_{a1}}{mE}$

$$\epsilon_{c1} = \frac{pD}{4t_1 E}\left(2 - \frac{1}{m}\right)$$

Circumferential stress developed in hemispherical portion

$$f_{c2}=\frac{pD}{4t_2}$$

Circumferential strain in hemispherical portion,

$$\epsilon_{c2}=\frac{pD}{4t_2E}\left(1-\frac{1}{m}\right)$$

Now for no distortion of the junction under pressure,

$$\epsilon_{c1}=\epsilon_{c2}$$

$$\frac{pD}{4t_1E}\left(2-\frac{1}{m}\right)=\frac{pD}{4t_2E}\left(1-\frac{1}{m}\right)$$

or

$$\frac{t_2}{t_1}=\frac{1-\frac{1}{m}}{2-\frac{1}{m}}. \qquad ...(12)$$

For maximum stress to be the same in both cylindrical and hemispherical portions

$$f_{c1}=f_{c2}$$

$$\frac{pD}{2t_1}=\frac{pD}{4t_2} \quad \text{or} \quad \frac{t_2}{t_1}=0{\cdot}5. \qquad ...(13)$$

Example 5·3-1. A thin cylindrical steel shell of diameter 150 mm and wall thickness 3 mm has hemispherical ends. Determine the thickness of hemispherical ends, if there is no distortion of the junction under pressure.

$$E_{steel}=208000 \text{ N/mm}^2$$

$$\frac{1}{m}=0{\cdot}3.$$

Solution. Thickness of cylindrical portion,

$$t_1=3 \text{ mm}$$

Thickness of hemispherical ends $=t_2$

For no distortion of the junction under pressure,

$$\frac{t_2}{t_1}=\frac{1-\frac{1}{m}}{2-\frac{1}{m}}=\frac{1-0{\cdot}3}{2-0{\cdot}3}$$

$$t_2=\frac{0{\cdot}7}{1{\cdot}7}\times 3.$$

Thickness of hemispherical ends $=1{\cdot}235$ mm.

Exercise 5·3-1. A thin cylindrical shell with hemispherical ends is of diameter 2 metres. It is subjected to an internal pressure of 4·5 N/mm². Determine the thickness of cylindrical and hemispherical portions if the maximum allowable stress is 90·0 N/mm².

[**Ans.** 50 mm, 25 mm]

5·4. WIRE WINDING OF THIN CYLINDERS

We have observed in the previous ariticles that the hoop stress developed in a thin cylinder is twice the axial stress and therefore, the chances of bursting the cylinder longitudinally are more than those for circumferential failure of the cylinder. Thus to increase the pressure-carrying capacity of the cylinder and to reduce the chances of longitudinal burst, the cylinder is strengthened longitudinally.

In order to achieve the above objective, the cylinder is wound with layers of wire kept under tension. In other words the cylinder wall is put under diametral compression, initially. When this wire wound cylinder is subjected to internal pressure, further hoop stress developed in

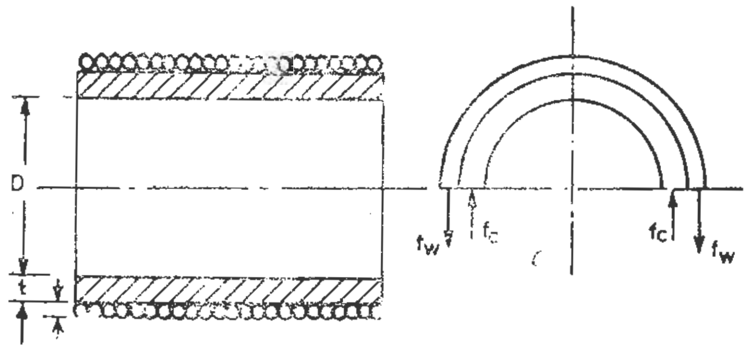


Fig. 5·9

the cylinder and wire is tensile. The resultant hoop stress in the cylinder is the sum of the initia compressive stress due to wire winding and further tensile stress due to internal pressure. Th resultant hoop stress in the wire is the sum of two tensile stresses developed due t wire winding under tension and internal pressure in the cylinder. Thus the pressure-carryin capacity of the cylinder is increased.

Consider a thin cylinder of diameter D, wall thickness t wound with a single layer c wire of diameter d. The wire is wound with an initial tension f_w.

Number of turns of wire per unit length, $n=1/d$

Say f_c is the compressive circumferential stress developed in the cylinder.

Tensile force exerted by wire per unit length,

$$=2n\times\frac{\pi}{4}\ d^2 f_w$$

Compressive force developed in the cylinder,

$$=2f_c\times t$$

For equilibrium, $$2f_c\ .\ t=2\ .\ n\ .\frac{\pi}{4}\ d^2 f_w$$

$$f_c=\frac{n\pi\ d^2}{4t}\ f_w.$$

or $$f_c=\frac{\pi d}{4t}\ f_w.\quad \text{(putting the value of } n\text{)} \qquad \ldots(1$$

Now when the wire wound cylinder is subjected to internal pressure p, say the axial and circumferential stresses developed in the cylinder are f_a' and f_c' and the stress developed in wire is f_w'.

Longitudinal bursting force $=p\times\frac{\pi}{4}D^2$

$=f_a'\times\pi Dt$ (for equilibrium)

or $$f_a'=\frac{pD}{4t}. \quad ...(15)$$

Diametral bursting force per unit length,

$=p\times D\times 1$

$=f_c'\times 2\times t+f_w'\times 2n\times\frac{\pi}{4}d^2$ (for equilibrium)

or $$pD=f_c'\times 2t+f_w'\frac{\pi d}{2} \quad ...(16)$$

Moreover the circumferential strain in the cylinder

=circumferential strain in wire

(for compatibility)

If E_c=Young's modulus for cylinder

E_w=Young's modulus for wire

Then $\frac{1}{m}$=Poisson's ratio for cylinder

$$\frac{f_c'}{E_c}-\frac{f_a'}{mE_c}=\frac{f_w'}{E_w}$$

$$\frac{f_c'}{E_c}-\frac{pD}{4tmE}=\frac{f_w'}{E_w}. \quad ...(17)$$

From the equations (16) and (17) stresses f_c' and f_w' can be determined.

Resultant stress in wire $=f_w+f_w'$

Resultant hoop stress in cylinder $=f_c'-f_c$ ...(18)

Example 5·4-1. A thin cylindrical shell of diameter 30 cm is closely wound around its circumference by a 2 mm diameter steel wire under a tension of 800 kg/cm². The cylinder is further subjected to an internal pressure of 20 kg/cm². Determine the wall thickness of the cylinder if the resultant hoop stress in the cylinder wall is 200 kg/cm² (Tensile) and the cylinder is made of copper.

E for copper =1050 tonnes/cm²

$1/m$ for copper =0·31

E for steel =2100 tonnes/cm².

Solution. Initial tension in wire,

f_w=800 kg/cm²

Diameter of the cylinder, D=30 cm

Wall thickness, t=?

Internal pressure, $p=20$ kg/cm²
Wire diameter, $d=0{\cdot}2$ cm.
f_c, initial hoop compression in cylinder

$$=\frac{\pi d}{4t}\times f_w=\frac{\pi\times 0{\cdot}2}{4\times t}\times 800$$

$$=\frac{125{\cdot}6}{t}\text{ kg/cm}^2$$

Due to internal pressure, axial stress developed in the cylinder,

$$f_a'=\frac{pD}{4t}=\frac{20\times 30}{4t}=\frac{150}{t}.$$

Now $$pD=\frac{\pi d}{2}f_w'+2f_c'\,.\,t$$

and $$\frac{f_w'}{E_w}=\frac{f_c'}{E_c}-\frac{1}{m}\frac{f_a'}{E_c}$$

where
f_c' hoop stress in cylinder
f_w' hoop stress in wire
E_w Young's modulus for wire
E_c Young's modulus for cylinder
$1/m$ Poisson's ratio for cylinder

So $$20\times 30=\frac{\pi\times 0{\cdot}2}{2}\times f_w'+2f_c'\,.\,\text{t}. \quad ...(1)$$

$$\frac{f_w'}{2100\times 1000}=\frac{f_c'}{1050\times 1000}-0{\cdot}31\times\frac{150}{t\times 1050\times 1000} \quad ...(2)$$

Moreover $$f_c'-f_c=200$$

$$f_c'-\frac{125{\cdot}6}{t}=200. \quad ...(3)$$

From (2) $$fw'=2\left[f_c'-\frac{46{\cdot}5}{t}\right]$$

Substituting in equation (1)

$$600=0{\cdot}314\times 2\left[f_c'-\frac{46{\cdot}5}{t}\right]+2f_c'\,.\,t$$

But $$f_c'=\left(200+\frac{125{\cdot}6}{t}\right)$$

Substituting above

$$600=0{\cdot}628\left[200+\frac{125{\cdot}6}{6}-\frac{46{\cdot}5}{t}\right]+2t\left[200+\frac{125{\cdot}6}{t}\right]$$

or $$400\,t^2-348{\cdot}8\,t+49{\cdot}675=0$$

$$t=0{\cdot}693\text{ cm.}$$

Exercise 5·4-1. A thin cylindrical shell of internal diameter 40 cm and wall thickness 10 mm is closely wound around its circumference by a 3 mm diameter steel wire under an initial tension of 10 N/mm². The cylinder is further subjected to an internal pressure of 2·4 N/mm². Determine the resultant hoop stress developed in the cylinder and the wire. The cylinder is also made of steel.

$$E_{steel}=208{,}000 \text{ N/mm}^2$$

$1/m$ for steel $=0{\cdot}30$. [**Ans.** 37·864 N/mm², 43·02 N/mm²]

Problem 5·1. A steam boiler 150 cm internal diameter is subjected to an internal pressure of 12 kg/cm². What will be the tension in the boiler per linear cm of the longitudinal joint in the boiler shell. Calculate the thickness of the plate if the maximum tensile stress in the plate section is not to exceed 1000 kg/cm², taking the efficiency of the longitudinal riveted joint as 75%.

Solution. Internal diameter of boiler shell,

$$D=150 \text{ cm}$$

Wall thickness, $t=?$

Internal pressure, $p=12 \text{ kg/cm}^2$

Circumferential stress developed,

$$f_c=\frac{pD}{2t}=\frac{150\times 12}{2\times t}=\frac{900}{t} \text{ kg/cm}^2$$

Axial stress developed, $f_a=\dfrac{pD}{4t}=\dfrac{12\times 150}{4\times t}$

$$=\frac{450}{t} \text{ kg/cm}^2.$$

Tension in the boiler per linear cm of the longitudinal joint

$$=f_c\times 2t\times 1$$

$$=\frac{900}{t}\times 2t\times 1=1800 \text{ kg.}$$

Efficiency of the longitudinal joint $=75\%$

Therefore allowable circumferential stress

$$=0{\cdot}75\times 1000=750 \text{ kg/cm}^2$$

$$\frac{900}{t}=750 \quad \text{or} \quad t=\mathbf{1{\cdot}2 \text{ cm.}}$$

Problem 5·2. A cylindrical tank 2 m inside diameter and 20 m high is filled with water of specific weight 10000 N/m³. The material of the tank is a structural steel with a yield strength of 250 N/mm². What is the necessary thickness at the bottom of the steel tank, if the efficiency of the longitudinal seam is 80% ? Take factor of safety as 4.

Solution. Weight density,

$$w=10000 \text{ N/m}^3$$

or $$w=10^{-5} \text{ N/mm}^3$$

The hydrostatic pressure at the bottom of the tank

$$p=wh$$

$$=10^{-5}\times 20\times 1000 \text{ N}$$

$$=0{\cdot}2 \text{ N/mm}^2$$

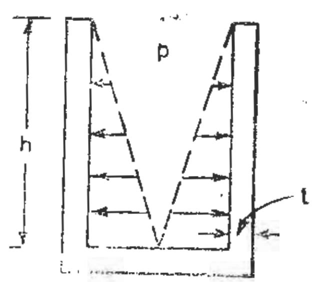


Fig. 5.10

Diameter of the tank, $D=2000$ mm
Yield strength of the material

$$=250 \text{ N/mm}^2$$

Allowable stress in the material

$$=\frac{250}{4}=62{\cdot}5 \text{ N/mm}^2$$

Efficiency of the longitudinal seam, $\eta=0{\cdot}8$
Allowable stress in the joint

$$=0{\cdot}8\times 62{\cdot}5=50 \text{ N/mm}^2$$

Now the maximum stress developed

$$=\text{Hoop stress}=\frac{pD}{2t}$$

$$\leqslant 50 \text{ N/mm}^2$$

or

$$50=\frac{0{\cdot}2\times 2000}{2\times t} \quad \text{or} \quad t=\frac{400}{2\times 50}$$

Thickness of the steel tank at the bottom $=4$ mm

Problem 5·3. To what depth would a copper float 30 cm in diameter and 3 mm thick has to be sunk in sea water in order that its diameter is decreased by 0·003 cm ?

E for copper $=1050$ tonnes/cm²

$\frac{1}{m}$ for copper $=0{\cdot}32$

Density of sea water $=1025$ kg/m³

Solution.

w, density of water $=1025 \text{ kg/m}^3=1025\times 10^{-6} \text{ kg/cm}^3$

Say depth through which float is sunk $=h$ cm

Pressure on the float, $p=wh$

$$=1025\times 10^{-6}\times h \text{ kg/cm}^2$$

Diameter of float, $D=30$ cm

Wall thickness, $t=0{\cdot}3$ cm

Circumferential stress, $f_c=\frac{pD}{4t}=\frac{1025\times 10^{-6}\times 30h}{4\times 0{\cdot}3}$

$$=\frac{1{\cdot}025h}{40}$$

Diametral strain, $\epsilon_c=\frac{f_c}{E}\left(1-\frac{1}{m}\right)$

$$=\frac{1{\cdot}025h}{40\times 1050\times 1000}(1-0{\cdot}32)$$

$$=\frac{0{\cdot}68\times 1{\cdot}025h}{40\times 1050{,}000}$$

Decrease in diameter, $\delta D = \epsilon_c \times D$

$$0{\cdot}003 = \frac{0{\cdot}68 \times 1{\cdot}025\, h}{40 \times 1050{,}000} \times 30$$

$$h = 6025{\cdot}8 \text{ cm} = \mathbf{60{\cdot}258 \text{ metres}}$$

Problem 5·4. A thin copper pipe 8 cm internal diameter, 2 mm wall thickness and 150 cm long is closed at the ends with plugs. The pipe is filled with water under pressure. Determine the increase in pressure when an additional 10 c.c. of water is pumped into the pipe.

E for copper $= 1050{,}000$ kg/cm²

$\frac{1}{m}$ for copper $= 0{\cdot}32$

K for water $= 21000$ kg/cm².

Solution. Initial volume of pipe

$$= \frac{\pi}{4} (8)^2 \times 150 = 7539{\cdot}84 \text{ cm}^3$$

Let the increase in fluid pressure,

$$= p \text{ kg/cm}^2$$

Additional volume of water pumped in,

$$\delta V = 10 \text{ cm}^3 = \delta V_1 + \delta V_2$$

δV_1 = increase in volume of cylinder

δV_2 = decrease in volume of water

$$\delta V_1 = \frac{pD}{4tE} \times V \left[5 - \frac{4}{m} \right]$$

$$= \frac{p \times 8 \times 7539{\cdot}84}{4 \times 0{\cdot}2 \times 1050{,}000} [5 - 4 \times 0{\cdot}32] = 0{\cdot}267\, p$$

$$\delta V_2 = (10 - 0{\cdot}267\, p) \text{ c.c.}$$

Bulk modulus of water, $K = \dfrac{p}{\delta V_2/(V + \delta V_1)}$

$$\frac{10 - 0{\cdot}267\, p}{7539{\cdot}84 + 0{\cdot}267\, p} = \frac{p}{21000}$$

$$p = 15{\cdot}96 \text{ kg/cm}^2.$$

Problem 5·5. The dimensions of a steel cylinder are length 200 cm, internal diameter 25 cm and wall thickness 1 cm. The cylinder is initially filled with water at atmospheric pressure. Considering this to be a thin cylinder, find the increase in volume when the water is pumped in so as to raise the internal pressure to 60 kg/cm². If the quantity of water which has to be pumped in so as to produce the required pressure is 350 c.c., determine the modulus of compressibility of water. Neglect the deformation at the ends.

E for steel $= 2100$ tonnes/cm²

$\frac{1}{m}$ for steel $= 0{\cdot}28$.

Solution. Initial volume,

$$V = \frac{\pi}{4} D^2 l = \frac{\pi}{4} (25)^2 \times 200 = 31250\, \pi \text{ cm}^3$$

Volumetric strain in cylinder,

$$\epsilon_v=\frac{pD}{4tE}\left(5-\frac{4}{m}\right)$$

$$=\frac{60\times25}{4\times1\times2100\times1000}(5-4\times0{\cdot}28)=0{\cdot}693\times10^{-3}$$

Increase in volume of cylinder,

$$\delta V_1=0{\cdot}693\times10^{-3}\times31250\,\pi=68{\cdot}03\text{ cm}^3$$

$$\delta V_2=350-68{\cdot}03=281{\cdot}97\text{ cm}^3$$

Modulus of compressibility,

$$K=\frac{p}{\delta V_2/(V+\delta V_1)}=\frac{60\times(31250\pi+68{\cdot}03)}{281{\cdot}97}$$

$$=20905\text{ kg/cm}^2.$$

Problem 5·6. A steel tube having a bore of 10 cm, wall thickness 1·5 mm is plugged at each end to form a closed cylinder with internal length of 30 cm. The tube is completely filled with oil and is subjected to a compressive load of 6 tonnes. Determine

(*a*) the pressure in kg/cm² produced on oil

(*b*) the resulting circumferential stress in tube wall.

K for oil $=28000$ kg/cm²

E for steel $=2100$ tonnes/cm²

$\frac{1}{m}$ for steel $=0{\cdot}28$.

Solution. Say the pressure developed on oil,

$=p$ kg/cm²

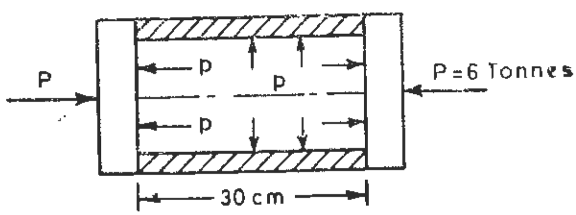


Fig. 5.11

Circumferential stress developed in cylinder,

$$f_c=\frac{pD}{2t}=\frac{p\times10}{2\times0{\cdot}15}=33{\cdot}33\,p$$

Say the axial stress developed $=f_a$

For equilibrium $\quad p\times\frac{\pi}{4}D^2+f_a\,.\,\pi\,Dt=6000$ kg

$$p\times\frac{\pi}{4}\times10^2+f_a\times\pi\times10\times0{\cdot}15=6000\text{ kg}$$

$$f_a=1273{\cdot}23-16{\cdot}66\,p$$

Axial strain, $\epsilon_a = \dfrac{f_a}{E} - \dfrac{f_c}{mE}$

Circumferential strain, $\epsilon_c = \dfrac{f_c}{E} - \dfrac{f_a}{mE}$

Volumetric strains, $\epsilon_v = 2\,\epsilon_c + \epsilon_a$

$$= \frac{2f_c}{E} - \frac{2f_a}{mE} + \frac{f_a}{E} - \frac{f_c}{mE}$$

$$= \frac{1}{E}\,[1{\cdot}72\,f_c + 0{\cdot}44\,f_a]$$

Now bulk modulus for oil,

$$K = \frac{p}{\epsilon_v}$$

Since ϵ_v in this case is negative

$$-\frac{p}{K} = \frac{1}{E}\,[1{\cdot}72\,f_c + 0{\cdot}44\,f_a]$$

$$-\frac{p}{28000} = \frac{1}{2100 \times 1000}\,[1{\cdot}72 \times 33{\cdot}33\,p + 0{\cdot}44\,(1273{\cdot}23 - 16{\cdot}66\,p)]$$

(f_a is negative)

$$-75\,p = 57{\cdot}327\,p - 560{\cdot}22 + 7{\cdot}33\,p$$

$$p = \frac{560{\cdot}22}{139{\cdot}657} = 4{\cdot}01 \text{ kg/cm}^2$$

Circumferential stress, $f_c = 33{\cdot}33\,p$

$= 133{\cdot}699$ kg/cm² (Tensile)

Problem 5·7. The ends of a thin cylindrical shell are closed by flat plates. It is subjected to an internal fluid pressure under the following conditions :

(*i*) The ends are free to move axially (along the axis of the cylinder).

(*ii*) The ends are rigidly stayed and no axial movement is permitted.

Determine the ratio of the increase in the volume of the shell under the above conditions. Take Poisson's ratio$=0{\cdot}25$.

Solution. Say internal diameter of cylinder $=D$

Wall thickness $=t$

Internal pressure $=p$

Young's modulus of elasticity $=E$

Poisson's ratio, $\dfrac{1}{m} = 0{\cdot}25$

V= Initial volume of the cylinder.

(*i*) When the ends are free to move axially

Increase in volume of shell,

$$\delta V' = \frac{pD}{4tE}\left(5 - \frac{4}{m}\right)V$$

$$=\frac{pD}{4tE}(5-4\times0{\cdot}25)\,V$$

$$=\frac{pD}{tE}\,V. \qquad \text{...(1)}$$

(*ii*) When the ends are stayed and no axial movement is permitted *i.e.*, axial strain$=0$.

This is possible only when compressive force P acts axially on the shell

f_a'=axial compressive stress due to P

$$=\frac{P}{\pi Dt}. \qquad \text{...(2)}$$

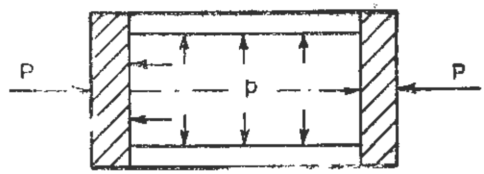


Fig. 5·12

Circumferential stress due to p,

$$f_c=\frac{pD}{2t}$$

Axial stress due to p, $f_a=\frac{pD}{4t}$

Axial strain, $\epsilon_a=\frac{1}{E}(f_a-f_a')-\frac{f_c}{mE}$

$$=\frac{pD}{4tE}-\frac{P}{\pi DtE}-\frac{0{\cdot}25\times pD}{2tE}$$

$$=\frac{pD}{8tE}-\frac{P}{\pi DtE}=0$$

or $$P=\frac{\pi pD^2}{8}. \qquad \text{...(3)}$$

Circumferential strain, $\epsilon_c=\frac{f_c}{E}-\frac{1}{mE}(f_a-f_a')$

$$=\frac{pD}{2tE}-\frac{0{\cdot}25}{E}\left(\frac{pD}{4t}-\frac{P}{\pi Dt}\right)$$

$$=\frac{pD}{2tE}-\frac{pD}{16tE}+\frac{P}{4\pi DtE}$$

$$=\frac{7}{16}\frac{pD}{tE}+\frac{1}{4\pi DtE}\left(\frac{\pi pD^2}{8}\right)$$

$$=\frac{7}{16}\frac{pD}{tE}+\frac{pD}{32tE}=\frac{15}{32}\frac{pD}{tE}$$

Volumetric strain, $\epsilon_v=2\epsilon_c+\epsilon_a=\frac{15}{16}\frac{pD}{tE}$ as $\epsilon_a=0$

Change in volume, $\delta V''=\frac{15}{16}\frac{pD}{tE}\times V$ ...(3)

Ratio $\frac{\delta V'}{\delta V''}=\frac{16}{15}$.

Problem 5·8. A closed pressure vessel of length 40 cm, thickness 5 mm, internal diameter 12 cm is subjected to an internal pressure of 80 kg/cm². Determine the normal and shear stresses in an element of the cylinder-wall on a plane at 30° to the longitudinal axis.

Solution. Fluid pressure,

$$p=80 \text{ kg/cm}^2$$

Internal diameter, $D=12$ cm

Wall thickness, $t=0·5$ cm

Circumferential stress, $f_c=\frac{pD}{2t}=\frac{80\times 12}{2\times 0·5}=960 \text{ kg/cm}^2$

Axial stress, $f_a=\frac{pD}{4t}=\frac{80\times 12}{4\times 0·5}=480 \text{ kg/cm}^2$

Normal stress on inclined plane,

$$f_n=\frac{f_c+f_a}{2}+\frac{f_c-f_a}{2}\cos(2\times 30^\circ)$$

$$=\frac{960+480}{2}+\frac{960-480}{2}\cos 60^\circ$$

$$=840 \text{ kg/cm}^2$$

Tangential stress on inclined plane,

$$f_t=\frac{f_c-f_a}{2}\sin(2\times 30^\circ)$$

$$=\frac{960-480}{2}\times 0·866=207·84 \text{ kg/cm}^2.$$

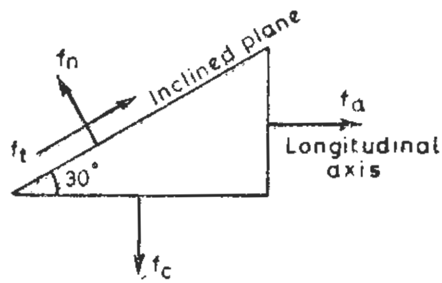


Fig. 5·13

Problem 5·9. A thin spherical shell made of copper is of 30 cm diameter with 1·5 mm wall thickness. It is full of water at atmospheric pressure. Find by how much the internal pressure will increase if 20 c.c. of water is pumped inside the shell.

$$E=100{,}000 \text{ N/mm}^2$$

$$\frac{1}{m}, \text{ Poisson's ratio}=0·29.$$

Bulk modulus of water $=2200$ N/mm²

Solution.

Diameter, $D=30$ cm $=300$ mm

Wall thickness, $t=1·5$ mm

Say, due to additional pumping in of water, the increase in internal pressure

$$=p \text{ N/mm}^2$$

Initial volume of shell, $V=\frac{\pi D^3}{6}=\frac{\pi\times 30^3}{6}=14137·2$ c.c.

Additional volume of water pumped,

$$\delta V=20 \text{ c.c.}$$

Total volumetric strain,

(water and shell) $\epsilon_v=\frac{20}{14137·2}=14·147\times 10^{-4}$

Volumetric strain on shell,

$$\epsilon_v' = \frac{3pD}{4tE}\left(1-\frac{1}{m}\right) = \frac{3p\times 300}{4\times 1{\cdot}5\times 10{,}000}(1-0{\cdot}29)$$

$$= 10{\cdot}65\ p\times 10^{-4}$$

Volumetric strain on water,

$$\epsilon_v'' = \frac{p}{K} = \frac{p}{2200} = 4{\cdot}545\times 10^{-4}\ p$$

Now $\epsilon_v' + \epsilon_v'' = \epsilon_v$

$$(10{\cdot}65\ p + 4{\cdot}545\ p)\times 10^{-4} = 14{\cdot}147\times 10^{-4}$$

or $$p = \frac{14{\cdot}147}{15{\cdot}195}$$

Increase in internal pressure

$$= 0{\cdot}931\ \text{N/mm}^2.$$

Problem 5·10. A copper tube 30 mm bore and 3 mm thick is plugged at its ends. It is just filled with water at atmospheric pressure. If an axial compressive load of 8 kN is applied to the plugs, find by how much the water pressure will increase. The plugs are assumed to be rigid and fixed to the tube.

$$E = 100{,}000\ \text{N/mm}^2$$

Poisson's ratio, $\frac{1}{m} = 0{\cdot}33$

Bulk modulus, $K = 2200\ \text{N/mm}^2$

Solution. Internal diameter of the the tube,

$$D = 30\ \text{mm}$$

Wall thickness, $t = 3$ mm

Axial force, $P = 8\ \text{kN} = 8000\ \text{N}$

Area of cross section of tube,

$$A = \pi Dt = \pi\times 30\times 3 = 282{\cdot}744\ \text{mm}^2$$

Axial compressive stress,

$$f = \frac{P}{A} = \frac{8000}{282{\cdot}744} = 28{\cdot}29\ \text{N/mm}^2$$

ϵ_a, axial strain, $= -\frac{f}{E}$ (compressive)

$$= -\frac{28{\cdot}29}{100{,}000} = -28{\cdot}29\times 10^{-5}$$

ϵ_c, diametral strain $= +\frac{f}{mE} = 28{\cdot}29\times 0{\cdot}33\times 10^{-5}$

Total volumetrie strain $= 2$ diametral strain $+$ axial strain

$$= 2\times 28{\cdot}29\times 0{\cdot}33\times 10^{-5} - 28{\cdot}29\times 10^{-5}$$

$$= -0{\cdot}34\times 28{\cdot}29\times 10^{-5} \quad \text{...(1)}$$

Now say the increase in internal pressure due to axial compressive force $= p$

Volumetric strain in cylinder

$$\epsilon_v' = \frac{pD}{4tE}\left(5-\frac{4}{m}\right) = \frac{p\times 30}{4\times 3\times 100{,}000}(5-4\times 0{\cdot}33)$$

$$=9{\cdot}2\times 10^{-5}\,p \qquad ...(2)$$

Volumetric strain on liquid,

$$\epsilon_v'' = \frac{p}{K} = \frac{p}{2200} = 0{\cdot}4545\times 10^{-3}\,p$$

$$=45{\cdot}45\times 10^{-5}\,p. \qquad ...(3)$$

Now due to the axial compressive force, there is compressive strain of $-0{\cdot}34\times 28{\cdot}29\times 10^{-5}$ *i.e.* if V is the original volume of water, then it means that V has been compressed to the volume $(V-9{\cdot}6188\times 10^{-5}\,V)$

So from equations (1), (2) and (3)

$$9{\cdot}2\times 10^{-5}p+45{\cdot}45\times 10^{-5}p=9{\cdot}6188\times 10^{-5}$$

$$54{\cdot}65\,p=9{\cdot}6188$$

$$p=\frac{9{\cdot}6181}{54{\cdot}65}=0{\cdot}176\ \text{N/mm}^2$$

Increase in internat pressure

$$=0{\cdot}176\ \text{kN/m}^2.$$

Problem 5·11. A solid cylindrical piece of Aluminium 75 mm long and 50 mm diameter is enclosed within a hollow pressure vessel. With the piece inside the vessel, 20×10^3 mm^3 of oil is required just to fill the pressure vessel. Measurement shows that 50 mm^3 of oil has to be pumped into the vessel to raise the oil pressure to 7N/mm^2.

The experiment is repeated using the same pressure vessel and oil, but without the test piece inside the vessel. This time, after initially filling the pressure vessel, a further 364 mm^3 of oil is needed to raise the pressure to 7N/mm^2. Find the bulk modulus of oil.

E for aluminium $=70$ GN/m^2

Poisson's ratio of aluminium $=0{\cdot}3$ (*P.U.*)

Solution. Length of aluminium piece = 75 mm

Diameter of aluminium piece = 50 mm

V_2, Volume of aluminium piece

$$=\frac{\pi}{4}(50)^2\times 75=147262{\cdot}5\ \text{mm}^3$$

Additional volume of oil to fill the vessel,

$$V_3=20\times 10^3\ \text{mm}^3=20{,}000\ \text{mm}^3$$

Therefore volume of vessel,

$$V_1=147262{\cdot}5+20{,}000$$

$$=167262{\cdot}5\ \text{mm}^3$$

Oil pressure, $p=7$ N/mm^2

Say the expansion in volume of vessel

$$=\delta V_1$$

Compression in volume of aluminium piece

$$\delta V_2 = \frac{p}{K_{aluminium}} \times V_2$$

Bulk Modulus,

$$K_{aluminium} = \frac{E}{3(1-2/m)} = \frac{70\ \text{GN/m}^2}{3(1-0{\cdot}6)}$$

$$= 58{\cdot}333\ \text{GN/m}^2$$

$$= 58{\cdot}333 \times 10^3\ \text{N/mm}^2$$

$$\delta V_2 = \frac{7}{58{\cdot}333 \times 10^3} \times \mathbf{147262{\cdot}5} = 17{\cdot}67\ \text{mm}^3$$

Compression in volume of the oil,

$$\delta V_3 = \frac{p}{K_{oil}} \times V_3$$

$$= \frac{7}{K} \times 20000 = \frac{140{,}000}{K}\ \text{mm}^3$$

Now $\delta V_1 + \delta V_2 + \delta V_3 = 50$

$$\delta V_1 + 17{\cdot}67 + \frac{14{,}0000}{K} = 50 \qquad \ldots(1)$$

Using the vessel without aluminium piece
Expansion in volume of vessel

$$= \delta V_1$$

Compression in volume of oil

$$= \frac{V_1}{K_{oil}} \times p = \frac{167262{\cdot}5}{K_{oil}} \times 7$$

or
$$\delta V_1 + \frac{167262{\cdot}5 \times 7}{K_{oil}} = 364 \qquad \ldots(2)$$

From equation (2) and (1),

$$\frac{1030837{\cdot}5}{K_{oil}} = 381{\cdot}67$$

$$K_{oil} = \frac{1030837{\cdot}5}{381{\cdot}67}$$

Bulk modulus of oil $= 2700\ \text{N/mm}^2$

Problem 5·12. A gun metal tube of 5 cm bore, wall thickness 1/8 cm is closely wound externally by a steel wire 0·5 mm diameter. Determine the tension under which the wire must be wound on the tube, if an internal radial pressure of 15 kg/cm² is required before the tube is subjected to the tensile stress in the circumferential direction.

E for gun metal $= 1020$ tonnes/cm²

$\frac{1}{m}$ for gun metal $= 0{\cdot}35$

E for steel $= 2100$ tonnes/cm²

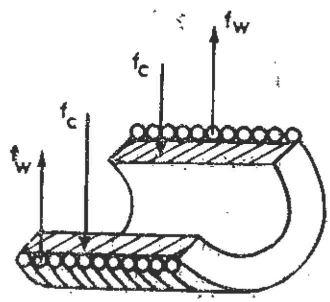


Fig. 5·14

Solution. Internal diameter of the tube, $D=5$ cm

Thickness of the tube, $t=0.125$ cm

Wire diameter, $d=0.05$ cm

Number of wires per cm length,

$$n=\frac{1}{0.05}=20$$

Say the initial tension in wire $=f_w$

Initial compression in tube due to wire winding $=f_c$

$$f_c=\frac{\pi d}{4t}\cdot f_w=\frac{\pi\times 0.05}{4\times 0.125}\times f_w$$

$$=0.31416\,f_w \quad \text{(compressive)}$$

When the tube is subjected to internal pressure

$p=15$ kg/cm²

f_c'=circumferential stress developed in tube

f_w'=further tension developed in steel wire

For equilibrium $\quad p\times D=f_c'\times 2t+2n\times\frac{\pi}{4}d^2 f_w'$

So $\quad 15\times 5=f_c'\times 2\times 0.125+2\times 20\times\frac{\pi}{4}\times(0.05)^2 f_w'$ ...(1)

Axial bursting stress $\quad f_a'=\frac{pD}{4t}=\frac{15\times 5}{4\times 0.125}$

$$=150 \text{ kg/cm}^2 \text{ (tensile)}$$

For compatibility of strain

$$\frac{f_c'}{E_c}-\frac{f_a'}{mE_c}=\frac{f_w'}{E_w}$$

$$\frac{f_c'}{1020\times 1000}-\frac{0.35\times 150}{1020\times 1000}=\frac{f_w'}{2100\times 1000}$$

or $\quad f_c'-52.5=0.4857\,f_w'$ ...(2)

or $\quad f_c'=52.5+0.4857\,f_w'$

Substituting in equation (1)

$$75=0.25\,(52.5+0.4857\,f_w')+0.0785\,f_w'$$

$$f_w'=\frac{75-13.125}{(0.1214+0.0785)}$$

$$=309.53 \text{ kg/cm}^2$$

$$f_c'=52.5+0.4857\times 309.53$$

$$=202.84 \text{ kg/cm}^2 \text{ tensile}$$

But $f_c' - f_c = 0$ (as given in the problem)

$$f_c = 202.84 \text{ kg/cm}^2 \quad \text{(compressive)}$$

$$f_w = \frac{f_c}{0.31416} = \frac{202.84}{0.31416}$$

Initial tension in wire $= \mathbf{645.66 \ kg/cm^2}$

Problem 5.13. A thin cylinder made of brone 250 mm internal diameter and 6 mm thick is wound with a single layer of steel tape 1.5 mm thick under a tensile stress of 100 N/mm². Find the maximum internal pressure if the hoop stress in the cylinder is not to exceed 50 N/mm². Determine also the final stress in the steel tape.

Poisson's ratio of bronze $= 0.33$

E for bronze $= 117000 \text{ N/mm}^2$

E for steel $= 208{,}000 \text{ N/mm}^2$.

Solution. Internal dia. of cylinder, $D = 250$ mm

Wall thickness of cylinder, $t = 6$ mm

Steel tape thickness, $t_w = 1.5$ mm

Init.al tension in tape, $f_w = 100 \text{ N/mm}^2$

Initial compressive stress in cylinder due to tape tension $= f_c$ (say)

Now $$f_c \,.\, t = f_w \,.\, t_w \quad \text{...(1)}$$

$$f_c \times 0.6 = 1.5 \times 100$$

$$f_c = \frac{100 \times 1.5}{6} = 25 \text{ N/mm}^2 \text{ (compressive)}$$

Say the internal pressure of liquid in cylinder $= p$

Axial stress due to $p = f_a' = \dfrac{pD}{4t} = \dfrac{p \times 250}{4 \times 6} = 10.4167\, p$

Circumferential stress in cylinder $= f_c'$

Additional stress in tape $= f_w'$

Now $2f_c' \times t + 2f_w' \times t_w = pD$ ...(2)

(considering unit length of cylinder)

or $$f_c' \times 6 \times 2 + f_w' \times 1.5 \times 2 = p \times 250$$

$$12 f_c' + 0.3\, f_w' = 250\, p \quad \text{...(3)}$$

Now diametral strain in cylinder $=$ diametral strain in tape

$$\frac{f_c'}{E_c} - \frac{1}{m}\frac{f_a'}{E_c} = \frac{f_w'}{E_w} \quad \text{...(4)}$$

where $$\frac{1}{m} = 0.33$$

$E_c = 117000 \text{ N/mm}^2$ (bronze cylinder)

$E_w = 208{,}000 \text{ N/mm}^2$ (steel wire)

$$\frac{f_c'}{117000} - \frac{0.33 \times 10.4167\, p}{117000} = \frac{f_w'}{208000}$$

$$f_c' - 3.4375\, p = 0.5625\, f_w'$$

or $$f_c' = 3.4375\, p + 0.5625\, f_w' \quad \text{...(5)}$$

Substituting in equation (3) above

$$12\,(3{\cdot}4375\,p+0{\cdot}5625\,f_w')+3\,f_w'=250\,p$$

$$9{\cdot}75\,f_w'=208{\cdot}75\,p$$

$$f_w'=21{\cdot}41\;p \qquad \ldots(6)$$

Moreover $\quad f_c+f_c'=50$

$$-25-f_c'=50$$

$$f_c'=75\ \text{N/mm}^2$$

From equation (5) $\quad 75=3{\cdot}4375\,p+0{\cdot}5625\,f_w'$

or $\quad f_w'=233{\cdot}333-6{\cdot}111\,p \qquad \ldots(7)$

Equating equation (6) and (7)

$$21{\cdot}41\,p=133{\cdot}333-6{\cdot}111\,p$$

Maximum internal pressure

$$p=\frac{133{\cdot}333}{27{\cdot}521}=4{\cdot}844\ \text{N/mm}^2$$

$$f_w'=133{\cdot}333-6{\cdot}111\times4{\cdot}844=103{\cdot}731\ \text{N/mm}^2$$

Final stress in steel tape $=f_w+f_w'=100+103{\cdot}731$

$$=203{\cdot}731\ \text{N/mm}^2.$$

Problem 5·14. A brass cylinder 120 mm outside diameter, wall thickness 10 mm is strengthened by a single layer of steel wire 1·5 mm diameter wound over it under a constant stress of 50 N/mm². If the cylinder is then subjected to an internal pressure of 18 N/mm² with rise in temperature of the cylinder by 80°C. Determine the final values of

(*i*) stress in the wire (*ii*) radial pressure between the wire and the cylinder, (*iii*) circumferential stress in the cylinder wall. The cylinder can be assumed to be a thin shell with closed ends.

$E_{steel}=208{,}000$ N/mm², $\quad \alpha_{steel}=11{\cdot}8\times10^{-6}/°\text{C}$

$E_{brass}=90{,}000$ N/mm² $\quad \alpha_{brass}=18{\cdot}6\times10^{-6}/°\text{C}$, Poisson's ratio for brass$=0{\cdot}32$.

Solution. Outside diameter of cylinder $=120$ mm

Wall thickness, $t=10$ mm

D, Inside diameter of the cylinder $=100$ mm

Wire diameter, $d=1{\cdot}5$ mm

Initial tension in wire, $f_w=50$ N/mm²

Say initial compressive stress in cylinder $=f_c$

Thus considering length l of the cylinder

$$2f_c\times l\times t=2\left(\frac{l}{d}\right)\left(\frac{\pi}{4}\,d^2\right)f_w$$

or $$f_c\,.\,t=\frac{\pi d}{4}\,.\,f_w,\ \text{putting value}$$

$$f_c\times10=\frac{\pi\times1{\cdot}5}{4}\times50$$

$$f_c=5{\cdot}8905\ \text{N/mm}^2\ \text{(compressive)} \qquad \ldots(1)$$

Now internal pressure $p=18$ N/mm²

Rise in temperature $=80°C$

$$\alpha_{brass}=18{\cdot}6\times10^{-6}/°C$$

$$\alpha_{steel}=11{\cdot}8\times10^{-6}/°C.$$

Considering the equilibrium

$$2\,f_c'\times l\times t+2\times f_w'\left(\frac{l}{d}\right)\left(\frac{\pi}{4}\,d^2\right)=pDl \quad ...(2)$$

$$2\,f_c'\times t+2\,f_w'\times\frac{\pi d}{4}=pD$$

$$2\times f_c'\times10+2\times f_w'\times\frac{\pi\times1{\cdot}5}{4}=18\times100 \quad ...(3)$$

$$20f_c'+2{\cdot}3562\,f_w'=1800$$

or
$$f_c'+0{\cdot}1178\,f_w'=90 \quad ...(4)$$

Equating the strains in cylinder and wire

$$\frac{f_c'}{E_c}-\frac{1}{m}\frac{f_a'}{E_c}+\alpha_{brass}\times80=\frac{f_w'}{E_w}+\alpha_{steel}\times80 \quad ...(5)$$

where
$$f_a'=\text{axial stress}$$

$$=\frac{pD}{4t}=\frac{18\times100}{4\times10}=45\text{ N/mm}^2$$

or
$$\frac{f_c'}{90{,}000}-\frac{0{\cdot}32\times45}{90{,}000}+18{\cdot}6\times10^{-6}\times80=\frac{f_w'}{208000}+11{\cdot}8\times10^{-6}\times80$$

$$f_c'-14{\cdot}4+133{\cdot}92=0{\cdot}4326\,f_w'+84{\cdot}96$$

$$f_c'=0{\cdot}4326\,f_w'-34{\cdot}56$$

Substituting the value of f_c' in equation (2)

$$0{\cdot}4326\,f_w'+0{\cdot}1178\,f_w'-34{\cdot}56=90$$

$$fw'=\frac{124{\cdot}56}{0{\cdot}5504}=226{\cdot}308\text{ N/mm}^2\quad\text{(tension)}$$

(*i*) Final stress in the wire

$$=fw+fw'=226{\cdot}308+50{\cdot}000$$

$$=276{\cdot}308\text{ N/mm}^2\quad\text{(tension).}$$

(*ii*) Final circumferential stress in cylinder $=f_c+f_c'$

where
$$f_c'=0{\cdot}4326\,f_w'-34{\cdot}56$$

$$=0{\cdot}4326\times276{\cdot}308-34{\cdot}56$$

$$=84{\cdot}9708\text{ N/mm}^2.$$

Final circumferential stress in cylinder

$$=84{\cdot}9708-5{\cdot}8905=79{\cdot}0803\text{ N/mm}^2.$$

Say the radial pressure between the wire and cylinder $= p_r$

(This is due to wire winding and temperature rise)

Final circumferential stress

$$= \frac{pD}{2t} - \frac{p_r D_0}{2t}$$

$$79.0803 = \frac{18 \times 100}{2 \times 10} - \frac{p_r \times 120}{2 \times 10}$$

or $\quad 6p_r = 90 - 79.0803$

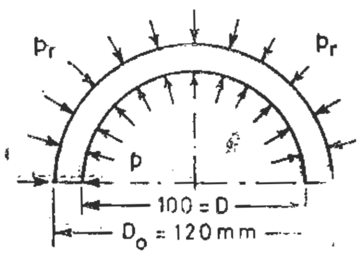


Fig. 5·15

Final radial pressure between wire and cylinder.

$$p_r = \frac{10.9197}{6} = 1.82 \text{ N/mm}^2.$$

Problem 5·15. A bronze sleeve of 20 cm internal diameter and 6 mm thick is pressed over a steel liner 20 cm external diameter and 15 mm thick with a force fit allowance of 0·08 mm on diameter. Considering both the bronze sleeve and steel liner as thin cylinders, determine :

(*a*) radial pressure at the common radius, (*b*) hoop stresses in both, (*c*) the percentage of fit allowance met by the sleeve.

Given $E_{bronze} = 120{,}000$ N/mm²

Poisson's ratio for bronze $= 0.33$

$E_{steel} = 208{,}000$ N/mm²

Poisson's ratio for steel $= 0.30$.

Solution.

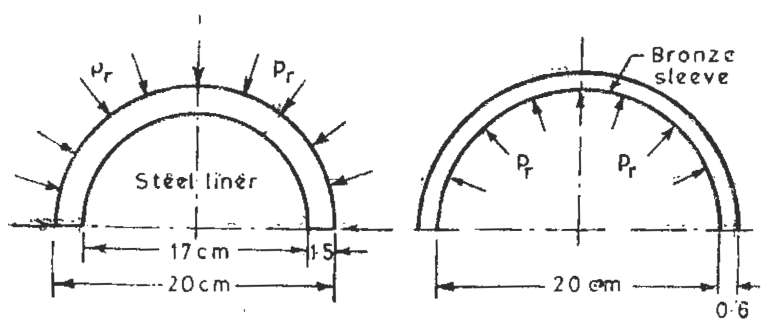


Fig. 5·16

Say the radial pressure at the common surface

$= p_r$ N/mm².

Bronze sleeve. Circumferential stress,

$$f_{cb} = \frac{p_r D}{2t_b} \quad \text{tensile}$$

where D = Inner dia of sleeve = 200 mm

t_b = wall thickness of bronze sleeve.

So $$f_{cb}=\frac{p_r\times 200}{2\times 6}=16\cdot667\ p_r\ \text{N/mm}^2\quad \text{(tensile)}$$

Steel liner. Circumferential stress,

$$f_{cs}=\frac{p_r\ D}{2t_s}$$

where t_s = wall thickness of steel liner.

$$f_{cs}=+\frac{p_r\times 200}{2\times 15}=+6\cdot667\ p_r\quad \text{(compressive)}$$

Now radial strain in bronze sleeve

$$=\frac{f_{cb}}{E_b}+\left(\frac{1}{m}\right)_{bronze}\times\frac{p_r}{E_b}$$

$$=\frac{16\cdot667\ p_r}{E_b}+\frac{0\cdot330\ p_r}{E_b}=\frac{16\cdot997\ p_r}{E_b}\qquad \ldots(1)$$

Radial strain in steel liner

$$=\frac{f_{cs}}{E_s}-\left(\frac{1}{m}\right)_s\frac{p_r}{E_s}$$

$$=\frac{6\cdot667\ p_r}{E_s}-\frac{0\cdot30\ p_r}{E_s}=\frac{6\cdot367\ p_r}{E_s}\qquad \ldots(2)$$

Total radial clearance = 0·04 mm

Common radius = 100 mm.

Therefore $$\left(\frac{16\cdot997\ p_r}{E_b}+\frac{6\cdot367\ p_r}{E_s}\right)\times 100=0\cdot04$$

or $$\frac{16\cdot997\times p_r}{120{,}000}+\frac{6\cdot367\ p_r}{208{,}000}=0\cdot0004$$

$$p_r\ (1\cdot4164+0\cdot3061)=4\qquad \ldots(3)$$

$$p_r=\frac{4}{1\cdot7225}$$

(*a*) Radial pressure at common radius $=2\cdot322\ \text{N/mm}^2$

(*b*) Hoop stress in sleeve $=16\cdot667\ p_r=2\cdot322\times 16\cdot667$

$=38\cdot70\ \text{N/mm}^2$ (tensile)

Hoop stress in liner $=6\cdot667\ p_r=15\cdot480\ \text{N/mm}^2$ (compressive)

% of fit allowance of sleeve $$=\frac{1\cdot4164}{1\cdot4164+0\cdot3061}\times 100=82\cdot23\%$$

(From equation (3) above).

Problem 5·16. A brass hoop of 40 cm inside diameter and 1 cm wall thickness fits snugly at 180°C over a steel hoop which is 1·5 cm thick. Both the hoops are 5 cm wide. If the temperature drops to 20°C, determine the circumferential stress in each hoop and the radial pressure at the common radius.

$$E_{steel}=20\times10^5 \text{ kg/cm}^2$$
$$E_{brass}=10\times10^5 \text{ kg/cm}^2$$
$$\alpha_{steel}=12\times10^{-6}/°\text{C}$$
$$\alpha_{brass}=20\times10^{-6}/°\text{C}.$$

Solution. Say radial pressure at common radius $=p$ kg/cm²

Temperature drop $=180-20=160°$C

Inside diameter of brass hoop, $D=40$ cm

Wall thickness of brass hoop $t_b=1$ cm

f_{cb}, Hoop stress in brass hoop $=\dfrac{pD}{2t_b}=\dfrac{p\times40}{2\times1}=20\,p$ kg/cm² (tensile)

Outside diameter of steel hoop $=40$ cm

Wall thickness of steel hoop $t_s=1{\cdot}5$ cm

f_{cs}, Hoop stress in steel hoop

$$=\frac{pD}{2t_s}=\frac{p\times40}{2\times1{\cdot}50}$$

$=13{\cdot}33\,p$ kg/cm² (compressive)

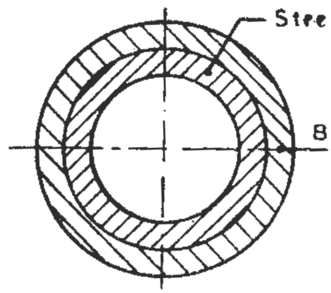


Fig. 5·17

Now, $\left(\dfrac{f_{cb}}{E_b}+\dfrac{f_{cs}}{E_s}\right)D$

$$=(20-12)\times10^{-6}\times160$$

$$\frac{20\,p}{10\times10^5}+\frac{13{\cdot}33\,p}{20\times10^5}$$

$$=8\times160\times10^{-6}$$

$$20\,p+6{\cdot}665\,p=1280$$

$$p=\frac{1280}{26{\cdot}665}$$

Radial pressure $=48{\cdot}00$ kg/cm²

Circumferential stress in brass hoop,

$$f_{cb}=20\,p=960 \text{ kg/cm}^2 \text{ (tensile)}$$

Circumferential stress in steel hoop,

$$f_{cs}=13{\cdot}333\,p=640 \text{ kg/cm}^2 \text{ (compressive).}$$

SUMMARY

1. Additional volume δV of liquid pumped inside the cylinder is equal to the sum of increase in volume of the cylinder δV_1 and decrease in the volume of the liquid δV_2.

2. In a thin cylindrical shell of diameter D, wall thickness t subjected to internal pressure p,

hoop stress, $f_c=\dfrac{pD}{2t}$; axial stress, $f_a=\dfrac{pD}{4t}$

Circumferential strain, $\epsilon_c = \frac{pD}{4tE}\left(2-\frac{1}{m}\right)$, Axial strain, $\epsilon_a = \frac{pD}{4tE}\left(1-\frac{2}{m}\right)$

where E=Young's modulus, $1/m$=Poisson's ratio

Change in diameter, $\delta D = \epsilon_c\, D$, Change in length, $\delta l = \epsilon_a \,.\, l$

Volumetric strain, $\epsilon_v = 2\epsilon_c + \epsilon_a$,

Change in volume of the cylinder, $\delta V_1 = \epsilon_v \,.\, V$

Change in volume of the liquid, $\delta V_2 = \frac{p}{K} \,.\, V$

where K=Bulk modulus,

V=Original volume of cylinder.

3. In a thin spherical shall of diameter D, wall thickness t subjected to internal fluid pressure p

Hoop stress, $f_c = \frac{pD}{4t}$

Circumferential strain $\epsilon_c = \frac{pD}{4tE}\left(1-\frac{1}{m}\right)$

Volumetric strain, $\epsilon_v = 3\epsilon_c$

Change in volume of shell, $\delta V_1 = \epsilon_v \,.\, V$.

4. If a wire of diameter d is wound over a thin cylindrical shell, under tension f_w, f_c, Initial compressive hoop stress in cylinder

$$= \frac{\pi d}{4t} \,.\, f_w$$

where t=wall thickness of cylinder.

If the cylinder is now subjected to internal pressure p and f_c' and f_w' are the tensile hoop stresses developed in cylinder and wire respectively, then

$$pD = f_c' \times 2t + f_w' \times \frac{\pi d}{2}$$

$$\frac{f_c'}{E_c} - \frac{pD}{4tmE_c} = \frac{f_w'}{E_w}$$

where D=diameter of cylinder

E_c, E_w=Young's modulus for cylinder and wire respectively

Resultant stresses in cylinder$=f_c+f_c'$

in wire$=f_w+f_w'$.

MULTIPLE CHOICE QUESTIONS

1. Thin cylindrical shell of dia 100 mm, wall thickness 2·5 mm, is subjected to an internal fluid pressure of 1·5 N/mm². The maximum stress developed in cylinder wall is

(a) 15 N/mm² (b) 30 N/mm²

(c) 60 N/mm² (d) 120 N/mm².

2. A thin cylindrical shell of dia D, wall thickness t is subjected to an internal fluid pressure p. If E is the Young's modulus and $1/m$ is the poisson's ratio for the material of the cylinder, the expression for volumetric strain of the cylinder is

(a) $\frac{pD}{4tE}\left(5-\frac{4}{m}\right)$ (b) $\frac{pD}{4tE}\left(4-\frac{5}{m}\right)$

(c) $\frac{pD}{2tE}\left(5-\frac{4}{m}\right)$ (d) $\frac{pD}{2tE}\left(4-\frac{5}{m}\right)$.

3. A thin spherical shell of diameter 200 mm, wall thickness 5 mm is subjected to an internal fluid pressure p. If the maximum allowable stress in the shell is not to exceed 120 N/mm², the magnitude of p—

(a) 3 N/mm² (b) 6 N/mm²

(c) 12 N/mm² (d) 24 N/mm²

4. A thin spherical shell of diameter D, wall thickness t is subjected to an internal fluid pressure p. If E is the Young's modulus and $1/m$ is the Poisson's ratio for the material of the shell, the expression for the change in diameter is

(a) $\frac{pD^2}{4tE}\left(1-\frac{2}{m}\right)$ (b) $\frac{pD^2}{4tE}\left(2-\frac{1}{m}\right)$

(c) $\frac{pD^2}{4tE}\left(1-\frac{1}{m}\right)$ (d) None of the above.

5. A thin cylindrical steel shell of diameter 400 mm and wall thickness 10 mm has spherical ends. If there is no distortion of the junction at pressure and Poisson's ratio for steel is 1/3 the thickness of the hemispherical end will be

(a) 6 mm (b) 5 mm

(c) 4 mm (d) None of the above.

6. A thin cylindrical shell of diameter 250 mm, wall thickness 6 mm is closely wound around its circumference by a 1·5 mm thick steel tape under a tension of 100 N/mm². The circumferential stress developed in the cylinder wall is—

(a) +25 N/mm² (b) −25 N/mm²

(c) +50 N/mm² (d) −50 N/mm².

7. A steam boiler of 150 cm internal diameter is subjected to an internal pressure of 20 kg/cm². If the efficiency of the longitudinal riveted joint is 80% and the maximum tensile stress in the plate section is not to exceed 1250 kg/cm², the thickness of the plate will be

(a) 6·0 cm (b) 3·0 cm

(c) 1·5 cm (d) 0·75 cm.

8. A cylindrical tank 1 m inside diameter and 20 m high is filled with water of specific weight 1000 kg/m³. If the thickness of the tank is 2·5 cm, the maximum stress developed in the wall of the tank is

(a) 40 kg/cm² (b) 20 kg/cm²

(c) 10 kg/cm² (d) 5 kg/cm².

9. A thin cylindrical shell of volume 2000 cm³ is filled with oil at atmospheric pressure. An additional 1 c.c. of oil is pumped inside the cylinder to produce an internal pressure of 10 kg/cm². If the effect of the expansion of the cylinder is neglected, then modulus of compressibility of water is

(a) 200 kg/cm² (b) 2000 kg/cm²

(c) 20,000 kg/cm² (d) 200,000 kg/cm².

10. A closed pressure vessel of length 40 cm, wall thickness 5 mm internal diameter 10 cm is subjected to an internal pressure of 80 kg/cm². The normal stress in an element of the cylinder on a plane at 30° to the longitudinal axis will be
    (*a*) 1400 kg/cm² (*b*) 700 kg/cm²
    (*c*) 350 kg/cm² (*d*) None of the above.

ANSWERS

1. (*b*)	2. (*a*)	3. (*c*)	4. (*c*)	5. (*c*)
6. (*b*)	7. (*c*)	8. (*a*)	9. (*c*)	10. (*b*).

EXERCISES

5·1. A steam boiler 1 m internal diameter is subjected to an internal pressure of 1 N/mm². What will be the tension in the boiler per linear cm of the longitudinal joint of the boiler shell.

Calculate the thickness of the plate if the maximum tensile stress in the plate section is not to exceed 120 N/mm² taking the efficiency of the longitudinal riveted joint as 80%.
[**Ans.** 20000 N, 10·417 mm]

5·2. A cylindrical tank 3 m inside diameter and 24 m high is filled with oil of specific weight 9000 N/m³. The material of the tank is a structural steel with a yield strength of 300 N/mm². What is the necessary thickness at the bottom of the steel tank ; if the efficiency of the longitudinal seam is 75% ? Take factor of safety as 5. [**Ans.** 8·64 mm]

5·3. To what depth would a copper float 250 mm diameter and 3 mm thick has to be sunk in sea water in order that its diameter is reduced by 0·012% ?

E for copper = 105000 N/mm²

$1/m$ for copper = 0·32

Density of sea water = 10250 N/m³. [**Ans.** 86·77 metres]

5·4. A thin copper pipe 100 mm internal diameter, 2·5 mm wall thickness and 200 cm long is closed at the ends with plugs. The pipe is filled with water under pressure. Determine the increase in pressure when an additional 20 cm³ of water is pumped into the pipe.

E for copper = 105,000 N/mm²

$1/m$ for copper = 0·32

K for water = 2100 N/mm² [**Ans.** 1·53 N/mm²]

5·5. The dimensions of a copper cylinder are length 2·5 m, internal diameter 200 mm and wall thickness 8 mm. The cylinder is initially filled with water at atmospheric pressure. Considering this to be a thin cylinder, find the increase in volume when the water is pumped in so as to raise the internal pressure to 3 N/mm². If the quantity of water which has to be pumped in is 170 c.c. determine the modulus of compressibility of water. Neglect the deformation at the ends. E_{copper} = 105000 N/mm², $1/m$ = 0·32. [**Ans.** 52·173 cc, 2001 N/mm²]

5·6. A steel tube having a bore of 150 mm, wall thickness 2 mm is plugged at each end to form a closed cylinder with internal length of 400 mm. The tube is completely filled with oil and is subjected to a compressive force of 40 kN. Determine

(*a*) the pressure produced on oil

(*b*) the resulting circumferential stress in the tube wall.

Given K for oil $=2200$ N/mm^2

E for steel$=210000$ N/mm^2

$1/m$ for steel$=0{\cdot}3$ **[Ans.** 0·112 N/mm^2, 4·2 N/mm^2]

5·7. The ends of a thin cylindrical steel shell are closed by flat plates. It is subjected to an internal fluid pressure under the following conditions—

(*i*) The ends are free to move axially (along the axis of the cylinder).

(*ii*) The ends are rigidly stayed and no axial movement is permitted.

Determine the ratio of the increase in volume of the shell under the above conditions. Take Poisson's ratio of steel$=0{\cdot}30$. **[Ans.** 1·044]

5·8. A closed pressure vessel of length 1 m, thickness 4 mm and internal diameter 160 mm is subjected to an internal pressure of 10 N/mm^2. Determine the normal and shear stresses in an element of the cylinder wall on a plane at 60° to the longitudinal axis of the cylinder. **[Ans.** 125 N/mm^2, 43·3 N/mm^2]

5·9. A thin spherical shell made of copper is of 0·5 m diameter with 5 mm wall thickness. It is full of water at atmospheric pressure. Find by how much the internal pressure will increase if 25 cm^3 of water is pumped inside the shell Take $E=205{,}000$ N/mm^2

$1/m=0{\cdot}30$, for the material of the shell.

Bulk modulus of water$=2100$ N/mm^2. **[Ans.** 0·52 N/mm^2]

5·10. A steel tube 55 mm bore and 2·5 mm thick is plugged at its ends. It is just filled with water at atmospheric pressure. If an axial compressive load of 2 tonnes is applied to the plugs find by how much the water pressure will increase. The plugs are assumed to be rigid and fixed to the tube.

$E_{steel}=2000$ tonnes/cm^2, $1/m$ for steel$=0{\cdot}3$

Bulk modulus for water, $K=21000$ kg/cm^2. **[Ans.** 1·783 kg/cm^2]

5·11. A solid cylindrical piece of copper 8 cm long and 4 cm diameter is enclosed within a hollow pressure vessel. With the piece inside the vessel, 500 c.c of oil is required just to fill the pressure vessel. Measurement shows that 1·5 c.c. of oil has to be pumped into the vessel to raise the oil pressure to 60 kg/cm^2.

The experiment is repeated nsing the same pressure vessel, and oil but without the test piece inside the pressure vessel. This time after initially filling the pressure vessel, a further 1·65 c.c. of oil is needed to raise the pressure to 60 kg/cm^2, Find the bulk modulus of oil.

E for copper$=1050$ tonnes/cm^2 $1/m$ for copper$=0{\cdot}32$.

[Ans. 23068 kg/cm^2]

5·12. A gun-metal tube of 60 mm bore and wall thickness 1·5 mm is closely wound externally by a steel wire of 1 mm diameter. Determine the tension under which the wire must be wound on the tube, if an internal radial pressure of 2 N/mm^2 is required before the tube is subjected to the tensile stress in the circumferential direction.

E for gun metal$=102\times10^3$ N/mm^2, $1/m$ for gun metal$=0{\cdot}35$,

E for steel$=210\times10^3$ N/mm^2. **[Ans.** 43·69 N/mm^2]

5·13. A thin cylinder made of bronze 30 cm internal diameter and 6 mm thick is wound with a single layer of steel tape 1 mm thick under a tensile stress of 150 N/mm^2. Find the maximum internal pressure if the hoop stress in the cylinder is not to exceed 120 N/mm^2. Determine also the final stress in the steel tape.

Given Poisson's ratio for bronze$=0{\cdot}33$.

E for bronze$=117000$ N/mm^2, E for steel$=208{,}000$ N/mm^2.

[Ans. 7·15 N/mm^2 ; 352·96 N/mm^2]

5·14. A copper sleeve of 15 cm internal diameter and 5 mm thick is pressed over a steel liner 15 cm external diameter and 2·0 cm thick with a force fit allowance of 0·05 mm on diameter. Considering both the copper sleeve and steel liner as thin cylinders, determine

(*a*) radial pressure at the common radius

(*b*) hoop stresses in both

(*c*) the percentage of fit allowance met by the sleeve.

Given $E_{copper}=102{,}000$ N/mm²,

Poisson's ratio for copper $=0·32$

$E_{steel}=208{,}000$ N/mm²

Poisson's ratio for steel $=0·30$.

[**Ans.** 1 N/mm², +15 N/mm², −3·75 N/mm², 90%]

5·15. A bronze cylinder 100 mm outside diameter and 5 mm wall thickness is strengthened by a single layer of steel wire 1 mm diameter wound over it under a constant stress of 105 N/mm². If the cylinder is then subjected to an internal pressure of 20 N/mm² with rise in temperature of the cylinder by 50°C. Determine the final values of

(*i*) stress in wire

(*ii*) radial pressure between the wire and the cylinder

(*iii*) circumferential stress in the cylinder wall

The cylinder can be assumed to be a thin shell with closed ends. Given

$E_{steel}=208{,}000$ N/mm², $E_{bronze}=104{,}000$ N/mm²,

$\alpha_{steel}=12\times10^{-6}/°C$ $\alpha_{bronze}=12\times10^{-6}/°C$.

Poisson's ratio for bronze $=0·32$.

[**Ans.** (*i*) 367·373 N/mm², (*ii*) 7·272 N/mm², (*iii*) 107·276 N/mm²]

5·16. A thin steel cylinder of inner diameter 42 mm and outer diameter 44 mm just fits over a copper cylinder of inner diameter 40 mm. Find the tangential stress in each cylindrical shell due to a temperature rise of 60°F.

Neglect the effects introduced by longitudinal expansion.

Given $E_{steel}=208{,}000$ N/mm², $E_{copper}=90{,}000$ N/mm²

$\alpha_{steel}=6·8\times10^{-6}/°F$ $\alpha_{copper}=9·3\times10^{-6}/°F$.

[**Ans.** −9·408 N/mm², +9·408 N/mm²]

6

Thick Cylinders

In the last chapter on thin shells, we determined the hoop stress in the section of the shell on the assumption that the stress remains constant across the thickness of the shell. For thin shells the ratio of D/t is large, or in other words the thickness is much smaller than the diameter ; variation of the hoop stress along the thickness is neligible and one can safely assume uniform hoop stress. But when thickness is considerable as in the case of thick shells, hoop stress can not be assumed uniform along the thickness and expression for hoop stress is derived which shows that the stress varies along the radial direction of the shell.

6·1. LAME'S EQUATIONS

Stresses in the section of the thick cylinder are determined on the basic assumption that sections which are perpendicular to the longitudinal axis of the cylinder before the application of the internal fluid pressure remain perpendicular to the axis of the cylinder after the cylinder is subjected to internal fluid pressure. Consider a cylinder of inner radius R_1, outer radius R_2 closed at the ends. This is filled with fluid at atmospheric pressure. Now additional

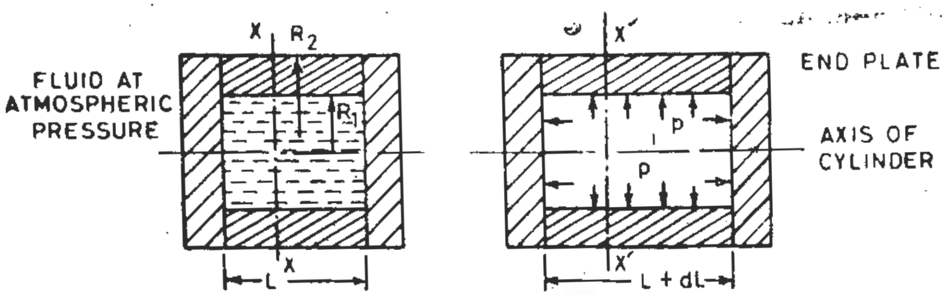


Fig. 6·1

fluid is pumped inside the cylinder so as to develop the internal fluid pressure p as shown in Fig. 6·1. A plane section $X-X$ perpendicular to the axis is shifted to the new position $X'-X'$ after the cylinder is subjected to internal pressure p. $X'-X'$ is also one plane perpendicular to the axis of the cylinder.

Considering the overall length, there is increase in length dl which is uniform throughout irrespective of the radius. This assumption also means that there is no distortion of the end plates. Thus the axial strain in the cylinder is the same at any radius of the cylinder.

Consider a transverse section of the cylinder (as shown in Fig. 6·2) subjected to internal fluid pressure p. This pressure acts radially on the inner surface of the cylinder and at the

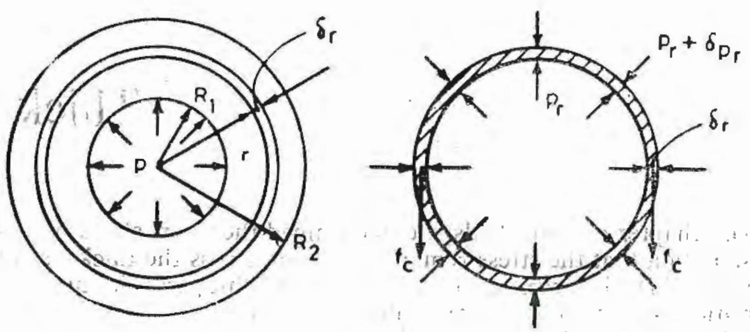


Fig. 6·2

outer surface of the cylinder this radial pressure is zero, showing there by that radial stress (or pressure) varies across the thickness of the cylinder.

When the cylinder is subjected to internal pressure p, it will try to expand the cylinder resulting in increase in length and increase in diameter. *i.e.*, axial and circumferential stresses are developed in the wall of the cylinder and both these stresses are tensile. Now consider a small elementary ring of radial thickness δr at a radius r from the axis of the cylinder say the stresses on an element of this ring are

(1) f_c, circumferential stress (tensile)

(2) f_a, axial stress (tensile)

(3) p_r at radius r and $p_r+\delta p_r$ at radius $r+\delta r$ as shown in Fig. 6·3, radial stress (compressive).

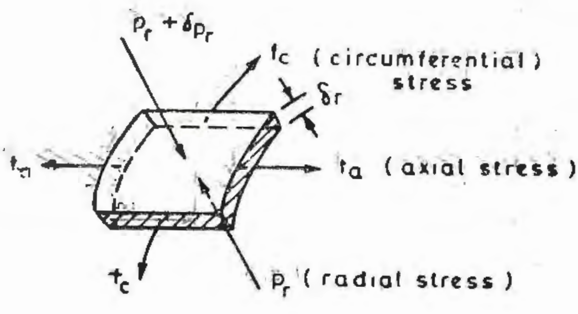


Fig. 6·3

Taking E as the Young's modulus and $\frac{1}{m}$ as the Poisson's ratio of the material of the cylinder.

Axial strain $\qquad = \frac{f_a}{E} - \frac{f_c}{mE} + \frac{p_r}{mE}$

$$= \frac{1}{E}\left[f_a - \frac{1}{m}\left(f_c - p_r \right)\right] \quad \text{...(1)}$$

where $\quad f_a$, axial stress $\quad = \dfrac{p\times \pi R_1^2}{\pi(R_2^2-R_1^2)} = \dfrac{pR_1^2}{R_2^2-R_1^2}$

i.e., f_a is constant across the thickness of the cylinder

E and $\dfrac{1}{m}$ are elastic constants of the material

So $\quad (f_c-p_r)=$a constant

$$=2A \text{ (say).} \qquad \ldots(2)$$

Now let us consider the equilibrium of the elementary ring under consideration.

The bursting force per unit length

$$=p_r\,.\,2\,.\,r-2\,(p_r+\delta p_r)(r+\delta r)$$

$$=2f_c\times\delta r$$

or

$$p_r\,.\,2r-2p_r\,.\,r-2\delta p_r\,r-2p_r\delta r-2\delta p_r\delta r$$

$$=2f_c\,.\,\delta r$$

where $\quad 2\delta p_r\delta r$ is a negligible term.

So $\quad -\delta p_r\,.\,r-p_r\,.\,\delta r=f_c\,.\,\delta r$

or

$$\frac{-r\delta p_r}{\delta r}=f_c+p_r. \qquad \ldots(3)$$

From equation (2) and (3) *i.e.* subtracting equation (2) from equation (3)

$$2p_r=-r\frac{\delta p_r}{\delta r}-2A$$

$$2A+2p_r=-r\,\frac{\delta p_r}{\delta r}$$

or

$$-2\frac{\delta r}{r}=\frac{\delta p_r}{A+p_r}$$

or in the limits

$$-2\,\frac{d_r}{r}=\frac{dp_r}{(A+p_r)}\,. \qquad \ldots(4)$$

Integrating both the sides

$$l_n(A+p_r)=-2l_n\,r+l_n\,B$$

where $\quad l_n\,B$ is a constant of integration

$$=l_n\,\frac{B}{r^2}$$

or

$$p_r+A=\frac{B}{r^2}$$

Radial stress, $\quad p_r=\dfrac{B}{r^2}-A. \qquad \ldots(5)$

From equation (2) circumferential stress, or hoop stress

$$f_c=\frac{B}{r^2}+A. \qquad \ldots(6)$$

In these equations A and B are called Lame's constants and these equations are called the Lame's equations.

The values of the constants are determined by using boundary conditions. Note that units of B will be those of force and units of A will be those of stress.

Boundary conditions

At radius $r=R_1$, radial pressure$=p$

At radius $r=R_2$, radial pressure$=0$

Using these conditions in equation (5)

So $$p=\frac{B}{R_1^2}-A$$

$$0=\frac{B}{R_2^2}-A$$

or $$A=\frac{B}{R_2^2}$$

$$p=\frac{B}{R_1^2}-\frac{B}{R_2^2}$$

or $$B=p\,\frac{R_1^2R_2^2}{R_2^2-R_1^2}. \qquad \ldots(7)$$

and $$A=\frac{B}{R_2^2}=\frac{p\,R_1^2}{R_2^2-R_1^2}. \qquad \ldots(8)$$

The expressions for f_c and p_r can now be written as

$$f_c=\frac{p}{r^2}\cdot\frac{R_1^2R_2^2}{R_2^2-R_1^2}+\frac{p\,R_1^2}{R_2^2-R_1^2}. \qquad \ldots(9)$$

$$p_r=\frac{p}{r^2}\times\frac{R_1^2R_2^2}{R_2^2-R_1^2}-\frac{p\,R_2^2}{R_2^2-R_1^2}. \qquad \ldots(10)$$

As the obvious from equation (9) f_c will be maximum at inner radius R_1 and is minimum at outer radius R_2

$$f_{c\,max}=\frac{p}{R_1^2}\times\frac{R_1^2R_2^2}{R_2^2-R_1^2}+\frac{p\,R_1^2}{R_2^2-R_1^2}$$

$$=p\left(\frac{R_2^2+R_1^2}{R_2^2-R_1^2}\right)\text{ tensile}$$

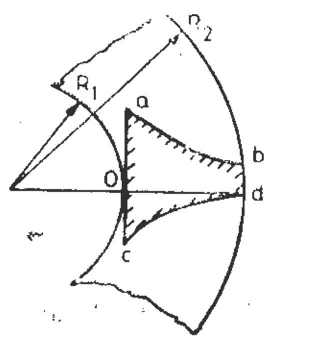
$$f_{c\,min}=\frac{p}{R_2^2}\times\frac{R_1^2R_2^2}{R_2^2-R_1^2}+\frac{p\,R_1^2}{R_2^2-R_1^2}$$

$$=p\left(\frac{2R_1^2}{R_2^2-R_1^2}\right)\text{ tensile}$$

Similarly p_r is maximum at inner radius R_1 and minimum at outer radius R_2

$p_{r\,max}=p$ compressive

$p_{r\,min}=0$ compressive



Stress distribution across cylinder thickness

Fig. 6·4

Fig. 6·4 shows the variation of f_c and p_r along the thickness of the cylinder

$$oa=f_{c\ max}$$
$$bd=f_{c\ min}$$
$$oc=p$$

Example 6·1-1. A cylindrical shell of inner radius 60 mm and outer radius 100 mm is subjected to an internal fluid pressure of 64 N/mm². Draw the distribution of stresses f_c and p_r along the thickness of the cylinder.

Solution. Inner radius, $R_1=60$ mm

Outer radius, $R_2=100$ mm.

Boundary conditions. $p_r=64$ N/mm² at $r=60$ mm

$=0$ at $r=100$ mm

or
$$\frac{B}{60^2}-A=64$$

$$\frac{B}{100^2}-A=0, \qquad A=\frac{B}{100^2}$$

$$\therefore \quad 64=\frac{B}{60^2}-\frac{B}{100^2}\ ; \quad B=\frac{64\times 60^2\times 100^2}{100^2-60^2}=360000 \text{ N}$$

$$A=36 \text{ N/mm}^2.$$

Circumferential stress. $f_c=\frac{B}{r^2}+A=\frac{360000}{r^2}+36$

At radius 60 mm, $f_{c\ 60}=\frac{360000}{(60)^2}+36=136$ N/mm²

Similarly $f_{c\ 70}=\frac{360000}{(70)^2}+36=109·47$ N/mm²

$$f_{c\ 80}=\frac{360000}{(80)^2}+36=92·25 \text{ N/mm}^2$$

$$f_{c\ 90}=\frac{360000}{(90)^2}+36=80.44 \text{ N/mm}^2$$

$$f_{c\ 100}=\frac{360000}{(100)^2}+36=72 \text{ N/mm}^2$$

Radial stress

$$p_r=\frac{B}{r^2}-A=\frac{360000}{r^2}-36$$

$$p_{r\ 60}=\frac{360000}{(60)^2}-36=64 \text{ N/mm}^2$$

$$p_{r\ 70}=\frac{360000}{(70)^2}-36=37·47 \text{ N/mm}^2$$

$$p_{r\ 80}=\frac{360000}{(80)^2}-36=20·25 \text{ N/mm}^2$$

$$p_{r\ 90}=\frac{360000}{(90)^2}-36=8·44 \text{ N/mm}^2$$

$$p_{r\ 100}=\frac{360000}{(100)^2}-36=0.$$

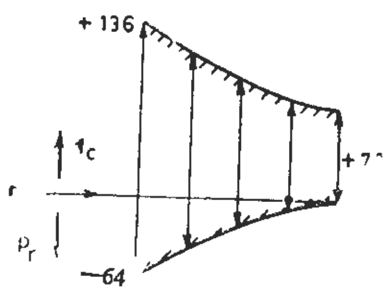


Fig. 6·5

The Fig. 6·5 shows the distribution of circumferential and radial stress along the radius 60 mm to 100 mm.

Example 6·1-2. A thick cylinder of inner radius R_1 is subjected to internal fluid pressure p. If the maximum hoop stress developed is 2·5 p, determine the external radius R_2.

Solution. Inner radius $=R_1$

Outer radius $=R_2$

Internal fluid pressure $=p$

Maximum hoop stress, $$f_{c\ max}=2\cdot5\ p=p\ \frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

or $$2\cdot5(R_2^2-R_1^2)=R_2^2+R_1^2$$
$$1\cdot5\ R_2^2=3\cdot5\ R_1^2$$
$$R_2=\sqrt{\frac{7}{3}}\ R_1=1\cdot527\ R_1.$$

Exercise 6·1-1. A cylindrical shell of inner radius 50 mm and outer radius 80 mm is subjected to an internal fluid pressure of 500 kg/cm². Draw the distribution of hoop and radial stresses along the thickness of the cylinder.

[**Ans.** $f_{c\ 50,\ 60,\ 70,\ 80}$=1141·02, 890·31, 739·14, 320·51 kg/cm²
$p_{r\ 50,\ 60,\ 70,\ 80}$=500·0, 249·29, 98·12, 0·00 kg/cm²]

Exercise 6·1-2. A thick cylinder of 120 mm internal diameter is subjected to an internal fluid pressure of 80 N/mm². If the maximum stress developed in the cylinder is not to exceed 270 N/mm², find the thickness of the cylinder. [**Ans.** 21·43 mm]

6·2. THICK CYLINDER SUBJECTED TO EXTERNAL FLUID PRESSURE

If a thick cylindrical shell is subjected to external fluid pressure as shown in Fig. 6·6, the effect of p will be to reduce the diameter of the shell or in other words compressive hoop or the circumferential stress will be developed in the cylinder.

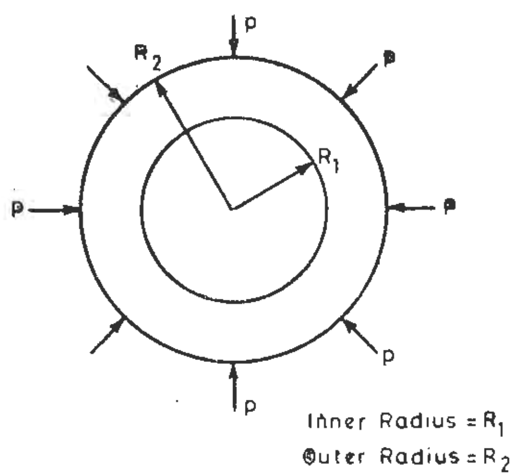


Fig. 6·6

Let us again consider Lame's equations taking A, B constants

$$f_c,\ \text{hoop stress} \qquad =\frac{B}{r^2}+A$$

$$p_r,\ \text{radial stress} \qquad =\frac{B}{r^2}-A.$$

Boundary conditions

At $\qquad r=R_1 \qquad p_r=0$

$\qquad\quad r=R_2 \qquad p_r=p$

or

$$0=\frac{B}{R_1^2}-A$$

$$p=\frac{B}{R_2^2}-A$$

From these equations $B=-p\,\dfrac{R_1^2R_2^2}{R_2^2-R_1^2}$

$$A=-p\,\frac{R_2^2}{R_2^2-R_1^2}.$$

Using these values of the constants, hoop stress can be determined

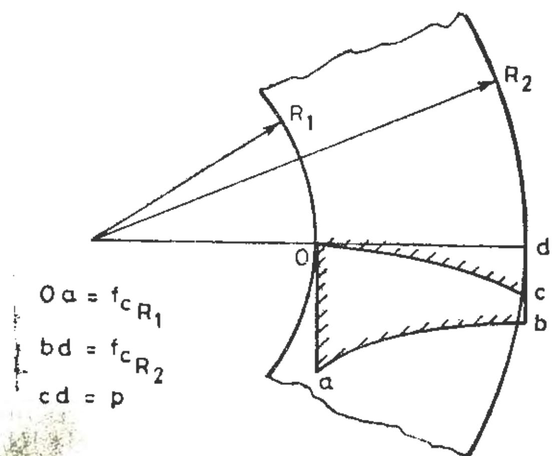
$$f_c=-\frac{p}{r^2}\,\frac{R_1^2R_2^2}{R_2^2-R_1^2}-p\,\frac{R_2^2}{R_2^2-R_1^2}$$

At $\qquad r=R_1,\ f_{c\,R_1}=-\dfrac{2p\,R_2^2}{R_2^2-R_1^2}$

At $\qquad r=R_2\ .\ f_{c_{R_2}}=-p\left(\dfrac{R_2^2+R_1^2}{R_2^2+R_1^2}\right)$.

This shows that magnitude of $f_{c_{R_1}}$ is greater than the magnitude of $f_{c_{R_2}}$.

Fig. 6·7, shows the distribution of hoop and radial stresses across the thickness of the cylinder. Both the stresses are compressive. *oabd*—stress distribution for hoop stress while *ocd*-stress distribution for radial stress.



Stress distribution along the cylinder thickness

Fig. 6·7

Example 6·2-1. A thick cylinder with inner diameter 14 cm and outer diameter 20 cm is subjected to a pressure of 200 kg/cm² on its outer surface. Determine the maximum values of the hoop stress developed.

Solution. Inner radius, $R_1=7$ cm

Outer radius, $R_2=10$ cm

External pressure, $p=200$ kg/cm²

Maximum hoop stress, $$f_{c_{R_1}}=-p\frac{2R_2^2}{R_2^2-R_1^2}$$

$$=-200\times\frac{2\times10^2}{10^2-7^2}=-784{\cdot}31 \text{ kg/cm}^2$$

Minimum hoop stress, $$f_{c_{R_2}}=-p\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

$$=-200\times\frac{10^2+7^2}{10^2-7^2}=-584{\cdot}31 \text{ kg/cm}^2$$

Example 6·2-2. A thick cylinder with external diameter 240 mm and internal diameter D is subjected to an external pressure of 56 N/mm². Determine the diameter D if the maximum hoop stress in the cylinder is not to exceed 220 N/mm².

Solution. External radius, $R_2=120$ mm

Internal radius, $R_1=\frac{D}{2}$

External pressure, $p=56$ N/mm²

Since maximum hoop stress occurs at inner radius and is compressive so

$$f_{c_{R_1}}=-220 \text{ N/mm}^2$$

$$=-p\times\frac{2R_2^2}{R_2^2-R_1^2}$$

$$-220=-56\times\frac{2\times120^2}{120^2-R_1^2}$$

$$\frac{220}{56}(120^2-R_1^2)=2\times120^2$$

$$R_1=84{\cdot}08 \text{ mm}$$

Diameter, $D=168{\cdot}16$ mm.

Exercise 6·2-1. A thick cylinder with inner diameter 110 mm and outer diameter 200 mm is subjected to an external fluid pressure of 60 MN/m². Determine the maximum and minimum hoop stresses developed in the cylinder. [**Ans.** $-172{\cdot}04$ MN/m², $-112{\cdot}04$ MN/m²]

Exercise 6·2-2. A thick cylinder with internal diameter 22 cm is subjected to an external pressure of 300 kg/cm². Determine the external diameter of the cylinder, if the maximum hoop stress in the cylinder is not to exceed 800 kg/cm². [**Ans.** 44 cm]

6·3. COMPOUND CYLINDERS

In the article 6·1, Fig. 6·4 we observe that maximum hoop stress occurs at the inner radius of the cylinder and the stress varies across the thickness of the cylinder *i.e.*, whole of the material, is not put to use uniformly. The pressure bearing capacity of the cylinder is

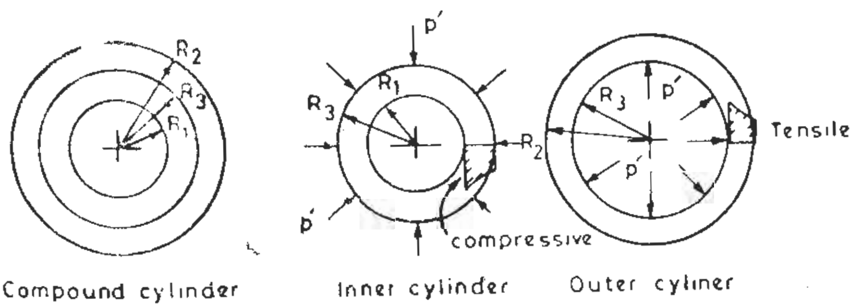


Fig. 6·8

limited by this maximum hoop stress which should not exceed the allowable stress for the material. Firstly to increase the pressure bearing capacity of the cylinder and secondly to reduce the variation in hoop stress across the thickness, two cylinders are compounded together. One cylinder is shrink fitted over another cylinder developing compressive hoop stress in the inner cylinder and tensile hoop stress in the outer cylinder. When this compound cylinder is subjected to an internal fluid pressure, hoop stress at the inner radius developed due to internal fluid pressure is tensile while the hoop stress due to shrink fitting is compressive. Thus the resultant stress at the inner radius is less than the hoop stress developed due to internal pressure only and consequently the pressure bearing capacity of the cylinder is increased, if the allowable stress in the cylinder remains the same as in a single cylinder.

Consider a compound cylinder as shown in Fig 6·8. A cylinder of inner radius R_1 and outer radius say R_3' is compounded with another cylinder of inner radius say R_3'' and outer radius R_2. Initially the inner radius of outer cylinder *i.e.*, R_3'' is smaller than the outer radius of inner cylinder *i.e.*, R_3'. The outer cylinder is now heated so that its inner radius becomes equal to the outer radius of the inner cylinder *i.e.*, R_3'' (after heating$=R_3'$) and then outer cylinder is pushed over the inner cylinder. After cooling down to room temperature, the outer cylinder tries to contract and exerts radial pressure over the inner cylinder and the inner cylinder offers equal and opposite reaction. The final radius at the junction of the cylinders is R_3 which is less than R_3' and greater than R_3''. Due to the shrink fitting radial pressure p' acts on the outer surface of the inner cylinder and on the inner surface of the outer cylinder. Let us determine the hoop stresses developed in both the cylinders due to shrinkage fitting. Let us take A_1, B_1 and A_2, B_2 Lame's constants for the inner and outer cylinders respectively.

Inner Cylinder. Boundary conditions :

At $r=R_1$, radial stress $p_r=0=\dfrac{B_1}{R_1^2}-A_1$

$$r=R_3\ ,\ p_r=p'=\frac{B_1}{R_3^2}-A_1$$

From these equations $B_1=-p'\dfrac{R_1^2\,R_3^2}{R_3^2-R_1^2}$

$$A_1=-p'\,\frac{R_3^2}{R_3^2-R_1^2}$$

Circumferential stress, $f_c' = \frac{B_1}{r^2} + A_1 = -\frac{p'}{r^2} \frac{R_1^2 R_3^2}{(R_3^2 - R_1^2)} - p' \frac{R_3^2}{(R_3^2 - R_1^2)}$

At radius R_1, $f_c'{}_{R_1} = -p' \frac{2R_3^2}{R_3^2 - R_1^2}$ compressive

At radius R_3, $f_c'{}_{R_3} = -p' \frac{R_3^2 + R_1^2}{R_3^2 - R_1^2}$ compressive.

Outer Cylinder. Boundary conditions :

At $r = R_3$, radial stress, $p_r = p' = \frac{B_2}{R_3^2} - A_2$

At $r = R_2$, radial stress, $p_r = 0 = \frac{B_2}{R_2^2} - A_2$

From these equations $B_2 = +p' \frac{R_2^2 R_3^2}{R_2^2 - R_3^2}$

$$A_2 = +p' \frac{R_3^2}{R_2^2 - R_3^2}.$$

Circumferential stress $f_c'' = \frac{B_2}{r^2} + A_2 = \frac{p'}{r^2} \frac{R_2^2 R_3^2}{(R_2^2 - R_3^2)} + p' \frac{R_3^2}{(R_2^2 - R_3^2)}$

At radius R_3, $f_c''{}_{R_3} = p'\left(\frac{R_2^2 + R_3^2}{R_2^2 - R_3^2}\right)$ tensile

At radius R_2, $f_c''{}_{R_2} = p'\left(\frac{2R_3^2}{R_2^2 - R_3^2}\right)$ tensile

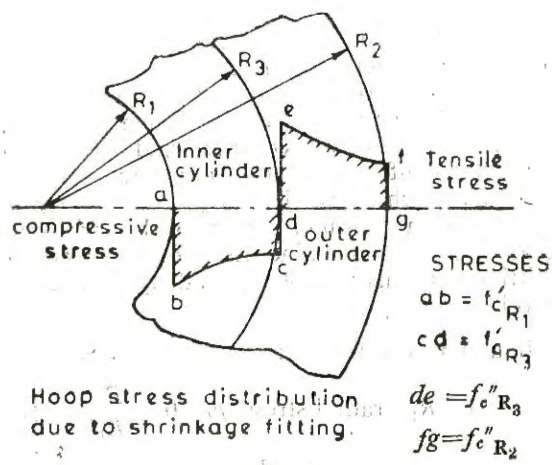


Fig. 6·9

Fig. 6·9 shows the distribution of hoop stress in the inner and outer cylinders across the thickness.

Now the compound cylinder is subjected to the internal fluid pressure p. Let us take A and B as Lame's constants.

Boundary conditions are $p_r = p$ at $r = R_1$, inner radius of compound cylinder

$p_r = 0$ at $r = R_2$, outer radius of compound cylinder.

or
$$p = \frac{B}{R_1^2} - A$$
$$o = \frac{B}{R_2^2} - A$$

or Constants
$$B = \frac{p\,R_1^2 R_2^2}{R_2^2 - R_1^2},\quad A = \frac{p\,R_1^2}{R_2^2 - R_1^2}$$

Hoop stress at any radius,
$$f_c = \frac{B}{r^2} + A$$
$$= \frac{p}{r^2} \times \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} + \frac{p\,R_1^2}{R_2^2 - R_1^2}$$

Hoop stress at inner radius R_1,
$$f_{cR_1} = \frac{p}{R_1^2} \times \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} + \frac{p\,R_1^2}{R_2^2 - R_1^2}$$
$$= p\left[\frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}\right]$$

Hoop stress at R_2,
$$f_{cR_3} = \frac{p}{R_3^2} \times \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} + \frac{p\,R_1^2}{R_2^2 - R_1^2}$$
$$= p\,\frac{R_1^2}{R_3^2}\left(\frac{R_2^2 + R_3^2}{R_2^2 - R_1^2}\right)$$

Hoop stress at R_3,
$$f_{cR_2} = \frac{p}{R_2^2} \times \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} + \frac{p\,R_1^2}{R_2^2 - R_1^2}$$
$$= \frac{2p\,R_1^2}{R_2^2 - R_1^2}.$$

Resultant Stresses

Inner cylinder.
$$f^{R}_{cR_1} = f_{cR_1} + f_c'{}_{R_1} = p\left[\frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}\right] - p'\frac{2R_3^2}{R_3^2 - R_1^2}$$
$$f^{R}_{cR_3} = f_{cR_3} + f_c'{}_{R_3} = \frac{p}{R_3^2} \times \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} + \frac{p\,R_1^2}{R_2^2 - R_1^2} - p' \times \frac{R_3^2 + R_1^2}{R_3^2 - R_1^2}.$$

Outer cylinder.
$$f^{R'}_{cR_3} = f_{cR_3} + f_c''{}_{R_3} = \frac{p}{R_3^2} \times \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} + \frac{p\,R_1^2}{R_2^2 - R_1^2} + p'\frac{R_2^2 + R_3^2}{R_2^2 - R_3^2}.$$
$$f^{R}_{cR_2} = f_{cR_2} + f_c''{}_{R_2} = \frac{2pR_1^2}{R_2^2 - R_1^2} + p'\,\frac{2R_3^2}{R_2^2 - R_3^2}.$$

Fig. 6·10 shows the stress distribution of the resultant hoop stress across the thickness of the compound cylinder.

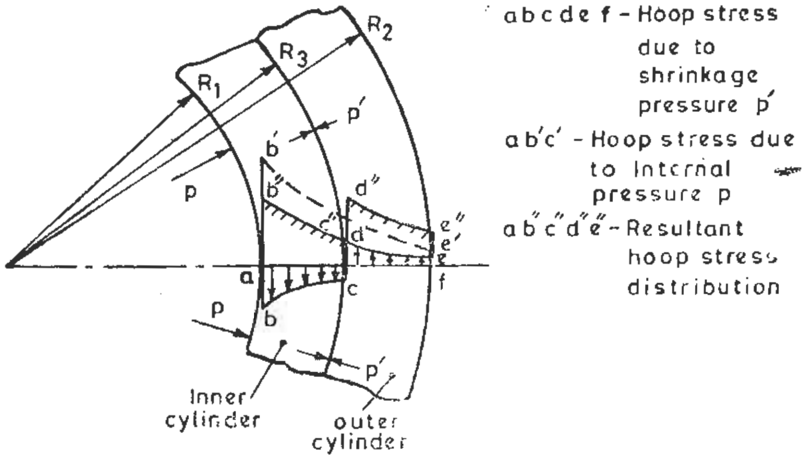


Fig. 6·10. Hoop stress distribution across the thickness of compound cylinder.

Example 6·3-1. A compound cylinder is obtained by shrink fitting of one cylinder of outer diameter 20 cm over another cylinder of inner diameter 14 cm, such that the diameter at the junction of the two cylinders is 17 cm. If the radial pressure developed at the junction is 50 N/mm², what are the hoop stresses at the inner and outer radii of both the cylinders.

Solution. Inner radius, $R_1=70$ mm

Outer radius, $R_2=100$ mm

Junction radius, $R_3=85$ mm

Junction pressure $p'=50$ N/mm².

Inner cylinder. Hoop stress,

$$f_c'{}_{R_1}=-p'\times\frac{2R_3^2}{R_3^2-R_1^3}=-50\times\frac{2\times85^2}{85^2-50^2}$$

$$=-152{\cdot}91 \text{ N/mm}^2.$$

$$f_c'{}_{R_3}=-p'\times\frac{R_3^2+R_1^2}{R_3^2-R_1^5}=-50\times\frac{85^2+50^3}{85^2-50^2}$$

$$=-102{\cdot}91 \text{ N/mm}^2.$$

Outer cylinder.

$$f_c''{}_{R_3}=+p'\times\frac{R_2^2+R_3^2}{R_2^2-R_3^2}=50\times\frac{100^2+85^2}{100^2-85^2}$$

$$=+310{\cdot}36 \text{ N/mm}^2$$

$$f_c''{}_{R_2}=+p'\times\frac{2R_3^2}{R_3^2-R_3^2}=50\times\frac{2\times85^2}{100^2-85^2}$$

$$=+260{\cdot}36 \text{ N/mm}^2.$$

Example 6·3-2. A compound cylinder is made by shrinking a cylinder of outer diameter 200 mm over another cylinder of inner diameter 100 mm. If the numerical value of the maximum hoop stress developed due to shrink fitting in both the cylinders is the same, find the junction diameter.

Solution. Inner radius, $R_1=50$ mm$=5$ cm

Outer radius, $R_2=100$ mm$=10$ cm

Say Junction radius $=R_3$

Junction pressure $=p'$

Maximum hoop stress in inner cylinder

$$=-p'\times\frac{2R_3^2}{R_3^2-R_1^2}$$

Maximum hoop stress in outer cylinder

$$=+p'\frac{R_2^2+R_3^2}{R_2^2-R_3^2}$$

or as given in the problem

$$p'\times\frac{2R_3^2}{R_3^2-R_1^2}=p'\times\frac{R_2^2+R_3^2}{R_2^2-R_3^2}$$

Substituting the values $\frac{2R_3^2}{R_3^2-5^2}=\frac{10^2+R_3^2}{10^2-R_3^2}$

or

$$2R_3^2\times 100-2R_3^4=100R_3^2-25R_3^2+R_3^4-2500$$

$$3R_3^4-125R_3^2-2500=0$$

$$R_3^2=\frac{125+\sqrt{125^2+12\times 2500}}{6}=\frac{125+213\cdot 6}{6}=56\cdot 433$$

$$R_3=7\cdot 512 \text{ cm.}$$

Junction diameter $=2\times 7\cdot 512$

$=15\cdot 024$ cm$=150\cdot 24$ mm.

Exercise 6·3-1. A compound cylinder is made by shrinking a cylinder of outer diameter 240 mm over another cylinder of inner diameter 160 mm such that the junction pressure is 60 N/mm². If the diameter at junction is 200 mm determine the values of hoop stress at the inner and outer radii of both the cylinders.

[**Ans.** $-333\cdot 33$, $-273\cdot 33$, $+332\cdot 73$, $+272\cdot 73$ N/mm²]

Exercise 6·3-2. A compound cylinder is made by shrinking one steel cylinder of outer radius 100 mm over another steel cylinder of inner radius 50 mm. The shrinkage allowance provided is such that the maximum hoop stress developed in both the cylinders is 30% of the yield strength of the material. If the yield strength of steel is 270 N/mm², find

(*i*) junction pressure

(*ii*) wall thickness of both the inner and outer cylinders.

[**Ans.** 60· 6 N/mm² ; 25·12 mm, 24·88 mm]

6·4. SHRINKAGE ALLOWANCE

Before the two cylinders are compounded, the inner radius of the outer cylinder R_3'' is less than the outer radius of the inner cylinder R_3' (as shown in Fig. 6·11). The outer cylinder

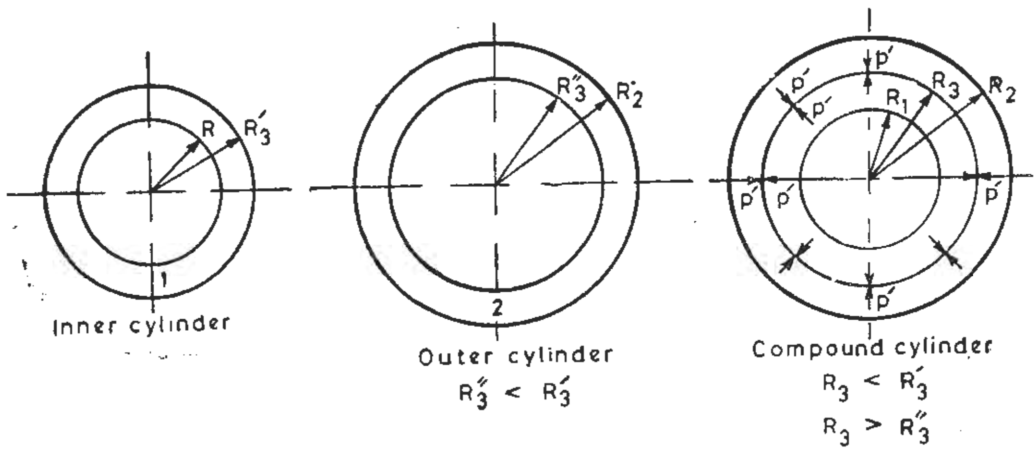


Fig. 6·11

is now heated upto a temperature such that R_3'' increases to R_3' and then it is pushed over the inner cylinder. When the outer cylinder is allowed to cool down to room temperature, it tries to contract exerting compressive radial stress p', on the outer surface of the inner cylinder. The inner cylinder, in turn, offers equal and opposite reaction exerting compressive radial stress p' on the inner surface of the outer cylinder. The final junction radius is R_3 which is smaller than R_3' and greater than R_3''.

Now hoop stress in inner cylinder at R_3,

$$f_c'{}_{R_3} = -p' \frac{R_3^2+R_1^2}{R_3^2-R_1^2}$$

Hoop stress in outer cylinder at R_3,

$$f_c''{}_{R_3} = +p' \frac{R_2^2+R_3^2}{R_2^2-R_3^2}$$

Say E_1=Young's modulus for inner cylinder

$\frac{1}{m_1}$ =Poisson's ratio for inner cylinder

E_2=Young's modulus for outer cylinder

$\frac{1}{m_2}$ =Poisson's ratio for outer cylinder

Then circumferential strain at R_3 in inner cylinder

$$\epsilon_c' = -\frac{p'(R_3^2+R_1^2)}{E_1(R_3^2-R_1^2)} + \frac{p'}{m_1 E_1}$$

Circumferential strain at R_3 in outer cylinder

$$\epsilon_c'' = +\frac{p'}{E_2}\left(\frac{R_2^2+R_3^2}{R_2^2-R_3^2}\right)+\frac{p'}{m_2E_2}$$

$$R_3'-R_3=\epsilon_c'\times R_3,$$

contraction in the outer radius of inner cylinder.

$$R_3-R_3''=\epsilon_c''\times R_3,$$

expansion in the inner radius of outer cylinder.

Now ϵ_c' is the compressive strain and ϵ_c'' is the tensile strain, so

The total shrinkage allowable

$$=\epsilon_c''R_3-\epsilon_c'R_3$$

$$\delta R_3=R_3\left[\frac{p'}{E_2}\left(\frac{R_2^2+R_3^2}{R_2^2-R_3^2}\right)+\frac{p'}{m_2E_2}\right]+R_3\left[\frac{p'}{E_1}\left(\frac{R_3^2+R_1^2}{R_3^2-R_1^2}\right)-\frac{p'}{m_1E_1}\right]$$

In case $E_1=E_2=E$

$$\frac{1}{m_1}=\frac{1}{m_2}=\frac{1}{m}$$

$$\delta R_3=R_3\left[\frac{p'}{E}\left(\frac{R_1^2+R_3^2}{R_2^2-R_3^2}\right)+\frac{p'}{E}\left(\frac{R_3^2+R_1^2}{R_3^2-R_1^2}\right)\right]$$

$$\delta R_3=\frac{R_3p'}{E}\left(\frac{R_2^2+R_3^2}{R_2^2-R_3^2}+\frac{R_3^2+R_1^2}{R_3^2-R_1^2}\right)$$

Or $\dfrac{\delta R_3}{R_3}=\dfrac{1}{E}$ (numerical sum of the hoop stresses at the common surface of two cylinders)

$=\dfrac{1}{E}$ (algebraic difference of the hoop stresses at the common surface of the two cylinders)

Example 6·4-1. A compound cylinder is formed by shrinking one cylinder on to another, the final dimensions being internal diameter 12 cm, external diameter 24 cm and diameter at junction 20 cm. After shrinking on the radial pressure at the common surface is 100 kg/cm². Calculate the necessary difference in diameters of the two cylinders at the common surface. Take E=2000 tonnes/cm². What is the minimum temperature through which outer cylinder should be heated before it can be slipped on? $\alpha=0{\cdot}000011$ per °C.

Solution. Inner radius, $R_1=6$ cm

Outer radius, $R_2=12$ cm

Junction radius, $R_3=10$ cm

Difference in radii, $$\delta R_3=\frac{R_3}{E}\times p'\left(\frac{R_2^2+R_3^2}{R_2^2-R_3^2}+\frac{R_3^2+R_1^2}{R_3^2-R_1^2}\right)$$

$$=\frac{10\times100}{2000\times1000}\left(\frac{12^2+10^2}{12^2-10^2}+\frac{10^2+6^2}{10^2-6^2}\right)$$

$$=\frac{1}{2000}(5{\cdot}545+2{\cdot}125)=3{\cdot}835\times10^{-3}\text{ cm.}$$

Difference in Diameter, $\delta D_3=2\times3{\cdot}835\times10^{-3}=0{\cdot}00767$ cm

Coefficient of linear expansion,

$$\alpha = 11 \times 10^{-6}/°C$$

Say the temperature rise $= T$ °C

$$\delta D_3 = D_3 \times \alpha \times T$$

$$0{\cdot}00767 = 20 \times 11 \times 10^{-6} \times T$$

Or
$$T = \frac{0{\cdot}00767 \times 10^6}{220} = \frac{7670}{220} = 34{\cdot}86 \text{ °C}$$

Example 6·4-2. A cylinder of outside diameter 350 mm is heated by 40°C above the room temperature, before it is slipped onto another cylinder of inside diameter 150 mm. If the junction diameter is 250 mm, what radial pressure is developed at the common surface after the outer cylinder cools down to room temperature.

Given
$$\alpha = 18 \times 10^{-6}/°C$$
$$E = 100 \times 10^3 \text{ N/mm}^2.$$

Solution. Inner radius,

$$R_1 = 75 \text{ mm}$$

Outer radius, $R_2 = 175$ mm

Junction radius, $R_3 = 125$ mm

Diameter difference, $\delta D_3 = D_3 \times \alpha \times T$ where T = temperature rise

$$= 250 \times 18 \times 10^{-6} \times 40 = 0{\cdot}18 \text{ mm}$$

or
$$\delta R_3 = 0{\cdot}09 \text{ mm}$$

Say the radial pressure at common surface

$$= p'$$

$$\delta R_3 = \frac{R_3\, p'}{E}\left[\frac{R_2^2 + R_3^2}{R_2^2 - R_3^2} + \frac{R_3^2 + R_1^2}{R_3^2 - R_1^2}\right]$$

$$0{\cdot}09 = \frac{125\, p'}{100 \times 1000}\left[\frac{175^2 + 125^2}{175^2 - 125^2} + \frac{125^2 + 75^2}{125^2 - 75^2}\right]$$

$$\frac{9000}{125} = p'\,[3{\cdot}083 + 2{\cdot}125]$$

$$p' = \frac{9000}{125 \times 5{\cdot}208} = 13{\cdot}825 \text{ N/mm}^2.$$

Exercise 6·4-1. A compound cylinder is formed by shrinking one cylinder onto another, the final dimensions being internal diameter 100 mm, external diameter 180 mm and diameter at common surface 140 mm, the radial pressure developed at the common surface is 15 N/mm². Calculate the necessary difference in diameters of the two cylinders at the common surface. Take $E = 102000$ N/mm². What is the minimum temperature through which the outer cylinder should be heated before it can be slipped on ? Take

$$\alpha = 17{\cdot}6 \times 10^{-6}/°C$$

[Ans. 0·1471 mm, 59·7°C**]**

Exercise 6·4-2. A cylinder of outside diameter 25 cm, is heated by 35°C above the room temperature before it is slipped on to another cylinder of inside diameter 15 cm. If the diameter at the junction after shrinking is 20 cm, what radial pressure is developed at the common surface ?

$$\alpha = 11 \times 10^{-6}/^\circ C$$

$$E = 2000 \text{ tonnes/cm}^2$$ [**Ans.** 94·757 kg/cm²]

6·5. HUB AND SHAFT ASSEMBLY

Generally the hub of a gear, a pulley or a flywheel is fitted on the shaft with the help of key inserted in the keyways provided on shaft and hub. But a keyway cut on a shaft or a hub reduces its strength, introduces stress concentration and the material becomes weak. To avoid these defects, the hub can be either force fitted or shrink fitted on the shaft as shown in Fig. 6·12.

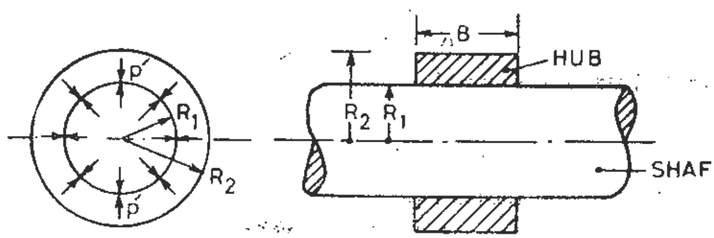


Fig. 6·12

Let us consider that a hub of outer radius R is force fitted over a shaft and the final radius at the junction of the two is r. Due to force fitting or shrink fitting, say the radial pressure developed at the common surface is p'. Assume Lame's constants A_1, B_1 for shaft and A_2, B_2 for hub.

Shaft. Boundary conditions are, at

$$r = R_1, \quad p_r = p' = \frac{B_1}{R_1^2} - A_1.$$

Since it is a solid shaft, *i.e.*, at the centre, radius is zero, and the radial or circumferential stress at the centre cannot be infinite. Therefore,

Constant $\quad B_1 = 0$

$$p' = -A_1, \text{ or } A_1 = -p_1'$$

Circumferential stress at any radius

$$f_c = \frac{B_1}{r^2} + A_1$$

But $\quad B_1 = 0$

$$f_c = +A_1 = -p' \text{ (compressive)}$$

i.e., in the solid shaft, both radial and circumferential stresses are compressive and are constant throughout.

Circumferential strain in shaft at common radius

$$\epsilon_c' = -\frac{p'}{E_1} + \frac{p'}{m_1 E_1} \quad \text{(compressive strain)}$$

where E_1 = Young's modulus of shaft material

$\frac{1}{m_1}$ = Poisson's ratio of shaft material.

Hub. Boundary conditions are at

$$r = R_1, \ p_r = p' = \frac{B_2}{R_1^2} - A_2$$

$$r = R_2 \ p_r = 0 = \frac{B_2}{R_2^2} - A_2$$

$$B_2 = p' \ \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}$$

$$A_2 = p' \ \frac{R_1^2}{R_2^2 - R_1^2}$$

Circumferential stress at any radius

$$= \frac{B_2}{r^2} + A_2$$

Circumferential stress at radius R_1,

$$f_c = p' \ \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} \quad \text{tensile}$$

Circumferential strain at radius R_1,

$$\epsilon_c'' = \frac{p'}{E_2} \times \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} + \frac{p'}{m_2 E_2} \quad \text{(tensile strain)}$$

where E_2 = Young's modulus of hub-material

$\frac{1}{m_2}$ = Poisson's ratio of hub-material.

Shrinkage allowance on radius,

$$\delta R_1 = \epsilon_c'' \times R_1 - \epsilon_c' \times R_1$$

Since ϵ_c' is compressive and ϵ_c'' is tensile strain

$$\delta R_1 = \frac{p' R_1}{E_2}\left(\frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} + \frac{1}{m_2}\right) + \frac{p' R_1}{E_1}\left(1 - \frac{1}{m_1}\right)$$

In a particular case where

$$E_1 = E_2 = E$$

$$\frac{1}{m_1} = \frac{1}{m_2} = \frac{1}{m}$$

$$\delta R_1 = \frac{p R_1}{E}\left(\frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} + 1\right)$$

$$= \frac{p' R_1}{E} \ \frac{(2R_2^2)}{R_2^2 - R_1^2}.$$

Example 6·5-1. A steel shaft of diameter 10 cm is driven into a steel hub. The driving allowance provided is 1/1000 of the diameter of the shaft. Determine the thickness of the hub if the maximum bursting stress in the hub is limited to 130 N/mm² and

$$E=208{,}000 \text{ N/mm}^2.$$

Solution. Radius of the shaft,

$$R_1=5 \text{ cm}=50 \text{ mm}$$

Say the outer radius of hub $=R_2$

Junction pressure $=p'$

$$\frac{\delta D_1}{D_1}=\frac{1}{1000}=\frac{\delta R_1}{R_1}$$

So

$$\frac{\delta R_1}{R_1}=\frac{1}{1000}=\frac{p'}{E}\left(\frac{2R_2^2}{R_2^2-R_1^2}\right)$$

$$p'=0{\cdot}001\times\frac{E\times(R_2^2-R_1^2)}{2R^2_2}$$

$$p'=0{\cdot}001\times\frac{208000\times(R_2^2-2500)}{2_2^2}$$

$$=\frac{104\ (R_2^2-2500)}{R_2^2} \qquad \ldots(1)$$

Now the maximum bursting stress in the hub

$$=p'\times\frac{R_2^2+R_1^2}{R_2^2-R_1^2}=130$$

$$p'\times\frac{R_2^2+2500}{R_2^2-2500}=130$$

$$p'=\frac{130\ (R_2^2-2500)}{R_2^2+2500} \qquad \ldots(2)$$

From equations (1) and (2)

$$\frac{104\ (R_2^2-2500)}{R_2^2}=\frac{130\ (R_2^2-2500)}{R_2^2+2500}$$

$$104\ (R_2^2+2500)=130\ R_2^2, \qquad 0{\cdot}8\ R_2^2+2000=R_2^2$$

or

$$0{\cdot}2\ R_2^2=2000, \qquad R_2^2=10{,}000$$

$$R_2=100 \text{ mm}$$

Thickness of the hub $=R_2-R_1=100-50=50$ mm.

Example 6·5-2. A steel shaft of 12 cm diameter is forced into a steel hub of 20 cm external diameter, so that the radial pressure developed at the common surface is 120 kg/cm². If $E=2100$ tonnes/cm², determine the force fit allowance on the diameter. What is the maximum hoop stress developed in the hub.

Solution. Shaft radius,

$$R_1=6 \text{ cm}$$

Outer radius of hub, $R_2=10$ cm

Junction pressure, $p'=120$ kg/cm²

Young's modulus, $E=2100\times1000$ kg/cm²

Allowance on radius, $\delta R_1 = \frac{p' R_1}{E}\left(\frac{2R_2^2}{R_2^2 - R_1^2}\right)$

$$= 120 \times \frac{6}{2100 \times 1000}\left(\frac{2 \times 10^2}{10^2 - 6^2}\right)$$

$$= 1.07 \times 10^{-3} \text{ cm}$$

So Allowance on diameter,

$$\delta D_1 = 2.14 \times 10^{-3} \text{ cm}$$

Maximum hoop stress in the hub

$$= p' \times \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}$$

$$= 120 \times \frac{10^2 + 6^2}{10^2 - 6^2} = 255.0 \text{ kg/cm}^2 \text{ (tensile).}$$

Exercise 6·5-1. A steel shaft of 80 mm diameter is driven into a steel hub. The driving allowance provided is 0·06 mm of the diameter of the shaft. Determine the thickness of the hub if the maximum bursting stress in the hub is limited to 100 MN/m².

$$E = 208 \text{ GN/m}^2$$

Note $100 \text{ MN/m}^2 = 100 \times 10^6 \text{ N/m}^2 = 100 \text{ N/mm}^2$

$208 \text{ GN/m}^2 = 208 \times 10^9 \text{ N/m}^2 = 208 \times 10^3 \text{ N/mm}^2$ [**Ans.** 35·317 mm]

Exercise 6·5-2. A steel shaft of 140 mm diameter is forced into a steel hub of 200 mm external diameter, so that the radial pressure developed at the common surface is 25 N/mm². If $E = 210 \times 1000$ N/mm², determine the force fit allowance on the diameter. What is the maximum bursting stress developed in the hub. [**Ans.** 0·064 mm, 73·04 N/mm²]

Problem 6·1. A steel cylinder 1 m inside diameter and 7 m long is subjected to an internal pressure of 10 MN/m². Determine the thickness of the cylinder if the maximum shear stress in the cylinder is not to exceed 40 MN/m². What will be the increase in the volume of the cylinder ?

$$E = 200 \text{ GN/m}^2$$

Poisson's ratio $= 0.30$.

Solution.

Inside diameter, $D = 1$ m

Length of cylinder, $L = 7$ m

Internal pressure, $p = 10$ MN/m²

Maximum shear stress $= 40$ MN/m²

$$= \frac{f_{c\,max} + p}{2} = \frac{f_{c\,max} + 10}{2}$$

or $f_{c\,max} = 70$ MN/m²

where $f_{c\,max}$, the maximum circumferential stress which occurs at inner radius

Say outer radius $= R_2$

Inner radius, $R_1 = 0.5$ m

Now $$f_{c\,max}=p\ \frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

$$70=10\ \frac{R_2^2+0{\cdot}5^2}{R_2^2-0{\cdot}5^2}$$

$$7R_2^2-1{\cdot}75=R_2^2+0{\cdot}25$$

$$6R_2^2=2{\cdot}00,\ R_2=0{\cdot}577 \text{ m}$$

Wall thickness $=0{\cdot}577-0{\cdot}500=0{\cdot}077$ m.

Now at the inner radius

$$f_{c\,max}=70 \text{ MN/m}^2 \quad \text{tensile}$$

$$p=10 \text{ MN/m}^2 \quad \text{compressive}$$

Axial stress, $$f_a=p\ \frac{R_1^2}{R_2^2-R_1^2}=\frac{10\times0{\cdot}5^2}{(0{\cdot}577)^2-(0{\cdot}5)^2}$$

$$=\frac{10\times0{\cdot}25}{0{\cdot}333-0{\cdot}250}=30{\cdot}12 \text{ MN/m}^2$$

Diameter strain, $$\epsilon_D=\frac{f_c}{E}-\frac{1}{m}\ \frac{f_a}{E}+\frac{1}{m}\ \frac{p}{E}$$

$$=\frac{70}{E}-\frac{0{\cdot}3\times30{\cdot}12}{E}+\frac{0{\cdot}3\times10}{E}$$

$$=\frac{63{\cdot}964}{E}$$

Axial strain, $$\epsilon_a=\frac{f_a}{E}-\frac{f_c}{mE}+\frac{p}{mE}$$

$$=\frac{30{\cdot}12}{E}-\frac{0{\cdot}3\times70}{E}+\frac{0{\cdot}3\times10}{E}$$

$$=\frac{12{\cdot}12}{E}$$

Volumetric strain, $$\epsilon_v=2\epsilon_D+\epsilon_a$$

$$=\frac{2\times63{\cdot}964}{E}+\frac{12{\cdot}12}{E}=\frac{140{\cdot}048}{E}$$

Original volume, $$V=\frac{\pi}{4}\ (1)^2\times7=5{\cdot}4978 \text{ m}^3$$

Increase in volume, $$\delta V=\epsilon_v\times V$$

$$=\frac{140{\cdot}048}{E}\times5{\cdot}4978 \text{ MN/m}^2$$

$$=\frac{140{\cdot}048\times5{\cdot}4978}{200}\times\frac{\text{MN/m}^2}{\text{GN/m}^2}$$

$$=\frac{140{\cdot}048\times5{\cdot}4978}{200\times1000}=0{\cdot}00385 \text{ m}^3.$$

Problem 6·2. A pressure vessel 20 cm internal radius, 25 cm external radius, 1 metre long is tested under a hydraulic pressure of 20 N/mm². Determine the change in internal and external diameters, if

$$E=208000 \text{ N/mm}^2 \qquad \frac{1}{m}, \text{ Poisson's ratio}=0\cdot3.$$

Solution. Pressure, $p=20$ N/mm²

Internal radius, $R_1=200$ mm

External radius, $R_2=250$ mm.

Hoop stress at inner radius,

$$f_{c_1}=p\cdot\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

$$=20\times\frac{250^2+200^2}{250^2-200^2}=91\cdot11 \text{ N/mm}^2 \text{ (tensile)}$$

Hoop stress at outer radius,

$$f_{c_2}=p\times\frac{2R_1^2}{R_2^2-R_1^2}$$

$$=20\times\frac{2\times200^2}{250^2-200^2}=71\cdot11 \text{ N/mm}^2 \text{ (tensile)}$$

Axial stress, $$f_a=\frac{pR_1^2}{R_2^2-R_1^2}$$

$$=\frac{20\times(200)^2}{250^2-200^2}=35\cdot55 \text{ N/mm}^2 \text{ (tensile)}$$

Diametral strain at inner radius

$$\epsilon_{D_1}=\frac{f_{c_1}}{E}-\frac{f_a}{mE}+\frac{p}{mE}$$

$$=\frac{91\cdot11}{E}-\frac{0\cdot3\times35\cdot55}{E}+\frac{20\times0\cdot3}{E}=\frac{86\cdot445}{E}$$

Change in internal diameter

$$=\epsilon_{D_1}\times400=\frac{86\cdot445}{208000}\times400=0\cdot166 \text{ mm}$$

Diametral strain at outer radius,

$$\epsilon_{D_2}=\frac{f_{c_2}}{E}-\frac{f_a}{mE}$$

$$=\frac{71\cdot11}{E}-\frac{0\cdot3\times35\cdot55}{E}=\frac{60\cdot445}{E}$$

Change in external diameter

$$=\epsilon_{D_2}\times500=\frac{60\cdot445\times500}{208000}=0\cdot145 \text{ mm.}$$

Problem 6·3. Strain gages are fixed on the outer surface of a thick cylinder with diameter ratio of 2·5. The cylinder is subjected to an internal pressure of 150 N/mm². The recorded strains are :

(*i*) Longitudinal strain $=59{\cdot}87\times10^{-6}$

(*ii*) Circumferential strain $=240{\cdot}65\times10^{-6}$.

Determine the Young's modulus of elasticity and Poisson's ratio of the material.

Solution. Say the inner radius of cylinder $=R_1$

Then outer radius of cylinder $=2{\cdot}5\ R_1$

Internal pressure, $p=150$ N/mm²

Axial stress,
$$f_a=\frac{pR_1^2}{R_2^2-R_1^2}$$
$$=\frac{150\times R_1^2}{6{\cdot}25\ R_1^2-R_1^2}=28{\cdot}57\ \text{N/mm}^2$$

Hoop stress at outer surface,
$$f_c=p\ \frac{2R_1^2}{R_2^2-R_1^2}$$
$$=2\times150\times\frac{R_1^2}{6{\cdot}25\ R_1^2-R_1^2}$$
$$=57{\cdot}14\ \text{N/mm}^2$$

Say Young's modulus $=E$

Poisson's ratio $=\dfrac{1}{m}$

Longitudinal strain,
$$\epsilon_a=\frac{f_a}{E}-\frac{1}{m}\ \frac{f_c}{E}$$
$$=\frac{28{\cdot}57}{E}-\frac{1}{m}\times\frac{57{\cdot}14}{E} \quad ..(1)$$

Circumferential strain,
$$\epsilon_c=\frac{f_c}{E}-\frac{1}{m}\ \frac{f_a}{E}$$
$$=\frac{57{\cdot}14}{E}-\frac{1}{m}\times\frac{28{\cdot}57}{E} \quad ...(2)$$

or
$$59{\cdot}87\times10^{-6}=\frac{28{\cdot}57}{E}\left(1-\frac{2}{m}\right) \quad ...(1)$$
$$240{\cdot}65\times10^{-6}=\frac{28{\cdot}57}{E}\left(2-\frac{1}{m}\right) \quad ...(2)$$

Dividing equation (2) by (1)
$$\frac{240{\cdot}65}{59{\cdot}87}=\frac{\left(2-\dfrac{1}{m}\right)}{\left(1-\dfrac{2}{m}\right)}$$

or $$4.02\left(1-\frac{2}{m}\right)=2-\frac{1}{m}$$

$$4.02-2=-\frac{1}{m}+\frac{8.04}{m}$$

$$\frac{2.02}{7.04}=\frac{1}{m}$$

or Poisson's ratio$=0.287$

Substituting the value of $\frac{1}{m}$ in equation (1)

$$59.87\times10^{-6}=\frac{28.57}{E}(1-2\times.287)=\frac{0.428\times28.57}{E}$$

or $$E=\frac{0.428\times28.57}{59.87}\times10^{6}\text{ N/mm}^2$$

$$=0.204\times10^{6}\text{ N/mm}^2$$

Young's modulus, $E=204{,}000\text{ N/mm}^2$

Problem 6.4. A thick cylinder 120 mm internal diameter and 180 mm external diameter is used for a working pressure of 15 N/mm². Because of external corrosion the outer diameter of the cylinder is machined to 178 mm. Determine by how much the internal pressure is to be reduced so that the maximum hoop stress remains the same as before.

Solution Inner radius, $R_1=60$ mm
Outer radius, $R_2=90$ mm
Radial pressure, $p_r=15$ N/mm²

Maximum hoop stress occurs at the inner radius

$$fc_{max}=p_r\,.\,\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

$$=15\times\frac{90^2+60^2}{90^2-60^2}=39\text{ N/mm}^2$$

When the external diameter is turned to 178 mm due to corrosion, say the pressure required is p_r'.

Inner radius, $R_1=60$ mm
Outer radius, $R_2'=89$ mm

Maximum hoop stress developed$=39$ N/mm² (as above)

$$39=p_r'\,.\,\frac{89^2+60^2}{89^2-60^2}$$

or $$p_r'=39\left(\frac{89^2-60^2}{89^2+60^2}\right)=14.627\text{ N/mm}^2$$

Reduction in internal pressure$=p_r-p_r'$

$$=15-14.627\text{ N/mm}^2=0.373\text{ N/mm}^2.$$

Problem 6.5. Two thick cylinders A and B are of the same dimensions. The external diameter is double the internal diameter. A is subjected to internal pressure p_1, while B is subjected to external pressure p_2 only. Find the ratio of p_1 and p_2 if the greatest circumferential strain developed in both is the same. Poisson's ratio of the material of both cylinders=0·3.

Solution. Cylinder A-subjected to internal pressure p_1 only. Greatest circumferential stress occurs at inner radius

$$f_{c_{max}}=p_1 \cdot \frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

where R_2=external radius

R_1=internal radius

$R_2=2R_1$ (as given)

So $$f_{c_{max}}=\frac{5}{3}\,p_1 \text{ (tensile)}$$

Axial stress in cylinder,

$$f_a=p_1\times\frac{\pi R_1^2}{\pi(R_2^2-R_1^2)}=+\frac{p_1}{3} \text{ (tensile)}$$

Greatest circumferential strain at R_1,

$$\epsilon_1=\frac{f_{c\,max}}{E}-\frac{f_a}{mE}+\frac{p_1}{mE}$$

$$=\frac{5p_1}{3E}-\frac{0\cdot3\,p_1}{3E}+\frac{0\cdot3\,p_1}{E}$$

$$=\frac{p_1}{E}\left[\frac{5}{3}-0\cdot1+0\cdot3\right]=\frac{1\cdot867\,p_1}{E} \text{ (tensile strain)}$$

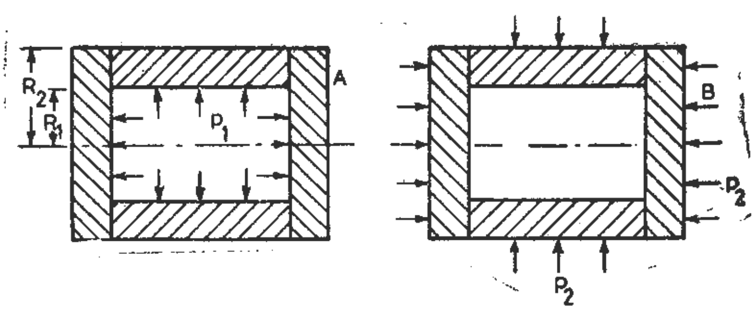


Fig. 6·13

Cylinder B-subjected to external pressure p_2 only. Greatest circumferential stress occurs at radius R_1

$$f_c'{}_{max}=-p_2\frac{2R_2^2}{R_2^2-R_1^2}=-p_2\times\frac{2\times(2R_1)^2}{(2R_1)^2-R_1^2}$$

$$=-\frac{8}{3}\,p_2 \text{ (compressive)}$$

Axial stress, $$f_a'=-p_2\,\frac{\pi R_2^2}{\pi R_2^2-R_1^2}=-\frac{4}{3}\,p_2 \text{ (compressive)}$$

Greatest circumferential strain,

$$\epsilon_2 = \frac{f_c'_{max}}{E} - \frac{f_a'}{mE} - \frac{p_2}{mE}$$

$$= -\frac{8}{3}\frac{p_2}{E} + \frac{4}{3} \times \frac{0{\cdot}3}{E} + \frac{p_2 \times 0{\cdot}3}{E}$$

(all the stresses $f_c'_{max}$, f_a', p_2 are compressive)

$$\epsilon_2 = \frac{p_2}{E}\left(-\frac{8}{3} + 0{\cdot}4 + 0{\cdot}3\right)$$

$$= -\frac{1{\cdot}967\, p_2}{E} \quad \text{(compressive strain)}$$

$$= \frac{1{\cdot}967\, p_2}{E} \quad \text{(numerical value)}$$

So $$\frac{1{\cdot}867\, p_1}{E} = \frac{1{\cdot}967\, p_2}{E}$$

or $$\frac{p_1}{p_2} = \frac{1{\cdot}967}{1{\cdot}867} = 1{\cdot}053$$

Problem 6·6. The maximum stress permissible in a thick cylinder of 5 cm internal diameter and 20 cm external diameter is 200 kg/cm². If the external radial pressure is 40 kg/cm², determine the intensity of the internal radial pressure.

Solution. Fig. 6·14 shows a cylindrical section subjected to external and internal radial pressures. Say the internal pressure $= p$.

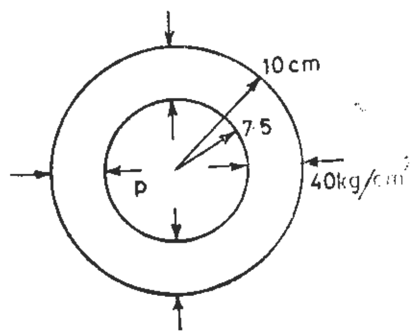


Fig. 6·14

Boundary Conditions

at radius $r = 7{\cdot}5$ cm, $p_r = p$

at radius $r = 10$ cm, $p = 40$ kg/cm²

Taking A and B as Lame's contants

$$p = \frac{B}{7{\cdot}5^2} - A \qquad \ldots(i)$$

$$40 = \frac{B}{10^2} - A \qquad \ldots(ii)$$

Solving the equation (*i*) and (*ii*) for A, B we get

$$B = (p-40)\ \frac{5625}{43{\cdot}75}$$

$$A = \left(\frac{56{\cdot}25\, p - 4000}{43{\cdot}75}\right)$$

Circumferential stress at any radius r,

$$f_c = \frac{B}{r^2} + A$$

$$= \frac{(p-40)(5625)}{43{\cdot}75\, r^2} + \frac{(56{\cdot}25\, p - 4000)}{43{\cdot}75}$$

r is in the denominator and to get the maximum value of f_c, r should be minimum *i.e.* 7·5 cm for the given case.

So
$$f_{c\ max}=\frac{(p-40)(5625)}{43{\cdot}75\times(7{\cdot}5)^2}+\left(\frac{56{\cdot}25\,p-4000}{43{\cdot}75}\right)$$
$$=(p-40)\,\frac{100}{43{\cdot}75}+\frac{56{\cdot}25\,p-4000}{43{\cdot}75}$$
$$=\left(\frac{156{\cdot}25\,p-8000}{43{\cdot}75}\right)$$

or
$$200=\frac{156{\cdot}25\,p-8000}{43{\cdot}75}$$

Internal pressure,
$$p=\frac{200\times43{\cdot}75+8000}{156{\cdot}25}=107{\cdot}2\ \text{kg/cm}^2$$

Problem 6·7. A thick cylinder of internal diameter D and wall thickness t is subjected to an internal pressure p. Determine the ratio of $\frac{t}{D}$ if the maximum hoop tension developed in the cylinder is 2·5 p.

Solution. Internal radius, $R_1=\frac{D}{2}$

External radius, $R_2=\left(\frac{D}{2}+t\right)$

Maximum hoop tension,

$$f_{c\ max}=p\,.\,\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

or
$$2{\cdot}5\,p=p\,.\,\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

or
$$2{\cdot}5\,(R_2^2-R_1^2)=R_2^2+R_1^2$$

or
$$1{\cdot}5\,R_2^2=3{\cdot}5\,R_1^2$$

Substituting the values of R_2 and R_1

$$1{\cdot}5\left(t+\frac{D}{2}\right)^2=3{\cdot}5\left(\frac{D}{2}\right)^2$$

or
$$t+\frac{D}{2}=\frac{D}{2}\times\sqrt{\frac{3{\cdot}5}{1{\cdot}5}}$$
$$t+\frac{D}{2}=\frac{D}{2}\times1{\cdot}526$$

Dividing throughout by D we get

$$\frac{t}{D}+0{\cdot}5=0{\cdot}763,\qquad \frac{t}{D}=0{\cdot}263.$$

Problem 6·8. A cylinder of internal diameter D and wall thickness t is subjected to internal pressure p. If it is assumed to be a thin cylindrical shell, what is the maximum value of $\frac{t}{D}$ if the error in the estimated value of maximum hoop stress is not to exceed by 10%.

Solution. Internal diameter $=D$

Wall thickness $=t$

External radius $=\frac{D}{2}+t$

$f_{c\,max}$ Maximum hoop stress as per Lame's theory

$$=p\times\frac{\left(\frac{D}{2}+t\right)^2+\left(\frac{D}{2}\right)^2}{\left(\frac{D}{2}+t\right)^2-\left(\frac{D}{2}\right)^2}$$

f_c, Hoop stress (considering thin shell) $=\frac{pD}{2t}$

Now $\frac{f_{c\,max}-f_c}{f_{c\,max}}=0·1$ (as given in the problem)

or $0·9\,f_{c\,max}=f_c$

or $$0.9\left[p\times\frac{\frac{D^2}{4}+Dt+t^2+\frac{D^2}{4}}{\frac{D^2}{4}+Dt+t^2-\frac{D^2}{4}}\right]=p\,\frac{D}{2t}$$

or $$0·9\left\{\frac{\frac{D^2}{2}+Dt+t^2}{Dt+t^2}\right\}=\frac{D}{2t}$$

or $$0·9\times 2t\left(\frac{D^2}{2}+Dt+t^2\right)=D^2t+Dt^2$$

$$0·9\,tD^2+1·8\,Dt^2+1·8\,t^3=D^2t+Dt^2$$

$$-0·1\,tD^2+0·8\,Dt^2+1·8\,t^3=0$$

or $$1·8\,t^2+0·8\,Dt-0·1\,D^2=0$$

$$t=\frac{-0·8\,D+\sqrt{0·64\,D^2+0·72\,D^2}}{3·6}$$

$$=\frac{-0·8\,D+1·166\,D}{3·6}=\frac{0·366\,D}{3·6}=0·1016\,D$$

Ratio, $\frac{t}{D}=0·1016.$

Problem 6·9. A thick cylinder of internal diameter D and wall thickness t is subjected to the internal pressure p. If the maximum hoop stress developed in the cylinder is 1·5 times the internal pressure, determine the ratio of t/D.

Find the increase in the internal and external diameters of such a cylinder with 160 mm internal diameter subjected to internal fluid pressure of 50 N/mm².

$$E=200\times 1000 \text{ N/mm}^2$$

$$\frac{1}{m}=0{\cdot}28.$$

Solution.

(*a*) Internal radius, $R_1=\dfrac{D}{2}$

External radius, $R_2=\dfrac{D}{2}+t$

Internal pressure $=p$

Maximum hoop stress $=1{\cdot}5\,p$

(Maximum hoop stress is developed at the inner radius of the cylinder)

$$f_{c\ max}=1{\cdot}5\ p=p\ \frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

or
$$1{\cdot}5=\frac{\left(\dfrac{D}{2}+t\right)^2+\dfrac{D^2}{2}}{\left(\dfrac{D}{2}+t\right)^2-\left(\dfrac{D}{2}\right)^2}$$

or
$$0{\cdot}5\left(\frac{D}{2}+t\right)^2=2{\cdot}5\left(\frac{D}{2}\right)^2,\qquad \frac{D}{2}+t=\frac{D}{2}\times\sqrt{5}$$

$$\frac{D}{2}+t=2{\cdot}236\times\frac{D}{2}$$

$$t=1{\cdot}236\ \frac{D}{2}$$

or
$$\frac{t}{D}=0{\cdot}618.$$

(*b*) Inner radius, $R_1=80$ mm

Wall thickness, $t=0{\cdot}618\times 160=98{\cdot}88$ mm

Outer radius, $R_2=R_1+t=178{\cdot}88$ mm

Internal pressure, $p=50$ N/mm²

Hoop stress at R_1, $f_{cR_1}=50\times\dfrac{R_2^2+R_1^2}{R_2^2-R_1^2}=50\times\dfrac{178{\cdot}88^2+80^2}{178{\cdot}88^2-80^2}$

$$=75{\cdot}00 \text{ N/mm}^2 \text{ (tensile)}$$

Hoop stress at R_2, $f_{cR_2}=50\times\dfrac{2R_1^2}{R_2^2-R_1^2}=50\times\dfrac{2\times 80^2}{178{\cdot}88^2-80^2}$

$$=25{\cdot}00 \text{ N/mm}^2 \text{ (tensile)}$$

Axial stress, $$f_a = p \times \frac{\pi R_1^2}{\pi(R_2^2 - R_1^2)} = 50 \times \frac{80^2}{(178 \cdot 88^2 - 80^2)}$$

$$= 12 \cdot 5 \text{ N/mm}^2 \text{ (tensile)}$$

Radial stress at inner radius

$$= 50 \text{ N/mm}^2 \text{ (compressive)}$$

Radial stress at outer radius

$$= 0$$

Circumferential strain at inner radius,

$$\epsilon_c' = \frac{75}{E} - \frac{12 \cdot 5}{mE} + \frac{50}{mE}$$

$$= \frac{1}{E} [75 - 12 \cdot 5 \times 0 \cdot 28 + 50 \times 0 \cdot 28]$$

$$= \frac{85 \cdot 5}{E}$$

Change in internal diameter,

$$\delta D_1 = \epsilon_c' \times D_1 = \frac{85 \cdot 5}{200 \times 1000} \times 160$$

$$= 0 \cdot 0684 \text{ mm}$$

Circumferential strain at outer radius,

$$\epsilon_c'' = \frac{25}{E} - \frac{12 \cdot 5}{mE}$$

$$= \frac{1}{E} [25 - 0 \cdot 28 \times 12 \cdot 5]$$

$$= \frac{21 \cdot 5}{E}$$

Change in external diameter,

$$\delta D_2 = \epsilon_c'' \times D_2$$

$$= \frac{21 \cdot 5 \, (2 \times 178 \cdot 88)}{200 \times 1000} = 0 \cdot 038 \text{ mm}$$

Problem 6·10. A thick cylinder of internal diameter 160 mm is subjected to an internal pressure of 5 N/mm². If the allowable stress for cylinder is 25 N/mm², determine the wall thickness of the cylinder. The cylinder is then strengthened by wire winding so that it can be safely subjected to an internal pressure of 8 N/mm². Find the radial pressure caused by wire winding.

Solution. Say the outer radius of the cylinder

$$= R_2$$

Inner radius of the cylinder,

$$R_1 = 80 \text{ mm}$$

Wall thickness, $=R_2-80$ mm

Internal pressure, $p=5$ N/mm²

The maximum hoop stress occurs at the inner radius of the cylinder, which is

$$f_{c\ max}=p\ \frac{R_2^2+R_1^2}{R_2^2-R_1^2} \leqslant 25 \text{ N/mm}^2$$

So $$25=5\ \frac{R_2^2+80^2}{R_2^2-80^2}$$

or $$R_2^2+80^2=5\ R_2^2-5\times 80^2 \text{ or } 4\ R_2^2=6\times 80^2$$

$$R_2=97{\cdot}98 \text{ mm}$$

Wall thickness, $=R_2-R_1=97{\cdot}98-80$
$=17{\cdot}98$ mm

Say the radial pressure developed after wire winding is p_r as shown in Fig. 6·15.

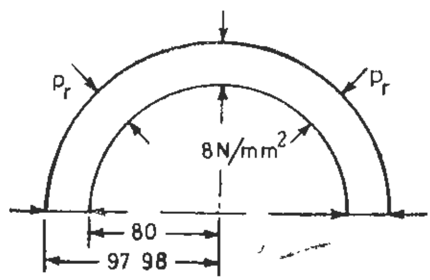


Fig. 6·15

Using lame's equations

$$p_r=\frac{B}{R_2^2}-A \quad ...(1)$$

$$8=\frac{B}{R_1^2}-A \quad ...(2)$$

$$f_{c\ max}=\frac{B}{R_1^2}+A \quad ...(3)$$

$$\leqslant 25 \text{ N/mm}^2$$

where A and B are constants

or $$8+25=\frac{2B}{R_1^2} \text{ From equations (2) and (3)}$$

$$B=16{\cdot}5\ R_1^2=105600$$

$$A=25-\frac{B}{R_1^2}=25-16{\cdot}5=8{\cdot}5$$

Substituting the values of the constants in equation (1)

Radial pressure,

$$p_r=\frac{105600}{(97{\cdot}98)^2}-8{\cdot}5=10{\cdot}999-8{\cdot}5=2{\cdot}499 \text{ N/mm}^2.$$

Problem 6·11. A compound cylinder is made by shrinking a tube of 150 mm outer diameter over another tube of 100 mm inner diameter. Find the common diameter if the greatest circumferential stress in the inner tube is numerically 0·70 times of that of the outer tube.

Solution.

Say the common radius $=R_2$

Junction pressure (due to shrinking)

$$=p$$

Inner radius, $R_1=50$ mm

Outer radius, $R_3=75$ mm

Greatest circumferential stress in the inner tube occurs at its inner radius, R_1

$$(f_{c\ max})_{inner\ tube}=-\frac{2p\ R_2^2}{R_2^2-R_1^2}=-\frac{2p\ R_2^2}{R_2^2-50^2} \quad ...(1)$$

The greatest circumferential stress in the outer tube occurs at its inner radius R_2

$$(f_{c\ max})_{outer\ tube}=\frac{p(R_3^2+R_2^2)}{R_3^2-R_2^2}=\frac{p(75^2+R_2^2)}{75^2-R_2^2} \quad ...(2)$$

Now $$\frac{2p\ R_2^2}{R_2^2-50^2}=\left[\frac{p\ (75^2+R_2^2)}{75^2-R_2^2}\right]\times 0{\cdot}7$$

or $$2R_2^2\ (75^2-R_2^2)=0{\cdot}7\ (75^2+R_2^2)(R_2^2-50^2)$$

$$R_2^4-3356{\cdot}48\ R_2^2-3647076{\cdot}4=0$$

$$R_2^2=4220{\cdot}59$$

Common radius, $R_2=64{\cdot}96$ mm

Common diameter, $=129{\cdot}92$ mm.

Problem 6·12. A steel cylinder of outer diameter 180 mm is shrunk or another cylinder of inner diameter 120 mm, the common diameter being 150 mm. If after shrinking on, the radial pressure at the common surface is 12 N/mm², determine the magnitude of the internal pressure p to which the compound cylinder can be subjected so that maximum hoop tensions in the inner and outer cylinders are equal.

Solution.

Inner radius, $R_1=60$ mm

Outer radius, $R_2=90$ mm

Junction radius, $R_3=75$ mm

Junction pressure, $p'=12$ N/mm²

Maximum hoop tension due to shrinkage pressure p' and internal pressure p occurs at the inner radius of both the cylinders. Let us first determine shrinkage stresses.

Shrinkage stresses. Hoop stress at R_1 in inner cylinder,

$$f_c'{}_{R_1}=-p'\times\frac{2R_3^2}{R_3^2-R_1^2}$$

$$=-12\times\frac{2\times 75^2}{75^2-80^2}=-66\ 67\ \text{N/mm}^2$$

Hoop stress at R_3 in outer cylinder,

$$f_c''_{R_3}=+p'\ \frac{R_2^2+R_3^2}{R_2^2-R_3^2}$$

$$=12\times\frac{90^2+75^2}{90^2-75^2}=66{\cdot}54\ \text{N/mm}^2$$

Stresses due to internal pressure p. Say Lame's constants for compound cylinder are A, B

Boundary conditions, at $r=R_1,\ p_r=p=\dfrac{B}{R_1^2}-A$

$$\text{at } r=R_2,\ p_r=0=\frac{B}{R_2^2}-A$$

$$B=p\ \frac{R_1^2\ R_2^2}{R_2^2-R_1^2}$$

$$A=p\ \frac{R_1^2}{R_2^2-R_1^2}$$

Hoop stress at any radius,

$$f_c=\frac{B}{r^2}+A=\frac{p}{r^2}\times\frac{R_1^2\ R_2^2}{R_2^2-R_1^2}+p\ \frac{R_1^2}{R_2^2-R_1^2}$$

$$=\frac{p}{r^2}\times\frac{60^2\times 90^2}{90^2-60^2}+p\times\frac{60^2}{90^2-60^2}$$

$$=\frac{p}{r^2}\times 6480+0{\cdot}8\ p$$

$$\text{At } r=R_1\ldots\ldots f_{c\,R_1}=\frac{p}{60^2}\times 6480+0{\cdot}8\ p$$

$$=1{\cdot}8\ p+0{\cdot}8\ p=2{\cdot}6\ p$$

$$\text{At } r=R_3\ldots\ldots f_{c\,R_3}=\frac{p}{75^2}\times 6480+0{\cdot}8\ p$$

$$=1{\cdot}152\ p+0{\cdot}8\ p=1{\cdot}952\ p$$

Now resultant hoop stresses at the inner radius of both the cylinder

$$f_c'''_{R_1}=f_c'_{R_1}+f_{c\,R_1}=-66{\cdot}67+2{\cdot}6\ p$$
$$f_c'''_{R_3}=f_c''_{R_3}+f_{c\,R_3}=66{\cdot}54+1{\cdot}952\ p$$

Now as per the condition given

$$-66{\cdot}67+2{\cdot}6\ p=66{\cdot}54+1{\cdot}952\ p$$

or $$0{\cdot}648\ p=133{\cdot}21$$

Required internal pressure,

$$p=205{\cdot}57\ \text{N/mm}^2$$

Problem 6·13. A compound cylinder has a bore of 120 mm, the outer diameter is 200 mm and the diameter at the common surface is 160 mm. Determine the radial pressure at the common surface which must be provided by the shrinkage fitting, if the resultant maximum hoop stress in the inner cylinder under a superimposed internal pressure of 50 N/mm² is to be half the value of the maximum hoop tension in the inner cylinder if this cylinder alone is subjected to an internal radial pressure of 50 N/mm².

Determine the resultant hoop stresses at the inner and outer radii of both the cylinders. Sketch the variation of resultant hoop stress along the thickness of the cylinder.

Solution.

Inner cylinder. Inner radius, $R_1=60$ mm

Outer radius, $R_3=80$ mm

Outer cylinder. Inner radius, $R_3=80$ mm

Outer radius, $R_2=100$ mm.

A. Inner cylinder alone is subjected to internal pressure of 50 N/mm²

$$f_{c\,max} \text{ at inner radius } R_1=50\times\frac{R_3^2+R_1^2}{R_3^2-R_1^2}$$

$$=50\times\left(\frac{80^2+60^2}{80^2-60^2}\right)=\frac{1250}{7}\text{ N/mm}^2\text{ (tensile)}$$

B. Let us say the junction pressure due to shrinkage is p_r at the common radius R_3 in the case of compound cylinder.

Hoop stress due to p_r at radius R_1 in the inner cylinder

$$f_c'_{R_1}=-p_r\cdot\frac{2R_3^2}{R_3^2-R_1^2}$$

$$=-p_r\cdot\frac{2\times 80^2}{80^2-60^2}=-\frac{32p_r}{7}\text{ (compressive)}$$

C. The compound cylinder is subjected to an internal fluid pressure of 50 N/mm².

Hoop stress at radius R_1,

$$f_c''_{R_1}=50\times\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

$$=50\times\frac{100^2+60^2}{100^2-60^2}=\frac{425}{4}\text{ N/mm}^2\text{ (tensile)}$$

Resultant hoop stress at radius R_1, (due to shrinkage and internal pressure)

$$f_{cR_1}=f_c'_{R_1}+f_c''_{R_1}$$

$$=-\frac{32}{7}p_r+\frac{425}{4}$$

$$=\frac{1}{2}\left(\frac{1250}{7}\right)\quad\text{as given in the problem}$$

or
$$-\frac{32}{7}p_r+\frac{425}{4}=\frac{625}{7}$$

$$\frac{425}{4}-\frac{625}{7}=\frac{32}{7}p_r$$

$$p_r=\frac{475}{28}\times\frac{7}{32}=3{\cdot}71\text{ N/mm}^2$$

i.e. Radial pressure at the common surface provided by the shrinkage fitting is 3·71 N/mm².

Now let us first determine the shrinkage stresses in both the cylinders.

Inner Cylinder. The junction pressure p_r is acting as the external pressure on this cylinder. The hoop stress developed will be compressive in nature.

$$f_c R_1 = -p_r \quad \frac{2R_3^2}{R_3^2 - R_1^2}$$

$$= -3{\cdot}71 \times \frac{2 \times 80^2}{80^2 - 60^2} = -16{\cdot}96 \text{ N/mm}^2$$

$$f_c R_3 = -p_r \,.\, \frac{R_3^2 + R_1^2}{R_3^2 - R_1^2}$$

$$= -3{\cdot}71 \times \frac{80^2 + 60^2}{80^2 - 60^2} = -13{\cdot}25 \text{ N/mm}^2.$$

Outer Cylinder. The junction pressure p_r is acting as the internal pressure on this cylinder, the hoop stress developed will be tensile in nature.

$$f'_c R_3 = +p_r \,.\, \frac{R_2^2 + R_3^2}{R_2^2 - R_3^2}$$

$$= 3{\cdot}71 \times \frac{100^2 + 80^2}{100^2 - 80^2} = +16{\cdot}90 \text{ N/mm}^2$$

$$f_c R_2 = +p_r .\ \frac{2R_3^2}{R_2^2 - R_3^2}$$

$$= 3{\cdot}71 \times \frac{2 \times 80^2}{100^2 - 80^2} = +13{\cdot}19 \text{ N/mm}^2$$

Now consider the compound cylinder with inner radius 60 mm, outer radius 100 mm subjected to internal fluid pressure of 50 N/mm². Let us take A and B as Lame's constants.

Boundary Conditions

At $r = R_1 = 60$ mm, $\quad p_r = 50 \text{ N/mm}^2 = \dfrac{B}{6r^2} - A$

At $r = R_2 = 100$ mm, $\quad p_r = 0 \qquad = \dfrac{B}{100^2} - A$

From these equations, the values of constants are

$$B = 281250 \text{ N}$$

$$A = 28{\cdot}125 \text{ N/mm}^2$$

Hoop stress at any radius r,

$$f_c = \frac{B}{r^2} + A = \frac{281250}{r^2} + 28{\cdot}125$$

at radius R₁, $\quad f_c''' R_1 = \dfrac{281250}{60^2} + 28{\cdot}125 = 106{\cdot}250 \text{ N/mm}^2$

at R_3, $\quad f_c''' R_3 = \dfrac{281250}{80^2} + 28{\cdot}125 = 72{\cdot}070 \text{ N/mm}^2$

at R_2, $\quad f_c''' R_2 = \dfrac{281250}{100^2} + 28{\cdot}125 = 56{\cdot}250 \text{ N/mm}^2$

Resultant Stresses

Inner cylinder $\quad f_c^i R_1 = f_c R_1 + f_c''' R_1 = -16{\cdot}96 + 106{\cdot}25 = 89{\cdot}29$ N/mm²

at R_3, $\quad f_c^i R_3 = f_c R_2 + f_c''' R_2 = -13{\cdot}25 + 72{\cdot}07 = 58{\cdot}82$ N/mm²

Outer cylinder $\quad f_c^{\circ} R_3 = f_c' R_3 + f_c''' R_3 = 16{\cdot}90 + 72{\cdot}070 = 88{\cdot}97$ N/mm²

at R_2 $\quad f_c^{\circ} R_2 = f_c R_2 + f_c''' R_2 = 13{\cdot}19 + 56{\cdot}250 = 69{\cdot}44$ N/mm².

Fig. 6·16 shows the sketch of the distribution of resultant hoop stress across the thickness of the compound cylinder.

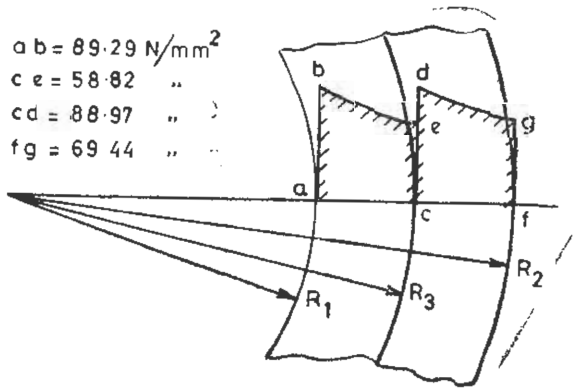


Fig. 6·16

Problem 6·14. A compound cylinder consists of a steel cylinder 18 cm internal and 25 cm external diameter and a bronze liner of 18 cm external and 15 cm internal diameter. Assuming the liner to be thin cylinder and that there is no stress in the compound cylinder due to fitting, determine the maximum direct stress and maximum shear stress in each material due to an internal pressure of 100 N/mm². Ignore the longitudinal stress and strain.

For steel $\quad E = 208000$ N/mm²

$$\frac{1}{m} = 0{\cdot}28$$

For bronze $\quad E = 112{,}000$ N/mm²

$$\frac{1}{m} = 0{\cdot}30$$

Solution.
Internal radius, $R_1 = 75$ mm
External radius, $R_3 = 125$ mm
Common radius, $R_2 = 90$ mm

For the compound cylinder radial pressure,

$$p = \frac{B}{r^2} - A$$

Hoop stress,

$$f_c = \frac{B}{r^2} + A.$$

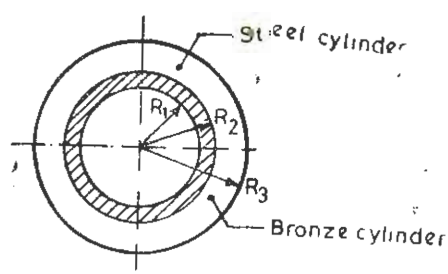


Fig. 6·17

where A and B are Lame's constants.

Now $p=100\ N/mm^2$ at $r=75$ mm

$p=0$ at $r=125$ mm

So $$100=\frac{B}{75^2}-A \quad \text{As there is no shrinkage stress} \quad ...(1)$$

$$0=\frac{B}{125^2}-A \quad ...(2)$$

From these equations $$B=\frac{100\times75^2\times125^2}{(125^2-75^2)}$$

$$A=\frac{100\times75^2}{(125^2-75^2)}$$

Radial pressure at the common radius,

$$p_r=\frac{B}{90^2}-A$$

$$=\frac{100\times75^2\times125^2}{90^2\,(125^2-75^2)}-\frac{100\times75^2}{(125^2-75^2)}$$

$$=108{\cdot}50-56{\cdot}25=52{\cdot}25\ N/mm^2$$

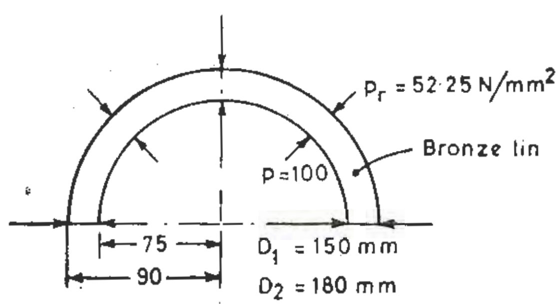
Treating the liner and cylinder separately. Liner is to be treated as thin cylinder.

Circumferential stress developed

$$f_c'=\frac{pD_1}{2t}-\frac{p_rD_2}{2t}$$

$$=\frac{100\times150}{2\times15}-\frac{52{\cdot}25\times180}{2\times15}$$

$$=500-313{\cdot}5=186{\cdot}5\ N/mm^2$$



t = 15 mm

Fig. 6·18

Maximum direct stress $=186{\cdot}5\ N/mm^2$ (tensile)

Maximum shear stress $=\dfrac{186{\cdot}5+100}{2}=143{\cdot}25\ N/mm^2$ As p is compressive

Steel cylinder

$$p_r = 52{\cdot}25 \text{ N/mm}^2$$

Maximum hoop stress occurs at the inner radius R_2

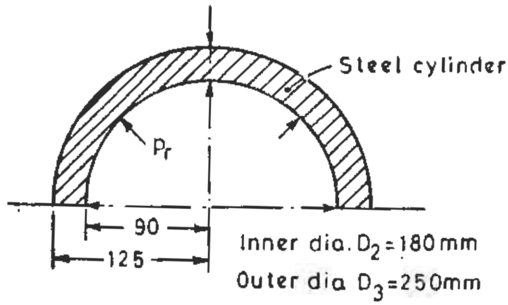


Fig. 6·19

$$f_c'' = p_r \cdot \frac{R_3^2 + R_2^2}{R_3^2 - R_2^2}$$

$$f_c'' = 52{\cdot}25 \times \frac{(125^2 + 90^2)}{(125^2 - 90^2)}$$

$$= \frac{52{\cdot}25 \times 23725}{7525}$$

Maximum direct stress $= 164{\cdot}73$ N/mm² (tensile)

Maximum shear stress $= \dfrac{164{\cdot}73 + 52{\cdot}25}{2} = 108{\cdot}49$ N/mm²

Problem 6·15. A steel tube of outside diameter 220 mm is shrunk on another tube of inside diameter 140 mm. The diameter at the junction is 180 mm after shrinking on. The shrinkage allowance provided on the radius of the inner tube is 0·08 mm. Determine

(*a*) junction pressure

(*b*) hoop stress at the outer and inner radii of the inner tube

(*c*) hoop stress at the outer and inner radii of the outer tube.

$$E = 210 \times 10^3 \text{ N/mm}^2.$$

Solution. After shrinking on of the outer cylinder over the inner cylinder, radial pressure acts on the outer surface of the inner cylinder and radial pressure of same magnitude acts on the inner surface of outer cylinder.

Inner radius, $R_1 = 70$ mm

Outer radius, $R_2 = 110$ mm

Junction radius, $R_3 = 90$ mm

Shrinkage allowance, $\delta R_3 = 0{\cdot}08$ mm

Say junction pressure $= p'$

Now
$$\delta R_3 = \frac{p' R_3}{E}\left[\frac{R_2^2+R_3^2}{R_2^2-R_3^2}+\frac{R_3^2+R_1^2}{R_3^2-R_1^2}\right]$$

$$0{\cdot}08 = \frac{p'\times 90}{210\times 1000}\left[\frac{110^2+90^2}{110^2-90^2}+\frac{90^2+70^2}{90^2-70^2}\right]$$

$$\frac{0{\cdot}08\times 210\times 1000}{90} = p'\left[5{\cdot}05+4{\cdot}0625\right]$$

$$\frac{186{\cdot}67}{9{\cdot}1125} = p'$$

Junction pressure, $p' = 20{\cdot}485 \text{ N/mm}^2$

Hoop stresses

Inner tube
$$f_c'{}_{R_1} = -\frac{2R_3^2}{R_3^2-R_1^2}\times p' = -\frac{2\times 90^2}{90^2-70^2}\times 20{\cdot}485$$

at inner radius,
$$= -103{\cdot}705 \text{ N/mm}^2$$

at the outer radius
$$f_c'{}_{R_3} = -\frac{R_3^2+R_1^2}{R_3^2-R_1^2}\times p'$$

$$= -\frac{90^2+70^2}{90^2-70^2}\times 20{\cdot}485 = -83{\cdot}22 \text{ N/mm}^2$$

Outer tube

at the inner radius R_3,
$$f_c''{}_{R_3} = \frac{R_2^2+R_3^2}{R_2^2-R_3^2}\times p'$$

$$= \frac{110^2+90^2}{110^2-90^2}\times 20{\cdot}485 = 103{\cdot}45 \text{ N/mm}^2$$

at the outer radius R_2,
$$f_c''{}_{R_2} = +\frac{2R_3^2}{R_2^2-R_3^2}\,p'$$

$$= +\frac{2\times 90^2}{110^2-90^2}\times 20{\cdot}485 = +82{\cdot}96 \text{ N/mm}^2$$

Problem 6·16. A steel cylinder 10 cm internal diameter and 15 cm external diameter is strengthened by shrinking another cylinder onto it, the internal diameter of which before heating is 14·992 cm. Determine the outer diameter of the outer cylinder if the pressure at the junction after shrinkage is 200 kg/cm².

Given E for steel = 2100 tonnes/cm².

Solution.

Inner radius, $R_1 = 5$ cm
Outer radius $= R_2$ (say)
Junction radius, $R_3 = 7{\cdot}5$ cm
Junction pressure, $p' = 200$ kg/cm².

Shrinkage allowance, $\delta D_3 = 15 - 14{\cdot}992 = 0{\cdot}008$ cm.

or $\delta R_3 = 0{\cdot}004$ cm.

$$0{\cdot}004 = \frac{R_3 p'}{E}\left(\frac{R_2^2+R_3^2}{R_2^2-R_3^2} + \frac{R_3^2+R_1^2}{R_3^2-R_1^2}\right)$$

$$0{\cdot}004 = \frac{7{\cdot}5\times 200}{2100\times 1000}\left(\frac{R_2^2+7{\cdot}5^2}{R_2^2-7{\cdot}5^2} + \frac{7{\cdot}5^2+5^2}{7{\cdot}5^2-5^2}\right)$$

$$\frac{0{\cdot}004\times 2100\times 1000}{7\,5\times 200} = \frac{R_2^2+56{\cdot}25}{R_2^2-56{\cdot}25} + 2{\cdot}6$$

$$5{\cdot}6 - 2{\cdot}6 = \frac{R_2^2+56{\cdot}25}{R_2^2-56{\cdot}25}$$

or $$3R_2^2 - 3\times 56{\cdot}25 = R_2^2 + 56{\cdot}25$$

$$2R_2^2 = 4\times 56{\cdot}25$$

$$R_2^2 = 112{\cdot}5$$

$$R_2 = 10{\cdot}60 \text{ cm}$$

i.e. outer diameter of outer cylinder $= 10{\cdot}60\times 2 = 21{\cdot}20$ cm.

Problem 6·17. A thick steel cylinder of inner diameter 120 mm and outer diameter 160 mm is subjected to an internal fluid pressure of 200 N/mm². A cylindrical jacket 20 mm thick of the same material is shrunk on to the cylinder so that the maximum hoop stress developed in the cylinder is not to exceed 280 N/mm². What should be the initial difference between the internal diameter of the jacket and external diameter of the cylinder.

$$E = 200\times 1000 \text{ N/mm}^2, \qquad \frac{1}{m} = 0{\cdot}3.$$

Solution.

Inner radius of cylinder, $R_1 = 0{\cdot}60$ mm
Then outer radius of jacket $= 100$ mm
Junction radius, $R_3 = 80$ mm
Internal pressure, $p = 200$ N/mm²
Say junction pressure $= p'$

Maximum hoop stress in the cylinder due to internal pressure, at R_1

$$= p\times\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

$$= 200\times\frac{100^2+60^2}{100^2-60^2} = 425 \text{ N/mm}^2.$$

But the maximum hoop stress is not to exceed 280 N/mm².

So hoop stress provided at R_1 by shrinkage

$$= 280 - 425 = -145 \text{ N/mm}^2 \text{ (compressive)}$$

$$= -p'\times\frac{2R_3^2}{R_3^2-R_1^2}$$

or $$p'\times\frac{2\times 80^2}{80^2-60^2}=145$$

$$p'=\frac{145\times 28}{128}=31{\cdot}72\ \text{N/mm}^2$$

Jacket and cylinder are made of the same material.

Shrinkage allowance, $$\delta R_3=\frac{p'R_3}{E}\left[\frac{R_2^2+R_3^2}{R_2^2-R_3^2}+\frac{R_3^2+R_1^2}{R_3^2-R_1^2}\right]$$

$$=\frac{31{\cdot}72\times 80}{200\times 1000}\left[\frac{100^2+80^2}{100^2-80^2}+\frac{80^2+60^2}{80^2-60^2}\right]$$

$$=12{\cdot}688\times 10^{-3}\,[4{\cdot}555+3{\cdot}571]=0{\cdot}103\ \text{mm}$$

$$\delta D_3=0{\cdot}206\ \text{mm}.$$

Initial difference between internal diameter of the jacket and external diameter of the cylinder $=0{\cdot}206$ mm.

Problem 6·18. A high tensile steel tyre of thickness 25 mm is shrunk on a cast iron rim of internal diameter 500 mm and external diameter 600 mm. Find the inside diameter of the steel tyre, if after shrinking on, the tyre exerts a radial pressure of 5 N/mm² on the cast iron rim.

$E_{steel}=210\times 10^3$ N/mm² $E_{CI}=100\times 10^3$ N/mm².

Poisson's ratio for steel$=0{\cdot}30$

Poisson's ratio for $CI=0{\cdot}25$

Solution.

Inner radius of rim, $R_1=250$ mm

Outer radius of tyre, $R_2=325$ mm

Junction radius, $R_3=300$ mm

Radial pressure, $p'=5$ N/mm².

Hoop stress in rim at R_3,

$$f_{cR_3}=-p'\times\frac{R_3^2+R_1^2}{R_3^2-R_1^2}$$

$$=-5\times\frac{300^5+250^2}{300^2-250^2}=-27{\cdot}72\ \text{N/mm}^2$$

Hoop stress in tyre at R_3,

$$f_{cR_3}'=+p'\times\frac{R_2^2+R_3^2}{R_2^2-R_3^2}$$

$$=5\times\frac{325^2+300^2}{325^2-300^2}=62{\cdot}6\ \text{N/mm}^2.$$

Circumferential strain at R_3, in rim

$$\epsilon_c'=-\frac{27{\cdot}72}{E_{CI}}+\frac{5\times 0{\cdot}25}{E_{CI}}$$

$$=\frac{1}{100\times 1000}\,[-27{\cdot}72+1{\cdot}25]=-26{\cdot}47\quad\text{(compressive)}$$

Circumferential strain at R_3 in tyre,

$$\epsilon_c'' = \frac{62.6}{Es} + \frac{0.3\times 5}{Es} = \frac{64.1}{210\times 1000}$$

Shrinkage allowance, $\delta R_3 = \epsilon_c'' \times R_3 - \epsilon_c' \times R_3$

$$= \left[\frac{64.1}{210\times 1000} + \frac{26.47}{100\times 1000}\right]\times 300 = 0.17 \text{ mm}$$

Inside diameter of the steel tyre

$$= 600 - 2\delta R_3 = 600 - 2\times 0.17$$
$$= 600 - 0.34 = 599.66 \text{ mm.}$$

Problem 6.19. A compound cylinder is formed by shrinking one cylinder over the another. The outer diameter of the compound cylinder is 24 cm, inner diameter is 16 cm and the diameter at the common surface is 20 cm. Determine (*a*) shrinkage allowance (*b*) the temperature rise of outer cylinder so that it passes on the inner cylinder, if the junction pressure after shrinking is 50 kg/cm².

Given $E = 2100$ tonnes/cm²

$\alpha = 6.2\times 10^{-6}$/°F

The compound cylinder is now subjected to an internal fluid pressure of 500 kg/cm², determine the maximum hoop tension in the cylinder. How much heavier a single cylinder of internal diameter 16 cm would be if it is subjected to the same internal pressure in order to withstand the same maximum hoop stress.

Solution.

Inner radius, $R_1 = 8$ cm

Outer radius, $R_2 = 12$ cm

Junction radius, $R_3 = 10$ cm

Junction pressure, $p' = 50$ kg/cm².

(*a*) **Shrinkage allowance**

Shrinkage allowance, $$\delta R = \frac{R_3 p'}{E}\left(\frac{R_2^2+R_3^2}{R_2^2-R_3^2} + \frac{R_3^2+R_1^2}{R_3^2-R_1^2}\right)$$

$$= \frac{10\times 50}{2100\times 1000}\left(\frac{12^2+10^2}{12^2-10^2} + \frac{10^2+8^2}{10^2-8^2}\right)$$

$$= \frac{1}{4200}(5.545+4.555) = 0.0024 \text{ cm on radius}$$

(*b*) **Temperature rise**

$$\delta R_3 = R_3 \times \alpha \times T$$
$$0.0024 = 10\times 6.2\times 10^{-6}\times T$$

or Temperature rise, $$T = \frac{0.0024\times 10^6}{62} = 38.7 \text{ °F}$$

Now the cylinder is subjected to an internal pressure of 500 kg/cm². To find out the maximum hoop tension, let us first find out the hoop stresses due to shrinkage pressure p' at the inner radii of both the cylinders.

Hoop stresses due to shrinkage pressure

In Inner cylinder at R_1, $f_c'{}_{R_1}=-p'\ \dfrac{2R_3^2}{R_3^2-R_1^2}$

$$=-50\times\frac{2\times 10^2}{10^2-8^2}=-277{\cdot}77 \text{ kg/cm}^2$$

In Outer cylinder at R_3,

$$f_c''{}_{R_3}=p'\times\frac{R_2^2+R_3^2}{R_2^2-R_3^2}$$

$$=50\times\frac{12^2+10^2}{12^2-10^2}=+277{\cdot}27 \text{ kg/cm}^2$$

Hoop stress due to the internal pressure

Now the compound cylinder is subjected to an internal pressure of 500 kg/cm². Let us assume A, B as Lame's constants.

Boundary conditions

Radial stress, $p_r=p=500$ kg/cm² at $r=R_1$

$p_r=0$ at $r=R_2$

or

$$500=\frac{B}{R_1^2}-A=\frac{B}{8^2}-A$$

$$0=\frac{B}{R_2^2}-A=\frac{B}{12^2}-A$$

$$B=\frac{500\times 12^2\times 8^2}{12^2-8^2}=57600 \text{ kg}$$

$$A=\frac{500\times 8^2}{12^2-8^2}=400 \text{ kg/cm}^2$$

Hoop stress at any radius r,

$$f_c=\frac{B}{r^2}+A$$

Hoop stress at R_1, $f_{c\,R_1}=\dfrac{57600}{8^2}+400$

$$=900+400=1300 \text{ kg/cm}^2$$

Hoop stress at R_3, $f_{c\,R_3}=\dfrac{57600}{10^2}+400$

$$=576+400=976 \text{ kg/cm}^2.$$

Resultant hoop stresses

$$\text{at } R_1,\ f^{R}_{c\,R_1}=f_c'{}_{R_1}+f_{c\,R_1}=-277{\cdot}77+1300=+1022{\cdot}23 \text{ kg/cm}^2$$

$$\text{at } R_3,\ f^{R}_{c\,R_3}=f_c''{}_{R_3}+f_{c\,R_3}=277{\cdot}27+976=+1253{\cdot}27 \text{ kg/cm}^2$$

In this particular case, maximum hoop stress occurs at the junction radius and it is equal to 1253·27 kg/cm².

Single cylinder. Single cylinder of inner radius 8 cm and outer radius say R_2, is subjected to internal fluid pressure of 500 kg/cm² such that the maximum hoop stress is 1253·27 kg/cm². In the case of a single cylinder maximum hoop stress occurs at the inner radius and it is equal to

$$f_{c\,max}=p\times\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

$$1253{\cdot}27=500\times\frac{R_2^2+8^2}{R_2^2-8^2}$$

$$2{\cdot}506\,R_2^2-2{\cdot}506\times 8^2=R_2^2+8^2$$

or

$$1{\cdot}506\,R_2^2=3{\cdot}506\times 8^2$$

$$R_2=12{\cdot}20 \text{ cm}$$

A_s, Area of cross section of single cylinder

$$=\pi(12{\cdot}20^2-8^2)=84{\cdot}84\,\pi \text{ cm}^2$$

A_c Area of cross section of compound cylinder

$$=\pi(12^2-8^2)=80\,\pi \text{ cm}^2$$

$$\frac{\text{Weight of single cylinder}}{\text{Weight of compound cylinder}}=\frac{\delta A_s L}{\delta A_c L}=\frac{A_s}{A_c}=\frac{84{\cdot}84\,\pi}{80\,\pi}=1{\cdot}0605$$

where L=length of the cylinder

δ=density of the material.

Therefore single cylinder is about 6·05% heavier than the compound cylinder.

Problem 6·20. A bronze liner of outside diameter 60 mm and inside diameter 39·94 mm is forced over a steel shaft of 40 mm diameter. Determine (*a*) the radial pressure between shaft and liner (*b*) the maximum circumferential stress in liner (*c*) the change in outside diameter of the liner.

$$E_s=208000 \text{ N/mm}^2$$

Steel, $\frac{1}{m}=0{\cdot}29$

$$E_b=125{,}000 \text{ N/mm}^2$$

Bronze, $\frac{1}{m}=0{\cdot}33.$

Solution. Diametral allowance between shaft and liner

$$=40-39{\cdot}94=0{\cdot}06 \text{ mm}$$

Say the radial pressure at the common radius

$$=p \text{ N/mm}^2$$

Shaft diameter, $D_1=40$ mm

Shaft radius $R_1=20$ mm

Liner outer radius, $R_2=30$ mm

Circumferential stress in shaft

$$=p \text{ (compressive)}$$

Circumferential stress in liner at inner radius,

$$f_c = p\ \frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

$$= p\times \frac{30^2+20^2}{30^2-20^2} = 2{\cdot}6\ p \text{ (tensile)}$$

Diametral strain in shaft (at common radius)

$$= \frac{p}{E_s} - \frac{1}{m}\ \frac{p}{E_s} = \frac{p}{E_s}\ (1-0\ 29)$$

$$= \frac{0{\cdot}71\ p}{E_s} \text{ (compressive)}$$

Diametral strain in liner (at common radius)

$$= \frac{f_c}{E_b} + \frac{1}{m}\ \frac{p}{E_b}$$

$$= \frac{p}{E_b}\ (2{\cdot}6+0{\cdot}33) = \frac{2{\cdot}93\ p}{E_b} \text{ (tensile)}$$

Total diametral allowance,

$$\frac{0{\cdot}71\ p}{E_s}\times 40 + \frac{2{\cdot}93\ p}{E_b}\times 40 = 0{\cdot}06$$

$$\frac{0{\cdot}71\ p\times 40}{208000} + \frac{2{\cdot}93\ p\times 40}{125000} = 0{\cdot}06$$

$$p\ (0{\cdot}1365+0{\cdot}9376) = 60$$

(*a*) Radial pressure, $p = \frac{60}{1{\cdot}0741} = 55{\cdot}86\ \text{N/mm}^2$

(*b*) Maximum circumferential stress in liner occurs at inner radius

$$f_{c\ max} = 2{\cdot}6\ p = 2{\cdot}6\times 55{\cdot}86 = 145{\cdot}23\ \text{N/mm}^2$$

(*c*) Circumferential stress at the outside radius of the liner,

$$f_{c\ min} = p\times \frac{2R_1^2}{R_2^2-R_1^2}$$

$$= p\times \frac{2\times 20^2}{30^2-20^2} = 1{\cdot}6\ p$$

$$= 1{\cdot}6\times 55{\cdot}86 = 89{\cdot}376\ \text{N/mm}^2$$

Change in outside diameter of liner

$$= \frac{f_{c\ min}}{E_b}\times 60$$

$$= \frac{89{\cdot}376}{125{,}000}\times 60 = 0{\cdot}043\ \text{mm}$$

Problem 6·21. A steel sleeve 1·5 cm (radial thickness) thick is pressed on to a solid steel shaft of 5 cm diameter, the junction pressure being p'. An axial tensile force of 10 tonnes is applied to the shaft. Determine the change in (*a*) radial pressure at the common surface (*b*) hoop tension in sleeve. If

$$p' = 250\ \text{kg/cm}^3$$

and $\frac{1}{m} = 0{\cdot}285$ for steel,

Solution.

Radius of shaft, $R_1=2{\cdot}5$ cm

Outer radius of sleeve, $R_2=2{\cdot}5+1{\cdot}5=4$ cm

Radial pressure at common surface,

$$p'=250 \text{ kg/cm}^2$$

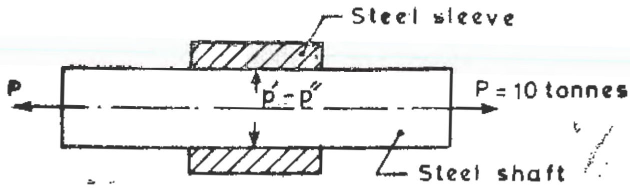


Fig. 6·20

Axial tensile stress, $$f_a=\frac{P}{\pi R_1^2}$$

$$=\frac{10\times 1000}{\pi\times 2{\cdot}5^2}=509{\cdot}29 \text{ kg/cm}^2 \text{ (tensile)}$$

Tensile stress in shaft introduces a lateral strain and its diameter will tend to decrease. Consequently the pressure at the common surface decreases. Say the decrease in radial pressure is p''.

Change in circumferential stress at outer surface of shaft

$$=-p''$$

Change in circumferential stress at inner radius of sleeve

$$=-p''\ \frac{R_2^2+R_1^2}{R_2^2-R_1^2}=-p''\times\frac{4^2+2{\cdot}5^2}{4^2-2{\cdot}5^2}$$

$$=-2{\cdot}28\ p''$$

Change in circumferential strain in shaft at R_1.

$$\epsilon_1=-\frac{p''}{E}+\frac{p''}{mE}-\frac{f_a}{mE}$$

Change in circumferential strain in sleeve at R_1,

$$\epsilon_2=\frac{2{\cdot}28\ p''}{E}+\frac{p''}{mE}$$

But $$\epsilon_1=\epsilon_2 \text{ (strain compatibility)}$$

$$-\frac{p''}{E}+\frac{p''}{mE}-\frac{f_a}{mE}=\frac{2{\cdot}28\ p''}{E}+\frac{p''}{mE}$$

$$-p''-0{\cdot}285\times 509{\cdot}29=2{\cdot}28\ p''$$

or $$3{\cdot}28\ p''=-0{\cdot}285\times 509{\cdot}29$$

$$p''=-44{\cdot}25 \text{ kg/cm}^2$$

i.e., Radial pressure is reduced by $44{\cdot}25$ kg/cm^2

Change in hoop tension in sleeve

$$=-44{\cdot}25\times 2{\cdot}28=-100{\cdot}89 \text{ kg/cm}^2$$

i.e., the hoop tension in sleeve is decreased by $100{\cdot}89$ kg/cm^2,

Problem 6·22. A steel sleeve is pressed on to a steel shaft of 5 cm diameter. The radial pressure between the steel shaft and sleeve is 200 kg/cm². and the hoop stress at the inner radius of the sleeve is 560 kg/cm². If an axial compressive force of 5 tonnes is now applied to the shaft, determine the change in the radial pressure.

$$E=2100 \text{ tonnes/cm}^2$$

$$\frac{1}{m}=0{\cdot}3$$

Solution. Radius of steel shaft, $R_1=2{\cdot}5$ cm

Say outer radius of steel sleeve $=R_2$

Junction pressure, $p'=200$ kg/cm²

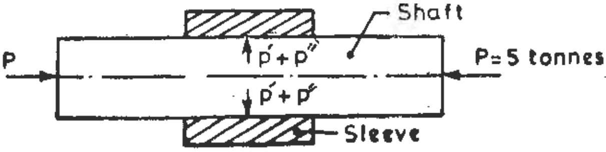


Fig. 6·21

Hoop stress at inner radius of sleeve=560 kg/cm²

Axial stress on shaft, $f_a=\dfrac{P}{\pi\times 2{\cdot}5^2}$

$$=\frac{5\times 1000}{\pi\times 6{\cdot}25}=254{\cdot}647 \text{ kg/cm}^2 \text{ (compressive)}$$

This axial compressive stress introduces lateral strain which is positive *i.e.* the radius of the shaft increases. But sleeve will resist this increase in radius and radial stress at the common surface increases.

Say the increase in radial pressure$=p''$

Due to p'', increase in circumferential stress at the inner radius of sleeve

$$=\frac{560}{200}\times p''=2{\cdot}8\, p''$$

Additional circumferential strain in sleeve at inner radius

$$\epsilon_2=\frac{2{\cdot}8\, p''}{E}+\frac{p''}{mE}$$

Additional circumferential strain in shaft at radius R_1

$$\epsilon_1=-\frac{p''}{E}+\frac{p''}{mE}+\frac{f_a}{mE}$$

where p''=circumferential stress in shaft (compressive)
p''=radial stress in shaft (compressive)
f_a=axial stress in shaft (compressive)

But $\epsilon_2=\epsilon_1$ (for strain compatibility)

$$\frac{2{\cdot}8\, p''}{E}+\frac{p''}{mE}=-\frac{p''}{E}+\frac{p''}{mE}+\frac{f_a}{mE}$$

$$2\cdot8\,p''=-p''+\frac{254\cdot647}{m}$$

$$3\cdot8\,p''=0\cdot3\times254\cdot647$$

$$p''=20\cdot10 \text{ kg/cm}^2$$

i.e., radial pressure at the common surface is increased by 20·10 kg/cm².

Problem 6·23. A steel rod 50 mm in diameter is forced into a bronze sleeve 80 m outside diameter, thereby producing a tension of 40 N/mm² at the outer surface of the sleev Determine (*a*) the radial pressure between the bronze sleeve and steel rod (*b*) the rise temperature which would eliminate the force fit.

$$E_s=E_{steel}=210\times10^3 \text{ N/mm}^2$$

$$E_B=E_{bronze}=114\times10^3 \text{ N/mm}^2$$

$$\frac{1}{m} \text{ for steel}=0\cdot28$$

$$\frac{1}{m} \text{ for bronze}=0\cdot33$$

$$\alpha_{steel}=11\cdot2\times10^{-6}/\text{C}^\circ$$

$$\alpha_{bronze}=18\times10^{-6}/\text{C}^\circ$$

Solution. Radius of steel rod, $R_1=25$ mm

Outer radius of bronze sleeve, $R_2=40$ mm

Hoop stress at the outer surface of sleeve

$$=p'\times\frac{2R_1^2}{R_2^2-R_1^2}$$

where p'=radial pressure between the sleeve and rod

$$40=p'\times\frac{2\times25^2}{40^2-25^2}$$

$$p'=\frac{40\times975}{1250}=31\cdot2 \text{ N/mm}^2$$

Hoop stress at the inner surface of the sleeve,

$$f_c''_{R_1}=p'\times\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

$$=31\cdot2\times\frac{40^2-25^2}{40^2-25^2}=71\cdot2 \text{ N/mm}^2$$

Circumferential strain at R_1, in sleeve,

$$\epsilon_c''=\frac{f_c''_{R_1}}{E_B}+\frac{p'}{m_BE_B}=\frac{71\cdot2}{114\times10^3}+\frac{31\cdot2\times0\cdot33}{114\times10^3}$$

$$=(\cdot624+\cdot090)\times10^{-3}=0\cdot714\times10^{-3}$$

Hoop stress at outer surface of rod

$$=-p'=-31\cdot2 \text{ N/mm}^2$$

Circumferential strain at R_1, in rod

$$\epsilon_0' = -\frac{p'}{E_s} + \frac{p'}{m_s E_s}$$

$$= -\frac{31{\cdot}2}{210\times10^3} + \frac{0{\cdot}28\times31{\cdot}2}{210\times10^3} = -0{\cdot}107\times10^{-3}$$

Force fit allowance, $\delta R_1 = \epsilon_c'' \times R_1 - \epsilon_0' \times R_1$

$$= (0{\cdot}714\times10^{-3} + 0{\cdot}107\times10^{-3})\times25$$

$$= 0{\cdot}821\times10^{-3}\times25 = 0{\cdot}0205 \text{ mm.}$$

Temperature Rise

Say the temperature of rod and sleeve is raised by T°F, the sleeve will expand more than the rod as $\alpha_B > \alpha_S$. When the differential expansion at radius R_1 equals the force fit allowance, then force fit will be eliminated.

$$\delta R_1 = R_1(\alpha_B - \alpha_S)T$$

$$0{\cdot}0205 = 25(18 - 11{\cdot}2)\times10^{-6}\times T$$

or Temperature Rise $= \dfrac{{\cdot}0205\times10^6}{95\times6{\cdot}8} = 120{\cdot}59$ °C

Problem 6·24. A bronze sleeve of outside diameter 80 mm is forced over a steel shaft of diameter 60 mm. The initial inside diameter of the sleeve is less than the diameter of the shaft by 0·06 mm. This compound rod is subjected to external pressure of 25 N/mm² and the temperature is raised by 80°C. Determine :

(*a*) radial pressure between the sleeve and shaft,

(*b*) maximum hoop stress developed in the sleeve.

Given for *steel* $E_S = 20800$ N/mm² ;

$$\frac{1}{m} = 0{\cdot}30\ ;\ \alpha = 11\times10^{-6}/°\text{C}.$$

For *bronze* $E_b = 105{,}000$ N/mm² ;

$$\frac{1}{m} = 0{\cdot}33\ ;\ \alpha = 19\times10^{-6}/°\text{C}.$$

Solution. Let us first determine the junction pressure developed due to the forcing-fit of sleeve over shaft.

Say pressure at common radius $= p_1$

Shaft diameter $= 60$ mm

Shaft radius, $R_1 = 30$ mm

Outside radius of sleeve, $R_2 = 40$ mm

Circumferential stress developed in sleeve,

$$f_{c1} = p_1 \times \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} = p_1 \times \frac{40^2 + 30^2}{40^2 - 30^2}$$

$$= \frac{25}{7} p_1 = 3{\cdot}57\ p_1 \text{ (tensile)}$$

$$f_{r2}=p_1\times\frac{2R_1^2}{R_2^2-R_1^2}=p_1\times\frac{2\times30^2}{40^2-30^2}$$

$$=2{\cdot}57\ p_1 \quad \text{(tensile)}$$

Circumferential stress developed in shaft

$$=p_1 \quad \text{(compressive)}$$

Diametral strain in shaft $=\dfrac{p_1}{E_s}-\dfrac{0{\cdot}3\ p_1}{E_s}=\dfrac{0{\cdot}7\ p_1}{E_s}$ (contraction)

Diametral strain in sleeve $=\dfrac{f_{r1}}{E_b}+\dfrac{0{\cdot}33p_1}{E_b}$

$$=\frac{3{\cdot}57\ p_1+0{\cdot}33\ p_1}{E_b}$$

$$=\frac{3{\cdot}90\ p_1}{E_b} \quad \text{(expansion)}$$

Now $$\left(\frac{0{\cdot}7\ p_1}{E_s}\right)\times40+\left(\frac{3{\cdot}90\ p_1}{E_b}\right)\times40={\cdot}06$$

$$\frac{0{\cdot}7\ p_1\times40}{208000}+\frac{3{\cdot}90\times40\ p_1}{105{,}000}={\cdot}06$$

$$p_1(0{\cdot}1346+1{\cdot}4857)=60$$

Radial pressure, $$p_1=\frac{60}{1{\cdot}6203}\ \text{N/mm}^2=37{\cdot}03\ \text{N/mm}^2$$

Circumferential stress in sleeve,

$$f_{r1}=3{\cdot}57\times37{\cdot}03=132{\cdot}19\ \text{N/mm}^2$$

$$f_{r2}=2{\cdot}57\times37{\cdot}03=95{\cdot}167\ \text{N/m}^2$$

Stresses due to external pressure and rise in temperature

Say the pressure at the common radius $=p_2$

Circumferential stress in shaft

$$=-p_2 \quad \text{(compressive)}$$

Using Lame's equations for the sleeve

$$25=\frac{B}{40^2}-A$$

$$p_2=\frac{B}{30^2}-A \quad \text{(given in problem)}$$

where A and B are constants.

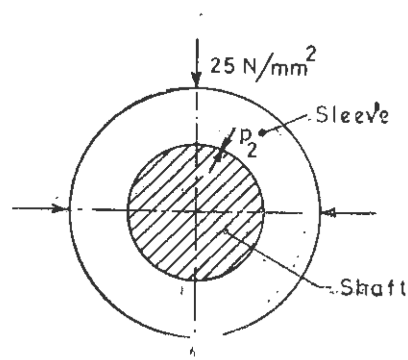


Fig. 6·22

$$(25-p_2)=\frac{B}{40^2}-\frac{B}{30^2}=\frac{B(30^2-40^2)}{30^2\times40^2}$$

or
$$B=-(25-p_2)\ \frac{30^2\times 40^2}{40^2-30^2}$$
$$A=\frac{B}{40^2}-25=-(25-p_2)\ \frac{30^2}{40^2-30^2}-25$$

Circumferential stress, at inner radius
$$f_{c1}'=\frac{B}{30^2}\ +A$$
$$=(p_2-25)\ \frac{40^2}{40^2-30^2}+(p_2-25)\ \frac{30^2}{40^2-30^2}-25$$
$$=(p_2-25)\ \frac{25}{7}-25=3{\cdot}57\ p_2-114{\cdot}28$$

At outer radius, $f_{c2}'=\frac{B}{40^2}+A$
$$=(p_2-25)\ \frac{30^2}{40^2-30^2}\ +(p_2-25)\ \frac{30^2}{40^2-30^2}-25$$
$$=(p_2-25)\ \frac{2\times 30^2}{40^2-30^2}-25=2{\cdot}57\ p_2-89{\cdot}28$$

Equating the strains at the common radius
$$\frac{p_2}{E_s}-\frac{0{\cdot}3\ p_2}{E_s}+11\times 10^{-6}\times 80$$
$$=\frac{3{\cdot}57\ p_2-114{\cdot}28}{E_b}+\frac{0{\cdot}33\ p_2}{E_b}+19\times 10^{-6}\times 80$$
$$\frac{0{\cdot}7\ p_2}{208000}+11\times 10^{-6}\times 80=\frac{3{\cdot}90\ p_2}{105000}-\frac{114{\cdot}28}{105000}+19\times 10^{-6}\times 80$$
$$0{\cdot}336\ p_2=3{\cdot}71\ p_2-108{\cdot}83+64$$
$$44{\cdot}83=3{\cdot}374\ p_2$$
$$p_2=13{\cdot}287\ \text{N/mm}^2$$

(*a*) Final radial pressure between sleeve and shaft
$$=13{\cdot}287+37{\cdot}03=50{\cdot}317\ \text{N/mm}^2.$$

Resultant circumferential stress at the inner radius of the sleeve
$$=132{\cdot}19+3{\cdot}57\ p_2-114{\cdot}28$$
$$=132{\cdot}19+3{\cdot}57\times 13{\cdot}287-114{\cdot}28$$
$$=132{\cdot}19+47{\cdot}434-114{\cdot}28$$
$$=65{\cdot}344\ \text{N/mm}^2$$

Resultant circumferential stress at the outer radius of the sleeve
$$=95{\cdot}167+2{\cdot}57\ p_2-89{\cdot}28$$
$$=95{\cdot}167+2{\cdot}57\times 13{\cdot}287-89{\cdot}28$$
$$=95{\cdot}167+34{\cdot}148-89{\cdot}28$$
$$=40{\cdot}035\ \text{N/mm}^2.$$

So the maximum hoop stress developed in sleeve
$$=65{\cdot}344\ \text{N/mm}^2.$$

Problem 6·25. A steel ring of internal radius r and external radius R is shrunk on to a solid steel shaft of radius $r+dr$. Prove that the intensity of pressure p at the mating surface is equal to $\left(1-\frac{r^2}{R^2}\right) E \frac{dr}{2r}$ where E is the modulus of elasticity of steel.

Solution. Radius of shaft $=r$

Outer radius of ring $=R$

Junction pressure $=p$

The mating surface between ring and shaft is at radius r. The ring and shaft are made of the same material.

Hoop stress in shaft at junction $=-p$ (compressive)

Hoop stress in ring at junction $=p\,\frac{R^2+r^2}{R^2-r^2}$ (tensile)

Shrinkage allowance, $dr=\frac{r}{E}\left[p\,\frac{R^2+r^2}{R^2-r^2}+p\right]$

$$dr=\frac{pr}{E}\left[\frac{R^2+r^2+R^2-r^2}{R^2-r^2}\right]=\frac{2pr}{E}\left[\frac{R^2}{R^2-r^2}\right]$$

or
$$p=\frac{dr}{2r}\times E\left[\frac{R^2-r^2}{R^2}\right]=E\,\frac{dr}{2r}\left[1-\frac{r^2}{R^2}\right]$$

Problem 6·26. A steel plug 80 mm in diameter is forced into a steel ring 120 mm external diameter and 50 mm wide. From a strain gage fixed on the outer surface of the ring in the circumferential direction, the strain is found to be $0{\cdot}41\times10^{-4}$. Considering that the coefficient of friction between the mating surfaces, $\mu=0{\cdot}2$, determine the axial force required to push the plug out of the ring. $E=210\times10^3$ N/mm².

Solution.

Inner radius, $R_1=40$ mm

Outer radius, $R_2=60$ mm

Breadth of the ring, $B=50$ mm

Say junction pressure $=p'$

Hoop stress or circumferential stress at outer surface of ring

$$=+p'\times\frac{2R_1^2}{R_2^2-R_1^2}$$
$$=p'\times\frac{2\times40^2}{60^2-40^2}=1{\cdot}6\,p'$$

Radial stress at outer surface of ring $=0$

So circumferential strain on outer surface of ring

$$=\frac{1{\cdot}6\,p'}{E}=0{\cdot}41\times10^{-4}$$
$$1{\cdot}6\,p'=0{\cdot}41\times10^{-4}\times210\times10^3$$
$$p'=5{\cdot}38\text{ N/mm}^2.$$

F_R, Radial force acting on the plug

$$=p'\times2\pi\,R_1\times B$$
$$=5{\cdot}38\times2\pi\times40\times50=67607{\cdot}23\text{ N}$$

Coefficient of friction, $\mu=0{\cdot}2$

Axial force, $F_A=\mu\, F_R=0{\cdot}2\times 67607{\cdot}23$ N

$=13521{\cdot}446$ N

Axial force required to push the plug out of the ring $=13{\cdot}521$ kN.

Problem 6·27. A cylindrical steel plug 8 cm in diameter is forced into a brass sleeve of 14 cm external diameter and 10 cm long. If the greatest hoop stress developed in the sleeve is 600 kg/cm², determine the torque required to turn the plug in the sleeve assuming $\mu=0{\cdot}18$, *i.e.*, the coefficient of friction between steel plug and brass sleeve.

Solution.

Inner radius, $R_1=4$ cm

Outer radius, $R_2=7$ cm

Say junction pressure $=p'$

Length of sleeve, $l=10$ cm

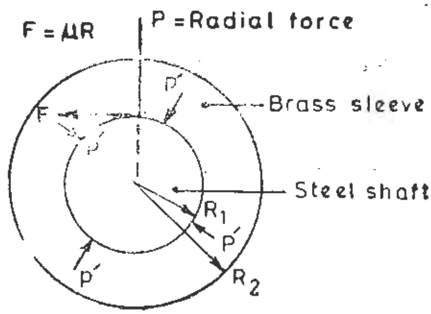


Fig. 6·23

Greatest hoop stress in sleeve occurs at radius R_1 and is equal to

$$f_{c\,max}=p'\ \frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

$$600=p'\times\frac{7^2+4^2}{7^2-4^2}$$

$$n'=\frac{600\times 33}{65}=304{\cdot}61\ \text{kg/cm}^2$$

Total radial force P acting throughout the mating surface

$=2\pi\ R_1\times l\times p'$

$=2\pi\times 4\times 10\times 304{\cdot}61=76557{\cdot}0$ kg

Tangential force, $F=\mu R$

$=0{\cdot}18\times 76557=13780{\cdot}26$ kg

Torque required to turn the plug in the sleeve

$T=F\times R_1=13780{\cdot}26\times 4$ kg-cm

$=551{\cdot}21$ kg-m

SUMMARY

1. Single thick cylinder of inner radius R_1, outer radius R_2 subjected to internal pressure p

Radial stress, $p_r=\dfrac{B}{r^2}-A$ (compressive)

Hoop stress, $f_c=\dfrac{B}{r^2}+A$ (tensile)

where A and B are Lame's constants

$$B=p.\left(\frac{R_1^2R_2^2}{R_2^2-R_1^2}\right),\qquad A=\frac{pR_1^2}{R_2^2-R_1^2}$$

Maximum hoop stress at R_1,

$$f_{c\ max}=p.\left(\frac{R_2^2+R_1^2}{R_2^2-R_1^2}\right)$$

Minimum hoop stress at R_2,

$$f_{c\ min}=p.\left(\frac{2R_1^2}{R_2^2-R_1^2}\right)$$

2. Single thick cylinder subjected to external pressure p, inner radius R_1, outer radius R_2

Lame's constants $\quad B=-p\,\frac{R_1^2R_2^2}{R_2^2-R_1^2} \qquad A=-p\left(\frac{R_2^2}{R_2^2-R_1^2}\right)$

Maximum hoop stress at R_1,

$$f_{c\ max}=-\frac{2p\ R_2^2}{R_2^2-R_1^2}$$

Minimum hoop stress at R_2,

$$f_{c\ min}=-p\left(\frac{R_2^2+R_1^2}{R_2^2-R_1^2}\right)$$

3. A cylinder of outer radius R_2 is shrunk over another cylinder of inner radius R_1; junction radius R_3 and junction pressure developed is p'. Compound cylinder subjected to internal pressure p. Hoop stresses due to shrinkage.

Inner cylinder $\quad f_c'{}_{R_1}=-p'\left(\frac{2R_3^2}{R_3^2-R_1^2}\right)$

$$f_c'{}_{R_3}=-p'\left(\frac{R_3^2+R_1^2}{R_3^2-R_1^2}\right)$$

Outer cylinder $\quad f_c''{}_{R_3}=+p'\left(\frac{R_2^2+R_3^2}{R_2^2-R_3^2}\right)$

$$f_c''{}_{R_2}=+p\left(\frac{2R_3^2}{R_2^2-R_3^2}\right)$$

Hoop stresses due to internal pressure p

$$f_{cR_1}=p\left(\frac{R_2^2+R_1^2}{R_2^2-R_1^2}\right)$$

$$f_{cR_3}=p\,\frac{R_1^2}{R_3^2}\left(\frac{R_2^2+R_3^2}{R_2^2-R_1^2}\right)$$

$$f_{cR_2}=\frac{2pR_1^2}{R_2^2-R_1^2}$$

Resultant hoop stress at any radius is obtained by combining the stresses due to shrinkage and internal pressure.

4. Shrinkage allowance at common radius in a compound cylinder where

R_1=Inner radius, R_2=outer radius, R_3=junction radius and p'=junction pressure.

E_1, $\frac{1}{m_1}$ and E_2, $\frac{1}{m_2}$ are the elastic constants for inner and outer cylinders respectively.

$$\delta R_3 = R_3\left[\frac{p'}{E_1}\left(\frac{R_3^2+R_1^2}{R_3^2-R_1^2}\right)-\frac{p'}{m_1E_1}\right]$$
$$+R_3\left[\frac{p'}{E_2}\left(\frac{R_2^2+R_3^2}{R_2^2-R_3^2}\right)+\frac{p'}{m_2E_2}\right]$$

when both the cylinders are of the same material

$$\delta R_3=\frac{R_3p'}{E}\left[\frac{R_3^2+R_1^2}{R_3^2-R_1^2}+\frac{R_2^2+R_3^2}{R_2^2-R_3^2}\right]$$

$\delta R_3=\alpha R_3T$ where α=coefficient of linear expansion of outer cylinder

T=Temperature rise.

5. A shaft of radius R_1 forced into the hub or liner of outside radius R_2, junction pressure developed p'.

Hoop stress in shaft $=-p'$ (compressive)

Maximum hoop stress in hub at R_1,

$$f_cR_1=p'\times\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

Minimum hoop stress in hub at R_2,

$$f_cR_2=p'\times\frac{2R_1^2}{R_2^2-R_1^2}$$

Force fit allowance on hub,

$$\delta R_1=\frac{p'R_1}{E}\left(\frac{2R_2^2}{R_2^2-R_1^2}\right)$$

when hub and shaft are of the same material.

MULTIPLE-CHOICE QUESTIONS

1. A thick cylinder of inner diameter 60 mm and outer diameter 100 mm is subjected to an internal fluid pressure of 64 N/mm², the maximum hoop tension developed in the cylinder is

(*a*) 32 N/mm² (*b*) 64 N/mm²
(*c*) 128 N/mm² (*d*) 136 N/mm².

2. A thick cylinder is subjected to an internal pressure of 50 N/mm², which produces maximum hoop tension at the inner radius of the cylinder and is equal to 90 N/mm². If the inner radius of the cylinder is 40 mm, the maximum shear stress developed in the cylinder is

(*a*) 140 N/mm² (*b*) 70 N/mm²
(*c*) 45 N/mm² (*d*) 20 N/mm².

3. In a thick cylinder of inner radius R_1, wall thickness t, an internal pressure p produces maximum hoop tension 1·25 p. The magnitude of wall thickness will be

(*a*) 2 R_1 (*b*) 1·5 R_1
(*c*) R_1 (*d*) 0·5 R_1.

4. The variation of hoop stress across the thickness of a thick cylinder is
   (a) Linear (b) Uniform
   (c) Parabolic (d) None of the above.

5. A thick cylindrical shell of inner radius 4 cm and outer radius 6 cm is subjected to external pressure of 20 N/mm². The maximum hoop stress developed is
   (a) −72 N/mm² (b) −52 N/mm²
   (c) +72 N/mm² (d) +52 N/mm².

6. The purpose of compounding cylinders is
   (a) To increase the pressure bearing capacity of a single cylinder
   (b) To make the hoop stress distribution uniform
   (c) To increase the strength of the cylinder
   (d) All the above.

7. A compound cylinder is made by shrinking a cylinder of outer R_2 over another cylinder of inner radius R_1 such that the junction pressure is p at the junction radius R_3. The shrinkage allowance over the diameter is given by

   (a) $\dfrac{2R_3\,p}{E}\left[\dfrac{R_2^2+R_3^2}{R_2^2-R_3^2}+\dfrac{2R_1^2}{R_3^2-R_1^2}\right]$

   (b) $\dfrac{2R_3\,p}{E}\left[\dfrac{2R_3^2}{R_3^2-R_2^2}+\dfrac{R_3^2+R_1^2}{R_3^2-R_1^2}\right]$

   (c) $\dfrac{2R_3\,p}{E}\left[\dfrac{R_2^2+R_3^2}{R_2^2-R_3^2}+\dfrac{R_3^2+R_1^2}{R_3^2-R_1^2}\right]$

   (d) $\dfrac{2R_3\,p}{E}\left[\dfrac{R_2^2+R_3^2}{R_2^2-R_3^2}-\dfrac{R_3^2+R_1^2}{R_3^2-R_1^2}\right]$.

   where E is the young's modulus.

8. A compound cylinder is obtained by shrinking on one steel cylinder over another steel cylinder. The circumferential stresses developed at the junction in the outer and inner cylinders are +840 kg/cm² and −660 kg/cm². If $E = 100 \times 1000$ kg/cm² and junction radius is 10 cm, then shrinkage allowance on diameter is
   (a) 0·3 cm (b) 0·15 cm
   (c) 0·036 cm (d) 0·018 cm.

9. A compound cylinder is obtained by shrinking on one cylinder over another, the dimensions of the compound cylinder are inner radius 3 cm, outer radius 5 cm and junction radius 4 cm. If the hoop stress developed in the outer cylinder at the junction radius is 287 kg/cm². Then the hoop stress developed in the inner cylinder at the junction radius is
   (a) −287 kg/cm² (b) −225 kg/cm²
   (c) −162 kg/cm² (d) None of the above.

10. A bronze sleeve of outer diameter 10 cm is forced over a solid steel shaft of 8 cm dia. If the maximum hoop tension developed in sleeve is 164 N/mm², maximum hoop stress tension developed in steel shaft is
   (a) +36 N/mm² (b) −72 N/mm²
   (c) −36 N/mm² (d) −18 N/mm².

11. A steel sleeve of outer diameter 10 cm is forced over a solid steel shaft of diameter 6 cm. If the junction pressure is 32 N/mm², the hoop stress at the outer radius of sleeve is

(*a*) 68 N/mm² (*b*) 36 N/mm²

(*c*) 32 N/mm² (*d*) 16 N/mm².

ANSWERS

1. (*d*).	2. (*b*).	3. (*a*).	4. (*c*).	5. (*a*).
6. (*a*).	7. (*c*).	8. (*a*).	9. (*b*).	10. (*c*).
11. (*b*).				

EXERCISES

6·1. A steel cylinder 50 cm inside diameter and 3 metre long is subjected to an internal pressure of 200 kg/cm². Determine the thickness of the cylinder if the maximum shear stress in the cylinder is not to exceed 450 kg/cm². What will be the increase in the volume of the cylinder.

$$E=2000\times 10^3 \text{ kg/cm}^2$$

$$\frac{1}{m}=0{\cdot}285.$$

[Ans. 8·54 cm, 435·6 c.c]

6·2. A pressure vessel 40 cm internal diameter and 100 cm external diameter is subjected to a hydraulic pressure of 500 kg/cm². Determine the change in internal and external diameters.

Given $E=2100$ tonnes/cm²,

$$\frac{1}{m}=0{\cdot}30.$$

[Ans. 0·01546 cm, 0·00771 cm]

6·3. Strain gauges are fixed on the outer surface of a thick cylinder with diameter ratio of 2. The cylinder is subjected to an internal pressure of 1000 kg/cm². The recorded strains are

Longitudinal strain $=560\times 10^{-6}$

Circumferential strain $=120\times 10^{-6}$

Determine the Young's modulus and Poisson's ratio of the material.

[Ans. 10×10^5 kg/cm², 0·32]

6·4. A thick cylinder 150 mm internal diameter and 200 mm external diameter is used for a working pressure of 200 kg/cm². Because of external corrosion, the outer diameter of the cylinder is machined to 197 mm. Determine by how much the internal pressure is to be reduced so that the maximum hoop stress developed remains the same as before.

[Ans. 10 kg/cm²]

6·5 Two thick cylinders *A* and *B* made of brass have the same dimensions ; the outer diameter is 1·8 times the inner diameter. The cylinder *A* is subjected to an internal pressure while *B* is subjected to an external radial pressure only. Determine the ratio of these pressures when the greatest circumferential strain is of the same numerical value for both.

Take Poisson's ratio of brass $=0{\cdot}32$.

$$\left[\textbf{Ans.}\ \ \frac{p_A}{p_B}=1{\cdot}18\right]$$

6·6. The maximum permissible stress in a thick cylinder of 50 mm internal radius and 80 mm external radius is 30 N/mm². If the external radial pressure is 5 N/mm², determine the intensity of internal radial pressure. [**Ans.** 20·337 N/mm²]

6·7. A think cylinder of internal diameter D and wall thickness t is subjected to an internal pressure p. Determine the ratio of t/D if the maximum hoop tension developed in the cylinder is 3·8 p. [**Ans.** 0·1546]

6·8. A cylinder of internal diameter D and wall thickness t is subjected to an internal pressure p. Considering this to be a thin cylindrical shell, what is the maximum value of t/D if the error in the estimated value of maximum hoop stress is not to exceed 5% ? [**Ans.** 0·050]

6·9 A thick cylinder of internal diameter D and wall thickness t is subjected to the internal pressure. It is maximum hoop stress developed in the cylinder is 2·6 times the internal pressure, determine the ratio of t/D.

Find the increase in the internal and external diameters of such a cylinder with 12 cm internal diameter subjected to internal fluid pressure of 600 kg/cm².

$$E=2100 \text{ tonnes/cm}^2$$

$$\frac{1}{m}=0{\cdot}3$$ [**Ans.** 0·5, 0·00912 cm, 0·00699 cm]

6·10. A thick cylinder of internal diameter 18 cm is subjected to an internal pressure of 80 kg/cm². If the allowable stress for the cylinder is 350 kg/cm², determine the wall thickness of the cylinder. The cylinder is now strengthened by wire winding so that it can be safely subjected to an internal pressure of 120 kg/cm². Find the radial pressure exerted by wire winding. [**Ans.** 2·358 cm ; 32·56 kg/cm²]

6·11. A compound cylinder is made by shrinking a cylinder of 18 cm outer diameter over another cylinder of 11 cm inner diameter Find the common diameter if the greatest circumferential stress in the inner cylinder is numerically 0·8 times of that of the outer cylinder. [**Ans.** 15·2 cm]

6·12. A steel cylinder of outer diameter 20 cm is shrunk on another cylinder of inner diameter 10 cm, the common diameter being 16 cm If after shrinking on, the radial pressure at the common surface is 150 kg/cm², determine the magnitude of the internal pressure p to which the compound cylinder can be subjected so that the maximum hoop tensions in the inner and outer cylinders are equal. [**Ans.** 1446·22 kg/cm²]

6·13. A compound cylinder has a bore of 16 cm, the outer diameter is 24 cm and diameter at the common surface which is 20 cm. Determine the radial pressure at the common surface which must be provided by the shrinkage fitting, if the resultant hoop stress in the inner cylinder under a superimposed internal pressure of 400 kg/cm² is to be 40% of the value of the maximum hoop tension in the inner cylinder, if this cylinder alone is subjected to an internal pressure of 400 kg/cm².

Determine the resultant hoop stresses at the inner and outer radii of both the cylinders.

[**Ans.** 56 kg/cm², inner cylinder 728·89, 525·69 kg/cm², outer cylinder 1091·345, 894·545 kg/cm²].

6·14. A compound cylinder consists of a steel cylinder 20 cm internal diameter and 30 cm external diameter and a bronze liner of 20 cm; external diameter and 18 cm internal diameter. Assuming the liner to be a thin cylinder and that there is no stress in the compound cylinder due to fitting, determine the maximum direct stress and maximum shear stress in each material due to an internal pressure of 400 kg/cm². Ignore the longitudinal stress and longitudinal strain.

E_{steel}=2100 tonnes/cm² E_{bronze}=1050 tonnes/cm²

$\frac{1}{m}$ steel=0·30 $\frac{1}{m}$ bronze=0·32

[**Ans.** liner 1280, 840 kg/cm², cylinder 605·176, 418·97 kg/cm²].

6·15. A steel tube of outside diameter 30 cm is shrunk an another tube of inside diameter 22 cm. The diameter at the junction is 26 cm after shrinking on. The shrinkage allowance provided on the radius of the outer tube is 0·1 mm. Determine

(*a*) Junction pressure

(*b*) Hoop stresses at the outer and inner radii of the inner tube

(*c*) Hoop stresses at the outer and inner radii of the outer tube

E=1050 tonnes/cm².

[**Ans.** (*a*) 61·81 kg/cm² (*b*) −435·24, −373·43 kg/cm² (*c*) +434·88, 373·07 kg/cm²]

6·16. A steel cylinder 80 mm internal diameter and 120 mm external diameter is strengthened by shrinking another steel cylinder onto it, the internal diameter of which before heating is 119·9 mm. Determine the outer diameter of the outer cylinder if the pressure at the junction after shrinkage is 25 N/mm².

E for steel$=210\times10^3$ N/mm² [**Ans.** 144·72 mm]

6·17. A thick steel cylinder of inner diameter 15 cm and outer diameter 20 cm is subjected to an internal fluid pressure of 1600 kg/cm². A cylindrical jacket 2·5 cm thick of the same material is shrunk on to the cylinder so that the maximum hoop stress developed in the cylinder is not to exceed 2400 kg/cm². What should be the initial difference between the inner diameter of the jacket and the outer diameter of the cylinder.

E=2000 tonnes/cm²

$\frac{1}{m}$=0·28 [**Ans.** 0·0589 cm]

6·18. A high tensile steel tyre 3 cm thick is shrunk on a cast iron rim of internal diameter 60 cm and external diameter 80 cm. Find the inside diameter of the steel tyre if after shrinking on, the tyre exerts a radial pressure of 100 kg/cm². Given :

E_{steel}=2100 tonnes/cm² $E_{C.I.}$=1000 tonnes/cm².

$\frac{1}{m}$ for steel=0·30, $\frac{1}{m}$ for C.I.=0·25 [**Ans.** 79·92 cm]

6·19. A compound cylinder is formed by shrinking one cylinder over another. The outer diameter of the compound cylinder is 200 mm, inner diameter 140 mm and the diameter at the common surface is 170 mm. Determine :

(*a*) Shrinkage allowance.

(*b*) Temperature rise of outer cylinder so that it passes on the inner cylinder, if the junction pressure after shrinking is 5 N/mm².

$E=210\times10^3$ N/mm²

$\alpha=6{\cdot}2\times10^{-6}$/°F.

the compound cylinder is now subjected to an internal fluid pressure of 40 N/mm², determine the maximum hoop tension in the cylinder. How much heavier a single cylinder of inner diameter 140 mm would be if it is subjected to the same internal pressure in order to withstand the same maximum hoop stress.

[**Ans.** 0·046 mm, 43·64 °F ; maximum hoop stress at inner radius of inner cylinder =85·785 N/mm² ; 67·8% heavier]

6·20. A bronze liner of outside diameter 100 mm and inside diameter 69·92 mm is forced over a steel shaft of 70 mm diameter. Determine (*a*) the radial pressure between the shaft and liner (*b*) the maximum circumferential stress in liner (*c*) change in outside diameter of the liner.

$$E_{steel}=2080 \text{ tonnes/cm}^2, \qquad E_{bronze}=1200 \text{ tonnes/cm}^2$$

$$\frac{1}{m}\text{steel}=0·29 \qquad \frac{1}{m}\text{ bronze}=0·32$$

[**Ans.** (*a*) 376·77 kg/cm² (*b*) 917·284 kg/cm² (*c*) 0·060 mm]

6·21. A steel sleeve 10 mm thick is pressed on to a solid steel shaft of 60 mm diameter. The junction pressure being p'. An axial tensile force of 50 kN is applied to the shaft. Determine the change in (*a*) radial pressure at the common surface (*b*) hoop tension in sleeve.

If $p'=30$ N/mm², $\frac{1}{m}$ for steel=0·30 [**Ans.** (*a*) 2·82 N/mm² (*b*) 5·996 N/mm²]

6·22. A bronze sleeve is pressed on to a steel shaft of 80 mm diameter. The radial pressure between steel shaft and sleeve is 15 N/mm² and the hoop stress at the inner radius of of the sleeve is 50 N/mm². If an axial compressive force of 60 kN is now applied to the shaft determine the change in radial pressure

$$E_{steel}=208\times10^3 \text{ N/mm}^2 \qquad \frac{1}{m}\text{ steel}=0·30$$

$$E_{bronze}=120\times10^3 \text{ N/mm}^2 \qquad \frac{1}{m}\text{ bronze}=0·33$$

[**Ans.** 0·85 N/mm²]

6·23. A steel rod 80 mm in diameter is forced into a bronze sleeve 120 mm in diameter, thereby producing a tension of 200 kg/cm² at the outer surface of the sleeve. Determine (*a*) the radial pressure between the bronze sleeve and steel rod (*b*) the rise in temperature which would eliminate the force fit.

$$E_s=2100 \text{ tonnes/cm}^2 \qquad E_B=1140 \text{ tonnes/cm}^2$$

$$\frac{1}{m}\text{ for steel}=0·28 \qquad \frac{1}{m}\text{ for bronze}=0·33$$

$$\alpha_{steel}=11·2\times10^{-6}/°C \qquad \alpha_{bronze}=18\times10^{-6}/°C$$

[**Ans.** 125 kg/cm², 53·54°C]

6·24. A bronze sleeve of outside diameter 12 cm is forced over a steel shaft of diameter 8 cm. The initial inside diameter of the sleeve is less than the diameter of the shaft by 0·1 mm. This compound rod is subjected to external pressure of 160 kg/cm² and the temperature is raised by 50°C. Determine (*a*) radial pressure between sleeve and shaft (*b*) maximum hoop stress developed in sleeve.

$$E_s=2080{,}000 \text{ kg/cm}^2, \qquad E_B=1050{,}000 \text{ kg/cm}^2$$

$$\frac{1}{m}\text{ steel}=0·30 \qquad \frac{1}{m}\text{ bronze}=0·33$$

$$\alpha_s=11\times10^{-6}/°C \qquad \alpha_B=19\times10^{-6}/°C$$

[**Ans.** 460·41 kg/cm², 621·066 kg/cm²]

6·25. A steel plug 10 cm in diameter is forced into a steel ring 12 cm external diameter and 10 cm wide. From a strain gauge fixed on the outer surface of the ring in the circumferential direction, the strain is found to be 50 microstrains. Considering that the coefficient of friction between the mating surfaces, $\mu = 0{\cdot}22$, determine the axial force required to push the plug out of the ring.

$E = 2100$ tonnes/cm^2 [**Ans.** 1596·56 kg]

6·26. A cylindrical steel plug 60 mm in diameter is forced into a brass sleeve of 100 mm external diameter and 5 cm wide. If the greatest hoop stress developed in the sleeve is 80 N/mm^2, determine the torque required to turn the plug in the sleeve assuming $\mu = 0{\cdot}2$ *i.e.* the coefficient of friction between steel plug and brass sleeve. [**Ans.** [2·128 k Nm]

7

Shear Force and Bending Moment Diagrams

In chapters 1 to 4 we have studied the effect of axial forces applied on machine members, producing tensile or compressive stresses and elongation or contraction along the length of the member. In this chapter we will study the effect of forces applied transverse to the axis of the member, producing bending in the member.

Any structural member sufficiently long as compared to its lateral dimensions, supported along the length and subjected to loads (forces) transverse to its longitudinal axis is called a beam. The applied transverse loads are such that they lie in the plane defined by an axis of symmetry of the cross section. Generally the beam is horizontal and the applied loads are in a vertical plane.

7·1. VARIOUS TYPES OF BEAMS

The ends of a beam can be simply supported, fixed or free as shown in the Fig. 7·1.

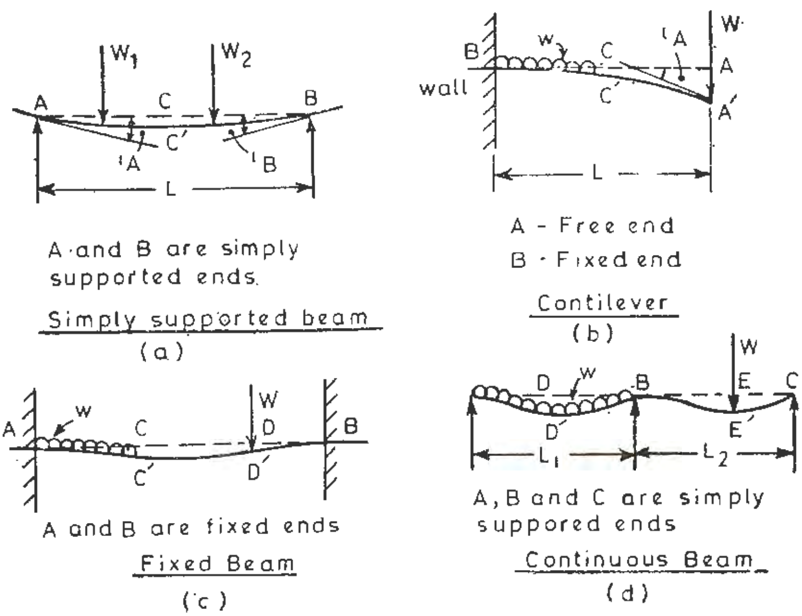


Fig. 7·1

Fig. 7·1 (*a*) shows a horizontal beam with longitudinal axis *ACB* supported on ends *A* and *B* subjected to vertical point loads or concentrated loads W_1 and W_2. After the application of these loads transverse to the axis, the beam bends and takes the shape *AC'B*. At the end *A*, the slope of the beam has changed from zero to i_A, but the vertical displacement of the point *A* is zero. Similarly at the end *B*, the slope of the beam has changed from zero to i_B but the vertical displacement of the point *B* is zero. The ends where slope changes but the vertical displacement or deflection remains zero, are called *supply supported ends*. With two simply supported ends, the beam is said to be a simply supported beam. The distance between the two supports *A* and *B* is called the span of the beam.

Fig. 7·1 (*b*) shows a horizontal beam with longitudinal axis *ACB* subjected to a vertical concentrated load *W* and a uniformly distributed load of intensity *w* per unit length on the portion *CB*. The end *A* of the beam is free and the end *B* is fixed in the wall. After the application of the loads, the beam has bent and axis has taken the shape *A'C'B*. At the end *A*, there is vertical displacement or vertical deflection *AA'* and slope has changed from zero to i_A. Such an end which is free to take any slope and any position is called a *free end*. At the end *B*, there is no vertical deflection and slope remains unchanged *i.e.*, slope remains zero before and after the application of transverse loads, such an end is called a *fixed end i.e.*, an end whose position and direction (*i.e.*, slope) remain unchanged.

A beam with one end free and the other end fixed is called *cantilever.*

Fig. 7·1 (*c*) shows a beam with both of its ends *A* and *B* fixed. *ACDB* is the longitudinal axis of the beam before the application of transverse load *W* at point *D* and a uniformly distributed load *w* over the portion *AC*, which changes to *AC'D'B* after the application of the loads. At ends *A* and *B*, there is no deflection and slope remains zero. Such a beam is called a *fixed beam.*

Fig. 7·1 (*d*) show a horizontal beam with longitudinal axis *ADBEC*, subjected to uniformly distributed load of intensity *w* per unit length over portion *AB* and a concentrated load *W* at point *E*. The ends *A* and *C* of the beam are simply supported. This beam is also supported at the point *C*. Such a beam which is supported on more than two supports is called a *continuous beam.*

In this chapter we will discuss the bending of simply supported beams and cantilevers. There will be detailed discussion on fixed and continuous beams in chapter 11.

7·2. SHEAR FORCE DIAGRAM OF SIMPLY SUPPORTED BEAM SUBJECTED TO CONCENTRATED LOADS

Fig. 7·2 shows a horizontal beam *ACDB*, of rectangular cross section, of length *L* and carrying a vertical concentrated load *W* at the centre of the beam and is simply supported at the ends *B* and *D*. To determine the reactions at the supports, let us take moments of the forces about the point *B*.

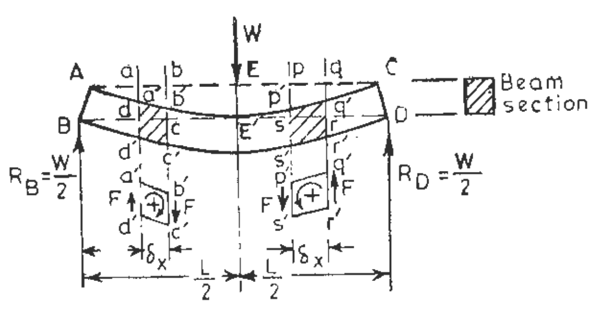


Fig. 7·2

$$W\times\frac{L}{2}-R_D\times L=0 \qquad \ldots(1)$$

$$W\downarrow=R_B\uparrow+R_D\uparrow \qquad \ldots(2)$$

For equilibrium, the vertical forces must balance and resultant moment of forces at any point is zero.

From equation (1) $\quad R_D=\frac{W}{2}\uparrow$

From equation (2) $\quad R_B=W-R_D$

$$=W-\frac{W}{2}=\frac{W}{2}\uparrow$$

Consider a small element of the beam (*abcd*) of length δx, at a distance of x from the end A. After bending, the element is distorted to the shape $a'b'c'd'$, which is the result of the application of a shear force.

Resultant force on the left side of the element

$$=\frac{W}{2}\uparrow$$

Resultant force on the right side of the element

$$=W-\frac{W}{2}=\frac{W}{2}\downarrow$$

This type of shear force which tends to rotate the element in a clockwise direction is called a positive shear force.

Consider another small element *pqrs* of length δx in the portion EC of the beam. After bending, the element is distorted to the shape $p'q'r's'$, which is the result of the application of a shear force on the element.

Resultant force on the left side of the element

$$=W\downarrow-\frac{W}{2}\uparrow=\frac{W}{2}\downarrow$$

Resultant force on the right side of the element

$$=\frac{W}{2}\uparrow$$

This type of the shear force which tends to rotate the element in an anticlockwise direction is called a negative shear force. In the limit $\delta x\rightarrow 0$, the shear force is defined at a certain cross section of the beam at a certain distance x from any end of the beam.

From this discussion, we can define that the magnitude of the shear force at any cross section of a beam is the unbalanced vertical force to the left or to the right side of the section.

Considering any cross section of the beam in the portion AE, the shear force on the left side of the section is $W/2 \uparrow$. Similarly considering any cross section of the beam in the portion EC, the shear force on the left side of the section is $W/2 \downarrow$ or in other words, shear force in portion AE is $+W/2$ and shear force in the portion EC is $-W/2$.

Fig. 7·3 (*a*) shows a beam of length L, simply supported at ends and carrying a concentrated load W at its centre. Reactions at both the ends $= W/2$ each.

Generally the depth of the beam is not shown while showing the transverse loads on any beam.

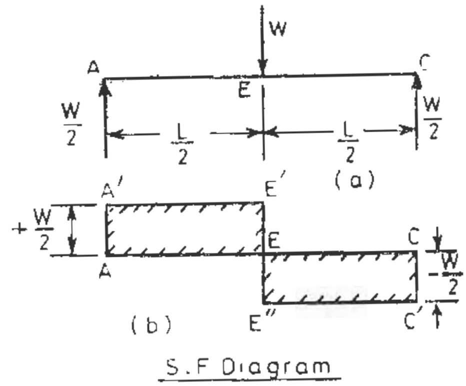


Fig. 7·3

Fig. 7·3 (*b*) shows the shear force diagram of the beam. To some suitable scales take $AE = EC = L/2$ and $AA' = +W/2$ and $CC' = -W/2$. The shear force remains constant along the portion AE and then along the portion EC. Then section lines are shown along the boundary of the shear force diagram as per the general convention.

Example 7·2-1. A beam 6 metres long simply supported at ends carries two concentrated loads of 4·5 tonnes and 3 tonnes at distances of 2 metres and 4 metres from one end. Draw the shear force diagram for the beam.

Solution. Fig. 7·4 (*a*) shows a beam $ABCD$, 6 m long supported at A and D. To determine support reactions let us take moments of the forces about the point A

$$4{\cdot}5\times 2 + 3\times 4 - R_D\times 6 = 0$$

$$R_D = 3{\cdot}5 \text{ Tonnes}$$

$$R_D + R_A = 4{\cdot}5 + 3\ T$$

$$R_A = 7{\cdot}5 - 3{\cdot}5 = 4\ T$$

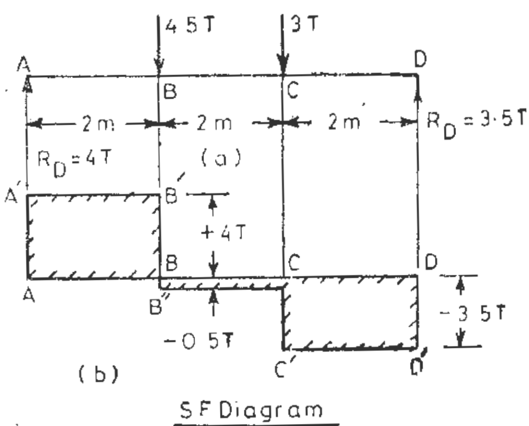


Fig. 7·4

Consider any section in the portion AB, and taking the resultant of the forces only on the left side of the section.

Portion AB. Shear force $F_1 = R_A = +4\ T \uparrow$

Portion BC. Shear force $F_2 = +R_A - 4{\cdot}5 = +4 - 4{\cdot}5 = -0{\cdot}5\ T \downarrow$

Portion CD. Shear force $F_3 = +4 - 4{\cdot}5 - 3 = -3{\cdot}5\ T \downarrow$.

Note that if the shear force on the left side of the section is vertically upwards ↑, It tends to rotate the element in a clockwise direction, it is a positive shear force. Similarly if the resultant vertical force on the left side of the section is downwards ↓, it tends to rotate the element in the anticlockwise direction. therefore it is a negative shear force. Fig. 7·4 (*b*) shows the SF diagram for the beam.

Exercise 7·2-1. Fig. 7·5 shows a beam 6 meters long supported at ends, carrying transverse loads of 600 kg and 400 kg at distances of 1 m and 3 m respectively from the end *A*. Determine the support reactions and draw the SF diagram.

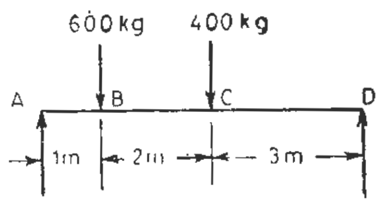


Fig. 7·5

[**Ans.** $R_A = 700$ kg, $R_D = 300$ kg, $F_{AB} = +700$ kg, $F_{BC} = +100$ kg $F_{CD} = -300$ kg]

7·3. SF DIAGRAM OF A SIMPLY SUPPORTED BEAM SUBJECTED TO UNIFORMLY DISTRIBUTED LOAD

Fig. 7·6 (*a*) shows a beam *AB* of length *L* simply supported at the ends and carrying a uniformly distributed load of *w* per unit length.

For reactions let us take moments of the forces about the point *A*.

Total vertical load on beam $= wL$

C.G. of the load wL lies at a distance of $L/2$ from the end *A*.

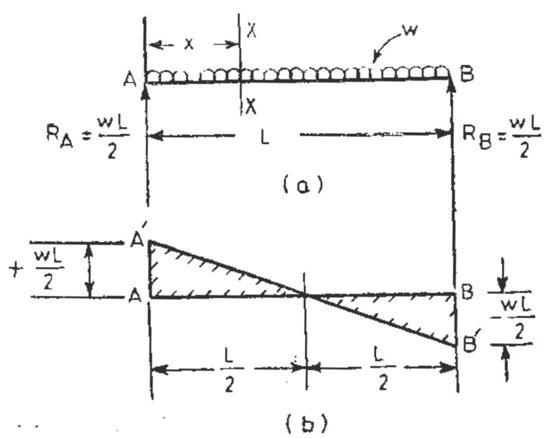


Fig. 7·6

So $$wL\times\frac{L}{2} - R_A\,.\,L = 0$$

or $$R_B=\frac{wL}{2}$$

and $$R_A+R_B=wL$$

$$R_A=wL-\frac{wL}{2}=\frac{wL}{2}$$

Consider a section X-X at a distance of x from the end A.

Shear force $$F_x=R_A-wx$$

$$=\frac{wL}{2}-wx$$

$$F_x=\frac{wL}{2} \text{ at } x=0$$

$$=+\frac{wL}{4} \text{ at } x=\frac{L}{4}$$

$$=0 \text{ at } x=\frac{L}{2}$$

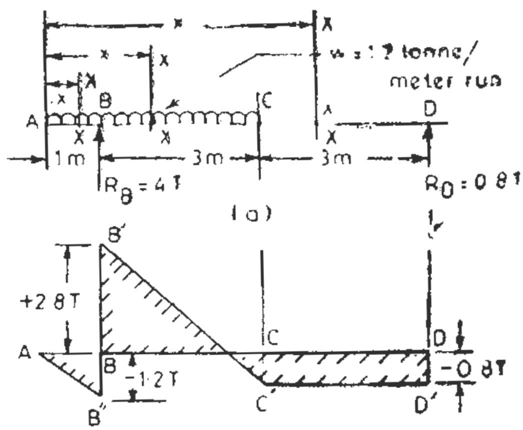
$$=-\frac{wL}{4} \text{ at } x=\frac{3L}{4}$$

$$=-\frac{wL}{2} \text{ at } x=L.$$

The shear force diagram for this case is shown in Fig. 7·5 (*b*).

Example 7·3-1. Fig. 7·7 (*a*) shows a beam 7 m long supported at a distance of 1 m from left hand end and at the other end. The beam carries a uniformly distributed load of 1·2 tonne/metre run over a length of 4 metres starting from left hand end.

Draw the SF diagram for the beam.



SF Diagram

(*b*)

Fig. 7·7

Solution. In this case there is an overhang of 1 metre of the beam from the left hand support.

Total vertical load on the beam $=1·2\times4=4·8$ tonnes.

C.G. of the uniformly distributed load (*udl*) lies at a distance of 2 m from the end A or 1 m from the support B.

To find out reactions, let us take moments of the forces about the point B.

$$4·8\times1-6\times R_D=0$$

or $$R_D=0·8 \text{ tonnes}$$

$$R_B=4·8-R_D=4 \text{ tonnes,}$$

Let us consider the portions AB, BC and CD separately taking x positive in the right direction and origin at A.

Portion AB. Considering resultant of the vertical forces on the left side of the section only.

Shear force $F_x = -wx$

$= 0$ at $x = 0$

$= -1.2 \times 0.5 = -0.6$ tonne at $x = 0.5$ m

$= -1.2 \times 1 = -1.2$ tonne at $x = 1$ m

Portion BC. Shear force

$F_x = -wx + R_B = -1.2x + 4$ tonne

$= -1.2 + 4 = +2.8$ tonne at $x = 1$

$= -1.2 \times 2 + 4 = +1.6$ tonne at $x = 2$

$= -1.2 \times 3 + 4 = +0.4$ tonne at $x = 3$

$= -1.2 \times 4 + 4 = -0.8$ tonne at $x = 4$

Portion CD. Shear force

$F_x = -4.8 + 4 = -0.8$ tonne (constant throughout the portion CD)

$= -0.8$ tonne at $x = 4$ to 7 m.

Note : T stands for tonnes.

Fig. 7·7 (b) shows the SF diagram.

Slope of SF diagram in portion AB,

$$\frac{dF_x}{dx} = -w$$

Slope of the SF diagram in portion BC,

$$\frac{dF_x}{dx} = -w$$

The slope of SF diagram in any portion of the beam gives the rate of loading w in that portion.

Exercise 7·3-1. A beam $ABCD$, 8 m long, supported at B and D carries a uniformly distributed load of 0·8 tonne/metre run as shown in Fig. 7·8. Determine the support reactions and draw the SF diagram.

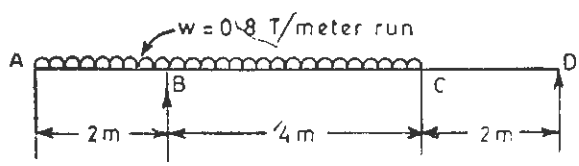


Fig. 7·8

[**Ans.** $R_B = 4.0$ tonne, $R_D = 0.8$ tonne, $F_{AB} = -0.8x$ tonne, $F_{BC} = -0.8x + 4$ tonne, $F_{AD} = -0.8$ tonne]

7·4. SF DIAGRAM OF A CANTILEVER SUBJECTED TO A CONCENTRATED LOAD

A cantilever AB of length L, free at end A and fixed at end B carries a concentrated load W at the free end, as shown in the Fig. 7·9.

There will be a reaction $R_B=W$ at the fixed end and a fixing couple WL exerted by the wall at the fixed end B, for equilibrium. At any section X—X at a distance of x from end A, there is a vertical force $W\downarrow$ on the left side of the section and a vertical force $W\uparrow$ on the right side of the section. This shear force is a negative SF tending to rotate the element of cantilever at the section in an anticlockwise direction.

Shear force is constant throughout the length L of the cantilever, as shown in the SF diagram Fig. 7·9 (b).

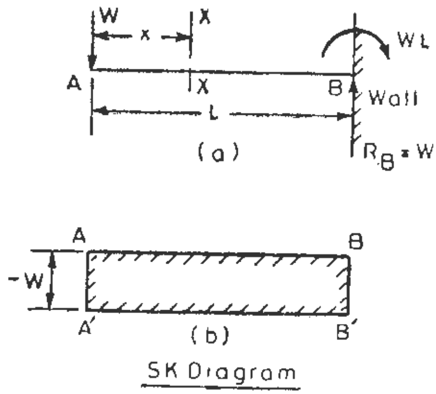


Fig. 7·9.

Example 7·4-1. Fig. 7·10 (a) shows a cantilever ABC, 5 metres long, free at end A and fixed at end C. A concentrated load 4 kN acts at A and 8 kN acts at B. Draw the SF diagram.

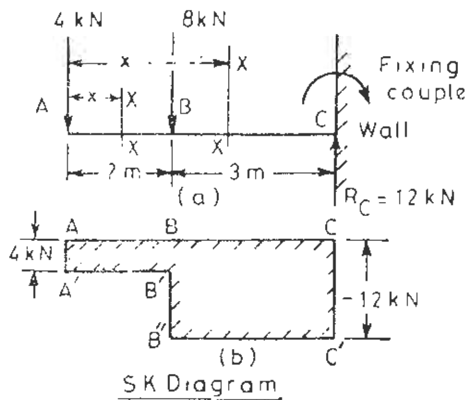


Fig. 7·10

Solution. For equilibrium, there will be reaction

$$R_C=4+8=12 \text{ kN}$$

and fixing couple offered by the wall

$$M_C=4\times5+8\times3=44 \text{ kNm at the fixed end } C.$$

Shear force in portion AB, $F_x=-4$ kN

Shear force in portion BC, $F_x=-4-8=-12$ kN

Fig. 7·10 (b) shows the SF diagram.

Exercise 7·4-1. A cantilever ABC 6 m long free at end A and fixed at end C, carries a concentrated load 400 kg at A and 500 kg at B (at a distance of 2m from end A). Draw the SF diagram. [**Ans.** $F_{AB}=-400$ kg, $F_{BC}=-900$ kg]

7·5. SF DIAGRAM OF A CANTILEVER SUBJECTED TO UNIFORMLY DISTRIBUTED LOAD

A cantilever AB of length L, free at end A and fixed at end B carries a uniformly distributed load w per unit length as shown in Fig. 7·11 (a).

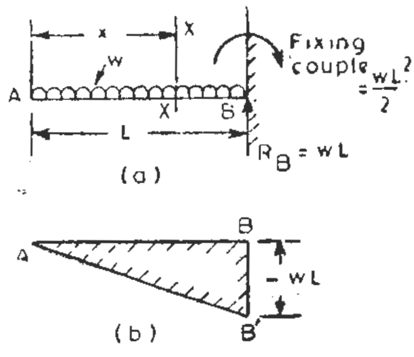


Fig. 7·11

Total vertical load on cantilever $= wL$

CG of the load lies at a distance of $\frac{L}{2}$ from the fixed end B.

For equilibrium, reaction at B,

$$R_B = wL$$

$$\text{Fixing couple offered by the wall at } B = wL \times \frac{L}{2} = \frac{wL^2}{2}$$

Consider a section X-X at a distance x from the end A

Shear force, $F_x = -wx$

$$= 0 \quad \text{at} \quad x = 0$$

$$= -\frac{wL}{4} \quad \text{at} \quad x = \frac{L}{4}$$

$$= -\frac{wL}{2} \quad \text{at} \quad x = \frac{L}{2}$$

$$= -wL \quad \text{at} \quad x = L$$

SF diagram is shown in the Fig. 7·11 (b).

Slope of the SF diagram, $\frac{dF_x}{dx} = -w$, *i.e.*, rate of loading.

Example 7·5-1. A cantilever 5 m long carries a uniformly distributed load of 200 kg/metre run from the free end upto the middle of its length as shown in the Fig. 7·12 (a). Draw the SF diagram.

Solution. Total load on the cantilever

$$= 200 \times 2·5 = 500 \text{ kg}$$

C.G. of the load lies at a distance of 1·25m from the end A or 3·75 m from the end C.

For equilibrium

Reaction at fixed end, $R_c = 500$ kg

Fixing couple offered by the wall

$$= 500\ (3·75)$$

$$= 1875 \text{ kg-m.}$$

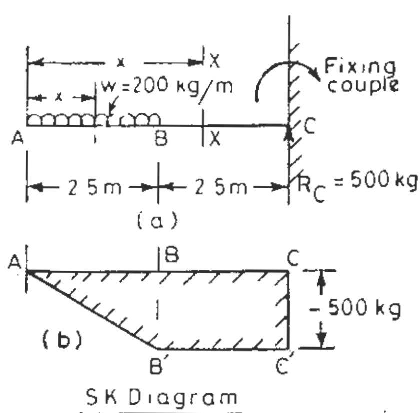


Fig. 7·12

For the SF diagram, consider a section X-X at a distance of x from the end A, in portion AB

Shear force,

$$F_x = -wx = -200\,x$$
$$= 0 \quad \text{at} \quad x = 0 \text{ m}$$
$$= -200 \times 0{\cdot}5 = -100 \text{ kg at } x = 0{\cdot}5 \text{ m}$$
$$= -200 \text{ kg at } x = 1 \text{ m}$$
$$= -400 \text{ kg at } x = 2 \text{ m}$$
$$= -500 \text{ kg at } x = 2{\cdot}5 \text{ m}$$

Again consider a section *X-X* at a distance of x from the end *A* in the portion *BC*.

$$F_x = -500 \text{ kg.}$$

This SF remains constant in the portion *BC*, *i.e.*, from *B* to *C*.

Slope of SF diagram, $\frac{dF_x}{dx} = -w$ (rate of loading) in portion *AB*.

Slope of SF diagram in portion *BC*, $\frac{dF_x}{dx} = 0$.

Exercise 7·5-1. A cantilever *ABC* 7 m long carries a uniformly distributed load of 2 kN/m run from the free end *A* upto *B*, 5 metres from end *A*. Draw the SF diagram.
[**Ans.** $F_{AB} = -2x$ kN, $F_{BC} = -10$ kN]

7·6. BENDING MOMENT DIAGRAM OF A SIMPLY SUPPORTED BEAM CARRYING A CONCENTRATED LOAD

A beam *AB* of length *L* simply supported at ends *A* and *B* carries a concentrated load *W* at its middle as shown in the Fig. 7·13 (*a*). Initially the beam *ACB* is straight and after the application of the load the beam bends to the shape *AC'B* (showing concavity upwards throughout the length of the beam.

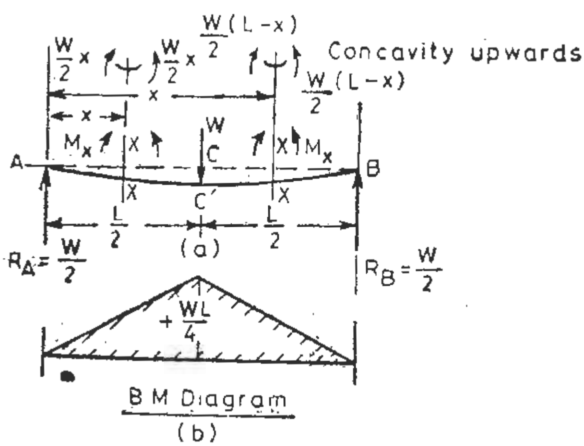


Fig. 7·13

To determine support reactions, let us take moments of the forces about the point *A*

$$W \times \frac{L}{2} - R_B \times L = 0 \qquad \text{For equilibrium}$$

$$R_B = \frac{W}{2} \uparrow$$

But $\qquad R_A+R_B=W \qquad$ for equilibrium

$$R_A=W-\frac{W}{2}=\frac{W}{2}\uparrow$$

Now consider a section $X-X$ at a distance of x from the end A in the portion AC of the beam.

Portion AC. Bending moment at any section

$$M_x=\circlearrowright+\frac{W}{2}x \quad \text{(clockwise)}$$

(taking moments of the forces on the left of the section).

Taking the moments of the forces on the right side of the section,

$$M_x=\frac{W}{2}(L-x)\circlearrowleft-W\left(\frac{L}{2}-x\right)\circlearrowleft$$

$$=+\frac{W}{2}x\circlearrowleft$$

A small length considered at the section will bend showing concavity upwards. (See the top of the Fig. 7·13 *a*).

Portion CB. Consider a section $X-X$ at a distance of x from the end A in the portion CB.

Bending moment (taking moments of the forces on the left side of the section),

$$M_x=\circlearrowright\frac{W}{2}(x)-\circlearrowleft W\left(x-\frac{L}{2}\right)$$

$$=\circlearrowright\frac{W}{2}(L-x)$$

Similarly taking moments of the forces on the right side of the section,

$$M_x=\frac{W}{2}(l-x)\circlearrowleft$$

A small length considered at this section will bend showing concavity upwards. (See the top of the Fig 7·13 (*a*)).

A bending moment which tends to bend the beam producing *concavity upwards* is said to be *a positive bending moment*, or in other words, if the resultant moment of the forces on the left of the section is clockwise, it is a positive bending moment.

Conversely a bending moment which tends to bend the beam producing convexity upwards is said to be a negative bending moment, or in other words, if the resultant moment of the forces on the left side of the section is anticlockwise, it is a negative bending moment.

Portion AC. Bending moment at any section,

$$M_x=\frac{W}{2}x$$

$$=0 \quad \text{at} \quad x=0$$

$$=\frac{WL}{8} \quad \text{at} \quad x=\frac{L}{4}$$

$$=\frac{WL}{4} \quad \text{at} \quad x=\frac{L}{2}$$

Slope of B.M. diagrams,

$$\frac{dM_x}{dx} = +\frac{W}{2} \qquad \text{(shear force in portion } AC\text{)}$$

Portion CB. Bending moment at any section,

$$M_x = \frac{W}{2}(l-x)$$

$$= \frac{WL}{4} \quad \text{at} \quad x = \frac{L}{2}$$

$$= \frac{WL}{8} \quad \text{at} \quad x = \frac{3L}{4}$$

$$= 0 \quad \text{at} \quad x = L$$

Slope of the B.M. diagram,

$$\frac{dM_x}{dx} = -\frac{W}{2} \qquad \text{(shear force in portion } CB\text{)}$$

Fig. 7·13 (*b*) shows the bending moment diagram. In this case maximum bending moment occurs at the centre of the beam and is equal to $\frac{WL}{4}$.

Example 7·6-1. A beam *ABCD*, 6 m long supported at *A* and *D* carries a concentrated load 3*T* at *B*, 2 metres from *A*, and another concentrated load 6*T* at *C*, 4 metres from end *A*. Draw the *BM* diagram.

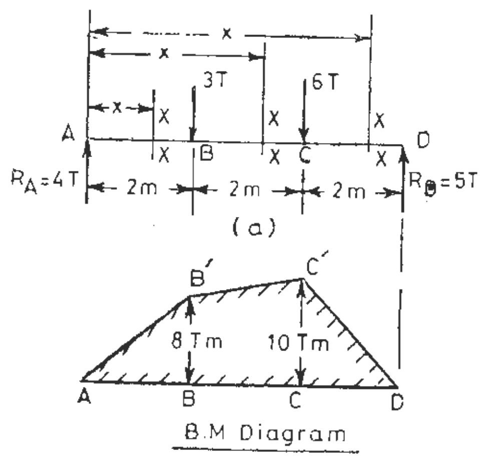


Fig. 7·14

Solution. To determine support reactions let us take moments of the forces about the point *A*

$$3\times2 \text{ Tm} + 6\times4 \text{ Tm} - R_D\times6 = 0$$

where Tm stands for tonne-metre

$$R_D = \frac{30}{6} = 5 \quad \text{Tonnes} \uparrow$$

But $\quad R_A + R_D = 3+6 = 9$ Tonnes

$$R_A = 9-5 = 4 \quad \text{Tonnes}$$

For bending moment, consider any section *X—X* at a distance of *x* metre from the end *A*

Portion AB

Bending moment, $M_x=+4x$ Tm (a clockwise moment on left of section)
$=0$ at $x=0$
$=+4$ Tm at $x=1$ m
$=+8$ Tm at $x=2$ m

Slope of BM diagram,

$$\frac{dM_x}{dx}=4T \quad (SF \text{ in portion } AC).$$

Portion BC

Bending moment, $M_x=+4x-3(x-2)$ Tm
$=8$ Tm at $x=2$ m
$=9$ Tm at $x=3$ m
$=10$ Tm at $x=4$ m

Slope of the BM diagram,

$$\frac{dM_x}{dx}=1 \quad T\text{-}(SF \text{ in portion } BC)$$

Portion CD. Bending moment at any section,

$M_x=+4x-3(x-2)-6(x-4)$
$=10$ Tm at $x=4$ m
$=5$ Tm at $x=5$ m
$=0$ Tm at $x=6$ m

Slope of the BM diagram,

$$\frac{dM_x}{dx}=4-3-6=-5 \text{ Tonne} \quad (SF \text{ in portion } CD)$$

It can be verified by the reader that SF in portion AB it is $+4T$, in portion BC is $+1T$ and in portion CD it is $-5\ T$, by drawing the SF diagram.

Fig. 7·14 (*b*) shows the BM diagram where AB', $B'C'$ and $C'D$ are the straight lines.

Exercise 7·6-1. A beam $ABCD$, 7 metres long, simply supported at A and D, carries a concentrated load of 21 kN at B, 1 metre from A and 28 kN at C, 5 metres from end A. Determine support reactions and draw the BM diagram.

[**Ans.** $R_A=26$ kN, $R_B=23$ kN, $M_B=+26$ kNm $M_C=+46$ kNm]

7·7. BENDING MOMENT DIAGRAM OF A S.S. BEAM SUBJECTED TO UNIFORMLY DISTRIBUTED LOAD

A beam AB, of length L, simply supported (S.S.) at ends A and B, carries a uniformly distributed load of w per unit length throughout its length. The beam is initially straight as ACB but after the application of the load, the beam bends showing concavity upwards.

Total load on the beam $=wL$

C.G. of the load lies at a distance of $\frac{L}{2}$ from the end A.

To determine support reactions, take moments of the forces about the point A

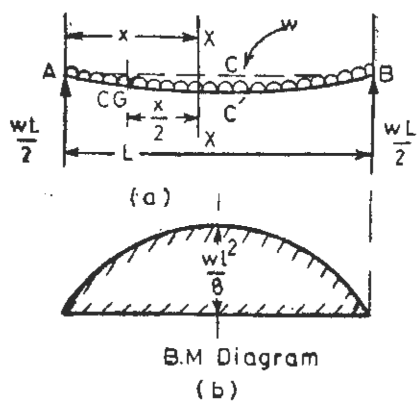


Fig. 7·15

$$wL \cdot \frac{L}{2} - R_B \cdot L = 0$$

$$R_B = \frac{wL}{2} \uparrow$$

But $$R_A + R_B = wL$$

$$R_A = wL - \frac{wL}{2} = \frac{wL}{2}$$

Now consider any section $X-X$ at a distance of x from the end A

Bending moment, $$M_x = +R_A \cdot x - wx \cdot \left(\frac{x}{2}\right)$$

Note that C.G. of the load $w.x$ lies at a distance of $\frac{x}{2}$ from the section $X-X$

Bending moment, $$M_x = +\frac{wL}{2}x - \frac{wx^2}{2}$$

$=0$	at	$x=0$
$=\frac{3}{32}wl^2$	at	$x=\frac{L}{4}$
$=\frac{wl^2}{8}$	at	$x=\frac{L}{2}$
$=\frac{3}{32}wl^2$	at	$x=\frac{3L}{4}$
$=0$	at	$x=L$

Slope of the B.M. diagram,

$$\frac{dM_x}{dx} = \frac{wL}{2} - wx \quad \text{(SF at any section, see article 7·3)}$$

$$\frac{dF}{dx} = \frac{d^2M}{dx^2} = -w \quad \text{(rate of loading)}$$

Fig. 7·15 (*b*) shows the B.M. diagram for the beam.

Example 7·7-1. A beam *ABCD*, 7 metres long, supported at *B*, 1 m from end *A* and at *D* carries a uniformly distributed load of 2 tonnes/metre run starting from end *A* upto the point 5 metre from end *A*. Draw the B.M. diagram.

Solution. The beam *ABCD* initially straight bends to the shape *A′BC′D* after the application of the transverse load.

Total load on the beam

$w \times 5 = 2 \times 5 = 10$ tonnes

The C.G. of the load lies at a distance of 2·5 m from end *A* or 1·5 m from the point *B*.

For support reactions, let us take the moments of the forces about the point *B*.

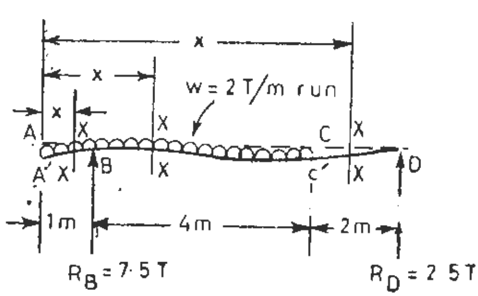


Fig. 7·16

$$10\times1{\cdot}5 - R_D\times6=0$$

or $$R_D=2{\cdot}5 \text{ tonnes}$$

But $$R_B+R_D=10 \text{ tonnes}$$

So $$R_B=10-2{\cdot}5=7{\cdot}5 \text{ tonnes.}$$

For bending moments, consider portions AB, BC and CD separately taking x positive towards right with origin at the point A. Take moments of the forces only on the left side of the section.

Portion AB

Bending moment, $$M_x=-wx\left(\frac{x}{2}\right)$$

$\left(\text{i.e. load}=wx \text{ and its C.G. lies at } \frac{x}{2} \text{ from the section } X-X\right)$

$$M_x=-\frac{wx^2}{2}$$

$=-x^2$ since $w=2$ Tm

$=0$ at $x=0$

$=-0{\cdot}25$ Tm at $x=0{\cdot}5$ m

$=-1$ Tm at $x=1$ m

Portion BC

Bending moment, $$M_x=-\frac{wx^2}{2}+R_B\,(x-1)$$

$$=-2\,.\,\frac{x^2}{2}+7{\cdot}5\,(x-1)$$

$=-x^2+7{\cdot}5(x-1)\ T\,.\,m$

$=-1\ T\,.\,m$ at $x=1$ m

$=+3{\cdot}5$ Tm at $x=2$ m

$=+6$ Tm at $x=3$ m

$=+6{\cdot}5$ Tm at $x=4$ m

$=+5$ Tm at $x=5$ m

Portion CD

Bending moment, $$M_x=-10(x-2{\cdot}5)+7{\cdot}5(x-4)\ \text{m}$$

(total $udl=10$ tonnes and its C.G. lies at a distance of 2·5 m from end A)

$=+5$ Tm at $x=5$ m

$=+2{\cdot}5$ Tm at $x=6$ m

$=0$ at $x=7$ m

Fig. 7·17 shows the BM diagram for the beam. At the point E in the BM diagram, BM=0 and BM changes sign from negative to positive. Such a point is called the *point of contraflexure*.

Point of contraflexure in this case lies in the portion BC

where $\quad M_x = -x^2 + 7{\cdot}5\,(x-1) = 0$

or $\quad x^2 - 7{\cdot}5\,x + 7{\cdot}5 = 0$

$$x = \frac{7{\cdot}5 - \sqrt{(7{\cdot}5)^2 - 4\times 7{\cdot}5}}{2}$$

$$= \frac{7{\cdot}50 - 5{\cdot}12}{2} = 1{\cdot}19 \text{ m}$$

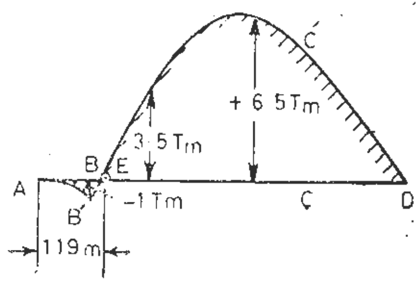


Fig. 7·17

i.e. point of contraflexure lies at a distance of 1·19 m from the end A.

Exercise 7·7-1. A beam ABC, 5 m long is supported at B, 1 m from end A and at C. The beam carries a uniformly distributed load of 10 kN/m run, throughout its length. Draw the BM diagram and determine the position and magnitude of maximum bending moment. Find the position of the point of contraflexure, if any.

[**Ans.** $M_{max} = 17{\cdot}578$ kNm at 3·125 m from A
Point of contraflexure lies at 1·25 m from end A]

7·8. BENDING MOMENT DIAGRAM OF A CANTILEVER SUBJECTED TO CONCENTRATED LOADS

Fig. 7·18 (a) shows a cantilever AB of length L subjected to a concentrated load W at the free end A. Initially the cantilever AB is straight, but after the application of the load W, the cantilever is bent to the shape $A'B$, showing convexity upwards.

For equilibrium, the reaction at B *i.e.* $R_B = W$ and fixing couple offered by the wall is wL.

Consider a section X-X at a distance of x from the end A. Taking moments of the forces on the left side of the section, $M_x = -Wx$ (the resultant anticlockwise moment on the left side of the section is negative.)

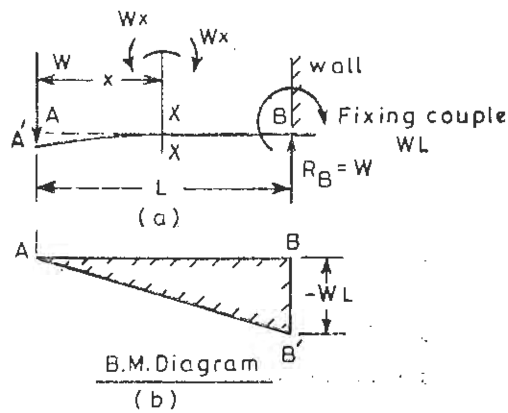


Fig. 7·18

Taking moments of the forces on the right side of the section

$$M_x = +W(l-x)\circlearrowright - wL\circlearrowleft$$

$$= -Wx \quad \text{(the resultant clockwise moment of the forces on the right side of the section is a negative BM).}$$

A small element considered at this section will bend with convexity upwards. As per the convention the BM which tends to produce convexity upwards is said to be a negative BM.

Now BM at any section,

$$M_x = -Wx$$
$$= 0 \quad \text{at} \quad x = 0$$
$$= -\frac{wL}{2} \quad \text{at} \quad x = \frac{L}{2}$$
$$= -wL \quad \text{at} \quad x = L$$

The slope of the BM diagram,

$$\frac{dM_x}{dx} = -W \quad \text{(shear force)}$$

The *BM* diagram for the cantilever is shown in Fig. 7·18 (*b*).

Example 7·8-1. A cantilever *ABC*, 6 metres long, fixed at *C*, carries a point load 10 kN at free end *A* and another point load 20 kN at *B*, 2 metres from *A*. Draw the BM diagram.

Solution. For equilibrium. Reaction at *C*, $R_c = 30$ kN

Consider section *X-X* at a distance of *x* from the free end *A* and taking moments of the forces only on the left side of the section.

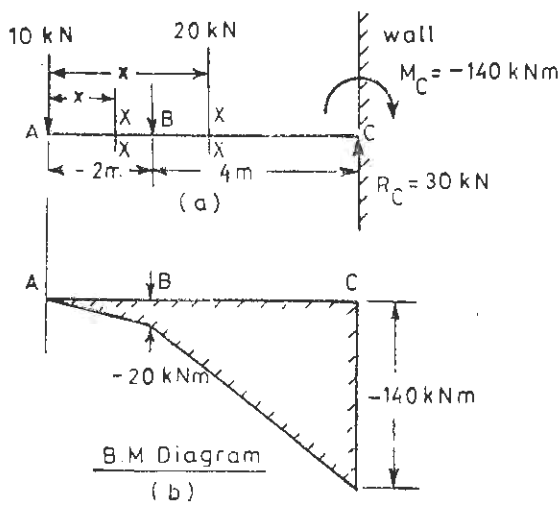


Fig. 7·19

Portion AB

BM at any section, $M_x = -10 \times x$ kNm (anticlockwise moment on the left side of the section is negative and produces convexity upwards)

$$M_x = 0 \quad \text{at} \quad x = 0$$
$$= -10 \text{ kNm} \quad \text{at} \quad x = 1 \text{ m}$$
$$= -20 \text{ kNm} \quad \text{at} \quad x = 2 \text{ m}$$

Portion BC

BM at any section, $M_x = -10\,x - 20\,(x-2)$ kNm

$$= -20 \text{ kNm} \quad \text{at} \quad x = 2 \text{ m}$$

$=-50$ kNm at $x=3$ m
$=-80$ kNm at $x=4$ m
$=-110$ kNm at $x=5$ m
$=-140$ kNm at $x=6$ m

The BM diagram for the cantilever is shown in Fig. 7·19 (*b*).

Exercise 7·8-1. A contilever *ABC* 5 m long carries a point load of 800 kg at its free end *A* and 600 kg at the middle of its length, *B*. Draw the *BM* diagram for the cantilever.
[**Ans.** $M_A=0$, $M_B=-2000$ kg/m, $M_C=-5500$ kg-m]

7·9. BENDING MOMENT DIAGRAM OF A CANTILEVER CARRYING A UNIFORMLY DISTRIBUTED LOAD

A cantilever *AB* of length *L*, free at end *A* and fixed at end *B* carries a uniformly distributed load *w* per unit length throughout its length. The beam is initially straight and bends showing convexity upwards after the application of load.

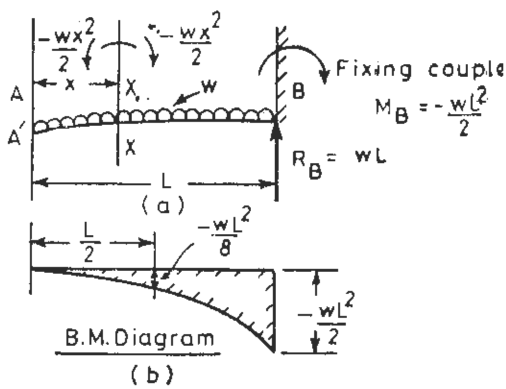


Fig. 7·20

Total load on cantilever $=wL$

For equilibrium, reaction at *B*,

$$R_B=wL$$

Bending moment at *B*,

$$M_B=-\frac{wL^2}{2}$$

Consider a section *X-X* at a distance of *x* from the end *A*.

Moment of the forces on the left side of the section,

$$M_x=-(wx)\left(\frac{x}{2}\right)$$

$$=-\frac{wx^2}{2} \quad \text{(anticlockwise moment)}$$

Moment of the forces on the right side of the section,

$$M_x=-\frac{w(L-x)^2}{2}+R_B\,(L-x)+M_B$$

$$=-\frac{wL^2}{2}-\frac{w(L-x)^2}{2}+wL\,(L-x)$$

$$=-\frac{wx^2}{2} \quad \text{(clockwise moment)}$$

Slope of the *BM* diagram,

$$\frac{dM_x}{dx}=-wx \quad \text{(SF on the cantilever)}$$

A small element considered at the section will bend showing convexity upwards, as shown on the top of the Fig 7·20 (*a*).

Now bending moment at any section,

$$M_x = -\frac{wx^2}{2}$$

$$= 0 \quad \text{at} \quad x = 0$$

$$= -\frac{wL^2}{8} \quad \text{at} \quad x = \frac{L}{2}$$

$$= -\frac{wL^2}{2} \quad \text{at} \quad x = L$$

Fig. 7·20 (*b*) shows the *BM* diagram of the cantilever.

Example 7·9-1. A beam *ABC*, 5 m long, free at end *A* and fixed at *C*, carries a uniformly distributed load of 1·5 tonne/metre run from the point *B*, 1 m from end *A*, upto the point *C*. Draw the BM diagram for the cantilever.

Solution. In this case, the cantilever does not carry any load between *A* to *B*, even if this part of the cantilever is removed, it will not affect the BM on the cantilever at any section.

Consider a section *X-X* at a distance of *x* from the end *A*, and taking moments of the forces *on the left side* of the section only.

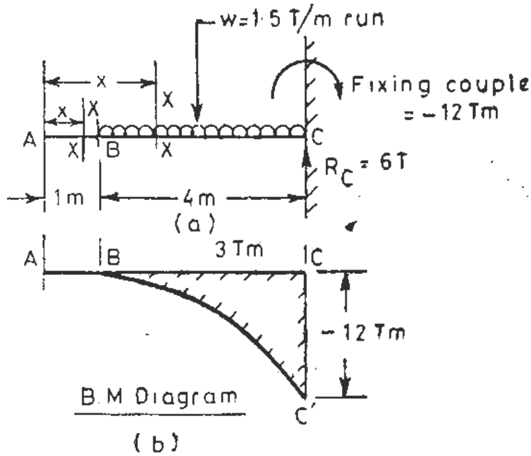


Fig. 7·21

Portion AB. BM at any section,

$$M_x = 0$$

$$= 0 \quad \text{at} \quad x = 0$$

$$= 0 \quad \text{at} \quad x = 1 \text{ m}$$

Portion BC. BM at any section,

$$M_x = -w(x-1)\frac{(x-1)}{2}$$

$$= -\frac{w}{2}(x-1)^2 \quad \text{(an anticlockwise moment)}$$

$$= -\frac{1{\cdot}5}{2}(x-1)^2$$

$$= 0 \quad \text{at} \quad x = 1 \text{ m}$$

$$= -0{\cdot}75 \text{ tonne-metre} \quad \text{at} \quad x = 2 \text{ m}$$

$$= -3{\cdot}0 \text{ tonne-metre} \quad \text{at} \quad x = 3 \text{ m}$$

$$= -12{\cdot}0 \text{ tonne-metre} \quad \text{at} \quad x = 5 \text{ m}$$

The BM diagram is shown in Fig. 7·21 (*b*).

Exercise 7·9-1. A cantilever *ABC* 7 m long, free at end *A* and fixed at end *C* carries a uniformly distributed load of 500 kg from the point *B*, 2 m from end *A* and upto the point *C*. Draw the BM diagram. [**Ans.** $M_A = M_B = 0$, $M_5 = -2250$ kg-m, $M_C = -6250$ kg-m]

7·10. SF AND BM DIAGRAMS OF A BEAM WITH VARIABLE LOADING

A beam AB of length L simply supported at ends A and B carries a varying load increasing from zero at A to w per unit length at B as shown in the Fig. 7·22 (a).

Total load on the beam $=\frac{wL}{2}$

CG of this load lies at a distance of $2L/3$ from end A or $L/3$ from end B (as is obvious for a triangle). To obtain support reactions take moments of the forces about the point A

$$\frac{wL}{2}\times\frac{2L}{3} \quad -R_B.L=0$$

or
$$R_B=\frac{wL}{3}$$

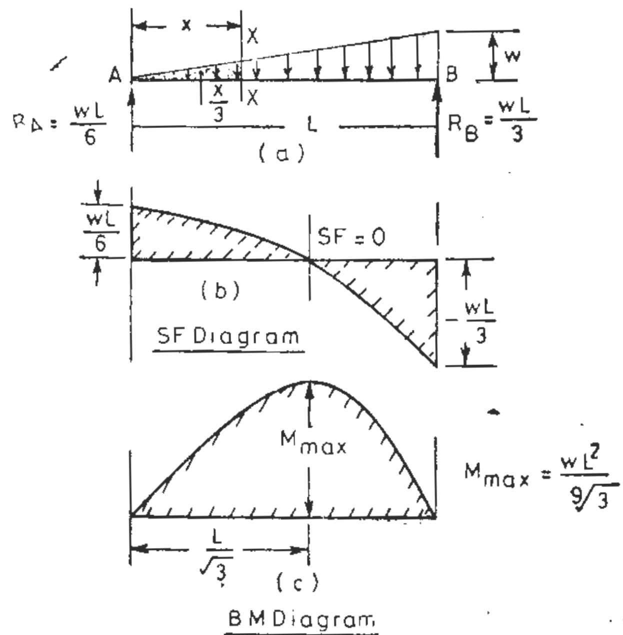


Fig. 7·22

But
$$R_A+R_B=\frac{wL}{2}$$

So
$$R_A=\frac{wL}{2}-\frac{wL}{3}=\frac{wL}{6}$$

Consider any section X-X at a distance of x from the end A.

Rate of loading $=\frac{wx}{L}$

Vertical load upto section

$$X\text{-}X=\frac{wx}{2L}\cdot x=\frac{wx^2}{2L}$$

Shear force,
$$F_x=R_A-\frac{wx^2}{2L}$$

(upwards force to the left side of the section is positive)

$$=\frac{wL}{6}-\frac{wx^2}{2L}=\frac{wL}{6} \quad \text{at} \quad x=0$$

$$=\frac{wL}{9} \quad \text{at} \quad x=\frac{L}{3}$$

$$=\frac{wL}{24} \quad \text{at} \quad x=\frac{L}{2}$$

$$=-\frac{wL}{18} \quad \text{at} \quad x=\frac{2L}{3}$$

$$=-\frac{wL}{3} \quad \text{at} \quad x=L$$

BM at any section, $M_x=+R_A \, . \, x-\frac{wx^2}{2L}\left(\frac{x}{3}\right)$

(clockwise moment on the left side of the section is positive)

where $\frac{wx}{L}$ =rate of loading at X

$\frac{wx^2}{2L}$ =vertical load upto X

$\frac{x}{3}$ =distance of CG of this load from section X-X

$$M_x=\frac{wLx}{6}-\frac{wx^3}{6L}$$

$$=0 \quad \text{at} \quad x=0$$

$$=\frac{4}{81}wL \quad \text{at} \quad x=\frac{L}{3}$$

$$=\frac{wL}{16} \quad \text{at} \quad x=\frac{L}{2}$$

$$=\frac{5}{81}wL \quad \text{at} \quad x=\frac{2L}{3}$$

$$=0 \quad \text{at} \quad x=L$$

To obtain maximum BM, put

$$\frac{dM_x}{dx}=0$$

i.e., $\frac{wL}{6}-\frac{wx^2}{2L}=0$ (a point at which shear force is zero)

or $$\frac{wL}{6}=\frac{wx^2}{2L}$$

or $$x^2=\frac{L^2}{3} \quad \text{or} \quad x=\frac{L}{\sqrt{3}}$$

$$M_{max}=\frac{wL}{6}\left(\frac{L}{\sqrt{3}}\right)-\frac{w}{6L}\left(\frac{L}{\sqrt{3}}\right)^3$$

$$=\frac{wL^2}{6\sqrt{3}}-\frac{w}{6}\times\frac{L^2}{3\sqrt{3}}=\frac{wL^2}{6\sqrt{3}}\left(1-\frac{1}{3}\right)=\frac{2wL^2}{3\times6\sqrt{3}}$$

$$=\frac{wL^2}{9\sqrt{3}} \quad \text{at} \quad x=\frac{L}{\sqrt{3}}$$

Maximum bending moment occurs at the point where shear force is zero.

Fig. 7·22 (*c*) shows the B.M. diagram for the beam.

Example 7·10-1. A beam 6 m long, simply supported at ends carries a linearly varying load with maximum rate at the centre of the beam *i.e.*, 1·5 tonne/m run as shown in Fig. 7·23 (*a*). Draw the SF and BM diagrams for the beam.

Solution. Total vertical load on the beam

$$=\frac{1{\cdot}5}{2}\times 6=4{\cdot}5 \text{ tonne}$$

The beam is symmetrically loaded so the reactions

$$R_A=R_B=\frac{4{\cdot}5}{2}=2{\cdot}25 \text{ tonnes} \uparrow$$

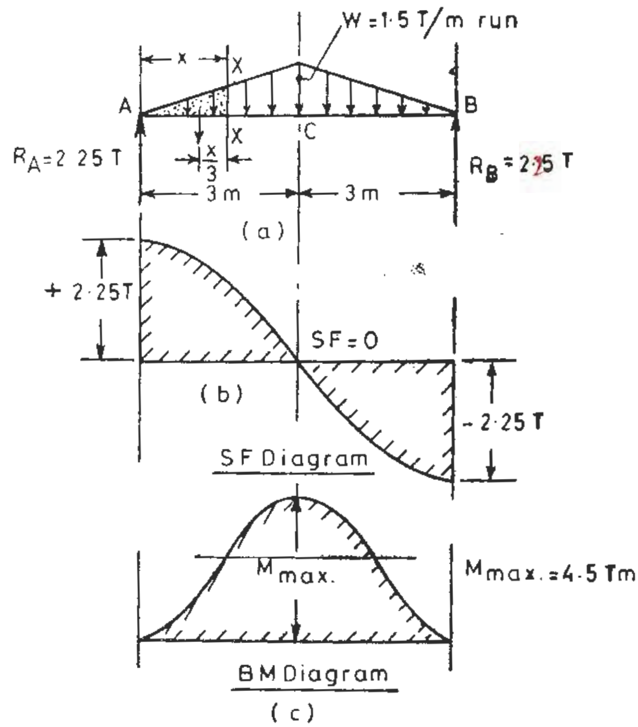


Fig. 7·23

SF diagram. Consider a section *X-X* at a distance of *x* from the end *A*.

Rate of loading at the section

$$=\frac{1{\cdot}5\,x}{3}=0{\cdot}5\,x$$

Vertical load upto x $$=\frac{0{\cdot}5\,x\,.\,x}{2}=0{\cdot}25\,x^2$$

Taking the resultant of the forces only on the left side of the section.

Portion AC. S.F. at any section,

$$F_x = 2{\cdot}25 - 0{\cdot}25\,x^2$$

$= 2{\cdot}25$ tonnes at $x=0$
$= 2{\cdot}00$ tonnes at $x=1$ m
$= 1{\cdot}25$ tonnes at $x=2$ m
$= 0$ tonnes at $x=3$ m

There is no necessity of determining SF in the portion CB, since the beam is symmetrically loaded.

(See article 7·2)

SF $= -1{\cdot}25$ tonnes at $x=4$ m
$= -2{\cdot}00$ tonnes at $x=5$ m
$= -2{\cdot}25$ tonnes at $x=6$ m

Fig. 7·23 (*b*) shows the SF diagram.

BM diagram. Taking moments of the forces on the left side of the section *X-X*.

Vertical load upto x $= 0{\cdot}25\,x^2$

Distance of CG of this load from section *XX*

$$= \frac{x}{3}$$

BM at any section, $M_x = +R_A \,.\, x - 0{\cdot}25\,x^2 \,.\, \frac{x}{3}$

$= 2{\cdot}25\,x - \frac{x^3}{12}$ T-m
$= 0$ at $x=0$
$= \frac{13}{6}$ T-m at $x=1$ m
$= \frac{46}{12}$ T-m at $x=2$ m
$= 4{\cdot}5$ T-m at $x=3$ m

Again there is no necessity of determining bending moments in the portion *CB*, as the beam is symmetrically loaded about its centre.

$M_x = \frac{46}{12}$ T-m at $x=4$ m
$M_x = \frac{13}{6}$ T-m at $x=5$ m
$M_x = 0$ at $x=6$ m

Maximum bending moment occurs at the centre where SF is zero.

Fig. 7·23 (*c*) shows the BM diagram of the beam.

Exercise 7·10-1. A beam *AB*, 6 metres long carries a linearly variable load from one end to the other end. At the end *A* the rate of loading is 1 tonne/m run and at the end *B* the rate of loading is 3 tonne/m run. Determine (*i*) support reactions (*ii*) magnitude and position of the maximum bending moment.

[**Ans.** $R_A = 5$ tonnes, $R_B = 7$ tonnes, $M_{max} = 9{\cdot}062$ T-m at $3{\cdot}245$ m from end *A*]

·11. SF AND BM DIAGRAMS OF A CANTILEVER WITH VARIABLE LOADING

A cantilever AB, free at end A, fixed .t end B, of length L, carries a linearly ariable load with rate of loading zero at A ncreasing to w at B.

Total load on cantilever

$$=\frac{wL}{2}\downarrow$$

For equilibrium, reaction at B

$$R_B=\frac{wL}{2}\uparrow$$

Consider a section X-X at a distance of from the end A.

Rate of loading at $x=\frac{wx}{L}$

Vertical load upto $x=\frac{wx}{L}\cdot\frac{x}{2}$

$$=\frac{wx^2}{2L}$$

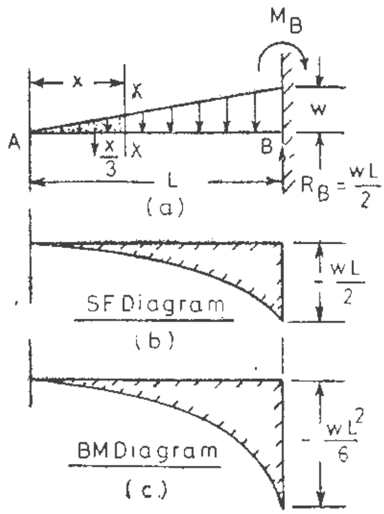


Fig. 7·24

F Diagram

At any section, $F_X=-\frac{wx^2}{2L}$ $\left(\begin{array}{l}\text{downward force on the left side of the}\\ \text{section is negative}\end{array}\right)$

$$=0 \quad \text{at} \quad x=0$$

$$=-\frac{wL}{8} \quad \text{at} \quad x=\frac{L}{2}$$

$$=-\frac{wL}{2} \quad \text{at} \quad x=L$$

Shear Force diagram is shown in Fig. 7·24 (b).

M Diagram

The CG of the load $\frac{wx^2}{2L}$ lies at a distance of $\frac{x}{3}$ from the section X-X.

BM at any section, $M_X=-\frac{wx^2}{2L}\cdot\frac{x}{3}$ $\left(\begin{array}{l}\text{Clockwise moment on the left side of}\\ \text{the section is negative}\end{array}\right)$

$$=-\frac{wx^3}{6L}$$

$$=0 \quad \text{at} \quad x=0$$

$$=-\frac{wL^2}{48} \quad \text{at} \quad x=\frac{L}{2}$$

$$=-\frac{wL^2}{6} \quad \text{at} \quad x=L$$

The BM diagram is shown in Fig. 7·24 (*c*).

Example 7·11-1. A cantilever *ACB*, 5 m long carries a linearly varying load starting from zero rate of loading at *A* to 600 kg/m run at *C*, 4 m from the end *A* as shown in Fig. 7·25 (*a*). Draw the SF and BM diagram. Total vertical load on cantilever

$$=\frac{600}{2}\times 4=1200 \text{ kg}$$

For equilibrium, reaction
$R_B=1200$ kg ↑

Consider a section *X-X* at a distance of *x* from the end *A*.

$$\text{Rate of loading at } X=\frac{600}{4}x$$

$$=150\,x$$

$$\text{Vertical load upto } X=150\,\frac{x\,.\,x}{2}$$

$$=75\,x^2$$

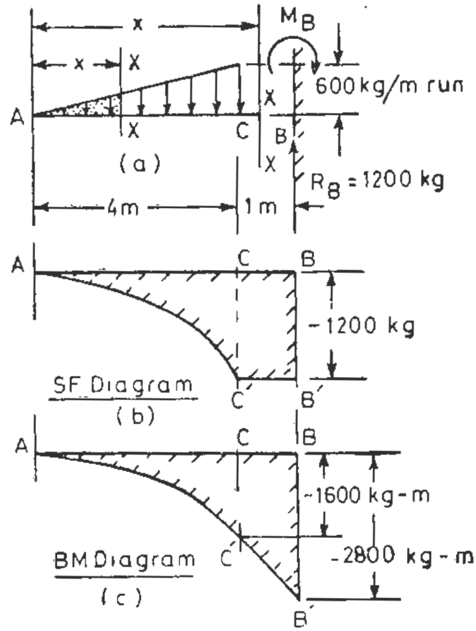


Fig. 7·25

SF Diagram

Portion AC

SF at any section *X*, $F_x=-75\,x^2$ (downward force on the left side of the section is negative)

$=0$ at $x=0$ m

$=-300$ kg at $x=2$ m

$=-1200$ kg at $x=4$ m

Portion CB

SF at any section, $F_x=-1200$ kg at $x=4$ m to 5 m.

The shear force diagram is shown by Fig. 7·25 (*b*).

BM Diagram

CG of the vertical load upto *X* lies at a distance of $\frac{x}{3}$ from the section *X-X*.

Portion AC

BM at any section *X*, $M_x=-75\,x^2\,.\,\frac{x}{3}$ (anticlockwise moment on the left side of the section is negative)

$=-25\,x^3$

$=0$ at $x=0$ m

$=-25$ kg m at $x=1$ m

$=-200$ kg m at $x=2$ m

$=-1600$ kg m at $x=4$ m

Portion CB

CG of the total vertical load lies at a distance of $\frac{8}{3}$ m from end A.

BM at any section $X-X$,

$$M_x = -1200\left(x-\frac{8}{3}\right) \text{ kg-m}$$

$$= -1600 \text{ kg-m} \quad \text{at} \quad x=4\text{m}$$

$$= -2200 \text{ kg-m} \quad \text{at} \quad x=4{\cdot}5 \text{ m}$$

$$= -2800 \text{ kg-m} \quad \text{at} \quad x=5 \text{ m}$$

Fig. 7·25 (c) shows the BM diagram for the cantilever.

Exercise 7·11-1. A cantilever 5 metres long carries a linearly varying load starting om zero rate of loading to 2·5 tonnes/metre run at the fixed end. Determine the shear force ıd bending moment at the fixed end of the cantilever.

[**Ans.** $F=-6{\cdot}25$ tonnes, $M=-30{\cdot}416$ tonne-metres]

12. SF AND BM DIAGRAM OF A BEAM SUBJECTED TO A MOMENT

A beam AB of length L hinged at both the ends is subjected to a turning moment M its centre. The hinged ends will prevent the lifting of the end A due to the application of e moment, M and the beam can take any slope or any direction at the hinged end.

To determine support reactions. Let us take moments of the forces about the point A

$$M \curvearrowright - R_B \times L \curvearrowleft = 0$$

$$R_B = \frac{M}{L} \uparrow$$

For equilibrium, $R_A = \frac{M}{L} \downarrow$ as shown in Fig. 7·26 (a)

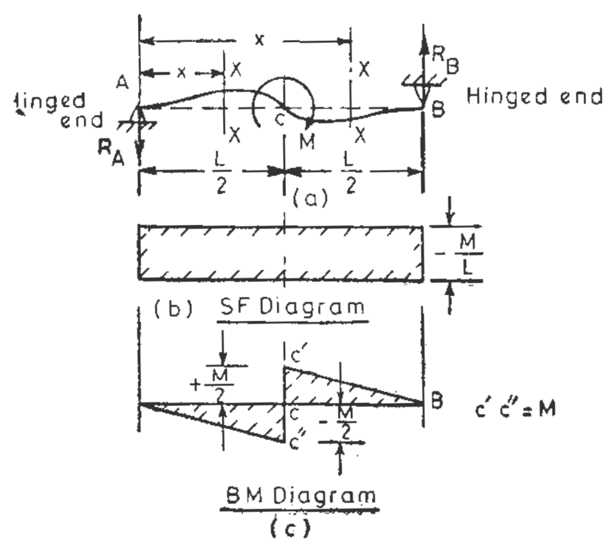


Fig. 7·26

SF diagram. Consider any section in the portion AC or CB.

Shear force $= -R_A$ (downward force on the left side of the section i negative)

$$= -\frac{M}{L}$$

Fig. 7·26 (b) shows the SF diagram of the beam,

BM diagram

Portion AC. BM at any section,

$M_x = -R_A \ x$ (anticlockwise moment on the left of th section is negative)

$$= -\frac{M}{L}x = 0 \quad \text{at} \quad x=0$$

$$= -\frac{M}{4} \quad \text{at} \quad x=\frac{L}{4}$$

$$= -\frac{M}{2} \quad \text{at} \quad x=\frac{L}{2}$$

Portion CB. BM at any section,

$$M_x = -R_A.x + M$$

$$= -\frac{M}{L}x + M$$

$$= +\frac{M}{2} \quad \text{at} \quad x=\frac{L}{2}$$

$$= +\frac{M}{4} \quad \text{at} \quad x=\frac{3L}{4}$$

$$= 0 \quad \text{at} \quad x=L$$

Fig. 7·26(c) shows the BM diagram for the beam.

Example 7·12-1. A beam 5 m long, hinged at both the ends is subjected to an anticlockwise moment M equal to 6 tonne-metres. At a point 3 m away from one end A. Draw the SF and BM diagrams.

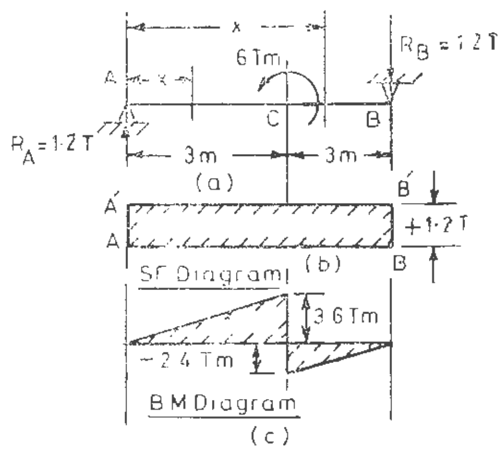


Fig. 7·27

Solution. Taking moments of the forces about the point A

$$6 \text{ Tm} = 5 \times R_B$$

or $\quad R_B = 1{\cdot}2 \ T \downarrow$

For equilibrium

$$R_A = 1{\cdot}2 \ T \uparrow$$

SF diagram. For portion AC or CD,

Shear force, $F_x = +1{\cdot}2\ T$ (tending to rotate the body in the clockwise direction)

Shear force is constant throughout the length of the beam. Fig. 7·27 (*b*) shows the SF diagram.

BM diagram. Consider a section $X-X$ at a distance of x from the end A and taking moments of the forces on the left side of section only.

Portion AC. BM at any section,

$$M_x = +1{\cdot}2 \times x$$
$$= 0 \quad \text{at} \quad x = 0 \text{ m}$$
$$= 1{\cdot}2 \text{ T-m} \quad \text{at} \quad x = 1 \text{ m}$$
$$= 2{\cdot}4 \text{ T-m} \quad \text{at} \quad x = 2 \text{ m}$$
$$= 3{\cdot}6 \text{ T-m} \quad \text{at} \quad x = 3 \text{ m}$$

Portion CB. BM at any section,

$$M_x = 1{\cdot}2\,x - 6$$
$$= -2{\cdot}4 \text{ T-m} \quad \text{at} \quad x = 3 \text{ m}$$
$$= -1{\cdot}2 \text{ T-m} \quad \text{at} \quad x = 4 \text{ m}$$
$$= 0 \quad \text{at} \quad x = 5 \text{ m}$$

Fig. 7·27 (*c*) shows the BM diagram for the beam.

Example 7·12-2. A cantilever AB, 6 m long, free at end A, fixed at the end B is subjected to a clockwise moment 8 kNm at the end A. Draw the SF and BM diagram.

Solution. For equilibrium, fixing couple,

or BM at B=6 kNm

Bending moment is constant throughout.

$$M_x = 6 \text{ kNm}$$

Shear force, $\dfrac{dM_x}{dx} = 0$ at any section.

Fig. 7·28 (*b*) shows the BM diagram of the cantilever.

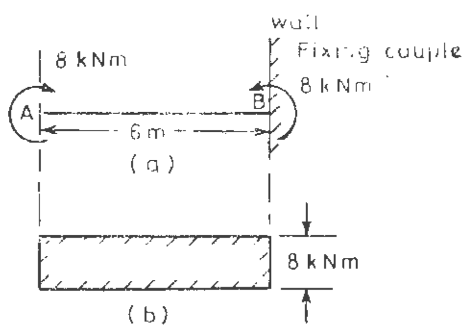


Fig. 7·28

7·13. RELATIONS BETWEEN RATE OF LOADING, SF AND BM

In the previous articles we have learnt about SF and BM diagrams of cantilever and beams (with and without overhangs) subjected to concentrated and distributed loads and we have observed that

(*i*) the portion in which SF is constant, BM curve is a straight line.

(*ii*) the portion in which SF is varying linearly, BM curve is parabolic.

(*iii*) maximum bending moment occurs at a point where either the SF is zero or the SF changes sign.

In other words, the curve for bending moment in any portion of the beam is one degree higher than the curve for S.F.

Fig. 7·29 shows a beam with a concentrated load W and a distributed load w. Any other beam or cantilever with any type of loading can also be considered. Consider a small section of the beam of length δx at a distance of x from the end A.

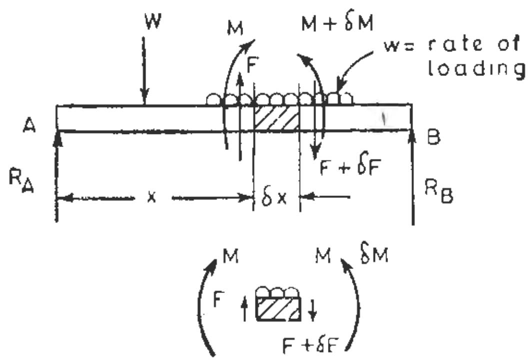


Fig. 7·29

Say on the left side of the section, SF$=F$ and BM$=M$.

On the right side of the section, SF$=F+\delta F$ and BM$=M+\delta M$.

Considering the equilibrium of forces

$$F=F+\delta F+w\delta x$$

or $$\delta F=-w\,\delta x$$

In the limits,

$$\frac{dF}{dx}=-w \qquad ...(i)$$

i.e. rate of change of SF at a section is equal to the rate of loading at the section.

Now taking the moments of the forces about the right hand end of the section,

$$M+\delta M=M+F\delta x-w.\delta x.\,\frac{\delta x}{n},$$

the value of n is 2 if the rate of loading is uniform

$$=M+F\delta x-\frac{w.\delta x^2}{2}$$

neglecting higher order of small quantities δx

$$\delta M=F\delta x$$

In the limits $$\frac{dM}{dx}=F \qquad ...(2)$$

i.e. the rate of change of BM at a section is equal to the shear force at the section.

Now (*i*) if rate of loading is zero, then

$$\frac{dF}{dx}=-w=0$$

or $$F=\text{a constant}$$

and $$\frac{dM}{dx}=F \quad \text{(a constant)}$$

$$\int dM=\int F\,dx$$

$$M=Fx+C$$ which is the equation of a stright line

(*ii*) If the rate of loading is uniform,

$$\frac{dF}{dx}=-w$$

$$\delta F=-w\delta x$$

Integrating we get $F=-wx+C_1$ where C_1 is the constant of integration

or $$\frac{dM}{dx}=-wx+C_1$$

Integrating further $$M=-\frac{wx^2}{2}+C_1x+C_2$$

where C_2 is another constant of integration

The BM curve is a parabola in this case.

Now for a maximum value of BM,

$$\frac{dM}{dx}=0 \text{ but } \frac{dM}{dx}=F$$ or the shear force is zero.

Example 7·13-1. A beam 10 m long simply supported at the ends carries transverse loads. The SF diagram for the beam is shown in Fig. 7·30(*a*). Draw the BM diagram for the beam.

Solution. Let us consider 3 portions of the beam *i.e.* *AB*, *BC* and *CD* separately. In *AB*, SF is constant and equal to $+5T$. In *BC*, SF is constant and is equal to 1T. In *CD*, SF is not constant but varies linearly as shown. This shows that portion *CD* of the beam carries a uniformly distributed load.

Since the beam is simply supported at *A* and *D*,

BM at $A=0$

BM at $D=0$

Now $\frac{dM}{dx}=F$ or $dM=F\,dx$

Integrating $$M=\int_A^B F\,dx$$

BM at B$-$BM at *A* $=$area of SF diagram between *B* and *A*

or $M_B-0=5\times2=10$ Tm

or $M_B=10$ Tm ...(1)

Similarly $M_C-M_B=$area of SF diagram between *C* and *B*

$M_C=M_B+2\times1=10+2=12$ Tm ...(2)

Say C' is the centre of the beam.

$$M_{C'} - M_C = \text{area of SF diagram between } C' \text{ and } C$$

$$M_{C'} = 12 + \tfrac{1}{2} \times 1 \times 1 = 12{\cdot}5 \text{ Tm}$$

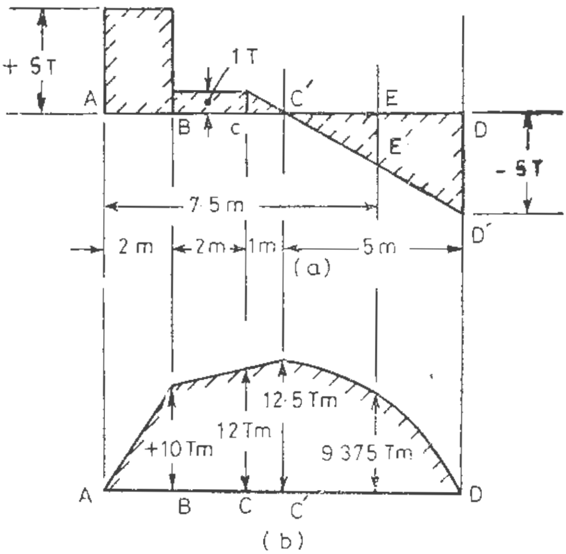


Fig. 7·30

Now in the portion $C'D$, SF is negative.

Let us take a point E at a distance of 7·5 m from A or 2·5 m from C'.

Now SF, $\quad EE' = -\dfrac{5}{2} = -2{\cdot}5\ T \quad$ (in SF diagram)

$$M_E - M_{C'} = \text{area of SF diagram between } E \text{ and } C'$$

$$= -2{\cdot}5 \times \frac{2{\cdot}5}{2}$$

$$M_E = 12{\cdot}5 - 3{\cdot}125 = 9{\cdot}375 \text{ Tm}$$

Similarly $\quad M_D - M_{C'} = \text{area of SF diagram between } D \text{ and } C'$

$$= -\frac{5{\cdot}0 \times 5}{2}$$

$$M_D = -12{\cdot}5 + 12{\cdot}5 = 0$$

The BM diagram is shown in Fig. 7·30(*b*).

Exercise 7·13-1. Fig. 7·31 shows SF diagram of a beam 8 m long, supported over a span of 6 m at B and E. Draw the BM diagram and determine the position of the point of contraflexure.

[**Ans.** $M_B = -20$ kNm, $M_C = +110$ k Nm, $M_D = +70$ kNm
Point of contraflexure lies at a distance of 2·307 m from end A.]

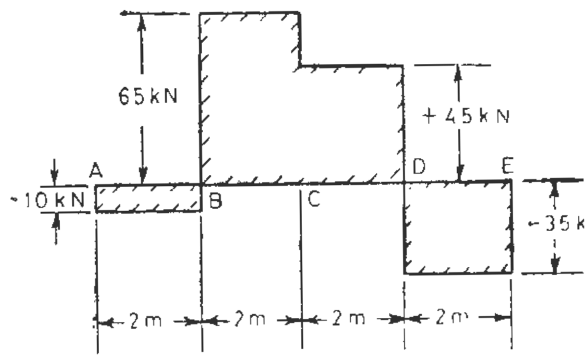


Fig. 7·31

7·14. GRAPHICAL METHOD

The graphical method for drawing SF and BM is complicated and time consuming.

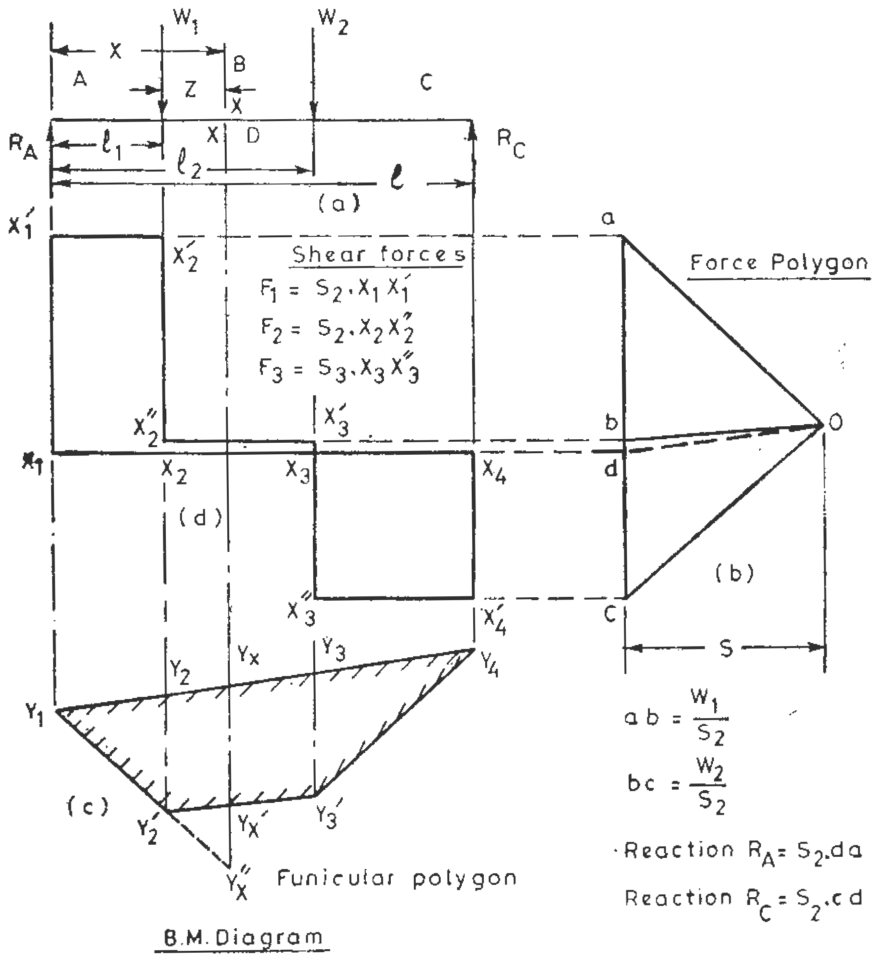


Fig. 7·32

However for the sake of explaination of a graphical method which may be useful in certain cases, the method is detailed as below (Consider a beam of length l, simply supported at its ends and carrying concentrated loads W_1 and W_2 at distances of l_1 and l_2 from one end of the beam. We have to draw SF and BM diagrams for the beam).

(*i*) To some suitable scales say 1 cm$=S_1$ m of beam length draw the load diagram for the beam as shown in the Fig. 7·32 (*a*).

(*ii*) Give Bow's notations to the spaces. AB represents load W_1, BC represents load W_2 then, CD represents reaction R_C and DA represents reaction R_A.

(*iii*) To some suitable scale say 1 cm$=S_2$ tonnes take $ab=W_1/S_2$, $bc=W_2/S_2$. Choose a pole O at a horizontal distance of S cm from the load line. Join oa, ob and oc as shown in Fig. 7·32 (*b*).

(*iv*) Draw vertical lines along reactions R_A, R_C and loads W_1 and W_2,

(*v*) Draw lines $Y_1\,Y_2'$ parallel to ao, $Y_2'\,Y_3'$ parallel to bo and $Y_3'\,Y_4$ parallel to co.

(*vi*) Join points Y_1 and Y_4. Then the funicular polygon drawn gives the BM diagram to some suitable scale.

(*vii*) From the point O draw a line od parallel to the line $Y_1\,Y_4$. Then cd represents the reaction R_C and da represents the reaction R_A.

or
$$R_C=S_2\,.\,cd$$
$$R_A=S_2\,.\,da$$

(*viii*) Draw a horizontal line from the point d which gives the base of SF diagram. Draw horizontal lines from the points a, b and c which intersect the vertical lines drawn from the reactions and the loads intersecting at X_1', X_2', X''_2, X_3', X_3'', X_4 as shown in Fig. 7·32 (*d*). The figure as shown gives the SF diagram for the beam.

SF in the portion of length l,
$$F_1=S_2.X_1\,X_1'=S_2\,R_A$$
SF in the portion of length l, to l_2,
$$F_2=S_2.X_2''\,X_2=S_2\,(R_A-W_1)$$
SF in the portion of length, l_2, l,
$$F_3=S_2\,.\,X_3X_3''=S_2\times R_C$$

(*ix*) The funicular polygon $Y_1\,Y_2'\,Y_3'\,Y_4$ gives the BM diagram. Let us consider a section X-X at a distance of x from the end A.

BM at the section, $M_x=R_A\,x-W_1\,z$

where z is the distance of the load W_1 from the section

then
$$M_x=(Y_x\,Y_x')\,S_1\,S_2\,S$$
where $Y_x\,Y_x'$ is the vertical projection along the section X-X, on the BM diagram.

or
$$(Y_x\,Y_x')\,S_1\,S_2\,S=R_A\,.\,x-W_1\,z$$
$$=S_1\,S_2\,.\,da\,.\,x-S_1\,S_2\,.\,ab\,.\,z=S_1\,S_2\,(da\,.\,x-ab\,.\,z)$$

or
$$(Y_x\,Y_x').S=da\,.\,x-ab\,.\,z$$

Extend the line $Y_1\,Y_2'$ to Y_x'' so as to meet the projection through the section X-X.
$$\Delta\,Y_1\,Y_x''\,Y_x \underset{\text{(similar)}}{\equiv} \Delta oad$$
$$\frac{da}{Y_xY_x''}=\frac{od}{Y_xY_1}=\frac{\text{horizontal projection of } od}{\text{horizontal projection of } Y_xY_1}=\frac{S}{x}$$
or
$$da\,.\,x=S\,.\,Y_x\,Y_x'' \qquad \ldots(1)$$

Similarly

$$\triangle Y_2' Y_x'' Y_x' \underset{\text{(similar)}}{\equiv} \triangle oab$$

$$\frac{ab}{Y_x'Y_x''} = \frac{ob}{Y_2'Y_x'} = \frac{\text{horizontal projection of } ob}{\text{horizontal projection of } Y_2'Y_x'} = \frac{S}{z}$$

or $$abz = S \,.\, Y_x' Y_x'' \qquad \text{...(2)}$$

From equations (1) and (2)

$$da \,.\, x - abz = S(Y_x Y_x'' - Y_x' Y_x'') = S \,.\, Y_x Y_x'$$

So the bending moment at any section

=(vertical intercept through the section on the funicular polygon) × length scale × load scale × S

S is the horizontal distance shown for the pole of the force polygon.

Example 7·14-1. A beam 10 m long, simply supported at the ends, carries concentrated loads of 4 tonnes, 5 tonnes and 3 tonnes at distances of 2 m, 5 m and 7 m from one end. With the help of graphical method determine the bending moments under the loads.

Solution. Let us take scales for length of the beam and for vertical loads as

1 cm = 1 metre length of beam

1 cm = 1 tonne load.

Draw the loading diagram as shown in Fig. 7·33 (*a*). Give Bow's notations to the spaces *i.e.*, *AB*, *BC*, *CD* representing loads 4 tonnes, 5 tonnes and 3 tonnes respectively. Draw vertical load line taking ab=4 cm, bc=5 cm and cd=3 cm. Choose a pole o at a horizontal distance of 5 cm from the vertical load line *abcd*. Join *ao*, *bo*, *co* and *do*.

Draw vertical projection lines through the reactions and loads. Draw lines $Y_1 Y_2' \parallel ao$, $Y_2' Y_3' \parallel bo$, $Y_3' Y_4' \parallel co$ and $Y_4' Y_5 \parallel do$ intersecting the projection lines at Y_1, Y_2', Y'_3, Y_4' and Y_5. Join $Y_1 Y_5$. Then the diagram $Y_1 Y_2' Y_3' Y_4' Y_5$ is the bending moment diagram. In the force polygon draw a line eo parallel to $Y_1 Y_5$. Then

Reaction, $ea = R_A = 6{\cdot}6$ cm $= 6{\cdot}6$ tonnes

$de = R_D = 5{\cdot}4$ cm $= 5{\cdot}4$ tonnes

Length scale, $S_1 = 1$ cm for 1 m length

Load scale, $S_2 = 1$ cm for 1 tonne load

Distance $S = 5$ cm

BM at the point 2, $M_2 = Y_2 Y_2' \,.\, S_1 S_2 S$

$= 2{\cdot}6 \times 1 \text{ m} \times 1 \text{ tonne} \times 5$

$= 13$ tonne-metres

at point 3, $M_3 = Y_3 Y_3' \,.\, S_1 S_2 S = 4{\cdot}1 \times 1 \times 1 \times 5 = 20{\cdot}5$ tonne-metres

at point 4, $M_4 = Y_4 Y_4' S_1 S_2 = 3{\cdot}15 \times 1 \times 1 \times 5 = 15{\cdot}75$ tonne-metres.

By Analytical Method. Reaction,

$R_A = 6{\cdot}6$ tonne, $R_D = 5{\cdot}4$ tonne (by taking moments)

BM at point 2, $M_2 = 6{\cdot}6 \times 2 = +13{\cdot}2$ tonne-metres

BM at point 3, $M_3 = 6{\cdot}6 \times 5 - 4 \times 3 = 21$ tonne-metres

BM at point 4, $M_4 = 6{\cdot}6 \times 7 - 4 \times 5 - 5 \times 2 = 16{\cdot}2$ tonne-metres

This shows that there are slight graphical errors in the answer.

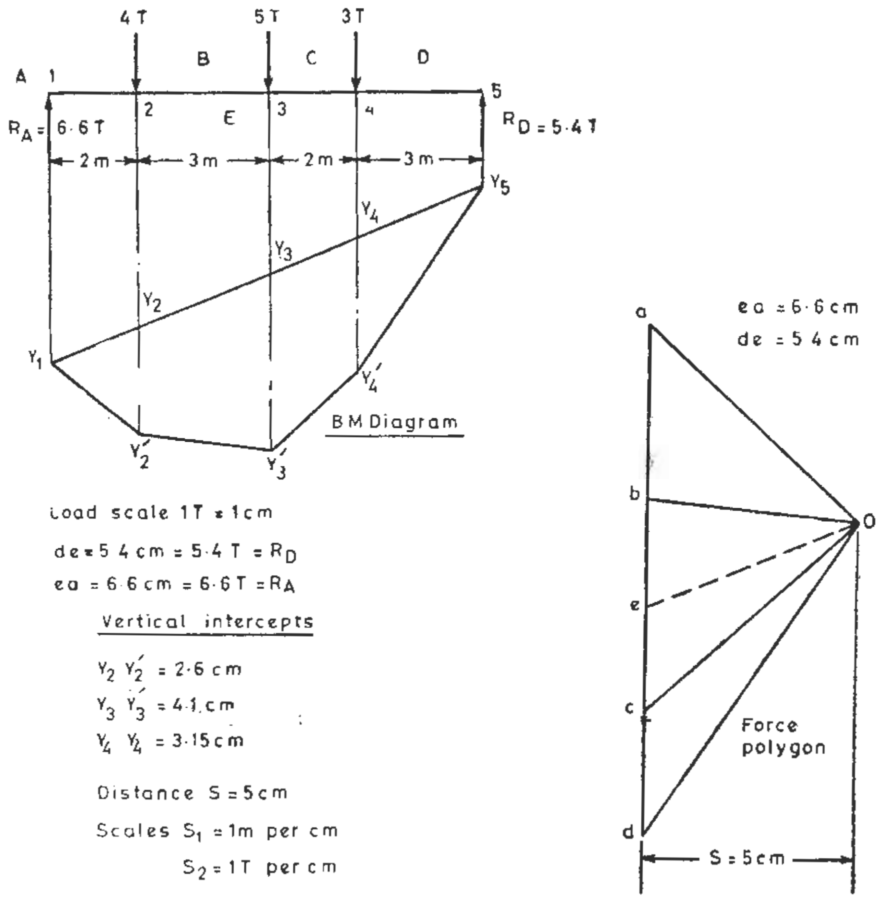


Fig. 7·33

Exercise 7·14-1. A beam 12 m long simply supported at the ends carries 3 concentrated loads of 60 kN each at distances of 3, 6 and 9 m from one end. Draw the load diagram, force polygon, SF diagram and BM diagram graphically and indicate the bending moments under the loads. [**Ans.** $M_2 = 270$ kNm, $M_3 = 360$ kNm, $M_4 = 270$ kNm]

Problem 7·1. A beam *ABCDE*, 14 m long supported over a length of 10 metres, over hang on both the sides being equal, carries a load 40 kN at one end, 40 kN at the other end and 80 kN at its centre. Draw the SF and BM diagrams. State the positions of the points of contraflexure and the maximum bending moment.

(*b*) Determine the position of the supports if the maximum positive BM is equal in magnitude to the maximum negative BM when the position of the loads remains unchanged.

Solution. Total vertical load

$=40+80+40=160$ kN

Since the beam is symmetrically loaded about its centre, reactions,

$$R_B=R_D=\frac{160}{2}=80 \text{ kN}$$

Consider a section XX at a distance of x from the end A and taking upward forces to be positive.

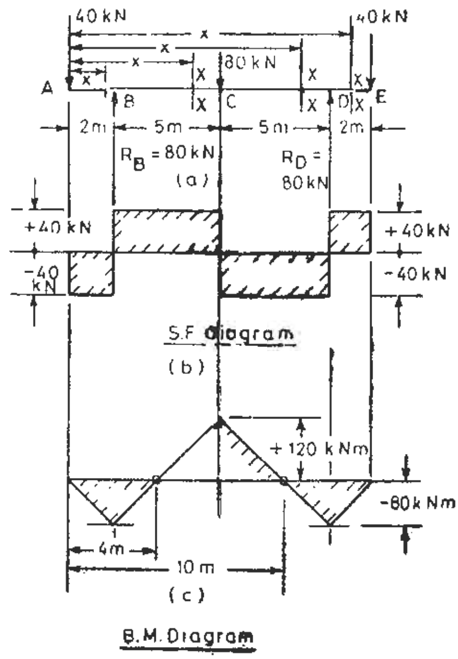


Fig. 7·34

SF diagram

Portion AB. $F_x=-40$ kN

(constant from $x=0$ to 2 m)

Portion BC.

$F_x=-40+80=+40$ kN

(constant from $x=2$ m to 7 m)

Portion CD.

$F_x=-40+80-80=-40$ kN

(constant from $x=7$ to 12 m)

Portion DE. $F_x=-40+80-80+80=+40$ kN (constant from $x=12$ m to $x=14$ m)

The SF diagram is shown in the Fig. 7·34 (*b*).

BM Diagram. Taking the clockwise moments on the left side of the section to be positive.

Portion AB. BM at any section,

$M_x=-40\,x$

$=0$ at $x=0$ m

$=-80$ kNm at $x=2$ m

Portion BC. $M_x=-40\,x+80\,(x-2)$

$=-80$ kNm at $x=2$m

$=+20$ kNm at $x=4{\cdot}5$ m

$=+120$ kNm at $x=7$ m.

Portion CD. $M_x=-40\,x+80\,(x-2)-80\,(x-7)$

$=+120$ kNm at $x=7$ m

$=+20$ kNm at $x=9{\cdot}5$ m

$=-80$ kNm at $x=12$ m.

Portion DE. $M_x=-40x+80\,(x-2)-80\,(x-7)+80\,(x-12)$

$=-80$ kNm at $x=12$ m

$=0$ kNm at $x=14$ m

Maximum bending moment 120 kNm occurs at the centre of the beam.

Points of contraflexure lie in the portions BC and CD.

For portion BC, $M_x = -40x + 80\,(x-2) = 0$

$$40x = 160$$

$$x = 4 \text{ m}$$

For portion CD, $M_x = -40x + 80x - 160 - 80x + 560$

$$= -40x + 400 = 0$$

$$x = 10 \text{ m.}$$

Fig. 7·34 (c) shows the BM diagram and points of contraflexure.

(b) We have seen that maximum negative BM occurs at the support B or D and maximum positive BM occurs at the centre C.

Say the beam is supported at a distance of a metre from both the sides as shown in the diagram 7·35.

Since the beam is symmetrically loaded about the centre.

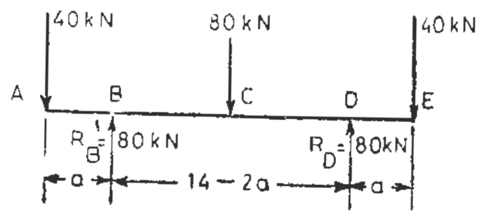


Fig. 7·35

Reactions, $R_B = R_D = \dfrac{40+80+40}{2} = 80$ kN

$$M_B = \text{BM at } B = -40 \times a \text{ kNm}$$

$$M_C = \text{BM at } C = -40 \times 7 + 80\,(7-a) = 280 - 80\,a$$

But $M_C = -M_B$

$$280 - 80\,a = 40\,a$$

$$\frac{280}{120} = a$$

or $a = 2{\cdot}333$ m

Problem 7·2. A beam ACB, hinged at the ends A and B, carries a uniformly distributed load of intensity w_1 per unit length acting downwards from the end A upto its centre C. Rest of the portion of the beam is covered with an upward uniformly distributed load of intensity w_2 per unit length.

(a) Draw the SF and BM diagram if $w_2 = 2w_1$

(b) Locate the position of the point of contraflexure.

Solution. Total vertical load on the beam

$$=\frac{w_1 l}{2}-\frac{w_2 l}{2}=\frac{w_1 l}{2}-\frac{2w_1 l}{2}$$

(as $w_2=2w_1$)

$$=\frac{w_1 l}{2}-w_1 l=-\frac{w_1 l}{2}$$

For support reactions, take moments of the forces about the point A

$$\frac{w_1 l}{2}\times\frac{l}{4}-\frac{w_2 l}{2}\times\frac{3l}{4}+R_B\times l=0$$

$$\frac{w_1 l^2}{8}-\frac{3w_1 l^2}{4}+R_B\times l=0$$

(since $w_2=2w_1$)

$$R_B=\frac{5}{8}w_1 l\downarrow$$

But $R_A+R_B=-\frac{w_1 l}{2}$

$$R_A-\frac{5}{8}w_1 l=-\frac{w_1 l}{2}$$

$$R_A=\frac{w_1 l}{8}\uparrow$$

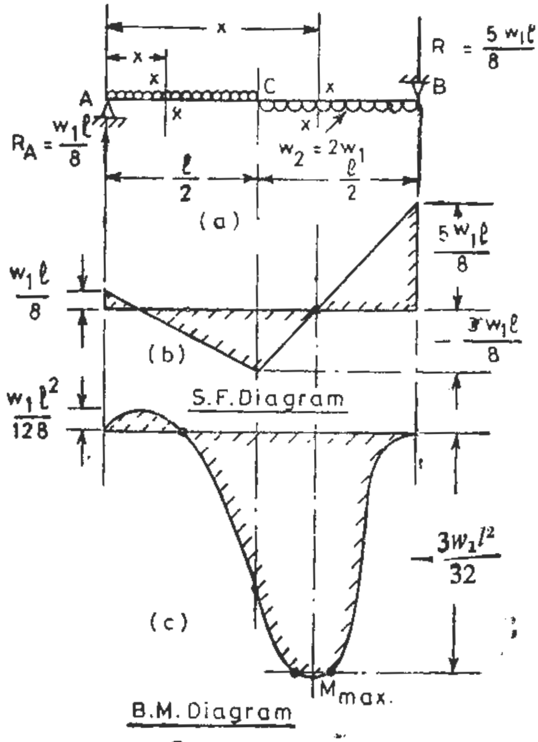


Fig. 7·36

Consider a section $X-X$ at a distance of x from end A and take upward forces on the left of the section to be positive and clockwise moments on the left of the section to be positive.

SF Diagram

Portion AC. SF at any section,

$$F_x=\frac{w_1 l}{8}-w_1 x$$

$$=\frac{w_1 l}{8}\quad\text{at}\quad x=0$$

$$=0\quad\text{at}\quad x=\frac{l}{8}$$

$$=-\frac{w_1 l}{8}\quad\text{at}\quad x=\frac{l}{4}$$

$$=-\frac{3w_1 l}{8}\quad\text{at}\quad x=\frac{l}{2}$$

Portion CB

$$F_x=+\frac{w_1 l}{8}-\frac{w_1 l}{2}+w_2\left(x-\frac{l}{2}\right)$$

$$=-\frac{3w_1 l}{8}+2w_1\left(x-\frac{l}{2}\right)$$

$$=-\frac{3w_1 l}{8}\quad\text{at}\quad x=\frac{l}{2}$$

$$=\frac{w_1 l}{8} \quad \text{at} \quad x=\frac{5l}{8}$$

$$=+\frac{w_1 l}{8} \quad \text{at} \quad x=\frac{3l}{4}$$

$$=+\frac{5w_1 l}{8} \quad \text{at} \quad x=l$$

BM diagram

Portion AC,

$$M_x=+\frac{w_1 l}{8}x-\frac{w_1 x^2}{2}$$

$$=0 \quad \text{at} \quad x=0$$

$$=+\frac{w_1 l^2}{128} \quad \text{at} \quad x=\frac{l}{8}$$

$$=0 \quad \text{at} \quad x=\frac{l}{4}$$

$$=-\frac{3}{128}w_1 l^2 \quad \text{at} \quad x=\frac{3l}{8}$$

$$=-\frac{w_1 l^2}{16} \quad \text{at} \quad x=\frac{l}{2}$$

Portion CB.

$$M_x=+\frac{w_1 l}{8}x-\frac{w_1 l}{2}\left(x-\frac{l}{4}\right)+w_2\left(x-\frac{l}{2}\right)^2$$

$$=\frac{w_1 l}{8}x-\frac{w_1 l}{2}\left(x-\frac{l}{4}\right)+\frac{2w_1}{2}\left(x-\frac{l}{2}\right)^2$$

$$=\frac{w_1 l}{8}x-\frac{w_1 l}{2}\left(x-\frac{l}{4}\right)+w_1\left(x-\frac{l}{2}\right)^2$$

$$=-\frac{3}{32}w_1 l^2 \quad \text{at} \quad x=\frac{5l}{8}$$

$$=-\frac{3}{32}w_1 l^2 \quad \text{at} \quad x=\frac{3l}{4}$$

$$=-\frac{1}{64}w_1 l^2 \quad \text{at} \quad x=\frac{7l}{8}$$

$$=0 \quad \text{at} \quad x=l$$

Fig. 7·36 (*c*) shows the BM diagram. The point of contraflexure lies at a distance of $\frac{l}{4}$ from the end *A*.

Problem 7·3. A beam 6 m long carries a uniformly distributed load of 2 Tm run. Counter clockwise moments of 4 Tm and 8 Tm are applied at the two ends. Draw the SF and BM diagrams. Find the magnitude of greatest BM and the position of the section where it occurs.

Solution. Fig. 7·37(*a*) shows the load on the beam *AB*, 6 m long.

Taking moments of the forces about the point *A*.

$$2\times6\times3=6\ R_B+4+8$$
$$36-12=6\ R_B$$
$$R_B=4T$$

But $R_A+R_B=2\times6=12\ T$
$$R_A=12-4=8T$$

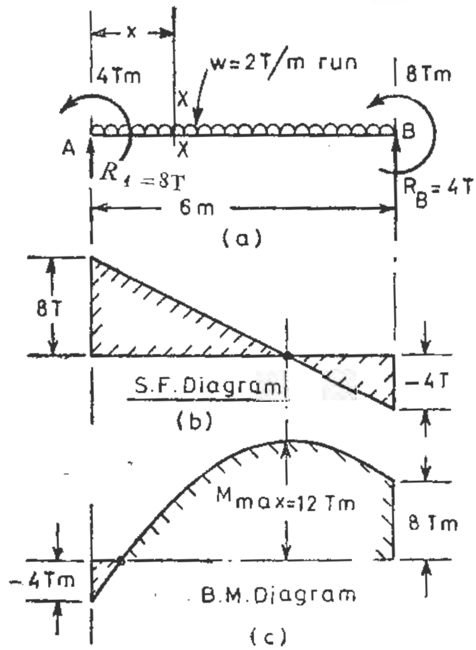


Fig. 7·37

For the SF diagram

Consider a section $X-X$ at a distance of x from the end A

Shear force at any section,

$$F_x=8-2x$$
$=8T$ at $x=0$
$=4T$ at $x=2$ m
$=0$ at $x=4$ m
$=-4T$ at $x=6$ m

BM diagram. Fig. 7·37 (*b*) shows the SF diagram.

Taking moments of the forces on the left side of the section and clockwise moments to be positive.

BM at any section, $M_x=R_A.x-4-\frac{w(x)^2}{2}$ where $w=2T/m$ and $R_A=8T$

$$M_x=8x-4-x^2$$
$=-4$ Tm at $x=0$ m
$=+8$ Tm at $x=2$ m
$=+12$ Tm at $x=4$ m
$=8$ Tm at $x=6$ m

$M_{max}=12$ Tm at $x=4$ m, where $F_x=0$ as is obvious from the SF and BM diagrams

Problem 7·4. A beam *ABCD*, 10 m long supported at *B*, 1 m from end *A* and at *C*, *x m* from end *D*. The beam carries a point load of 1000 kg at end *A* and a uniformly distributed load of 400 kg per metre run throughout its length. Determine the value of x if the centre of the beam becomes the point of contraflexure. Draw the BM diagram.

Solution. Total vertical load on beam

$$=1000+400\times10=5000\text{ kg.}$$

Reactions $R_B+R_C=5000$ kg

Taking moments of the forces about the point *A*

$$400\times10\times5=R_B\times1+R_C(10-x)$$
$$20{,}000=R_B+(5000-R_B)(10-x)$$
$$=R_B+5000\ (10-x)-R_B(10-x)$$

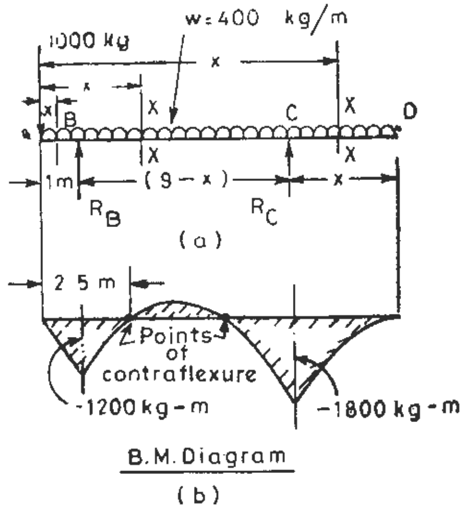


Fig. 7·38

$$20{,}000 = R_B + 5000(10-x) - R_B(10-x)$$

$$= R_B + 5000(10-x) - 10\ R_B + R_B.x$$

$$= -9R_B + R_B.x + 5000(10-x)$$

$$R_B = \frac{5000(10-x) - 20{,}000}{(9-x)} \qquad \text{...(1}$$

BM at the centre of the beam $= 0$

$$= -5 \times 1000 + 4 \times R_B - 400 \ \times \frac{x^2}{2}$$

$$= -5000 + 4\ R_B - \frac{400 \times 5^2}{2}$$

$$0 = -5000 + 4\ R_B - 5000$$

or $$4\ R_B = 10{,}000$$

$$4\left[\frac{5000(10-x)-20000}{(9-x)}\right] = 10000 \qquad \text{(Putting the value of } R_B \text{ in equation 1}$$

$$2(10-x) - 8 = 9 - x$$

$$20 - 2x - 8 = 9 - x$$

$$-x = -3$$

$$x = 3 \text{ metres.}$$

So reactions, $$R_B = \frac{5000(10-x)-20000}{(9-x)}$$

$$= \frac{5000 \times 7 - 20000}{6} = 2500 \text{ kg}$$

and $$R_C = 5000 - 2500 = 2500 \text{ kg}$$

BM diagram

Portion AB. At any section,

$$M_x = -1000\,x - \frac{wx^2}{2}$$

$$= -1000\,x - \frac{400\,x^2}{2}$$

$$= -1000\,x - 200\,x^2$$

$$= 0 \quad \text{at} \quad x = 0 \text{ m}$$

$$= -550 \text{ kg-m} \quad \text{at} \quad x = 0{\cdot}5 \text{ m}$$

$$= -1200 \text{ kg-m} \quad \text{at} \quad x = 1 \text{ m}$$

Portion BC $\quad M_x = -1000\,x + R_B\,(x-1) - \dfrac{wx^2}{2}$

$$= -1000\,x + 2500\,(x-1) - 200\,x^2$$

$$= -1200 \text{ kg-m} \quad \text{at} \quad x = 1 \text{ m}$$

$$= +200 \text{ kg-m} \quad \text{at} \quad x = 3 \text{ m}$$

$$= +300 \text{ kg-m} \quad \text{at} \quad x = 4 \text{ m}$$

$$= 0 \text{ kg-m} \quad \text{at} \quad x = 5 \text{ m}$$

$$= -700 \text{ kg-m} \quad \text{at} \quad x = 6 \text{ m}$$

$$= -1800 \text{ kg-m} \quad \text{at} \quad x = 7 \text{ m}$$

Portion CB. $\quad M_x = -1000\,x - 200\,x^2 + 2500\,(x-1) + 2500\,(x-7)$

$$= -800 \text{ kg-m} \quad \text{at} \quad x = 8 \text{ m}$$

$$= -200 \text{ kg-m} \quad \text{at} \quad x = 9 \text{ m}$$

$$= 0 \quad \text{at} \quad x = 10 \text{ m}$$

The BM diagram is shown in Fig. 7·38 (*b*). In this case there are two points of contraflexure lying in portion *BC*. To obtain the position of the second, let us put $M_x = 0$ for the portion *BC*.

$$-1000\,x + 2500\,(x-1) - 200\,x^2 = 0$$

$$1500\,x - 2500 - 200\,x^2 = 0$$

$$2\,x^2 - 15\,x - 25 = 0$$

$$x = \frac{15 \pm \sqrt{225 - 200}}{4}$$

$$= \frac{15 \pm 5}{4} = 2{\cdot}5 \text{ m}, \quad 5 \text{ m}$$

Second point of contraflexure lies at a distance of 2·5 from end *A*.

Problem 7·5. A beam *ABCDE*, 12 m long, cantilevered over the portion $AB = 4$ m ng, supported at points *B* and *E*, $BE = 8$ m long, carries a concentrated load 2 tonnes at end 3 tonnes at *C*, 2 m from *B* and 4 tonnes at *D*, 2 m from *E* In addition it carries a uniformly tributed load of 1 tonne/metre run over the portion *CD*. Draw the SF and BM diagram. termine the position of point of contraflexure if any.

Solution.

Calculation. The transverse loads on the beam are shown in Fig. 7·39 (*a*).

For equilibrium

$$R_B+R_E=2+3+4+4\times1=13 \text{ tonnes}$$

For support reactions let us take moments of the forces about the point *A*.

$$4\times R_B+12\ R_E-3\times6$$
$$-4\ (6+2)-4\times10=0$$

or $$4R_B+12\ R_E=18+32+40=90$$
$$R_B+3R_E=22{\cdot}5$$

But $$R_B+R_E=13$$
$$R_E=13-R_B$$

So $$R_B+3(13-R_B)=22{\cdot}5$$
$$-2R_B=22{\cdot}5-39$$
$$R_B=8{\cdot}25 \text{ tonnes,}$$
$$R_B=13-8{\cdot}25=4{\cdot}75 \text{ tonnes}$$

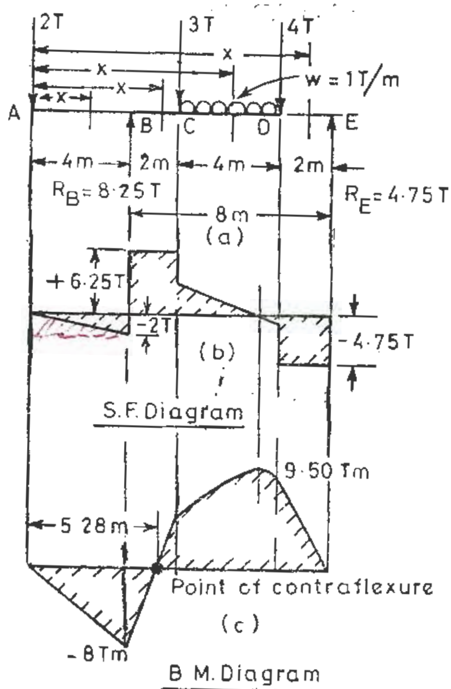


Fig. 7·39

SF diagram. Considering forces on the left side of the section and taking upward forces (tending to rotate the body in the clockwise direction) to be positive.

Shear force at any section,

Portion AB. $F_x=-2$ tonnes (constant in the portion)

Portion BC. $F_x=-2+8{\cdot}25=+6{\cdot}25$ tonnes (remains constant)

Portion CD. $F_x=-2+8{\cdot}25-3-w\ (x-6)$ where $w=1$ tonne/m

$=3{\cdot}25-(x-6)$

$=3{\cdot}25$ tonnes at $x=6$ m

$=1{\cdot}25$ tonnes at $x=8$ m

$=-0{\cdot}75$ tonnes at $x=10$ m

Portion DE. $F_x=-2+8{\cdot}25-3-4-4$

$=-4{\cdot}75$ tonne (constant in the portion *DE*).

Note that at $x=4$ m, there is shear force -2 tonnes and then $+6{\cdot}25$ tonnes *i.* considering a section very near to point *B* but on its left side, SF$=-2$ tonnes, then anoth section very near to the point *B* but on its right side, SF$=+6{\cdot}25$ tonnes. Similarly there a two values of SF at each of the points *C* and *D*.

BM diagram. Considering the moments of the forces on the left side of the secti and taking clockwise moments to be positive.

Portion AB. BM at any section,

$$M_x = -2x$$
$$= 0 \quad \text{at} \quad x = 0 \text{ m}$$
$$= -4 \text{ Tm} \quad \text{at} \quad x = 2 \text{ m}$$
$$= -8 \text{ Tm} \quad \text{at} \quad x = 4 \text{ m.}$$

Portion BC.

$$M_x = -2x + 8{\cdot}25\,(x-4)$$
$$= -8 \text{ Tm} \quad \text{at} \quad x = 4 \text{ m}$$
$$= -1{\cdot}75 \text{ Tm} \quad \text{at} \quad x = 5 \text{ m}$$
$$= +4{\cdot}5 \text{ Tm} \quad \text{at} \quad x = 6 \text{ m}$$

Portion CD.

$$M_x = -2x + 8{\cdot}25\,(x-4) - 3\,(x-6) - \frac{w\,(x-6)^2}{2}$$

where $w = 1$ tonne/m

$$= -2x + 8{\cdot}25\,(x-4) - 3\,(x-6) - 0{\cdot}5\,(x-6)^2$$
$$= +4{\cdot}5 \text{ Tm} \quad \text{at} \quad x = 6 \text{ m}$$
$$= +9 \text{ Tm} \quad \text{at} \quad x = 8 \text{ m}$$
$$= 9{\cdot}5 \text{ Tm} \quad \text{at} \quad x = 10 \text{ m}$$

Portion DE.

$$M_x = -2x + 8{\cdot}25\,(x-4) - 3\,(x-6) - 4\,(x-8) - 4\,(x-10)$$

(Note that $4\,(x-8)$ is the BM due to the uniformly distributed load)

$$= 9{\cdot}5 \text{ Tm} \quad \text{at} \quad x = 10 \text{ m}$$
$$= 4{\cdot}75 \text{ Tm} \quad \text{at} \quad x = 11 \text{ m}$$
$$= 0 \text{ Tm} \quad \text{at} \quad x = 12 \text{ m.}$$

Point of contraflexure lies in the portion BC, as is obvious from the BM diagram,

i.e.,

$$M_x = -2x + 8{\cdot}25\,(x-4) = 0$$
$$-2x + 8{\cdot}25x - 33 = 0$$
$$6{\cdot}25x = 33 \text{ m}$$
$$x = \frac{33}{6{\cdot}25} = 5{\cdot}28 \text{ m.}$$

i.e., Point of contraflexure lies at a distance of 5·28 m from the end A.

BM diagram is shown in Fig. 7·39 (c).

Problem 7·6. A beam 8 m long, supported over a span of 4 m and having equal overhang on both the sides, carries a concentrated load of 60 kN at one end and another concentrated load of 40 kN at the other end. In addition there is a uniformly distributed load of 20 kN/m run over 4 metres length starting from a point 2 m away from the end carrying the 60 kN load. Draw the SF and BM diagrams. Determine the magnitude and position of the greatest bending moment. What is the position of the point of inflexion ?

Solution. Total vertical load on the beam

$$= 60 + 40 + 20 \times 4 = 180 \text{ kN}$$

Support reactions,

$$R_B + R_C = 180 \text{ kN}$$

(as shown in the Fig. 7·40 (a)].

For support reactions, let us take moments of the forces about the point A.

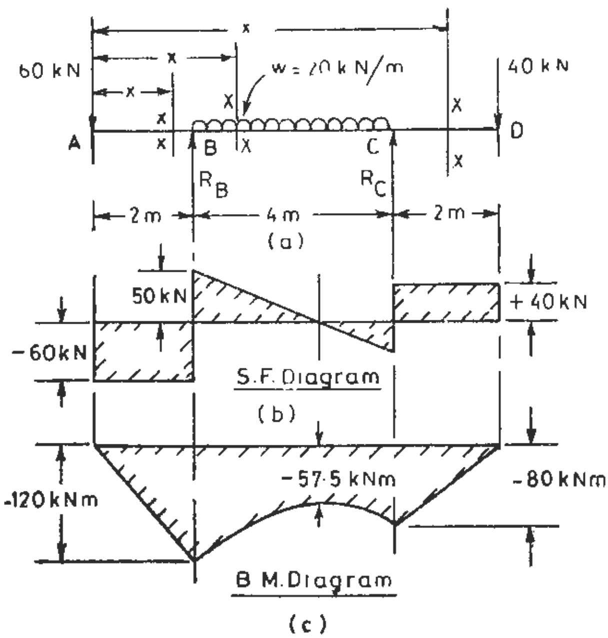


Fig. 7·40

$$20\times4\times4 + 40\times8 - 2\,R_B - 6\,R_C = 0$$

or $$640 = 6\,R_C + 2\,R_B$$

But $$R_B + R_C = 180 \text{ kN}$$

$$R_B = (180 - R_C)$$

or $$640 = 6\,R_C + 2\,(180 - R_C)$$

$$640 - 360 = 4\,R_C$$

$$R_C = 70 \text{ kN}$$

$$R_B = 180 - 70 = 110 \text{ kN}$$

SF Diagram. Take a section at a distance of x from the end A and vertically upwards forces on the left side of the section to be positive.

Portion AB. SF at any section,

$$F_x = -60 \text{ kN (constant at } x=0 \text{ to } x=2 \text{ m)}$$

Portion BC. $F_x = -60 + 110 - w\,(x-2)$ where $w = 20$ kN/m

$$= 50 - 20\,(x-2)$$

$$= +50 \text{ kN at } x = 2 \text{ m}$$

$$= +10 \text{ kN at } x = 4 \text{ m}$$

$$= -30 \text{ kN at } x = 6 \text{ m}$$

Portion CD. $F_x = -60 + 110 - 20 \times 4 + 70$

$= +40$ kN (constant at $x=6$ m to $x=8$ m)

The Fig. 7·40 (*b*) shows the SF diagram.

BM Diagram. Taking clockwise moments on the left side of the section to be positive.

Portion AB. BM at any section,

$$M_x = -60\,x$$

$= 0$ at $x = 0$ m

$= -60$ kNm at $x = 1$ m

$= -120$ kNm at $x = 2$ m.

Portion BC.

$$M_x = -60x + 110\,(x-2) - \frac{w}{2}\,(x-2)^2$$

$= -60x + 110\,(x-2) - 10\,(x-2)^2$

$= -120$ kNm at $x = 2$ m as $w = 20$ kN/m

$= -80$ kNm at $x = 3$ m

$= -60$ kNm at $x = 4$ m

$= -60$ kNm at $x = 5$ m

$= -80$ kNm at $x = 6$ m.

Portion CD.

$$M_x = -60x + 110\,(x-2) - 20 \times 4\,(x-4) + 70\,(x-6)$$

$= -60\,x + 110\,(x-2) - 80\,(x-4) + 70\,(x-6)$

[Note that at $(x-4)$ C.G. of uniformly distributed load lies].

$= -80$ kNm at $x = 6$ m

$= -40$ kNm at $x = 7$ m

$= 0$ at $x = 8$ m.

Fig. 7·40 (*c*) shows the BM diagram. In this case the maximum bending moment occurs at the point *B*, where the SF has changed sign.

$$M_{max} = -120 \text{ kNm}$$

The section where SF is zero, there is maximum bending moment.

Portion BC. Considering again,

$$F_x = 50 - 20\,(x-2)$$

$= 50 - 20x + 40$

or $x = 4·5$ m, where SF$=0$

$$M_x = -60\,x + 110\,(x-2) - 10\,(x-2)^2 \quad \text{at} \quad x = 4·5 \text{ m}$$

$$M_{min} = -60 \times 4·5 + 110\,(4·5 - 2) - 10\,(4·5 - 2)^2$$

$= -270 + 275 - 62·5$

$= -57·5$ kNm.

Note that there is no point of contraflexure or the point of inflexion in this case.

Problem 7·7. A propped cantilever *ABCD* 10 m long, carries the transverse loads as shown in Fig. 7·41 (*a*). Draw the SF and BM diagrams. Find the position and magnitude of the maximum bending moment. Determine also the position of the point of contraflexure if any.

Solution. Total vertical load on the cantilever

$$=3+5+1{\cdot}4\times7=17{\cdot}8 \text{ tonne}$$

Propping force at the end A

$$=5 \text{ tonnes}$$

Reaction at D, $R_D=17{\cdot}8-5=12{\cdot}8$ tonnes.

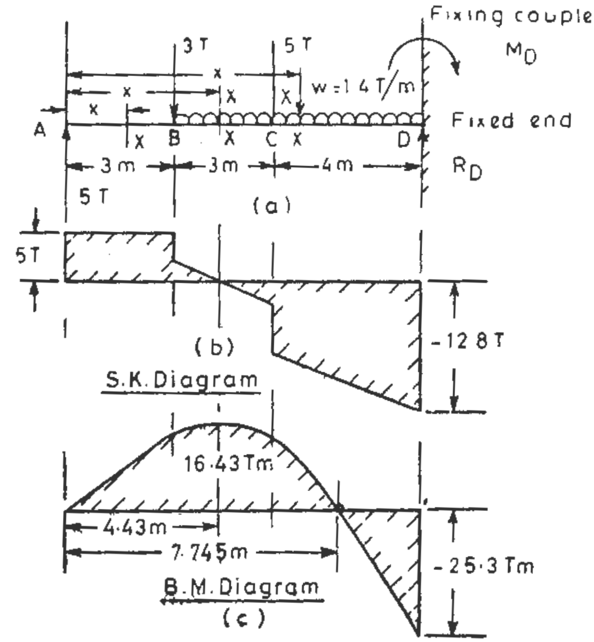


Fig. 7 41

SF diagram. Consider section X-X at a distance of x from the end A. Considering the forces only on the left side of the section, the vertically upward force is positive.

Portion AB. SF at any section,

$F_x=+5$ tonnes at $x=0$ to 3 m

(Constant throughout the portion AB).

Portion BC. SF at any section,

$F_x=5-3-w\,(x-3)$, where $w=1{\cdot}4$ tonne/m

$=2-1{\cdot}4\,(x-3)$

$=2$ tonnes at $x=3$ m

$=+0{\cdot}6$ tonne at $x=4$ m

$=-0{\cdot}8$ tonnes at $x=5$ m

$=-2{\cdot}2$ tonnes at $x=6$ m.

Portion CD. SF at any section,

$F_x=5-3-5-w\,(x-3)=-3-1{\cdot}4\,(x-3)$

$=-7{\cdot}2$ tonnes at $x=6$ m

$=-10$ tonnes at $x=8$ m

$=-12{\cdot}8$ tonnes at $x=10$ m

BM Diagram. (Taking clockwise moments on the left side of the section as positive).

Portion AB. BM at any section,

$$M_x = +5x$$
$$= 0 \quad \text{at} \quad x = 0$$
$$= 15 \text{ Tm} \quad \text{at} \quad x = 3 \text{ m}.$$

Portion BC. $M_x = 5x - 3(x-3) - \frac{w(x-3)^2}{2}$, where $w = 1{\cdot}4$ Tm

$$= 5x - 3(x-3) - 0{\cdot}7(x-3)^2$$
$$= 15 \text{ Tm} \quad \text{at} \quad x = 3 \text{ m}$$
$$= 22{\cdot}5 - 4{\cdot}5 - 1{\cdot}575 \text{ Tm} \quad \text{at} \quad x = 4{\cdot}5 \text{ m}$$
$$= 16{\cdot}425 \text{ Tm} \quad \text{at} \quad x = 4{\cdot}5 \text{ m}$$
$$= 14{\cdot}7 \text{ Tm} \quad \text{at} \quad x = 6 \text{ m}.$$

Portion CD. $M_x = 5x - 3(x-3) - 5(x-6) - 0{\cdot}7(x-3)^2$

$$= 14{\cdot}7 \text{ Tm} \quad \text{at} \quad x = 6 \text{ m}$$
$$= -2{\cdot}5 \text{ Tm} \quad \text{at} \quad x = 8 \text{ m}$$
$$= -25{\cdot}3 \text{ Tm} \quad \text{at} \quad x = 10 \text{ m}.$$

M_{max} BM. Occurs at the section where SF$=0$ in portion BC

i.e., $2 - 1{\cdot}4(x-3) = 0, \; x = 4{\cdot}43$ m

Putting the value of x in the expression for M_x

$$M_{max} = 5 \times 4{\cdot}43 - 3(4{\cdot}43 - 3) - 0{\cdot}7(4{\cdot}43 - 3)^2$$
$$= 22{\cdot}15 - 4{\cdot}29 - 1{\cdot}43 = 16{\cdot}43 \text{ Tm}$$

The BM diagram is shown in Fig. 7·41 (*c*).

The point of contraflexure lies in portion *CD*.

So M_x in portion $CD = 5x - 3(x-3) - 5(x-6) - 0{\cdot}7(x-3)^2 = 0$

$$5x - 3x + 9 - 5x + 30 - 0{\cdot}7(x^2 - 6x + 9) = 0$$
$$-3x + 39 - 0{\cdot}7x^2 + 4{\cdot}2x - 6{\cdot}3 = 0$$
$$0{\cdot}7x^2 - 1{\cdot}2x - 32{\cdot}7 = 0$$

or $$x = 7{\cdot}745 \text{ m}$$

Problem 7·8. A beam 6 m long, simply supported over a span of 5 m ; carries the transverse loads as shown in the Fig. 7·42 (*a*). Draw the SF and BM diagrams and find the position of the point of contraflexure if any.

Solution. A force of 50 kN can be applied at the point *B*, both in downward and upwards directions, so as not to disturb the equilibrium of the beam. In other words, a force of 50 kN acting at a lever of 1 m length (at point *B'*) can be replaced by a force of 50 kN at *B* and a couple 50 kN×1 m at *B*, as shown is the diagram 7·42 (*b*). The transverse loads shown on the beam are equivalent to the transverse loads and couples shown in Fig. 7·42 (*b*).

Total vertical load on the beam

$$= 50 + 10 \times 3 = 80 \text{ kN}$$

Reactions $R_A + R_D = 80$ kN

Taking moments of the forces about the point *A*

$$50 \times 1 \curvearrowright + 50 \curvearrowright + 10 \times 3(3 + 1{\cdot}5) \curvearrowright - 5\, R_D \curvearrowleft = 0$$

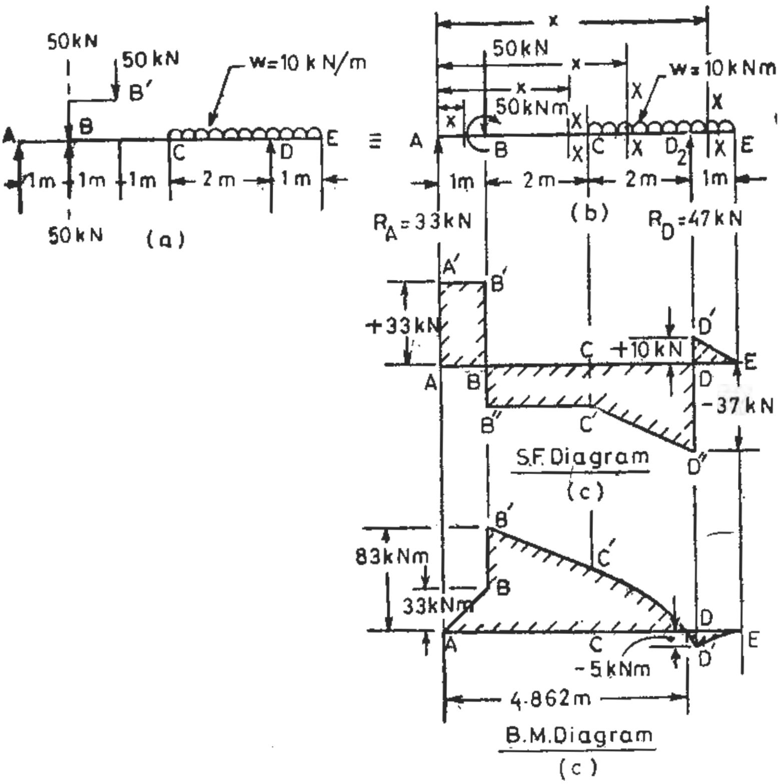


Fig. 7·42

Note that total u.d.l $=10\times3=30$ kN (udl=uniformly distributed load)

its CG lies at a distance of $3+1\cdot5=4\cdot5$ m from end A

or $235=5\,R_D$

$R_D=47$ kN

$R_A=80-47=33$ kN

Consider a section at a distance of x from the end A and taking the forces only on the left side of the section.

SF diagram

Portion AB. SF at any section,

$F_x=+33$ kN (at $x=0$ to 1 m)

(upward force on the left side of the section is positive)

Portion AC. SF at any section,

$F_x=+33-50$

$=-17$ kN at $x=1$ to 3 m

Portion CD $F_x=33-50-w(x-3)$ where $w=10$ k N/m

$=-17-10(x-3)$

$=-17$ kN at $x=3$ m

$=-27$ kN at $x=4$ m

$=-37$ kN at $x=5$ m

Portion DE. $F_x = +33-50+47-w(x-3)$

$=30-10(x-3)$

$=10 \text{ kN}$ at $x=5$ m

$=0 \text{ kN}$ at $x=6$ m

BM diagram. Taking the clockwise moments on the left side of the section to be positive.

Portion AB. BM at any section,

$M_x = +33\,x$

$=0$ at $x=0$ m

$=33$ kNm at $x=1$ m

Portion BC. $M_x = 33\,x+50-50\,(x-1)$

$=83$ kNm at $x=1$ m

$=66$ kNm at $x=2$ m

$=49$ kNm at $x=3$ m

Portion CD. $M_x = 33\,x+50-50(x-1) - \dfrac{w(x-3)^2}{2}$ where $w=10$ kN/m

$=33\,x+50-50(x-1)-5(x-3)^2$

$=49$ kNm at $x=3$ m

$=27$ kNm at $x=4$ m

$=-5$ kNm at $x=5$ m

Portion DE. $M_x = 33\,x+50-50(x-1) - \dfrac{w(x-3)^2}{2} + 47(x-5)$

$=33x+50-50(x-1)-5(x-3)^2+47(x-5)$

$=-5$ kNm at $x=5$ m

$=0$ at $x=6$ m

Note that maximum bending moment occurs at the point B where SF changes sign *i.e.* from +33kN to −17 kN and $M_{max}=83$ kNm.

Point of contraflexure lies in the portion CD

where $M_x = 33\,x+50-50(x-1)-5(x-3)^2$

$=33\,x+50-50x+50-5(x^2-6x+9)$

$=-17\,x-5x^2+30x+55$

$=-5x^2+13\,x+55$

$=0$ for point of contraflexure

or $5x^2-13-55=0$

$$x=\frac{13+\sqrt{169+4\times5\times55}}{10}$$

$$=\frac{13+35\text{·}62}{10}=4\text{·}862 \text{ m}$$

Point of contraflexure lies at a distance of 4·862 m from end A.

Problem 7·9. A cantilever 7 m long, carries a uniformly distributed load of 100 kg/m run and a concentrated load of 700 kg at the end of a lever at B, 2 m from free end A, as shown in Fig. 7·43 (a). Draw the SF diagram and BM diagram. Determine

(i) magnitude and position of maximum bending moment

(ii) position of the point of contraflexure.

Solution. The loading on the cantilever shown in Fig. 7·43 (a) is equivalent to that shown in Fig. (b). The portion AB is ineffective.

Consider a section $X—X$ at a distance of x from the end B. Take downward forces on the left side of the section to be negative and anticlockwise moments on the left side of the section to be negative.

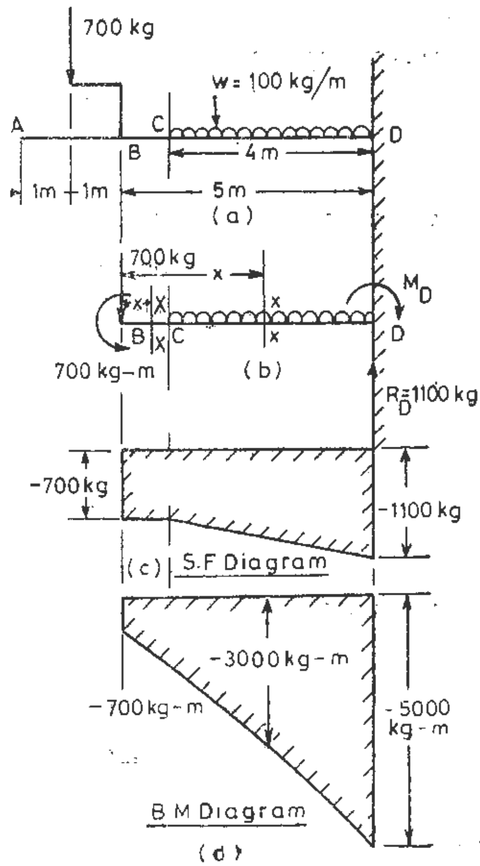


Fig. 7·43

SF diagram

Portion BC. SF at any section,

$F_x = -700$ kg.

(constant from $x=0$ to $x=1$ m)

Portion CD

$$F_x = -700 - w(x-1)$$
$$= -700 - 100(x-1)$$
$$= -700 \text{ kg at } x=1 \text{ m}$$
$$= -900 \text{ kg at } x=3 \text{ m}$$
$$= -1100 \text{ kg at } x=5 \text{ m}$$

BM diagram. Fig. 7·43 (c) shows the SF diagram.

Portion BC. BM at any section,

$$M_x = -700 - 700\,x$$
$$= -700 \text{ kg-m at } x=0$$
$$= -1400 \text{ kg-m at } x=1 \text{ m}$$

Portion CD. $M_x = -700 - 700\,x - \frac{w}{2}(x-1)^2$ where $w = 100$ kg/m

$$= -700 - 700\,x - 50(x-1)^2$$
$$= -1400 \text{ kg-m at } x=1 \text{ m}$$
$$= -3000 \text{ kg-m at } x=3 \text{ m}$$
$$= -5000 \text{ kg-m at } x=5 \text{ m}$$

Fig. 7·43 (d) shows the BM diagram. The maximum bending moment -5000 kg-m occurs at the fixed end and there is no point of contraflexure in the cantilever.

Problem 7·10. A simply supported beam ACB carries a linearly varying distributed load as shown in the Fig. 7·44. The maximum intensity of loading is w at each end of the beam. Determine the magnitude and position of the maximum bending moment.

Solution. Total vertical downward load on beam

$$\frac{wl}{2}\times\frac{1}{2}=\frac{wl}{4}\downarrow$$

Total vertical upward force on the beam

$$=\frac{wl}{2}\times\frac{1}{2}=\frac{wl}{4}\uparrow$$

CG of $\frac{wl}{4}\downarrow$ lies at a distance of $\frac{l}{6}$ from end A and CG of $\frac{wl}{4}\uparrow$ lies at a distance of $\frac{5l}{6}$ from end A.

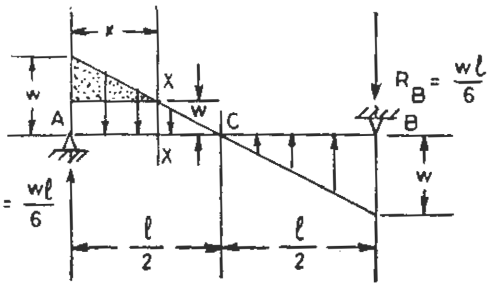


Fig. 7·44

For support reactions, take moments of the forces about the point A

$$\frac{wl}{4}\times\frac{l}{6}\circlearrowright-\frac{wl}{4}\times\frac{5l}{6}\circlearrowleft-R_B\times l\circlearrowright=0$$

or $$R_B=\frac{wl}{6}\downarrow$$

For equilibrium, $$R_A=\frac{wl}{6}\uparrow$$

Consider a section $X-X$ at a distance of x from the end A.

Rate of loading, $$w'=\frac{2w}{l}\left(\frac{l}{2}-x\right)$$

$$=\frac{w}{l}\ (l-2x) \qquad \ldots(1)$$

Total load upto x can be considered in two parts,

$W_1=w'x$ with CG at $\frac{x}{2}$ from $X-X$

$W_2=(w-w')\ \frac{x}{2}$ with CG at $\frac{2x}{3}$ from $X-X$

So $$W_1=\frac{wx}{l}\ (l-2x)$$

and $$W_2=\left[\ w-\frac{w}{l}\ (l-2x)\ \right]\frac{x}{2}=\frac{2wx}{l}\times\frac{x}{2}=\frac{wx^2}{l}.$$

Portion AC. Taking clockwise moments on the left side of the section to be positive. BM at any section X-X,

$$M_x=\frac{wl}{6}\ x-W_1\ \frac{x}{2}-W_2\ \frac{2x}{3}$$

or $$M_x=\frac{wl}{6}\ x-\frac{w}{l}x(l-2x)\frac{x}{2}-\frac{wx^2}{l}\times\frac{2x}{3}$$

$$=\frac{wlx}{6}-\frac{wx^2}{2l}(l-2x)-\frac{2wx^3}{3l}$$

$$=\frac{wlx}{6}-\frac{wx^2}{2}+\frac{wx^3}{l}-\frac{2}{3}\frac{wx^3}{l}$$

$$=\frac{wlx}{6}-\frac{wx^2}{2}+\frac{wx^3}{3l}$$

Taking $x=\frac{l}{2}$, $M_x=\frac{wl^2}{12}-\frac{wl^2}{8}+\frac{wl^2}{24}=0$

i.e., the point of contraflexure lies at the centre of the beam. For maximum bending moment in the portion AC

$$\frac{dM_x}{dx}=0=\frac{wl}{6}-wx+\frac{wx^2}{l}=0$$

or $$\frac{l}{6}-x+\frac{x^2}{l}=0$$

or $$x^2-lx+\frac{l^2}{6}=0$$

$$x=\frac{l\pm\sqrt{l^2-\frac{4l^2}{6}}}{2}=\frac{l-\frac{l}{\sqrt{3}}}{2}$$

$$=0{\cdot}211\ l.$$

Maximum bending moment

$$M_{max}=\frac{wl}{6}\ (0{\cdot}211\ l)-\frac{w}{2}\ (0{\cdot}211\ l)^2+\frac{w}{3l}\ (0{\cdot}211\ l)^3$$

$$=wl^2\ [0{\cdot}0351-0{\cdot}0222+0{\cdot}0031]$$

$$=0{\cdot}016\ wl^2.$$

The beam is symmetrically loaded about its centre, though in the opposite direction. The maximum negative BM will occur at a distance of 0·211 l from the end B.

Problem 7·11. A beam $ABCD$, 8 m long, supported at B and D 6 m apart carries a concentrated load of 16 kN at end A and 96 kN load distributed over a length of 4 m from C to D. Point C is at a distance of 2 m from B. Rate of loading varies from p at C to q at D. Determine p and q such that reactions at B and D are equal. Draw the SF and BM diagrams. Find the position of the point of contraflexure.

Solution. Total vertical load on the beam

$$=16+96=112 \text{ kN}$$

Support reactions $R_B=R_D=\frac{112}{2}=56$ kN

(as given in the problem).

Taking moment of the forces about the point A

$$56\times2+56\times8=p\times4\ (6)+(q-p)\ \left(\frac{4}{2}\right)\left(4+\frac{8}{3}\right)$$

$$112+448=24\,p+(q-p)\left(\frac{40}{3}\right)$$

$$560=24\,p+\frac{40}{3}\,q-\frac{40}{3}\,p \qquad \ldots(1)$$

But $$\frac{(p+q)}{2}\times 4=96 \text{ (as given)}$$

or $$p+q=48, \quad q=48-p \qquad \ldots(2)$$

Substituting the value of q in equation (1)

$$24\,p-\frac{40}{3}\,p+\frac{40}{3}\,(48-p)=560$$

$$24\,p-\frac{40}{3}\,p+640-\frac{40}{3}\,p=560$$

$$-\frac{8}{3}\,p=-80$$

$$p=30 \text{ kN/m, and } q=18 \text{ kN/m.}$$

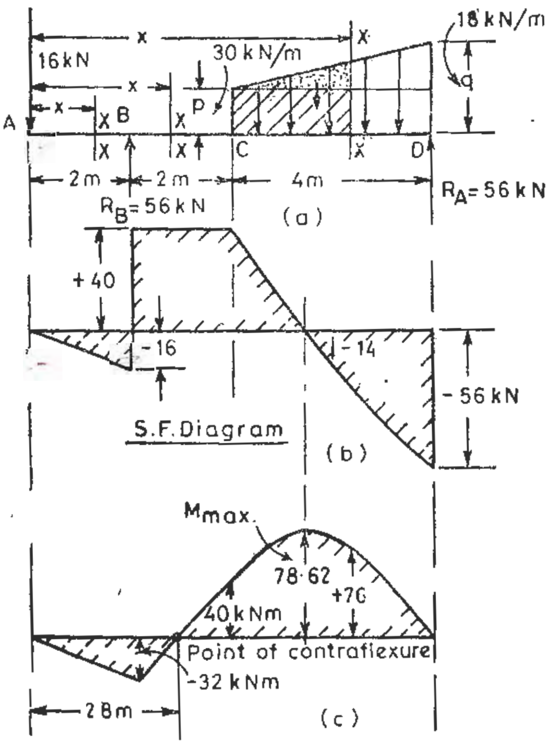


Fig. 7·45

SF diagram. Taking resultant of the forces only on the left side of the section.

Portion AB. SF at any section,

$$F_x=-16 \text{ kN} \quad \text{at} \quad x=0 \text{ to } 2 \text{ m.}$$

Portion BC. SF at any section,

$$F_x = -16+56 = +40 \text{ kN} \quad \text{at} \quad x=2 \text{ to } 4 \text{ m}.$$

Portion CD. Rate of loading at the section,

$$= p + \frac{q-p}{4}(x-4)$$

$$= 30 + \frac{18-30}{4}(x-4) = 30-3(x-4)$$

Downward distributed load

$$= \left(\frac{p+30-3(x-4)}{2}\right)(x-4)$$

$$= \left(\frac{30+30-3(x-4)}{2}\right)(x-4)$$

$$= [30-1\cdot5(x-4)](x-4) = 30(x-4)-1\cdot5(x-4)^2$$

Shear force at any section,

$$F_x = -16+56-[30(x-4)-1\cdot5(x-4)^2]$$

$$= 40-30(x-4)+1\cdot5(x-4)^2$$

$$= 40 \text{ kN} \quad \text{at} \quad x=4\text{m}$$

$$= -14 \text{ kN at } x=6\text{m}$$

$$0 = -56 \text{ kN at } x=8\text{m}.$$

BM diagram. Taking moments of the forces only on the left side of the section. (Clockwise moments are positive).

Portion AB. BM at any section,

$$M_x = -16x \text{ kNm}$$

$$= 0 \quad \text{at} \quad x=0 \text{ m}$$

$$= -16 \text{ kNm} \quad \text{at} \quad x=1 \text{ m}$$

$$= -32 \text{ kNm} \quad \text{at} \quad x=2 \text{ m}.$$

Portion BC.

$$M_x = -16x+56(x-2)$$

$$= -32 \text{ kNm} \quad \text{at} \quad x=2 \text{ m}$$

$$= +8 \text{ kNm} \quad \text{at} \quad x=3 \text{ m}$$

$$= +48 \text{ kNm} \quad \text{at} \quad x=4 \text{ m}.$$

Portion CD.

$$M_x = -16x+56(x-2)-p(x-4)\left(\frac{x-4}{2}\right) - \frac{(q-p)(x-4)}{2}\left(\frac{x-4}{3}\right)$$

Note that moment of the distributed load is considered in two parts as shown in Fig. 7·45 (*a*).

$$M_x = -16x+56(x-2)-\frac{p}{2}(x-4)^2-\frac{(q-p)}{6}(x-4)^2$$

Putting the values of p and q as 30 and 18 respectively

$$= -16x+56(x-2)-15(x-4)^2+2(x-4)^2 \quad \text{.. (1)}$$

$$= +48 \text{ kNm} \quad \text{at} \quad x=4 \text{ m}$$

$$=+76 \text{ kNm} \quad \text{at} \quad x=6 \text{ m}$$
$$=0 \quad \text{at} \quad x=8 \text{ m}.$$

Maximum bending moment occurs where $F_x=0$, *i.e.*, in portion BC

$$F_x=40-30\,(x-4)+1{\cdot}5\,(x-4)^2=0$$
$$40-30x+120+1{\cdot}5x^2-12x+24=0$$
$$1{\cdot}5x^2-42x+184=0$$

$$x=\frac{42-\sqrt{(42)^2-4\times 1{\cdot}5\times 184}}{3}=\frac{42-25{\cdot}69}{3}=5{\cdot}43 \text{ m}$$

Substituting the value of x in eqn. (1)

$$M_{max}=-16\times 5{\cdot}43+56\,(543-2)-15\,(5{\cdot}43-4)^2+2\,(5{\cdot}43-4)^3$$
$$=-86{\cdot}88+192{\cdot}08-30{\cdot}67+4{\cdot}09=78{\cdot}62 \text{ kNm}$$

Point of contraflexure lies in the portion BC as is obvious from the BM diagram

So $\qquad -16x+56x-112=0$

$$40x=112,\ x=2{\cdot}8 \text{ m}.$$

The BM diagram is shown in Fig. 7·45 (*c*).

Problem 7·12. A horizontal girder 10 m long is hinged at one end and rests freely on a roller support at a distance of 7 m from the hinged end. The beam carries a uniformly distributed load of intensity 1000 kg/m run from the end A for a length of 5 metres, a point load 2000 kg inclined at 30° to the vertical at a point D, 6 metres from end A and a point load of 3000 kg inclined at 45° to the vertical, at the point E, 1 m from the hinged end as shown. Determine the support reactions. Draw the SF and BM diagrams.

Solution. Let us first resolve the inclined loads into vertical and horizontal components

Vertical component of 2000kg load

$$=2000\times\cos 30°=1732\downarrow$$

Horizontal component of 2000 kg load

$$=2000\times\sin 30°=\overrightarrow{1000} \text{ kg}$$

Vertical component of 3000 kg lood

$$=3000\cos 45°=2121 \text{ kg}\downarrow$$

Horizontal component of 3000 kg load

$$=3000\times\sin 45°=\overleftarrow{2121} \text{ kg}$$

The beam hinged at end F,

Horizontal reaction at $F=\overrightarrow{2121}-\overleftarrow{1000},\ R_{FH}=\overrightarrow{1121}$

For vertical components of reactions, let us take moments of the forces about the point A.

$$1000\times 5\times 2{\cdot}5\circlearrowright+1732\times 6\circlearrowright+2121\times 9\circlearrowright=3\,R_B\circlearrowleft+10\,R_{FV}\circlearrowleft$$
$$12500+10392+19089=3\,R_B+10\,R_{FV}$$
$$41981=3\,R_B+10\,R_{FV}$$

For equilibrium

$$R_B+R_{FV}=5\times 1000+1732+2121=8853 \text{ kg}$$

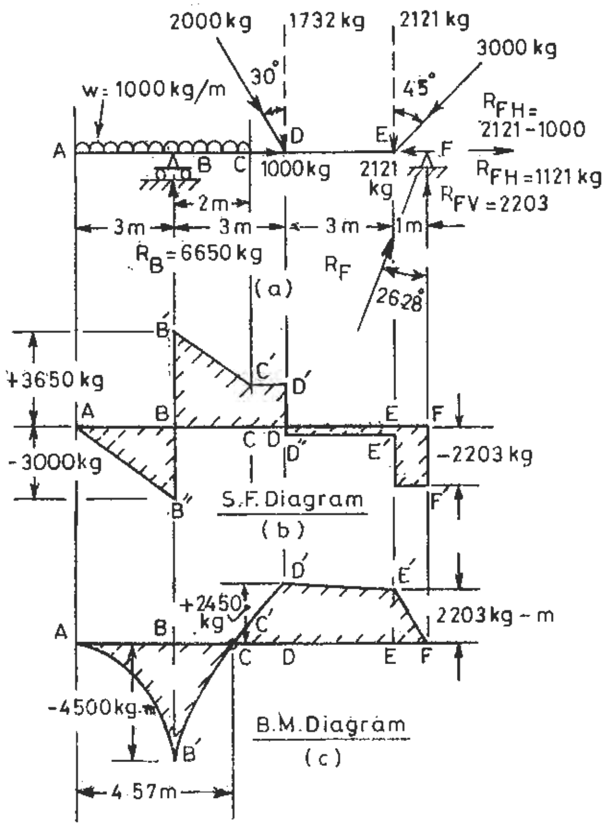


Fig. 7·46

So $\qquad R_B=(8853-R_{FV})$

$$10\,R_{FV}+3\,(8853-R_{FV})=41981$$

$$7R_{FV}=15422 \text{ kg-m}$$

$$R_{FV}=2203 \text{ kg}, \quad R_B=6650 \text{ kg}$$

Total reaction at F $\quad =R_F=\sqrt{(2203)^2+(1121)^2}=2472$ kg

Angle of Inclination of R_F to the vertical,

$$\tan\theta=\frac{R_{FH}}{R_{FV}}=\frac{1121}{2203}=0{\cdot}5088$$

or $\qquad \theta=26°\ 58'$.

For SF and BM diagrams consider a section X-X at a distance of x from the end A in portions AB, BC, CD, DE and EF respectively. Take upward forces on the left of the section to be positive, and clockwise moment on the left side of the section to be positive.

SF diagram.

Portion AB. SF at any section,

$$F_x=-wx \qquad \text{where } w=1000 \text{ kg/m}$$

$$=0 \quad \text{at} \quad x=0 \text{ m}$$

$$=-3000 \text{ kg at } x=3 \text{ m}$$

Portion BC. $F_x = -wx + 6650 = -1000x + 6650$
$= +3650$ kg at $x = 3$ m
$= +1650$ kg at $x = 5$ m.

Portion CD. $F_x = -1000 \times 5 + 6650 = 1650$ (constant throughout this portion).

Portion DE. $F_x = -5000 + 66\ 0 - 1732 = -82$ kg (constant throughout this portion).

Portion EF. $F_x = -5000 + 6650 - 1732 - 2121 = -2203$ kg (constant throughout this portion).

Fig. 7·46 (*b*) shows the SF diagram.

B.M. Diagram

Portion AB. B.M. at any section,

$$M_x = -\frac{wx^2}{2} \quad \text{where } w = 1000 \text{ kg/m}$$
$= -500\ x^2$
$= 0$ at $x = 0$ m
$= -500$ kg-m at $x = 1$ m
$= -2000$ kg-m at $x = 2$ m
$= -4500$ kg-m at $x = 3$ m.

Portion BC. $M_x = -\frac{wx^2}{2} + R_B(x-3) = -500\ x^2 + 6650\ (x-3)$
$= -4500$ kg-m at $x = 3$ m
$= -1350$ kg-m at $x = 4$ m
$= +800$ kg-m at $x = 5$ m.

Portion CD. $M_x = -5000\ (x - 2{\cdot}5) + 6650\ (x - 3)$
$= +800$ kg-m at $x = 5$ m
$= +2450$ kg-m at $x = 6$ m.

Portion DE. $M_x = -5000\ (x - 2{\cdot}5) + 6650(x-3) - 1732(x-6)$
$= +2450$ kg-m at $x = 6$ m
$= +2368$ kg-m at $x = 7$ m
$= +2204$ kg-m at $x = 9$ m
(There is slight error due to calculations *i e.* in place of 2203 kg-m we are getting 2204 kg-m).

Portion EF. $M_x = -5000\ (x - 2{\cdot}5) + 6650(x-3) - 1732(x-6) - 2121(x-9)$
$= 2204$ kg-m at $x = 9$ m
$= 0$ at $x = 10$ m

Fig. 7·46 (*c*) show the Bill diagram. The maximum bending moment -4500 kg-m occurs at the support *B* where *SF* has changed sign.

Point of contraflexure lies in the portion *BC*

Putting $-500x^2 + 6650(x-3) = 0$

$500x^2 - 6650x + 19950 = 0$

$x^2 - 13{\cdot}3 + 39{\cdot}9 = 0$

$$x=\frac{13\cdot3-\sqrt{(13\cdot3)^2-4\times39\cdot9}}{2}=\frac{13\cdot3-4\cdot16}{2}$$

$=4\cdot57$ m.

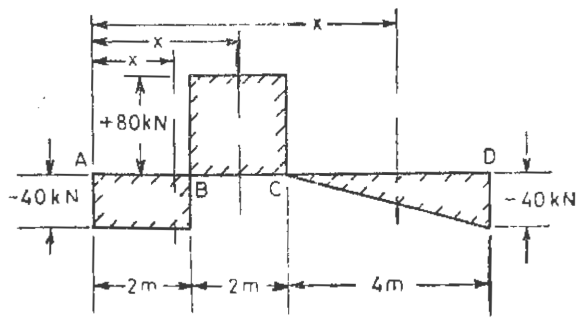
Point of contraflexure lies at a distance of 4·57 m from end A

Problem 7·13. A beam $ABCD$, 8 metres long, supported over a length of 6 metres at points B and D has the SF diagram as shown in Fig. 7·47. Determine the various loads acting on the beam. Then draw the BM diagram and find.

(1) Magnitude and position of the greatest bending moment.

(2) Position of the point of contraflexure if any.

Solution. The beam is supported on the points B and D, there will be support reactions say R_B and R_D. Consider the three portions of the beam is AB, BC and CD. Consider



S K Diagram

Fig. 7·47

a section X-X at a distance of x from the end A. Taking upward force on the left side of the section to be positive.

Portion AB. $F_x=-40$ kN (constant in the portion AB).

This shows that there is a vertical load of 40 kN acting on the point A.

Portion BC. $F_x=+80$ kN (constant in the portion BC).

At the point B, SF has changed from -40 kN to $+80$ kN showing thereby that

$$-40+R_B=80 \text{ kN}$$

or $$R_B=120 \text{ kN}$$

Reaction at the support B,

$$R_B=120 \text{ kN}.$$

Portion CD. At the point C,

$$\text{SF}=0.$$

Which shows that a vertical load of 80 kN is acting on this point.

From C to D, the SF is not constant but has a straight line relation, gradual decrease in SF. Showing uniformly distributed load over the portion CD.

Say the rate of loading $=w$

Then SF, $F_C - F_D = -w \times 4$ (4 m is the length of the portion CD) where

$0 - 40 \text{ kN} = -4w$ $\qquad F_C$ = shear force at C and

or $w = 10$ kN/metre run. $\qquad F_D$ = shear force at D

At the point D, there is a SF

$$F_D = -40 \text{ kN.}$$

(Note that vertically upward force on the right side of the section is taken as a negative SF).

So reaction at D, $R_D = 40$ kN.

The load diagram on the beam is as shown in Fig. 7·48 (*a*).

Total vertical load on the beam

$$= 40 + 80 + 4 \times 10 = 160 \text{ kN}$$

Reactions, $R_B + R_D = 120 + 40 = 160$ kN

Which shows that load diagram is correct.

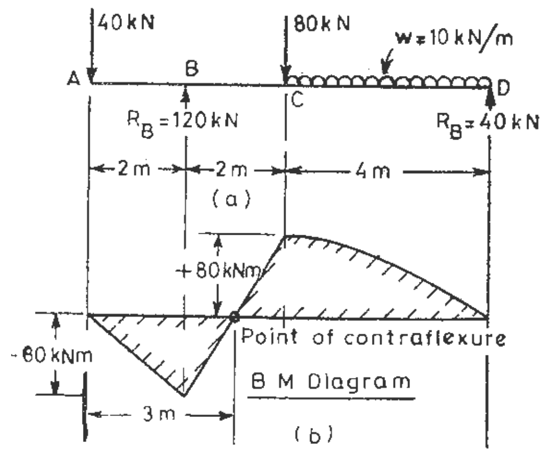


Fig. 7·48

B.M. Diagram. Taking clockwise moments on the left side of the section to be positive.

Portion AB. B.M. at any section,

$$M_x = -40\,x$$
$$= 0 \quad \text{at } x = 0 \text{ m}$$
$$= -80 \text{ kNm} \quad \text{at } x = 2 \text{ m.}$$

Portion BC.

$$M_x = -40x + 120\,(x-2)$$
$$= -80 \text{ kNm} \quad \text{at } x = 2 \text{ m}$$
$$= +80 \text{ kNm} \quad \text{at } x = 4 \text{ m.}$$

Portion CD.

$$M_x = -40x + 120(x-2) - 80(x-4) - \frac{w\,(x-4)^2}{2}$$

where $w = 10$ kN/m

$$= 80 - 5(x-4)^2$$
$$= 80 \text{ kNm} \quad \text{at } x = 4 \text{ m}$$
$$= 75 \text{ kNm} \quad \text{at } x = 5 \text{ m}$$

$$=60 \text{ kNm at } x=6 \text{ m}$$
$$=35 \text{ kNm at } x=7 \text{ m}$$
$$=0 \text{ kNm at } x=8 \text{ m}.$$

Fig. 7·48 (*b*) shows the B.M. diagram. Maximum bending moment ± 80 kN m occurs at distances of 2 and 4 m from the end *A*. Point of contraflexure lies at a distance of 3 m from the end *A*.

Problem 7·14. A beam *ABCD*, 10 m long and hinged at its ends is subjected to clockwise couples 60 kNm and 80 kNm at distances of 3 m and 7 m from the left hand end support. Draw the SF and BM diagrams and determine the position of the point of contra-flexure if any.

Solution. Taking moments of the forces about the point *A*

$$60\circlearrowright + 80\circlearrowright - R_D \times 10\circlearrowleft = 0$$

$$R_D = 14 \text{ kN} \uparrow$$

For equilibrium $\qquad R_A = 14 \text{ kN} \downarrow$

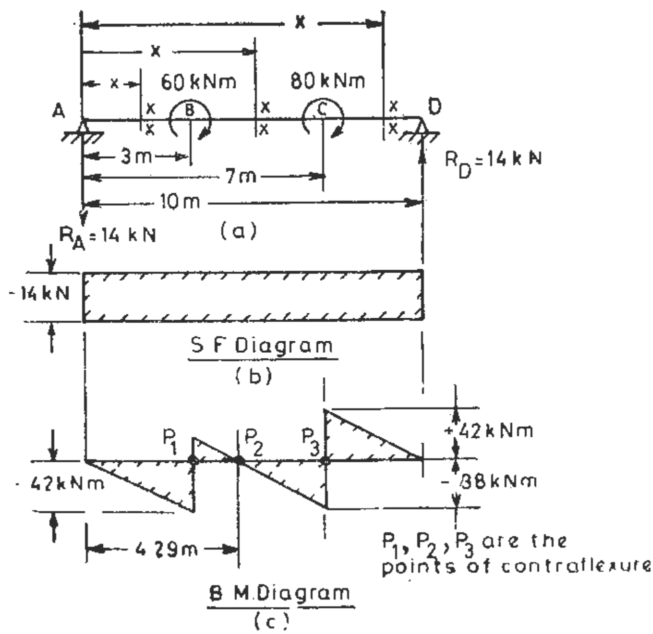


Fig. 7·49

SF Diagram. Consider a section *X-X* at a distance of x from the end *A*. Taking upward forces on the left side of the section to be positive.

Portion AB. SF at any section,

$$F_x = -14 \text{ kN}$$

(constant from $x=0$ to 3 m).

Portion BC. $\qquad F_x = -14$ kN (constant from $x=3$ m to 7 m)

Portion CD. $F_x = -14$ kN (constant from $x=7$ m to 10 m)

Fig. 7·49 (*b*) shows the SF diagram.

BM Diagram. (Clockwise moments on the left side of the section are positive).

Portion AB. BM at any section,

$$M_x = -14\,x$$
$$= 0 \text{ at } x = 0$$
$$= -42 \text{ kNm at } x = 3 \text{ m.}$$

Portion BC.
$$M_x = -14x + 60 \text{ kNm}$$
$$= +18 \text{ kNm at } x = 3 \text{ m}$$
$$= -10 \text{ kNm at } x = 5 \text{ m}$$
$$= -38 \text{ kNm at } x = 7 \text{ m.}$$

Point of contraflexure lies in this portion also as the BM has changed sign.

Putting $M_x = 0$

or $-14x + 60 = 0$

We get, $x = 4·29$ m (from end A).

Portion CD.
$$M_x = -14x + 60 + 80$$
$$= +42 \text{ kNm at } x = 7 \text{ m}$$
$$= +28 \text{ kNm at } x = 8 \text{ m}$$
$$= 0 \quad \text{at } x = 10 \text{ m.}$$

The BM diagram is shown in the Fig. 7·49 (*c*).

As is obvious from the BM diagram there are 3 points of contraflexure lying at distances of 3 m, 4·29 m and 7 m from the end A.

Problem 7·15. A beam $ABCD$, hinged at one end and simply supported at other carries the loads/forces as shown in the Fig. 7·50. Draw the SF and BM diagrams.

Solution. The load diagram is equivalent to the diagram shown below in Fig. (*b*), which can be obtained as follows :

(*i*) At point B, an inclined load of 5 tonnes is resolved into two components of $3T\downarrow$ and $\overrightarrow{4T}$. The component $4T$ at a lever of 1 m length is equivalent to an anticlockwise moment $4T$-m and a force of $\overleftarrow{4T}$ at the point B.

(*ii*) At the point C, Forces $\overrightarrow{4T}$ at lever length of 1 m and $\overleftarrow{3T}$ at lever length of 2 m are equivalent to a force $\overrightarrow{1T}$ and a clockwise moment 10 Tm at the point C. The end A of the beam is simply supported, while the end D is hinged.

So horizontal reaction at D,

$$R_{DH} = 4 - 1 = \overrightarrow{3T}$$

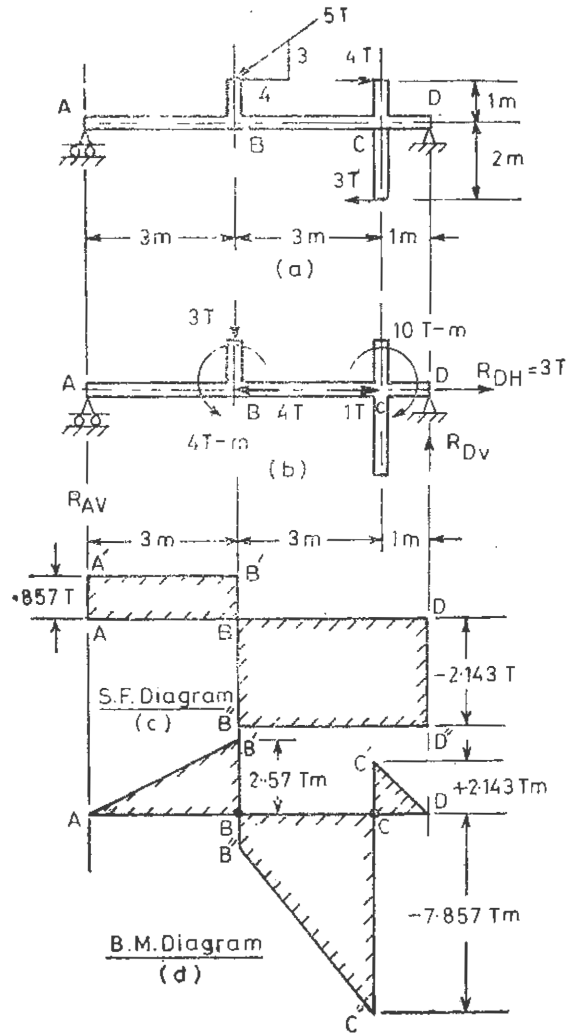


Fig. 7·50

For support reactions, take moments about the point A.

$$3\times 3 \circlearrowright -4 \circlearrowleft +10 \circlearrowright -7\times R_{DV} \circlearrowleft =0$$

$$15=7\ R_{DV}$$

$$R_{DV}=2{\cdot}143\ T.$$

But $\quad R_{DV}+R_{AV}=3T$

$$R_{AV}=3-2{\cdot}143=0{\cdot}857\ T$$

SF diagram. (Taking upward forces on the left of a section to be positive)

Portion AB. $\quad F_x=+0{\cdot}857\ T \quad$ (constant from A to B)

Portion BC. $\quad F_x=+0{\cdot}857\ T-3T=-2{\cdot}143\ T \quad$ (constant from B to C)

Portion CD. $\quad F_x=+0{\cdot}857T-3T=-2{\cdot}143\ T \quad$ (constant from C to D)

Fig. 7·50(c) shows the SF diagram.

BM diagram. Taking clockwise moment on the left side of the section to be positive.

Portion AB. BM at any section,

$$M_x = +0{\cdot}857\ x$$
$$= 0 \quad \text{at} \quad x = 0$$
$$= +2{\cdot}570\ \text{Tm} \quad \text{at} \quad x = 3\ \text{m}$$

Portion BC.

$$M_x = 0{\cdot}857\ x - 4 - 3\ (x - 3)$$
$$= -1{\cdot}43\ \text{Tm} \quad \text{at} \quad x = 3\ \text{m}$$
$$= -3{\cdot}572\ \text{Tm} \quad \text{at} \quad x = 4\ \text{m}$$
$$= -5{\cdot}715\ \text{Tm} \quad \text{at} \quad x = 5\ \text{m}$$
$$= -7{\cdot}857\ \text{Tm} \quad \text{at} \quad x = 6\ \text{m}$$

Portion CD.

$$M_x = +0{\cdot}857 - 4 - 3(x - 3) + 10$$
$$= +2{\cdot}143\ \text{Tm} \quad \text{at} \quad x = 6\ \text{m}$$
$$= 0\ \quad \text{Tm} \quad \text{at} \quad x = 7\ \text{m}$$

Fig. 7·50 (*d*) shows the BM diagram with two points of contraflexure at points *B* and *C*.

SUMMARY

1. Resultant of forces parallel to the section of the beam carrying transverse loads on the left or on the right side of the section is called shear foree.

2. On the left side of the section vertically upward force is a positive shear force. On the right side of the section, vertically downward force is a positive shear force.

3. Resultant moment of the forces on the left or on the right side of a section is called Bending Moment.

4. Clockwise moments on the left side of the section are positive BM. Anticlockwise moments on the right side of the section are positive BM.

5. For a cantilever of length L, carrying load W at free end, maximum bending moment $-WL$ occurs at the fixed end.

6. For a beam of length L, simply supported at it ends, carrying a concentrated load W at its centre, the maximum bending moment $\frac{WL}{4}$ occurs at the centre of the beam.

7. For a eantilever of length L, carrying uniformly distributed load w per unit length, maximum bending moment $\frac{wL^2}{2}$ occurs at the fixed end.

8. For a beam of length L simply supported at its ends carrying uniformly distributed load w per unit length, maximum bending moment $\frac{wL^2}{8}$ occurs at the centre of the beam.

9. Maximum bending moment in a beam occurs at a point where shear force either is zero or shear force changes sign.

10. A point of contraflexure in a beam occurs at a point where bending moment changes sign.

11. For a beam of length L, hinged at both the ends subjected to a turning moment M, the reactions at ends are $\pm\frac{M}{L}$. The shear force remains constant throughout the length of the beam.

12 If for a certain portion of a beam, bending moment is constant, then shear force is zero.

13. For a beam carrying transverse point loads and distributed loads

(i) $\frac{dF}{dx}=-w,$

i.e., rate of change of S.F. is equal to the rate of loading at a particular section

(ii) $\frac{dM}{dx}=F,$

i.e., rate of change of B.M. is equal to the shear force at a particular section

MULTIPLE CHOICE QUESTIONS

1. A cantilever 5 m long, carries a point load of 5 tonnes at its free end and a uniformly distributed load of 2 tonnes/metre run throughout its length, the maximum bending moment on the cantilever is

(*a*) 100 tonne-metres (*b*) 50 tonne-metres
(*c*) 25 tonne metres (*d*) None of the above.

2. A cantilever 8 m long carries a point load of 5 tonnes at its free end and 5 tonnes at its middle. The bending moment at the middle of the cantilever is

(*a*) 10 tonne-metres (*b*) 20 tonne-metres
(*c*) 40 tonne-metres (*d*) None of the above.

3. A cantilever 10 metres long, carries a uniformly distributed load of 10 kN/metre run starting from free end upto the middle of its length. The BM at the fixed end of the cantilever is

(*a*) 25 kNm (*b*) 50 kNm
(*c*) 75 kNm (*d*) 100 kNm.

4. A cantilever 8 m long carries throughout its length a uniformly distributed load of w kg/m run. If the maximum bending moment is 3200 kg-metre, the rate of loading w is

(*a*) 100 kg/m (*b*) 50 kg/m
(*c*) 25 kg/m (*d*) None of the above.

5. A cantilever 6 m long, carries a point load of 100 kN at its free end and another point load W at the middle of its length. If the maximum BM on cantilever is 900 kNm, the valve of load W is

(*a*) 50 kN (*b*) 100 kN
(*c*) 150 kN (*d*) 200 kN.

6. A cantilever 10 m long carries a uniformly distributed load of 20 kN/m run throughout its length. If it is propped by a force P at its free end so that the centre of the cantilever becomes the point of inflexion, the magnitude of P is

(*a*) 200 kN (*b*) 150 kN
(*c*) 100 kN (*d*) 50 kN

7. A beam 8 m long, simply supported at its ends, carries a point load of 800 kg at a distance of 3 m from one end. The BM under the load is
   (*a*) 4000 kg-metre (*b*) 1600 kg-metre
   (*c*) 1500 kg-metre (*d*) 1000 kg-metre.

8. A beam 10 m long supported over 8 m span, having equal over hang on both the sides, carries loads of 8 tonnes each at its ends and a load of 2 tonnes at its centre, the points of contraflexure lie at
   (*a*) at the supports (*b*) at the centre
   (*c*) at 2 m from each end (*d*) None of the above.

9. A beam 8 m long, supported over a span of 6 m, carries a concentrated load of 20 kN at its centre. The maximum bending in the beam is
   (*a*) 80 k Nm (*b*) 60 k Nm
   (*c*) 40 k Nm (*d*) 30 k Nm.

10. A beam 8 m long, simply supported at the ends, carries a uniformly distributed load of $2T/m$ from one end to a distance of 2 m, and from the other end to a distance of 2 m. The SF at the centre of the beam is
   (*a*) 4 T (*b*) 2 T
   (*c*) 1 T (*d*) 0.

11. A beam carries transverse loads and is simply supported with over hang on both the sides. The point of contraflexure is a point where—
   (*a*) Shear force is maximum (*b*) Shear force is zero.
   (*c*) Bending moment changes sign (*d*) Bending moment is maximum.

12. A beam 10 m long hinged at both the ends is subjected to a clockwise turning moment of 40 k Nm at a distance of 3 m from one end. The SF at the centre of the beam is
   (*a*) 0 kN (*b*) 2 kN
   (*c*) 4 kN (*d*) 8 kN.

13. A beam carries transverse loads. Its SF and BM diagrams are drawn. In a portion of the beam where SF is zero, the bending moment is
   (*a*) maximum (*b*) minimum
   (*c*) constant (*d*) zero.

14. A beam 10 m long is supported over 6 m span with equal over hang on both the sides. It carries point loads of 40 kN each at its ends and a point load of 80 kN at its centre. The points of contraflexure lie at a distance of x metres form each end. The value of x is
   (*a*) 2 m (*b*) 3 m
   (*c*) 4 m (*d*) 5 m.

15. A beam 10 m long carries point loads. When SF diagram is drawn, there are two rectangles of the size 10 kN$\times$2 m, one is starting from end and above the base. The other starting from the other end but below the base line. The BM at the centre of the beam is
   (*a*) 50 kNm (*b*) 40 kNm
   (*c*) 30 kNm (*d*) 20 kNm.

ANSWERS

1. (*b*)	2. (*b*)	3. (*c*)	4. (*a*)	5. (*b*)
6. (*d*)	7. (*c*)	8. (*d*)	9. (*d*)	10. (*d*)
11. (*c*)	12. (*c*)	13. (*c*)	14. (*c*)	15. (*d*)

EXERCISE

7·1. A beam *ABCDE*, 16 m long supported over a span *ABCD* of 12 metre carries concentrated loads of 6 tonnes at *B*, 4 m from *A*, 5 tonnes at *C*, 8 m from *A* and 4 tonnes at *E*. Draw the SF and BM diagrams stating (*i*) the position and magnitude of maximum BM (*ii*) the position of the point of contraflexure.

[**Ans.** Reactions, $R_A=4{\cdot}33$ tonnes, $R_D=10{\cdot}67$ tonnes, $M_{max}=17{\cdot}32$ tonnes/m at a distance of 4 m from end *A*, points of contraflexure lies at a distance of 9·6 m from *A*].

7·2. A beam *AB*, hinged at the ends *A* and *B*, of length *l* carries a uniformly distributed load of intensity *w* acting downwards on half of its length and an upward uniformly distributed load of intensity *w* acts on the remaining half of the beam (*a*). Draw the SF and BM diagrams. (*b*) Locate the position of the point of inflexion, if any. (*c*) What is the maximum bending moment and where it occurs.

[**Ans.** Reactions $\pm wl/4$, (*b*) Point of inflexion lies at the centre of the beam, (*c*) Maximum bending moment $\pm wl^2/32$ occurs at $l/4$ from both the ends].

7·3. A beam 8 m long carries a uniformly distributed load of 10 kN/m run, throughout its length. Clockwise moments of 50 kN m and 30 kN m are applied at the two ends. Determine the support reactions. Find the magnitude and position of the greatest bending moment.

[**Ans.** Support reactions 30 kN and 50 kN $M_{max}=95$ kN m at 3 metres from one end]

7·4. A beam *ABCD*, 8 m long supported at *B*, 1 m from *A* and at *C x* metre from *D*. The beam carries a point load of 4 kN at end *A* and a uniformly distributed load of 2 kN/m run throughout its length. Determine the value of *x*, if the centre of the beam becomes the point of contraflexure. Draw the BM diagram and find the position of any other point of contraflexure.

[**Ans.** $x=16/7$ m, $R_B=32/3$ kN, $R_C=28/3$ kN ; other point of contraflexure lies at 7/3 m from end *A*]

7·5. A beam *ABCDE*, 12 m long cantilevered over the portion $AB=4$ m long, supported at points *B* and *E*, $BE=8$ m, carries a concentrated load 2 kN at *A*, 2 kN at *C*, 2 m from *A* and 2 kN at *D*, 2 m from *E*. In addition it carries a uniformly distributed load of 1 kN/m over the portion *CD*. Draw the SF and BM diagrams, indicating the values of BM at *B*, *C* and *D*. Find the position of the point of contraflexure.

[**Ans.** $R_B=7$ kN, $R_E=3$ kN, $M_B=-8$ kNm, $M_C=+2$ kNm, $M_D=+6$ kNm ; Point of contraflexure lies at a distance of 5·6 m from *A*]

7·6. A beam *ABCDE*, 14 metres long supported at *B* and *D*, the overhang on both the sides being 3 metres, carries the transverse loads as shown in the Fig. 7·51.

(*a*) Draw the SF and BM diagrams

(*b*) Find the position of the point of inflexion.

(*c*) Determine the position and magnitude of maximum BM.

[**Ans.** $R_B=6$ tonne, $R_D=8$ tonnes, (*b*) There is no point of inflexion in the beam, (*c*) M_{max} 15 tonne-metres occurs at the support *D*]

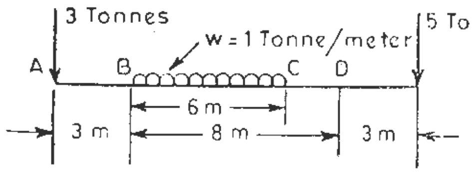


Fig. 7·51

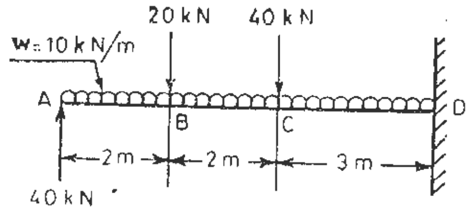


Fig. 7·52

7·7. A propped contilever *ABCD*, 7 m long carries the transverse loads as shown in the Fig. 7·52. Draw the BM diagram and determine (*i*) magnitude and position of the maximum BM (*ii*) Position of the point of contraflexure.

[**Ans.** $M_{max}=-185$ kNm at the fixed end. Point of contraflexure lies at a distance of 4·633 m from end *A*]

7·8. A beam 8 m long simply supported over a span of 6 m, carries the transverse loads as shown in the Fig. 7·53. Draw the shear force and bending moment diagrams. Determine (*i*) the position of the point of contraflexure if any (*ii*) position and magnitude of the maximum bending moment.

[**Ans.** Point of contraflexure lies at a distance of 5·48 m from one end, $M_{max}=-15·833$ tonne metres at a distance of 1 m from one end]

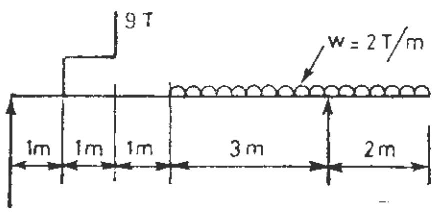


Fig. 7·53

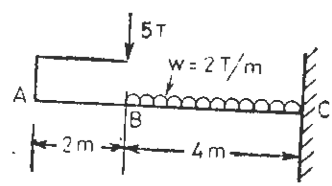


Fig. 7·54

7·9. A cantilever 6 m long, carries a uniformly distributed load of 2 tonne/m run from *B*, 2 m from free end *A*, upto the fixed end *C* and a concentrated load 5 tonnes at the end of a lever at *A*, as shown in the Fig 7·54. Determine

(*i*) the point where shear force is zero.

(*ii*) magnitude and position of the maximum bending moment.

(*iii*) position of the point of inflexion.

[**Ans.** (*i*) at no point SF is zero
(*ii*) -36 tonne-metres at the fixed end
(*iii*) at 2 m from end *A*]

7·10. A beam *ACB* of length *l* hinged at both the ends carries a linearly varying distributed load as shown in Fig. 7·55. Determine the maximum bending moment and its position. Locate also the point of contraflexure.

[**Ans.** $M_{max}=\pm wl^2/36\sqrt{3}$ at $l/2\sqrt{3}$ from both the ends. Point of contraflexure lies at the centre]

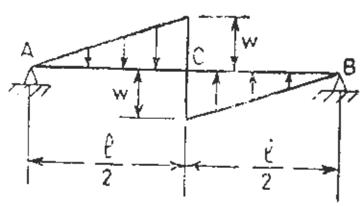


Fig. 7·55

7·11. A beam $ABCD$, 12 m long supported at B and D, 8 m apart carries a concentrated load of 3 tonnes at A and 18 tonnes load distributed from C to D. Point C is at a distance of 4 m from support B. Rate of loading varies from p at C to q at D. Determine the values of p and q such that reactions at B and D are equal. Draw the SF and BM diagrams. Find the magnitude and position of the maximum bending moment. Where is the point of contraflexure ?

[**Ans.** $R_B = R_D = 10·5$ tonne, $p = 9$ tonne-metre, $q = 0$,

$M_B = -12$ tonne-metres, $M_C = +18$ tonne-metre

$M_{max} = 21·38$ tonne-metres at 8·945 m from end A.

Point of contraflexure lies at a distance of 5·6 m from A]

7·12. A horizontal girder, $ABCD$, 8 m long is hinged at end A and rests freely on a roller support at D. The girder is loaded with vertical and inclined loads as shown in Fig. 7·56. Assuming the direction of the reaction at D to be vertical, determine (*a*) magnitude of reaction at D (*b*) magnitude and direction of reaction at A. Draw the BM diagram to a suitable scale.

[**Ans.** $R_D = 6·121$ tonnes, $R_A = 7·447$ tonnes, inclined at an angle 34° 46′ to the vertical]

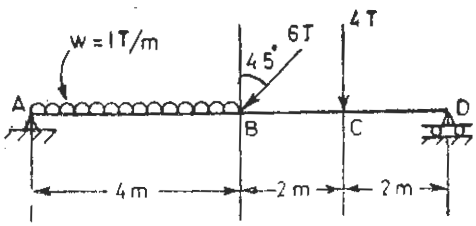


Fig. 7·56

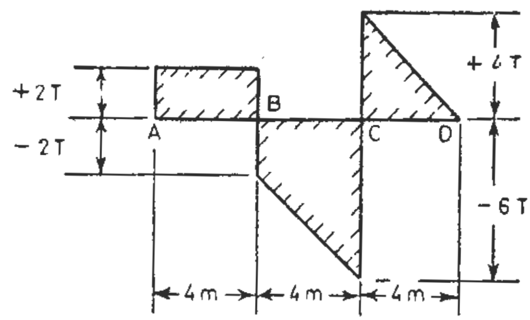


Fig. 7·57

7·13. The SF diagram of a beam $ABCD$, 12 m long, supported at the points A and C is shown in the Fig. 7·57. Draw (*a*) the load diagram, (*b*) the BM diagram of the beam. (*c*) Determine the position of the point of contraflexure.

Ans $M_B = +8$ tonne metres, $M_C = -8$ tonne metre. Point of contraflexure lies at 6·47m from A]

7·14. A beam $ABCD$, 8 metres long and hinged at ends is subjected to two couples $M_1 = 4$ Tm and $M_2 = 8$ T-m at points B and C. Both the couples are in anticlockwise direction and the points B and C are at 2 m and 6 m respectively from the end A (as shown in Fig. 7·58). Draw the SF and BM diagrams and find the position of the points of contraflexure.

[**Ans.** Points of contraflexure lie at distances of 2 m, 2·667 m and 6 m from end A]

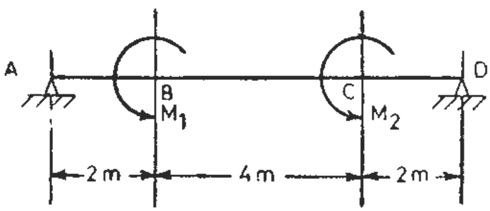


Fig. 7·58

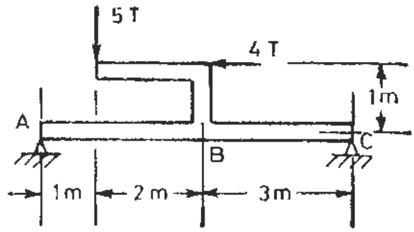


Fig. 7·59

7·15. A beam ABC, 6 m long, hinged at C and simply supported at A carries load/force at a lever 1 m×2 m as shown in the Fig. 7·59. Determine reactions at A and C. Draw SF and BM diagrams. Find also the position of the point of contraflexure if any.

[**Ans.** R_A=4·833 tonnes, R_{CH}=4 tonnes, R_{CV}=0·167 tonnes ;
M_B=+14·5 tonne-metres, =0·5 tonne-metres,
No point of contraflexure any where]

8

Theory of Simple Bending

In the last chapter we have studied about the Shear Force and Bending Moment diagrams of cantilevers and beams subjected to transverse loads. Shear stress is developed across the section of the beam due to the shear force on the section and longitudinal or direct stress is developed on the section of the beam due to the bending moment on the section. An element of the beam may be subjected to positive bending moment (*i.e.* a bending moment which produces concavity upwards in the beam) or a negative bending moment (*i.e.* a bending moment which produces convexity upwards) as shown in the Fig. 8·1. An element of the beam initially straight, bends to the shape $a'b'd'c'$ due to *a positive bending moment.*

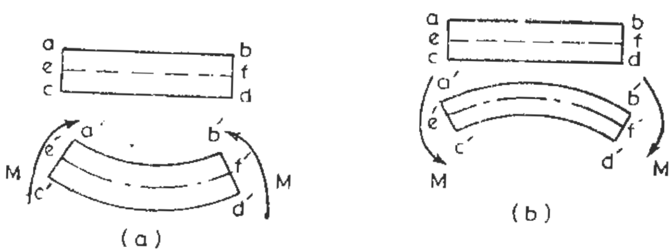


Fig. 8·1

As is obvious, the upper layer ab gets contracted to $a'b'$ *i.e.* $ab>a'b'$ and the lower layer gets extended to $c'd'$ *i.e.* $c'd'>cd$. There is a layer (shown dotted) which neither contracts nor extends *i.e.* $ef=e'f'$. There will be compressive strain and compressive stress in the upper layers and tensile strain and tensile stress in the lower layers. Similarly when the element of the beam is subjected to a *negative BM*, the upper layers will extend and lower layers will contract *i.e.* $a'b'>ab$ and $c'd'<cd$ as shown in Fig. 8·1 (*b*). Again there is a layer ef which neither extends nor contracts *i.e.* $e'f'=ef$.

The layer which neither contracts nor extends due to bending moment and does not have any strain or any stress in it is called a neutral layer. It will be shown that this neutral layer passes through the centroidal axis of the sections of the beam.

There is a definite relationship between the direct stress f developed due to bending and the bending moment M, which will be derived in this chapter. To develop the relationship between f and M certain assumptions are taken.

8·1. ASSUMPTIONS FOR THE THEORY OF SIMPLE BENDING

For developing the theory of simple bending or for working out relationship between the stress f and the bending moment M, following assumptions are taken :

(*i*) The beam is initially straight before the application of transverse loads on the beam.

(*ii*) The material of the beam is homogeneous and isotropic. *i.e.* the material possesses the same elastic properties in all directions through out the length and breadth of the beam.

(*iii*) Elastic limit is not exceeded *i.e.* if the beam is unloaded it returns to its original shape and dimensions.

(*iv*) Transverse sections which are plane before bending remain plane after bending. Fig. 8·2 explains the meaning of this assumption. Transverse sections of the beam such as *ab* which is in one plane, after bending changes the direction $a'b'$ but $a'b'$ section remains in one plane. Similarly the section *cd* in one plane after change remains in one plane $c'd'$. In other words this assumtion means that transverse sections of the beam are not distorted in shape after bending.

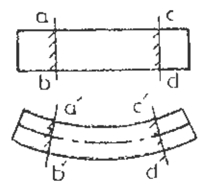


Fig. 8·2

(*v*) Each layer of the beam is free to expand or contract independently of the layers above or below it.

(*vi*) The value of the Young's modulus of elasticity E of the material is the same in tension and in compression.

(*vii*) The beam section is symmetrical about the plane of bending *i.e.* about the plane passing through the neutral layer.

8·2. THEORY OF SIMPLE BENDING

Consider an element *ABCD* of the beam of small length δx as shown in Fig. 8·3 (*a*). After the application of the transverse loads on the beam, the beam bends and say that on this small element, the bending moment M is positive *i.e.* producing concavity upwards. The beam section can be of any shape. Say the beam section is trapezoidal as shown. Due to bending moment, the upper layer *AB* contracts and lower layer *CD* extends. A layer *EF* which neither contracts nor extends is called the neutral layer. After bending *AB* changes to $A_1 B_1$, *EF* changes to $E_1 F_1$ and *CD* changes to $C_1 D_1$. Such that $A_1B_1<AB$, $C_1D_1>CD$ and $E_1F_1=EF$. Say for the small infinitesimal length δx, the bent length can be considered as a part of a circle of definite radius, as shown in Fig. 8·3. (*b*) Say the centre of circle or centre of curvature is O. Radius of the circle or radius of curvature upto the neural layer E_1F_1 is R.

Consider a layer *GH* at a distance of y from the neutral layer, which is reduced in length to G_1H_1 after the bending of the beam.

Strain in the layer $GH=\dfrac{\text{Final length—Original length}}{\text{Original length}}$

$$=\frac{G_1H_1-GH}{GH}$$

But initially $GH=EF=E_1F_1$

Strain in the layer GH, $\epsilon=\dfrac{G_1H_1-E_1F_1}{E_1F_1}$

where $E_1F_1 = R\theta$

$G_1H_1 = (R-y)\theta$

The distance between the layers *GH* and *EF* *i.e.* y is changed to y'. But the change in this thickness is negligible and $y' \simeq y$.

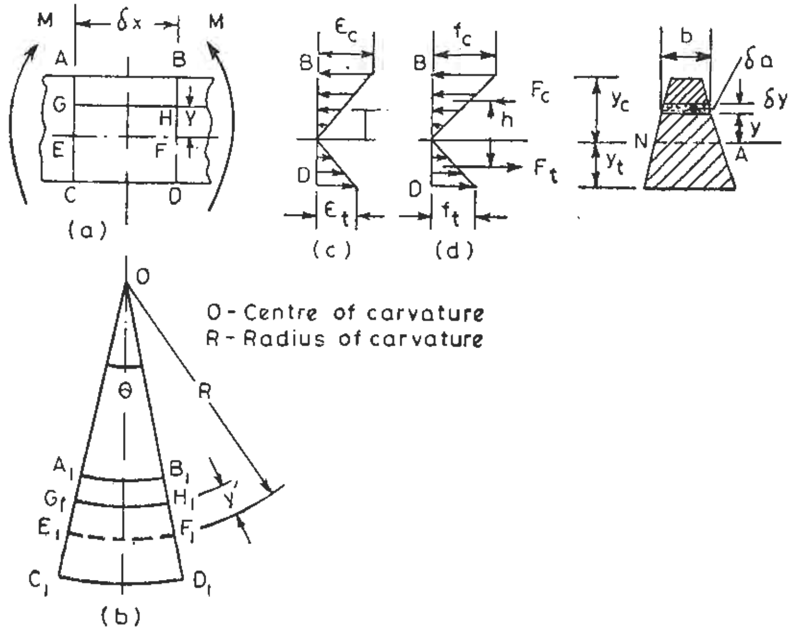


Fig. 8·3

So the strain in the layer *GH*,

$$\epsilon = \frac{(R-y)\,\theta - R\theta}{R\theta}$$

$$= \frac{-y}{R}$$

or $$\epsilon \propto y \qquad \ldots(1)$$

The strain in the layer is a compressive strain or a negative strain.

It can be deduced from this equation that *strain in any layer is proportional to its distance from the neutral layer*. The strain is compressive or tensile depends upon the position of the layer *i.e.* whether the layer is above or below the neutral layer. In this particular case, maximum negative or compressive strain will be at the top and maximum positive or tensile strain will be at the bottom layers.

Say y_c=distance of the top layer from the neural layer

y_t=distance of the bottom layer from the neutral layer

Then ϵ_c, maximum compressive strain$= \dfrac{y_c}{R}$

ϵ_t, maximum tensile strain $= \dfrac{y_t}{R}$

The variation of the strain is shown by the Fig. 8·3 (*c*). The strain distribution is linear across the thickness of the beam.

8·3. NEUTRAL AXIS

The intersection of the plane of the neutral layer with the cross section of the beam is called the neutral axis, as shown in the Fig. 8·30 by *NA* across the section of the beam.

The strain in *any layer* is directly proportional to its distance from the neutral axis

Say the stress in the layer GH

$$=f$$

But $f=\epsilon E$

where E=Young's modulus of the material

So
$$f=-\frac{y}{R}\times E$$

Consider an elementary area, $b\,\delta y$ as shown in the Fig. 8·3. Force on the layer GH of thickness, δy

$$\delta F=f\,.\,b\,\delta y=f\,\delta a$$
$$=-\frac{y}{R}\,.\,E\,.\,\delta a=-\frac{E}{R}\,.\,y\,\delta a$$

Total force on the section,

$$F=\int \delta F=-\frac{E}{R}\int_{y_t}^{-y_c} y\,.\,\delta a$$

For equilibrium resultant force on the section is zero. *i.e.* Total compressive force F_c acting on the section above the neutral axis is equal to the total tensile force F_t acting on the section below the neutral axis.

i.e. $F_c-F_t=0$

Therefore $F=0$ or $-\frac{E}{R}\int_{y_t}^{-y_c} y\,\delta a=0$

Now $E\neq 0,\quad R\neq 0$

So
$$\int_{y_t}^{-y_c} y\,\delta a=A\,\bar{y}=0,$$

i.e. the first moment of area about the neutral axis is zero

Therefore $\bar{y}=0$ because $A\neq 0$

The first moment of area of section about its centroidal axis is zero. This shows that neutral axis of the beam passes through the centroid of the section. In other words, the neutral layer along the length of the beam passes through the centroids of all the sections along the length of the beam.

Example 8·3-1. A brass strip 80 mm wide and 30 mm thick is bent into an arc of radius 60 m. What is the maximum stress developed in the strip if

$$E_{brass}=1\times10^5 \text{ MN/m}^2.$$

Solution. Say the maximum stress developed$=f$

Strip is of rectangular section, neutral axis will pass through the centre of the thickness,

So, $y=\pm\frac{t}{2}$ where t=thickness

$=\pm15$ mm (since it is a rectangular section)

Radius of the arc, $R=60$ m
$=60000$ mm

E for brass $=1\times10^5 \text{MN/m}^2$
$=1\times10^5 \text{ N/mm}^2$

Now
$$\frac{f}{y}=\frac{E}{R}$$

$$f=\frac{Ey}{R}=\pm\frac{15\times1\times10^5}{60000}$$
$$=\pm25\cdot0 \text{ N/mm}^2.$$

Example 8·3-2. A mild steel beam of depth 200 mm is bent into an arc of a circle of radius R. What is the minimum value of R if the stress in beam is not to exceed 600 kg/cm². The beam section is symmetrical about the neutral layer. $E=2\times10^6$ kg/cm².

Solution. f, maximum stress

$=\pm600$ kg/cm²

y, distance of extreme layers from Neutral axis

$=\pm100$ mm (as the beam section is symmetrical about NA)
$=\pm10$ cm

$$\frac{f}{y}=\frac{E}{R}$$

Radius of curvature, $R=\frac{Ey}{f}=\frac{2\times10^6\times10}{600}=3\cdot3333\times10^4$ cm

$=333\cdot33$ m.

Exercise 8·3-1. A round steel bar of diameter 50 mm is bent into an arc of radius 50 m. What is the maximum stress developed in the bar.

Given, $E=2\times10^6$ kg/cm². [**Ans.** ±500 kg/cm²]

Exercise 8·3-2. To what radius an aluminium strip 100 mm wide and 20 mm thick can be bent if the maximum stress in strip is not to exceed 50 N/mm²? E for aluminium $=70\times10^3$ N/mm². [**Ans.** 14 m]

8·4. MOMENT OF RESISTANCE

Refering to Fig. 8·3 (*d*)

Force on elementary δa,

$$\delta F=-\frac{E}{R}\,y\,.\,\delta a$$

Moment of the force δF about the neutral axis,

$$\delta M = -y\,\delta F$$

$$= -\frac{E}{R}\,y\,\delta a(-y) = \frac{E}{R}\,y^2\delta a.$$

Total moment about the neutral axis

$$= \frac{E}{R}\int_{y_t}^{-y_c} y^2\delta a.$$

Moment of resistance due to internal stresses developed in the section of the beam,

$$M_r = \frac{E}{R}\int_{y_t}^{-y_c} y^2\delta a = \text{Applied Moment, } M$$

But $\quad \displaystyle\int_{y_t}^{-y_c} y^2\delta a$ = second moment of the area about the neutral axis

$$= I_{NA}$$

or $$M = \frac{E}{R} \times I_{NA}$$

Say CG is the centroid of the section and XX and YY are the horizontal and vertical axis passing through the centroid. Neutral axis of the beam passes through the centroid of the section, as shown in the Fig. 8·4.

So, $\quad I_{NA} = I_{xx}$

$$\frac{E}{R}\,.\,I_{xx} = M$$

or $$\frac{M}{I_{xx}} = \frac{E}{R}. \qquad \ldots(3)$$

Morever $\quad f = \dfrac{y}{R}\,.\,E$

or $$\frac{f}{y} = \frac{E}{R}.$$

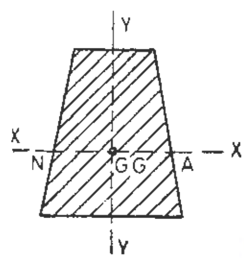


Fig. 8·4

The value of y is negative when it is taken towards the centre of curvature from the neutral axis, and y is positive when the distance of the layer is taken away from the centre of curvature from the neutral axis.

From the above equations

$$\frac{M}{I_{xx}} = \frac{E}{R} = \frac{f}{y}. \quad \text{(flexure formula)} \qquad \ldots(4)$$

Moreover we see that F_c and F_t *i.e.* total compressive force on the section on one side of neutral axis is equal to the total tensile force on the section on the other side of the neutral axis and they constitute a couple of arm h [see Fig. 8·3 (*d*)].

Moment of resistance $M_r = F_c \times h = F_t \times h =$ Applied Moment, M.

Fig. 8·3 (d) shows the stress distribution across the section. The axis of the resultant forces F_c and F_t and the length of the arm of the couple can be determined.

Equation (4) in terms of maximum tensile and maximum compressive stresses can be written as

$$f_c = -\frac{My_c}{I_{xx}} = -\frac{M}{Z_c}$$

$$f_t = \frac{My_t}{I_{xx}} = \frac{M}{Z_t}.$$

Where Z stands for the modulus of the section and is equal to the moment of inertia of the section about the neutral axis divided by the extreme value of y.

If the section of the beam is symmetrical about the neutral axis *i.e.* about its centre of gravity, and if d is the depth of the section then

$$y_c = -\frac{d}{2},\ y_t = +\frac{d}{2}$$

$$Z_c = Z_t = Z = \frac{I_{xx}}{d/2}$$

Then, the expression for the moment of resistance,

$$M = fZ.$$

8·5. MOMENT OF INERTIA OF SECTIONS

Before we proceed to determine the stresses in the beams due to bending moment, let us revise the information on the position of centroid and moment of inertia of different sections, most commonly used for beams.

(i) **Rectangular Section.** Fig. 8·5 shows a rectangular section of breadth B and depth D. CG lies at distances of

$$\left(\bar{x} = \frac{B}{2},\ \bar{y} = \frac{D}{2}\right) \text{ and}$$

$$I_{xx} = \frac{BD^3}{12},\ I_{yy} = \frac{DB^3}{12}.$$

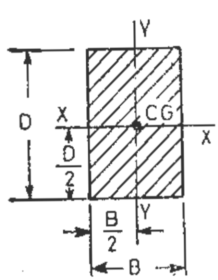


Fig. 8·5

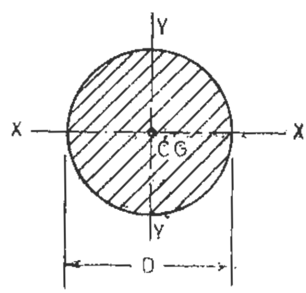


Fig. 8·6

(*ii*) **Circular Section.** Fig. 8 6 shows a circular section of diameter D, with CG at its centre

$$I_{xx}=I_{yy}=\frac{\pi D^4}{64}.$$

(*iii*) **Semi-Circular Section.** Fig. 8·7 shows a semi circular section of radius, R. CG lies along YY axis and at a distance of $\frac{4R}{3\pi}$ from the diametral axis OO as shown.

$$I_{oo}=\frac{\pi D^4}{128}$$

$$I_{xx}=I_{oo}-\frac{D^4}{18\pi}.$$

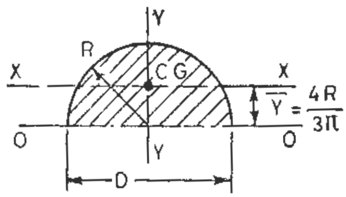


Fig. 8.7

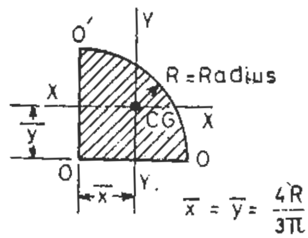


Fig. 8·8

(*iv*) **Quarter-Circular Section.** Fig. 8·8 shows a quarter-circular section of radius R with C.G. lying at

$$\bar{x}=\bar{y}=\frac{4R}{3\pi}$$

$$I_{oo}=I_{oo'}=\frac{\pi D^4}{256}$$

$$I_{xx}=I_{yy}=I_{oo}-\frac{D^4}{36\pi}.$$

(*v*) **Triangular Section.** Fig. 8·9 shows a triangular section DEF of base B and Height H. Its C.G. will pass through X-X axis at a distance of $\frac{H}{3}$ from the base. Say NF equal to H_1, is the altitude on side DE from the point F. PQ is parallel to the side DE at a distance of $\frac{H_1}{3}$ from the side. CG lies at the inter section of X-X axis and line PQ.

$$I_{xx}=\frac{BH^3}{36}$$

$$I_{EF}=I_{Base}=\frac{BH^3}{12}.$$

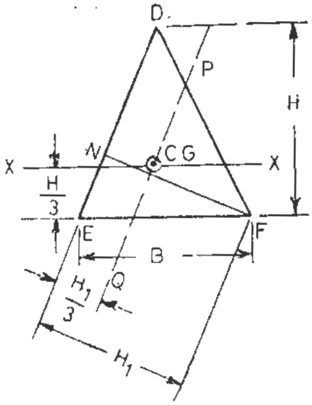


Fig. 8·9

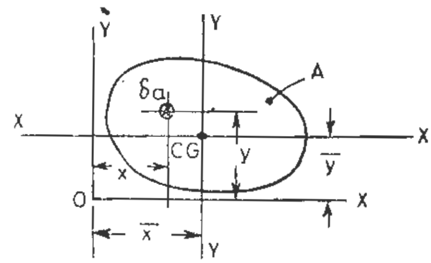


Fig. 8·10

(*vi*) **Any Section.** Consider a section of any shape as shown in Fig. 8·10.

Total area can be considered as a summation of small areas $\delta a_1, \delta a_2, \delta a_3 \ldots\ldots \delta an$.

Say the co-ordinates of a very small area δa are x and y.

Then moment of inertia, $I_{xx} = \int y^2 da$

$$I_{yy} = \int x^2 da.$$

Location of CG,
$$\bar{x} = \frac{\int x da}{A}$$

$$\bar{y} = \frac{\int y da}{A}.$$

Perpendicular Axis Theorem. Fig. 8·11 shows any area with its CG lying at *O*. *X-X* and *Y-Y* are the horizontal and vertical axes passing through the centroid *O*. If *O-O* is the polar axis of the area, then

Moment of Inertia $= I_{oo}$

$$= I_{xx} + I_{yy}$$

I_{oo} is generally called the polar moment of inertia.

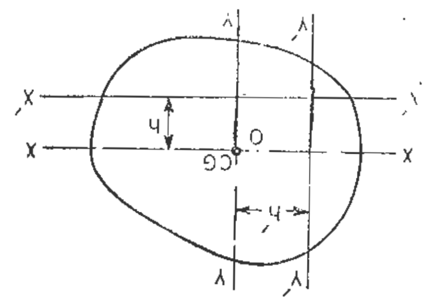


Fig. 8·11

Parallel Axis Theorem. Fig. 8·11 show a plane lamina with its C.G. at *O*. *X-X* and *Y-Y* are the horizontal and vertical axes passing through the centroid *O*. Axis *X'-X'* is parallel to *X-X* axis and is at a distance of *h*. Axis *Y'Y'* is parallel to *Y-Y* axis and is at a distance of *h'* from *YY*.

Moment of inertia, $I_{Y'Y'} = I_{YY} + Ah'^2$

Moment of inertia, $I_{X'X'} = I_{XX} + Ah^2$

where A = area of the plane lamina,

8·6. BEAMS OF RECTANGULAR SECTION

Fig. 8·12 shows a beam of rectangular section subjected to bending moment M. The breadth of the section is B and depth is D. The neutral axis passes through the centroid of the section, and the rectangular section is symmetrical about its CG.

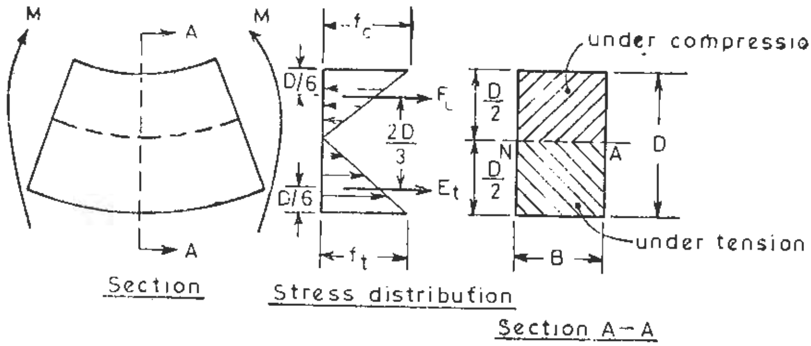


Fig. 8·12

Therefore distance of extreme layers from the neutral axis

$$=y=\pm\ \frac{D}{2}$$

The longitudinal stresses or the direct stresses developed on the extreme layers are :qual and opposite.

So $\qquad f_c=-f_t$ (as shown).

The stress distribution across the depth of the section is linear. Upper half of the ection comes under compression and the lower half comes under tension.

Compressive force on upper half,

$$F_c=\left(\frac{0+f_c}{2}\right)\frac{BD}{2}=\frac{f_c\,.\,BD}{4}$$

Tensile force on lower half,

$$F_t=\left(\frac{0+f_t}{2}\right)\frac{BD}{2}=\frac{f_t\,.\,BD}{4}.$$

The resultant force, F_c passes through an axis at a distance of $1/3\times D/2$ from the top ıyer and the resultant force F_t passes through an axis at a distance of $D/6$ from the bottom .yer.

F_t and F_c form a couple of arm $2D/3$

Moment of resistance, $M=F_t\times\dfrac{2D}{3}=F_c\times\dfrac{2D}{3}$

$$=f_t\,.\,\frac{BD}{4}\times\frac{2D}{3}=f_c\times\frac{BD}{4}\cdot\frac{2D}{3}$$

$$=f_t \cdot \frac{BD^2}{6}=f_c \ \frac{BD^2}{6}$$

$$=f_t \ . \ Z_t=f_c \ . \ Z_c$$

The section modulus Z_t= The section modulus Z_c

$$=\frac{BD^2}{6}.$$

Example 8·6-1. A timber joist of rectangular section 10 cm × 20 cm deep is simply supported over a span of 4 metres and carries a uniformly distributed load of 1 tonne/metre run. Calculate the skin stresses at the centre of the beam.

Solution.

Length of the beam, $L=4$ metre

Rate of loading, $w-1$ tonne/metre

BM at the centre of the beam,

$$M=\frac{wl^2}{8}=\frac{1\times4\times4}{8}=2 \text{ tonne-metres}$$

$$=2\times10^5 \text{ kg-cm}$$

Breadth of the section, $B=10$ cm

Depth of the section, $D=20$ cm

Section modulus, $Z=\frac{BD^2}{6}=\frac{10\times20^3}{6}=1{\cdot}333\times10^4 \text{ cm}^3$

Skin stresses, $f_c=\frac{M}{Z}=\frac{2\times10^5}{1{\cdot}333\times10^4}=15 \text{ kg/cm}^2$ (compressive)

$$f_t=\frac{M}{Z}=15 \text{ kg/cm}^2 \text{ (tensile)}.$$

Example 8·6-2. A steel beam of hollow square section with outer side 50 mm and inner side 40 mm is fixed as a cantilever with a length of 3 metres. Now much concentrated load can be applied at the free end of the cantilever, if the maximum stress is not to exceed 60 N/mm^3.

Solution. Length of the cantilever,

$$L=3 \text{ m}=3000 \text{ mm}$$

Say the load at free end,

$$=W \text{ Newtons.}$$

The maximum bending moment WL occurs at the fixed end of the contilever, and so the maximum stress in the cantilever section will be developed at the fixed end. The Fig. 8·13 shows the hollow square section and the stress distribution with extreme stresses ±60 N/mm^2.

Distance of extreme layers from Neutral axis,

$$y=\pm25 \text{ mm}$$

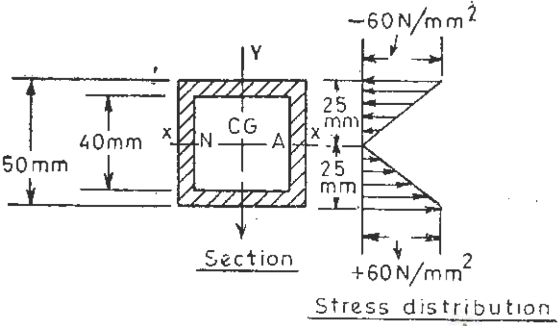


Fig. 8·13

Moment of Inertia, $I_{NA}=I_{xx}=\dfrac{50^4}{12}-\dfrac{40^4}{12}$

$$=\frac{10^4}{12}\times 369 \text{ mm}^4$$

Maximum stress, $f=60$ N/mm²

$y=25$ mm

Therefore $M=\dfrac{f}{y}\times I_{xx}=\dfrac{60}{25}\times\dfrac{10^4\times 369}{12}$

$$=73{\cdot}8\times 10^4 \text{ Nmm}$$

$$=WL=3000\ W \text{ Nmm}$$

Concentrated load at the free end,

$$W=\frac{73{\cdot}1\times 10^4}{3000}=246 \text{ N.}$$

Exercise 8·6-1. A timber joist of square section 200 mm × 200 mm is fixed as a cantilever with a length of 3 metres. What uniformly distributed load can be applied throughout the length of the beam if the maximum stress is not to exceed 5 N/mm² ? [**Ans.** 1·48 kN/m]

Exercise 8·6-2. A steel beam of hollow rectangular section with outer sides 50 mm wide, 100 mm deep and inner sides 30 mm wide and 80 mm deep is simply supported over a span of 6 metres. How much central load the beam can carry if the maximum stress is not to exceed 75 MN/m². [**Ans.** 2·88 kN]

8·7. CIRCULAR SECTION

Fig. 8·14 shows a circular section of the beam subjected to bending moment M. The diameter of the circular section is D and its CG lies at the centre of the circle. Under the action of the bending moment shown, upper half of the section comes under tension and lower half comes under compression.

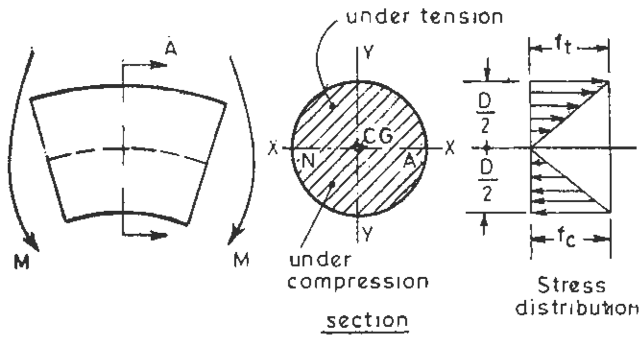


Fig. 8·14

Moment of inertia,

$$I_{xx} \text{ or } I_{NA} = \frac{\pi D^4}{64}$$

Distance of extreme layers from neutral layer

$$= \pm \frac{D}{2}$$

Stresses developed in extreme layers,

$$f_t \text{ or } f_c = \pm \frac{M}{I_{xx}} \cdot \frac{D}{2}$$

$$= \pm \frac{M}{\pi D^4} \times 64 \times \frac{D}{2} = \pm \frac{32\,M}{\pi D^3} = \frac{M}{Z}$$

Section Modulus, Z_c=Section modulus, $Z_t = Z = \frac{\pi D^3}{32}$.

Example 8·7-1. A cast iron water pipe 50 cm bore and 2 cm thick is supported over a span of 10 metres. Find the maximum stress in the metal when the pipe is running full.

Density of cast iron $= 7300$ kg/m³

Density of water $= 1000$ kg/m³.

Solution. The Fig. 8·15 shows a section of cast iron pipe with a bore of 50 cm and outside dia. 54 cm.

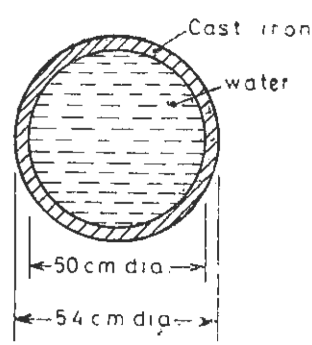


Fig. 8·15

Area of cross section of pipe

$$= \frac{\pi}{4}(54^2 - 50^2) = 326·72 \text{ cm}^2$$

Area of cross section of pipe carrying water

$$= \frac{\pi}{4} \times 50^2 = 1963·5 \text{ cm}^2$$

Density of cast iron $=0{\cdot}0073\ kg/cm^3$

Density of water $=0{\cdot}001\ kg/cm^3$

Weight of C.I. pipe per metre length

$$=326{\cdot}72\times{}'00\times0{\cdot}0073=238{\cdot}50\ kg$$

Weight of water carried per metre length

$$=1963{\cdot}5\times100\times0\ 001=196{\cdot}35\ kg$$

Total weight of pipe per metre length

$$=238{\cdot}50+196{\cdot}35=434{\cdot}85\ kg$$

Span length, $l=10\ m$

Load on the beam per unit length,

$$w=434{\cdot}85\ kg/m$$

Maximum bending moment occurs at the centre of the beam,

$$M_{max}=\frac{wl^2}{8}=\frac{434{\cdot}85\times10\times10}{8}=5435{\cdot}62\ kg\text{-}m$$

$$=543562\ kg\text{-}cm$$

I_{xx}, moment of inertia of CI section,

$$=\frac{\pi}{64}(D^4-d^4)$$

$$=\frac{\pi}{64}(54^4-50^4)=110596{\cdot}9\ cm^4$$

Distance of extreme layers from the neutral axis,

$$y_c \text{ or } y_t=\pm27\ cm$$

Stress developed in metal,

$$f_t=\frac{M_{max}}{I_{xx}}\cdot y_t$$

$$=\frac{543562\times27}{110596{\cdot}9}=132{\cdot}7\ kg/cm^2$$

$$=-f_c.$$

Exercise 8·7·1. A steel tube 8 mm bore and 1 mm wall thickness is fully charged with mercury and forms the part of an apparatus of a laboratory. The tube is 600 mm long and is supported over a span of 500 mm. What is the maximum stress in the tube due to bending. Given

Density of steel $=0{\cdot}0078\ kg/cm^3$

Density of mercury $=0{\cdot}0136\ kg/cm^3$ [**Ans.** $446{\cdot}91\ kg/cm^2$]

8·8. I-SECTION

Fig. 8·16 shows a symmetrical I section most commonly used as a structural member. There are two flanges on the top and bottom of the dimensions $B \times t$ and one web in the centre of dimensions $b \times d$. Since the section is symmetrical, its CG is located at the CG of the web as shown.

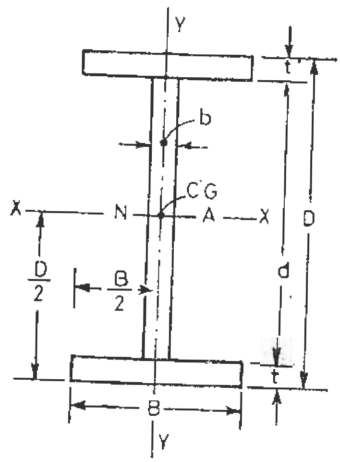


Fig. 8·16

Moment of Inertia,

$$I_{xx} = \frac{BD^3}{12} - \frac{(B-b)\, d^3}{12}$$

Distance of the extreme layers from the neutral axis

$$= \pm \frac{D}{2}$$

Section modulus, $\quad Z_t = Z_c = Z$

$$= \frac{I_{xx}}{D/2} = \frac{BD^3 - (B-b)\, d^3}{6D}$$

In this case most of the bending moment is resisted by the flanges.

Example 8·8·1. A beam of I section 30 cm × 12 cm, has flanges 2 cm thick and web 1 cm thick. Compare its flexural strength with that of a beam of rectangular section of the same weight, the depth being twice the width. What will be the maximum stress developed in *I* section for a bending moment 30 kNm.

Solution. Fig. 8·17 shows I section of given dimensions.

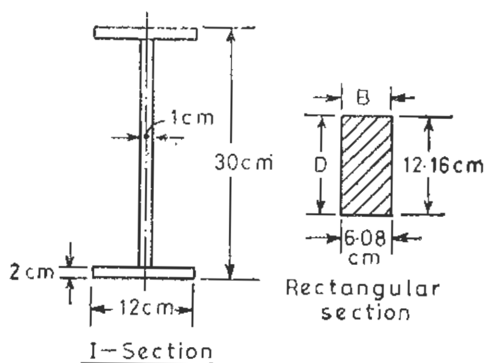


Fig. 8·17

Area of cross section

$= 12 \times 2 + 12 \times 2 + (30-4) \times 1$

$= 74 \text{ cm}^2.$

Rectangular beam is of the same weight and same material.

So Area of cross section of rectangular section

$= 74 \text{ cm}^2 = B \times D$

or $\quad = B \times 2B = 2B^2$

or $\quad B = \sqrt{\frac{74}{2}} = 6{\cdot}08 \text{ cm}$

$D = 6{\cdot}08 \times 2 = 12{\cdot}16$ cm

I-Section. $\quad B = 12$ cm

$b = 1$ cm

$D = 30$ cm

$d = 30 - 4 = 26$ cm

Section modulus, $Z_I = \frac{12\times 30^3-(12-1)26^3}{6\times 30} = \frac{324000-193336}{180}$

$= 725{\cdot}9 \text{ cm}^3 = 725{\cdot}9\times 10^3 \text{ mm}^3$

Rectangular section, $Z_R = \frac{BD^2}{6}$ where $B=6{\cdot}08$, $D=12{\cdot}16$

$= \frac{6{\cdot}08\times 12{\cdot}16^2}{6} = 149{\cdot}84 \text{ cm}^3$

The flexural strength of a beam is directly proportional to its section modulus

$$\frac{Z_I}{Z_R} = \frac{725{\cdot}9}{149{\cdot}84} = 4{\cdot}84$$

Bending moment on I section beam,

$M = 30$ kNm

$= 30\times 10^6$ Nmm

$= f_{max} Z = f_{max}\times 725{\cdot}9\times 10^3.$

Maximum stress developed

$$= \frac{30\times 10^6}{725{\cdot}9\times 10^3} = 41{\cdot}32 \text{ N/mm}^2$$

Exercise 8·8-1. A beam is of I section 20×15 cm with thickness of the flanges 2·5 cm and thickness of the web 1·5 cm. Compare its flexural strength with that of a beam of circular section of the same weight and same material. What will be the maximum stress developed in I section for a bending moment of 2·5 tonne-metres. [**Ans.** 7·14, 403·024 kg/cm²]

8·9. T-SECTION

Fig. 8·18 shows a T section of breadth B and depth D. The thickness of the flange is t_1 and thickness of web is t_2. The section is symmetrical about $Y-Y$ axis as shown but unsymmetrical about $X-X$ axis passing through the centroid of the section. To determine distance of the CG from the lower edge.

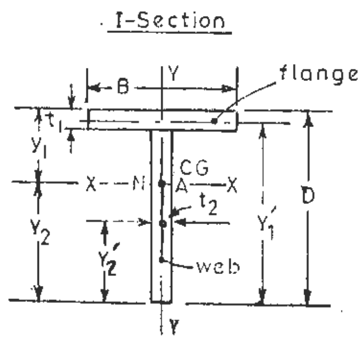


Fig. 8·18

Let us take, area of flange, $a_1 = Bt_1$

y_1', distance of C.G. of the flange from lower edge

$$= \left(D - \frac{t_1}{2}\right)$$

Area of web, $a_2 = t_1(D-t_1)$

y_2', distance of C.G. of the web from lower edge

$$= \left(\frac{D-t_1}{2}\right)$$

Distance of the C.G. of the T-section from lower edge,

$$y_2 = \frac{a_1 y_1' + a_2 y_2'}{a_1 + a_2}$$

Then $y_1 = D - y_2$

Neutral axis passes through the centroid of the section.

So moment of inertia,

$$I_{xx} = \frac{Bt_1^3}{12} + Bt_1\left(y_1 - \frac{t_1}{2}\right)^2 + \frac{t_2(D-t_1)^3}{12} + t_2(D-t_1)(y_2 - y_2')^2$$

(Using parallel axis theorem)

Section modulus, $Z_1 = \frac{I_{xx}}{y_1}$, $Z_2 = \frac{I_{xx}}{y_2}$

Example 8·9-1. Find the position of the C.G. and calculate I_{xx} for 15 cm × 10 cm × 1·5 cm. *T* section shown in Fig. 8·19. A cantilever of length 3 metres and of the section shown with flange at the top carries a load *W* at its free end. What can be the maximum value of *W* so that the stress in the section must not exceed 50 N/mm².

Solution. Area of flange,

$a_1 = 10 \times 1·5 = 15$ cm²

y_1' from bottom edge

$= 15 - 0·75 = 14·25$ cm

Area of web,

$a_2 = 1·5 \times (15 - 1·5)$ Bottom edge

$= 20·25$ cm²

y_2', from bottom edge

$= \frac{15 - 1·5}{2} = 6·75$ cm.

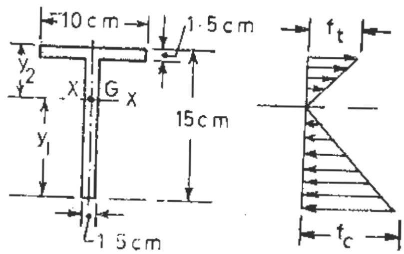


Fig. 8·19

y_1, distance of C.G. of the section from bottom edge

$$= \frac{15 \times 14·25 + 20·25 \times 6·75}{15 + 20·25} = 9·94 \text{ cm}$$

y_2, distance of C.G. of the section from top edge

$= 15 - 9·94 = 5·06$ cm

Moment of Inertia, $$I_{xx} = \frac{10 \times 1·5^3}{12} + 10 \times 1·5\left(5·06 - \frac{1·5}{2}\right)^2 + \frac{1·5 \times 13·5^3}{12} + 1·5 \times 13·5\,(9·94 - 6·75)^2$$

$= 2·8125 + 278·6415 + 307·5469 + 206·0660$

$= 795·07$ cm⁴

Section modulus, $Z_1 = \frac{I_{xx}}{y_1} = \frac{795·07}{9·94} = 79·987 \text{ cm}^3 = 79·987 \times 10^3 \text{ mm}^3$

$Z_2 = \frac{I_{xx}}{y_2} = \frac{795·07}{5·06} = 157·13 \text{ cm}^3$

The stress due to bending in a layer is proportional to its distance from the neutral layer. Therefore maximum stress in the section will be developed at the lower edge. The cantilever is fixed such that flange is at the top. Cantilever carries a load W at free end. So the flange will be in tension and the lower portion of the web *i.e.* below the neutral axis will be in compression. So f_c should not exceed 50 N/mm².

Max. B.M. on cantilever $=WL$

(at the fixed end) $=3000\ W$ Nmm if W is in Newtons

$$=f_c.Z_1$$

$$3000\ W=50\times 79{\cdot}987\times 10^3$$

$$W=\frac{50\times 79{\cdot}987}{3}=1333 \text{ N}$$

$$=1{\cdot}333 \text{ kN}.$$

Exercise 8·9-1. A beam of T section, 4 m long carries a uniformly distributed load w per metre run throughout its length. The beam is simply supported at its ends. The T section is 20×10×1·2 cm. What is the maximum value of w so that stress in the section does not exceed 600 kg/cm². [**Ans.** 340·8 kg/metre run]

8·10. L-SECTION

Fig. 8·20 shows on unequal L-section, of breadth B and depth D and thickness t.

a_1, area of leg $L_1=B.t$

Distance of C.G. of area a_1, from edge PO

$$=\frac{B}{2}$$

Area of leg L_2, $a_2=(D-t)\ t$

Distance of C.G. of area a_2 from edge PO

$$=\frac{t}{2}$$

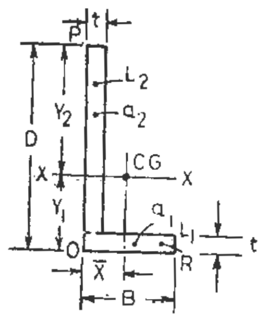


Fig. 8·20

Distance of C.G. of L section from PO,

$$\bar{x}=\frac{a_1\left(\frac{B}{2}\right)+a_2\left(\frac{t}{2}\right)}{a_1+a_2}$$

Distance of C.G. of area a_1 from edge $OR=\frac{t}{2}$

Distance of C.G. of area a_2 from edge $OR=\left(\frac{D-t}{2}+t\right)$

$$=\left(\frac{D+t}{2}\right)$$

Distance of C.G. of L section from edge OR,

$$y_1=\frac{a_1\frac{t}{2}+a_2\left(\frac{D+t}{2}\right)}{a_1+a_2}$$

$$y_2=D-y_1$$

Moment of Inertia, $$I_{xx}=\frac{t}{12}(D-t)^3+t(D-t)\left(y_2-\frac{D-t}{2}\right)^2+\frac{Bt^3}{12}+Bt\left(y_1-\frac{t}{2}\right)^2$$

Section modulus, $$Z_1=\frac{I_{xx}}{y_1}$$

$$Z_2=\frac{I_{xx}}{y_2}$$

Example 8·10-1. Find the position of C.G. and calculate moment of inertia I_{xx} of an unequal angle section 10 cm × 8 cm × 1 cm. A beam of this angle section is used as a cantilever of length 3 metre subjected to a turning moment M at its free end. What is the maximum value of M if the stress in the section is not to exceed 70 MN/m².

Solution. L section is also called an angle section.

area, $a_1=8$ cm²

area, $a_2=(10-1)\times 1=9$ cm²

$$\bar{x}=\frac{8\times 4+9\times 0{\cdot}5}{8+9}=\frac{36{\cdot}5}{17}$$

$$=2{\cdot}147 \text{ cm}$$

$$y_1=\frac{8\times 0{\cdot}5+9(1+4{\cdot}5)}{8+9}$$

$$=3{\cdot}147 \text{ cm}$$

$$y_2=10-3{\cdot}147$$

$$=6{\cdot}853 \text{ cm}$$

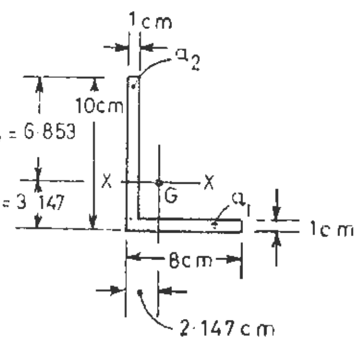


Fig. 8·21

Moment of inertia, $$I_{xx}=\frac{8\times 1^3}{12}+8(3{\cdot}147-0{\cdot}5)^2+\frac{1\times 9^3}{12}+9(6{\cdot}853-4{\cdot}5)^2$$

$$=0{\cdot}667+56{\cdot}053+60{\cdot}750+49{\cdot}829$$

$$=167{\cdot}299 \text{ cm}^4$$

$$=167{\cdot}299\times 10^4 \text{ mm}^4$$

Admissible stress, $$f=70 \text{ MN/m}^2$$

$$=70 \text{ N/mm}^2$$

The stress in a layer due to bending is directly proportional to its distance from the neutral layer, therefore in this case maximum stress will occur at the top edge because $y_2>y_1$.

Maximum admissible turning moment, M

$$=f\times\frac{I_{xx}}{y_2}$$

$$=\frac{70\times 167{\cdot}299\times 10^4}{6{\cdot}853}$$

$$=1708{\cdot}88\times 10^4 \text{ Nmm}$$

$$=17{\cdot}08 \text{ kNm.}$$

Exercise 8·10-1. Determine the position of the centroid of an unequal angle 120 mm×60 mm×8 mm. A beam of this section is simply supported over a span of 8 m. A load 1 kN acts at a distance of 2 metre from one end of the beam. What is the maximum stress developed in the angle section.

[**Ans.** $\bar{x}=13{\cdot}07$ mm, $\bar{y}=43{\cdot}07$ mm, $I_{xx}=206{\cdot}44\times 10^4$ mm^4
$f_{max}=55{\cdot}898$ N/mm^2]

Note that in this case, maximum bending moment occurs under the load.

8·11 CHANNEL SECTION

Fig. 8·22 shows a channel section of depth D, breath B and thickness t. CG lies at a distance of $\bar{x}$ from edge PR and at distance of $D/2$ from the edge RN or the edge PM.

$$\bar{x}=\frac{B\,.\,t\,.\,\frac{B}{2}+B\,.\,t\,.\,\frac{B}{2}+(D-2t)\,t\,.\,\frac{t}{2}}{B\,t+B\,t+(D-2t)\,t}$$

$$=\frac{t\,B^2+\frac{t^2}{2}\,(D-2t)}{2\,Bt+t\,(D-2t)}$$

$$=\frac{B^2+\frac{t}{2}\,(D-2t)}{2B+(D-2t)}$$

$$y_1=y_2=\frac{D}{2}$$

Moment of Inertia,

$$I_{xx}=\frac{BD^3}{12}-\frac{(B-t)(D-2t)^3}{12}$$

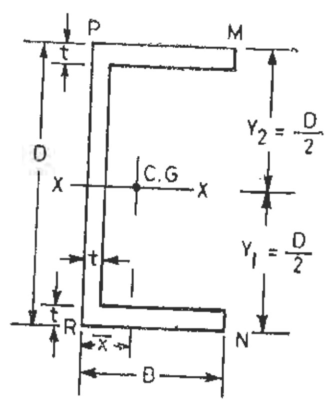


Fig. 8·22

Section modulus,

$$Z_1=Z_2=\frac{I_{xx}}{D/2}=\frac{BD^3-(B-t)(D-2t)^3}{6D}.$$

Example 8·11-1. The thickness of flange and web of a channel section are 10 mm and 8 mm respectively, while its breadth and depth are 50mm and 100 mm. Find the position of the CG of the section and its I_{xx}. If a beam of this channel section is used, what maximum bending moment can be applied if the stress is not to exceed 0·5 tonne/cm^2.

Solution. Considering the channels and web separately as shown in the Fig. 8·23.

$$\bar{x}=\frac{50\times10\times25+50\times10\times25+80\times8\times4}{50\times10+50\times10+80\times8}$$

$$=\frac{25000+2560}{1640}=16{\cdot}80 \text{ mm}$$

Moment of Inertia,

$$I_{xx}=\frac{50\times100^3}{12}-\frac{42\times80^3}{12}$$

$$=237{\cdot}467\times10^4 \text{ mm}^4$$

$$=237{\cdot}467 \text{ cm}^4$$

Allowable stress due to bending

$$=0{\cdot}5 \text{ tonne/cm}^2$$

Maximum allowable B.M.

$$M_{max}=0{\cdot}5\times\frac{I_{xx}}{D/2}=\frac{0{\cdot}5\times237{\cdot}467}{5{\cdot}0}$$

$$\left[\text{where} \quad \frac{D}{2}=5 \text{ cm}\right]$$

$$=23{\cdot}7467 \text{ tonne-cm}=0{\cdot}237467 \text{ T-m.}$$

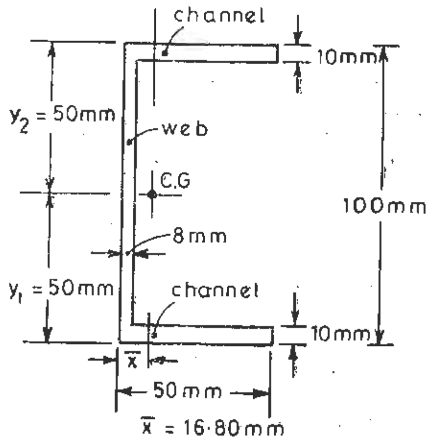


Fig. 8·23

Exercise 8·11-1. A 20 cm×8 cm channel with thickness of web 10 mm and thickness of flanges 12 mm is used as a beam with the 20 cm base vertical. At a certain cross section it has to resist a bending moment of 14 kNm. Calculate the maximum intensity of stress due to bending at that section. [**Ans.** 65 N/mm²]

8·12. UNEQUAL I SECTION

Fig. 8·24 shows an unequal I section. The top flange $B_1\times t_1$, bottom flange $B_2\times t_2$ and web $t_3\times d$ are symmetrical about the Y-Y axis. So C.G. of the section lies along Y-Y axis. The overall depth of the section is D.

To determine position of C.G. along the Y-Y axis, take area of top flange,

$$a_1=B_1\,t_1$$

C.G. of a_1 from bottom edge PQ

$$=D-\frac{t_1}{2}$$

Area of bottom flange, $a_2=B_2\,t_2$

C.G. of a_2 from edge $PQ=\frac{t_2}{2}$

Area of the web, $\quad a_3=t_3\,.\,d$

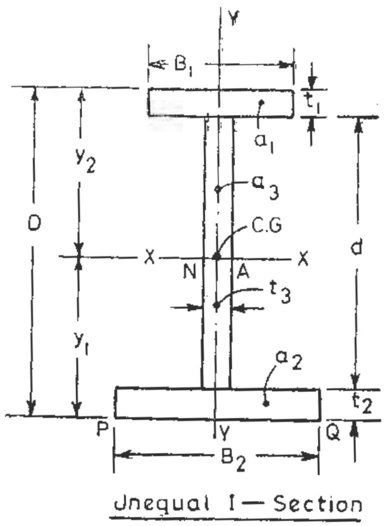


Fig. 8·24

C.G. of a_3 from edge $PQ = t_2 + \frac{d}{2}$

then
$$y_1 = \frac{B_1\, t_1 \left(D - \frac{t_1}{2}\right) + B_2\, t_2 \,.\, \frac{t_2}{2} + t_3 \,.\, d \left(t_2 + \frac{d}{2}\right)}{B_1\, t_1 + B_2\, l_2 + t_3\, d}$$

$$y_2 = D - y_1$$

Moment of Inertia,
$$I_{xx} = \frac{B_1\, t_1^3}{12} + B_1\, t_1 \left(y_2 - \frac{t_1}{2}\right)^2$$
$$+ \frac{t_3\, d^3}{12} + t_3\, d \left(t_2 + \frac{d}{2} - y_1\right)^2$$
$$+ \frac{B_2\, t_2^3}{12} + B_2\, t_2 \left(y_1 - \frac{t_2}{2}\right)^2$$

Section modulus, $Z_1 = \frac{I_{xx}}{y_1}$, and $Z_2 = \frac{I_{xx}}{y_2}$.

Example 8·12-1. A CI beam of I section with top flange 15 cm×1 cm, bottom flange 20 cm×2 cm and web 27 cm×1 cm is supported over a span of 6 metres. If the permissible stresses are 1 tonne/cm² in compression and 0·25 tonne/cm² in tension, what uniformly distributed load can be safely applied on the beam.

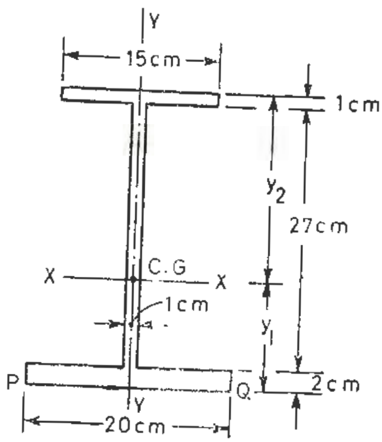


Fig. 8·25

Solution.

Top flange area, $a_1=15$ cm²

Bottom flange area, $a_2=40$ cm²

Web area, $a_3=27$ cm²

Section is symmetrical about Y-Y axis, so C.G. will lie along this axis.

The distance of C.G. of I section from bottom edge PQ,

$$y_1=\frac{15\times(27+2+0{\cdot}5)+40\,(1)+27\,(2+13{\cdot}5)}{15+40+27}$$

$$=\frac{442{\cdot}5+40+418{\cdot}5}{82}=\frac{901}{82}=10{\cdot}988 \text{ cm}$$

$$y_2=30-10{\cdot}988=19{\cdot}012 \text{ cm}.$$

Moment of Inertia,
$$I_{xx}=\frac{15\times1^3}{12}+15\,(19{\cdot}012-0{\cdot}5)^2$$
$$+\frac{1\times27^3}{12}+27\,(2+13{\cdot}5-10{\cdot}988)^2$$
$$+\frac{20\times2^3}{12}+40\,(10{\cdot}988-1)^2$$
$$=1{\cdot}250+5140{\cdot}412+1640{\cdot}25+549{\cdot}670+13{\cdot}333+3990{\cdot}406$$
$$=11335{\cdot}32 \text{ cm}^4$$

Section modulus,
$$Z_1=\frac{I_{xx}}{y_1}=\frac{11335{\cdot}32}{10{\cdot}988}=1031{\cdot}61 \text{ cm}^3$$
$$Z=\frac{I_{xx}}{y_2}=\frac{11335{\cdot}32}{19{\cdot}012}=596{\cdot}22 \text{ cm}^3.$$

Now the stress due to bending in any layer is proportional to its distance from the neutral axis. As the allowable stress in tension is much less than the allowable stress in compression and $y_1 < y_2$, the bottom flange should come under tension.

Taking $f_t=0{\cdot}25$ tonne-cm²

Bending moment $=f_t\,.\,Z_1$

$=0{\cdot}25\times1031{\cdot}61=257{\cdot}90$ tonne/cm

$=2{\cdot}579$ tonne-metres

Taking $f_c = 1$ tonne-cm³

Bending moment $= f_c \times Z_2$

$= 1 \times 596{\cdot}22 = 596{\cdot}22$ tonne-cm

$= 5{\cdot}9622$ tonne-metres

Therefore allowable moment,

$$= 2{\cdot}579 \text{ tonne-metre} = \frac{wl^2}{8}$$

where w = rate of loading

l = length of beam, 6 m

$$w \times \frac{36}{8} = 2{\cdot}579$$

Rate of loading, $w = \dfrac{2{\cdot}579}{4{\cdot}5} = 0{\cdot}5731$ tonne/metre

$= 573{\cdot}1$ kg/metre run.

Exercise 8·12-1. The cross section of a cast iron beam is an I section with **top flange** 15 cm × 5 cm, web 22 cm × 4 cm and bottom flange 25 cm × 8 cm. The loading **being in** the plane of the web. The upper portion of the section is in compression. If the allowable maximum stresses are 60 N/mm² in tension and 150 N/mm² in compression, find the moment of resistance ot the section. [**Ans.** $I_{xx} = 52559$ cm⁴, 233·25 kNm]

8·13. MODULUS OF RUPTURE

Flexure formula $M/I_{xx} = f/y = E/R$ is derived on the assumption that stress developed in beam due to bending does not exceed the proportional limit stress. It is also proved that stress strain in any layer is proportional to its distance from the neutral layer. But if the stress developed in any layer exceeds the proportional limit stress as obtained from the stress strain diagram of the material Fig. 8·26 (*a*), though the strain in any layer remains proportional to its distance from the neutral layer but the stress in the layer does not remain

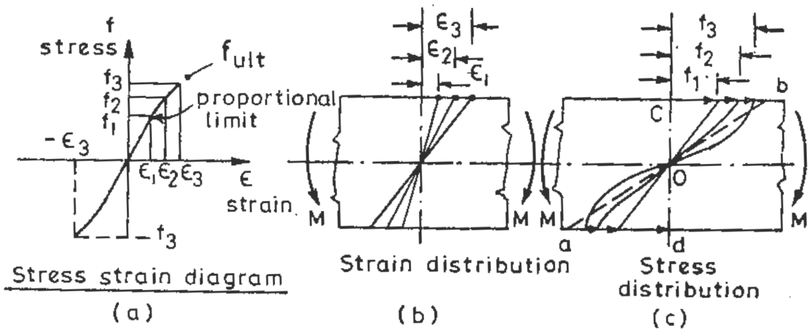


Fig. 8·26

proportional to its distance from the neutral layer. Consider a beam of rectangular section subjected to a bending moment M. The stress-strain diagram is the same in tension and in compression as shown. From the stress-strain diagram, ϵ_1 is the strain at the proportional limit upto which stress is proportional to strain, we get linear stress distribution diagram as shown in diagram (*c*). At the strain ϵ_2 in the extreme layer, stress is f_2, stress distribution diagram is linear upto certain distance from neutral layer but beyond which the diagram is non linear and stress is f_2. Similarly f_3 is the stress in the extreme fibres corresponding to strain ϵ_3 as shown. If the bending moment applied is such that stress in the extreme layer reaches the ultimate stress, the beam is supposed to have failed.

If M_{ult}=ultimate bending moment determined experimentally.

Modulus of rupture=ultimate stress in extreme fibres calculated on the basis of flexural formula.

This modulus of rupture is higher than the true stress.

Modulus of rupture $=\dfrac{6\,M_{ult}}{bd^2}$

where b=breadth of rectangular section

d=depth of rectangular section.

The theoretical value of modulus of rupture is given by cb while the true ultimate stress is f_3 as shown in Fig. 8·26 (c).

Example 8·13-1. 15 cm × 15 cm pine beam was supported at the ends on a 4·5 m span and loaded at the third points. The beam failed when a 0·8 tonne load was placed at 1·5 m from each end. Find the modulus of rupture.

Solution. The beam is of square section 15 cm side

Section modulus, $Z=\dfrac{a^3}{6}=\dfrac{15^3}{6}$ cm³.

The beam is of 4·5 m length and carries loads of 0·8 tonne each at a distance of 1·5 m from each end.

Reactions $R_A=R_B=0·8$ tonne

(because of symmetry)

B.M. at B and $C=0·8\times1·5$

$=1·2$ tonne-metres

The B.M. diagram is as shown in the Fig. 8·27.

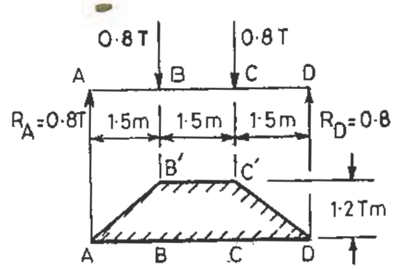


Fig. 8·27

Mult. $=1·2$ tonne-metre

$=f_m\times$section modulus

$$1·2\times1000\times100 \text{ kg—cm}=f_m\times\frac{15^3}{6}$$

modulus of rupture, $f_m=\dfrac{1·2\times10^5\times6}{15^3}=213·3$ kg/cm².

Exercise 8 13-1. A 10 cm × 15 cm wooden beam 3 m long was tested to failure by applying a concentrated load at the middle of the span. Find the modulus of rupture if the maximum load was 1520 kg. [**Ans.** 304 kg/cm²]

8·14. BUILT UP SECTIONS

A number of compound sections can be built up using standard rolled sections such I, T, (channel) and Angle (L) sections and flat plates making beams and columns of required strength and stiffness. In the previous articles we have seen different sections with sharp edges and corners. To reduce the effect of stress concentration at the corners and to eliminate sharp outer edges, the steel sections of different shapes are rolled out having fillet radius at the corners and rounded edges. The dimensions of standard rolled steel sections are given in Indian standards. The properties of standard sections such as I_{xx}, I_{yy} and distance of centroid from edges are also provided in these tables. A few examples are given below.

An I section ISLB 150, a channel section ISLC 75, an equal angle section ISA 6060 and an unequal section ISA 5030 are shown in the Fig 8·28.

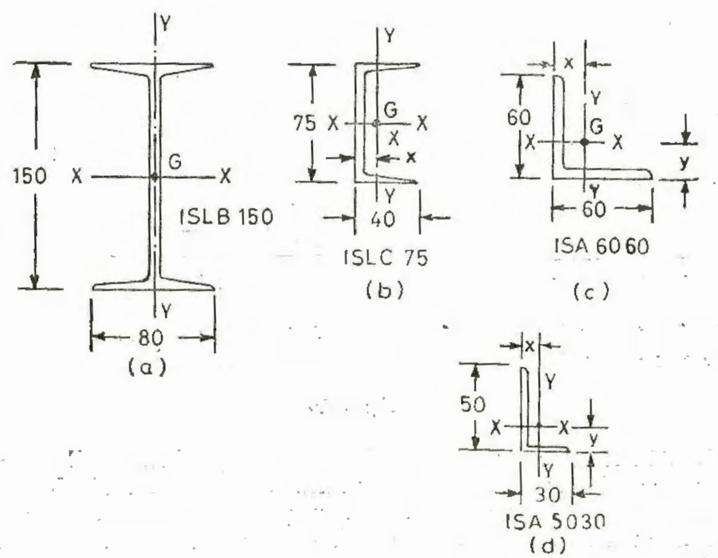


Fig. 8·28

I section ISLB 150 has weight 14·2 kg/m, Area 18·08 cm², Depth 150 mm, width 80 mm, Flange thickness 6·8 mm, web thickness 4·8 mm, moment of inertia $I_{xx}=688\cdot2$ cm^4 and $I_{yy}=55\cdot2$ cm^4. Channel section ISLC 75 has weight 5·7 kg/metre. Area=7·26 cm², Depth 75 mm, flange width 40 mm, flange thickness 6·0 mm, web thickness 3·7 mm, $I_{xx}=66\cdot1$ cm^4, $I_{yy}=11\cdot5$ cm^4, distance of C.G. of the section from outer edge of web, $x=1.35$ cm. ISA 6060 has weight 4·5 kg/m, Area=5·75 cm², $I_{xx}=I_{yy}=19\cdot2$ cm^4, $x=y=1\cdot65$ cm for a thickness of 5 mm. The thickness of the section can also be 6, 8 and 10 mm and correspondingly the properties change ISA 5030 has size 50×30 mm, weight 1·8 kg/m, Area 2·34 cm², $I_{xx}=5\cdot9$ cm^4, $I_{yy}=1\cdot6$ cm^4, $x=0\cdot65$ and $y=1\cdot63$ for thickness of 3 mm. For other thicknesses of 4, 5 and 6 mm, the properties of the section change correspondingly.

Combining the standard sections with plates built up sections are made. A few examples are given in the Fig. 8·29.

The built up sections are made with the help of riveting or welding of plates with the standard sections. The CG of a built up section is found out and then moments of inertia I_{xx} and I_{yy} are determined taking the help of parallel axis theorem.

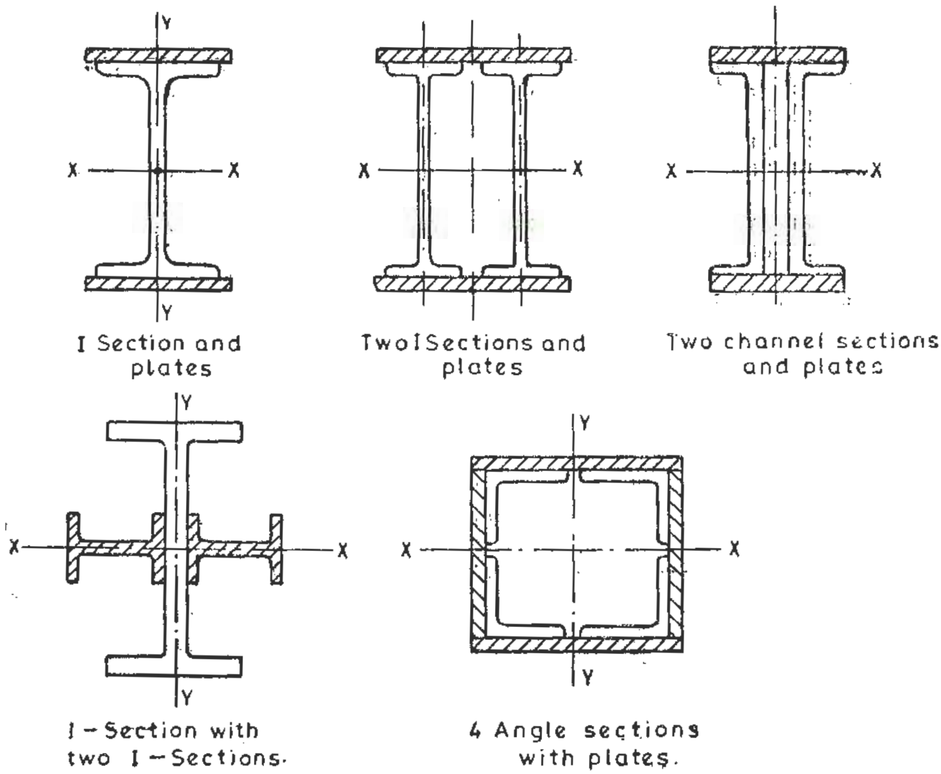


Fig. 8·29

Example 8·14-1. A compound section is built up of two rolled steel beams ISJB 150 standard section placed side by side with two plates 10 mm thick and 150 mm wide each riveted to top and bottom flanges. The rolled sections are placed symmetrically about the centre of the plates. Calculate the I_{xx} and I_{yy} for the built up section. For each ISJB 150.

A', Area$=9{\cdot}01$ cm^2 Depth$=150$ mm, Flange width$=50$ mm, $I_{xx}'=322{\cdot}1$ cm^4 and $I_{yy}'=9{\cdot}2$ cm^4.

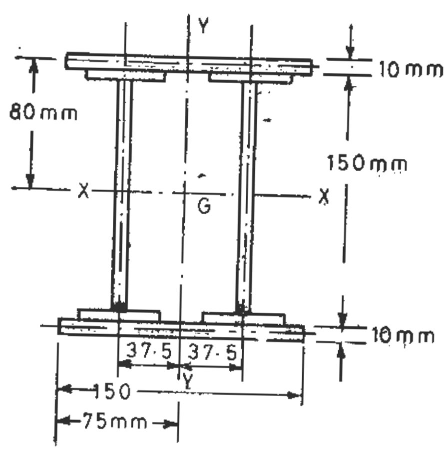


Fig. 8·30

Fig. 8·30 shows the built up section having 2 I-sections and two plates on top and bottom. Since the I-section or beam section are placed symmetrically about the centre of the plates 150 mm×10 mm, the CG of the built up section will lie at the centre G as shown. The X-X and Y-Y axis of the built up section are shown, passing through G.

Distance of CG of I section from Y-Y axis=37·5 mm.

(Since the I sections are symmetrically placed about the centre of the plates).

Distance of CG of the plates from X-X axis=75+5=80 mm.

Using the parallel axis theorem

$$I_{xx}=2Ix'x'+2\times\frac{15\times1^3}{12}+2\times15\times1\times8^2$$
$$=2\times322{\cdot}1+2{\cdot}5+1920=2566{\cdot}7\text{ cm}^4$$
$$Iyy=2Iy'y'+2A'(3{\cdot}75)^2+2\times1\times\frac{15^2}{12}$$
$$=2\times9{\cdot}2+2\times9{\cdot}01\times3{\cdot}75^2+562{\cdot}5$$
$$=18{\cdot}4+253{\cdot}40+562{\cdot}5=834{\cdot}3\text{ cm}^4.$$

Note that I section is also called a beam section.

Example 8·14-2. A box section is made by joining 4 equal angle sections ISA 75 75 and two top plates 200 mm×10 mm and two side plates 180 mm×10 mm as shown in the Fig. 8·31. Determine the moment of inertia I_{xx} and I_{yy}. Properties of equal angle sections are

A', Area $=7{\cdot}27\text{ cm}^2$

$$I_{xx}'=I_{yy}'=38{\cdot}7\text{ cm}^4$$
$$x'=y'=2{\cdot}02\text{ cm}.$$

The box section made is shown in Fig. 8·31. The equal angles are placed symmetrically about the centroid G of the whole of the section.

Distance of G' of angle section from X-X axis

$$=10-1-2{\cdot}02=6{\cdot}98\text{ cm}$$

Distance of G' of angle section from YY axis

$$=10-1-2{\cdot}02=6{\cdot}98\text{ cm}$$

Distance of CG of top plates from X-X axis

$$=100-5=95\text{ mm}=9{\cdot}5\text{ cm}$$

Distance of CG of sides plates from YY axis

$$=100-5=95\text{ mm}=9{\cdot}5\text{ cm}.$$

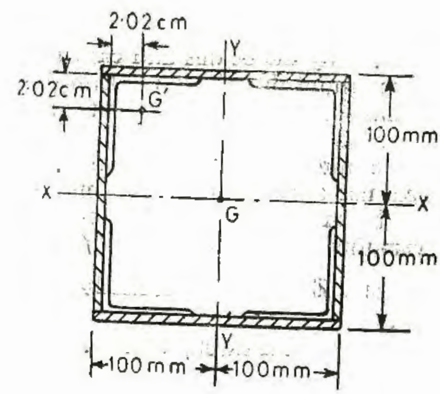


Fig. 8·31

Using parallel axis theorem

$$I_{xx}=\frac{2\times20\times1^3}{12}+2\times20\times1\times9{\cdot}5^2+\frac{2\times1\times18^3}{12}+4\times Ix'x'$$
$$+4\times A'\times9{\cdot}5^2$$

$$=3{\cdot}333+3610+972+4\times38{\cdot}7+4\times7{\cdot}27\times9{\cdot}5^2$$
$$=4585{\cdot}333+154{\cdot}8+2624{\cdot}47$$
$$=7364{\cdot}603 \text{ cm}^4$$

$$I_{yy}=\frac{2\times1\times20^3}{12}+\frac{2\times18\times1^3}{12}+2\times18\times9{\cdot}5^2+4\ I_{yy}'+4A'\times9{\cdot}5^2$$
$$=1333{\cdot}333+3+3249+4\times38{\cdot}7+4\times7{\cdot}27\times9{\cdot}5^2$$
$$=7364{\cdot}603 \text{ cm}^4.$$

We have obtained $I_{yy}=I_{xx}$ because the section is symmetrical about X-X and YY-axis and forms a square type box section.

Exercise 8·14-1. Two channel sections ISJC 100 placed back to back at a distance 30 mm are joined by two plates 120 mm$\times$15 mm at the bottom and top flanges. Determine the moment of inertia I_{xx} and I_{yy}. Properties of a channel section are, Area$=7{\cdot}41$ cm^2, Depth 100 mm, flange width$=45$ mm, $I_{x'x'}=123{\cdot}8$ cm^4, $I_{y'y'}=14{\cdot}9$ cm^4, $x'=1{\cdot}40$ (distance of C.G. from outer edge of web). Determine moment of inertia I_{xx} and I_{yy}.

[**Ans.** 1444·6 cm^4, 586·43 cm^4]

Exercise 8·14-2. A beam section ISLB 250 is riveted to the beam sections ISMB 100. The smaller sections are riveted by coinciding the XX axis of ISLB 250 to the YY axis of ISMB 100. Determine the moment of inertia I_{xx} and I_{yy} of the built up section. Properties of ISLB 250 are, web thickness$=6{\cdot}1$ mm, Area 35·53 mm^2, depth 250 mm, width 125 mm, $I_{x'x'}=3717{\cdot}8$ cm^4, $I_{y'y'}=193{\cdot}4$ cm^4. Properties of ISMB 100 are, area$=14{\cdot}60$ cm^2, depth 100 mm, width$=75$ mm, $I_{x'x'}=257{\cdot}5$ cm^4 and $I_{y'y'}=40{\cdot}8$ cm^4.

[**Ans.** 3799·4 cm^4, 1530·17 cm^4]

8·15. BEAMS OF UNIFORM STRENGTH

Generally the beams and cantilevers which are commonly employed are of uniform section throughout their length. Limiting stresses are reached only at the section subjected to the maximum bending moment. At all other sections of the beam, maximum allowable stress is not reached and the material is not put to its most economical use and as a result a considerable amount of the material is understressed and is wasted. To achieve the most economical use of the material, beams and cantilevers of uniform strength throughout their length can be suitably designed, so that the maximum stress developed anywhere along the length remains the same. Stress f at any section subjected to bending moment M is equal to $\frac{M}{Z}$, where Z is the section modulus. Therefore to achieve uniform strength it is necessary to have $\frac{M}{Z}$ constant throughout. Plate girders, carriage springs, tapered masts, electric poles are some of the examples in which concept of uniform strength has been used to some extent.

Let us consider a cantilever of length l, of rectangular section and carrying a concentrated load at the free end. M, Bending moment at any section$=Wx$ (numerically)

Z, section modulus $=\dfrac{bd^2}{6}$

or $$\frac{M}{Z}=\frac{6Wx}{bd^2}. \quad \ldots(1)$$

In this case either b is varied and d is kept constant throughout or b is kept constant and d is varied along the length.

(i) $$f=\frac{6Wx}{bd^2}=\frac{6W}{d^2}\left(\frac{x}{b}\right).$$

To achieve uniform strength, b is uniformly increased from zero at one end to maximum at the fixed end as shown in Fig. 8·32 where $b_x=\frac{6Wx}{fd^2}$ and the breadth at the fixed end, $B=\frac{6Wl}{fd^2}$.

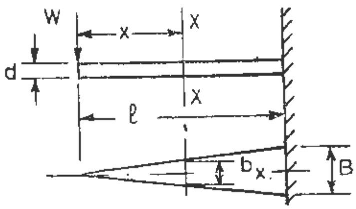


Fig. 8·32

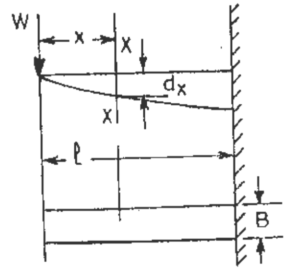


Fig. 8·33

Secondly, the width of the cantilever section is constant say B throughout and the depth varies. Then $f=\frac{6Wx}{Bd^2}$ or $d_x=\sqrt{\frac{6Wx}{Bf}}$ as shown in Fig. 8·33 and at the fixed end, depth of the section $=\sqrt{\frac{6Wl}{Bf}}$.

Let us consider that the cantilever carries a uniformly distributed load w per unit length. The bending moment at any section$=\frac{wx^2}{2}$ (numerically). Say the depth of the rectangular section is kept constant as d and breadth varies for the uniform strength f. Then

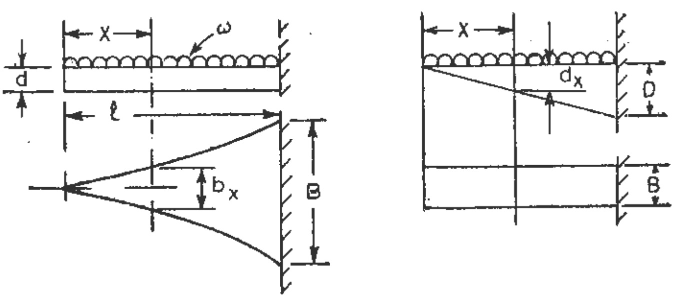


Fig. 8·34

breadth at any section $b_x=\frac{6wx^2}{fd^2}$ and breadth, B at the fixed end will be $\frac{6wl^2}{fd^2}$. Secondly, we consider breadth to be constant say B and depth variable *i.e.* $d_x^2=\frac{6wx^2}{fB}$ or depth at any section, $d_x=\sqrt{\frac{6w}{fB}}x$ and at the fixed end $D=\sqrt{\frac{6w}{fB}}$. l as shown in Fig. 8·34.

Similarly we can consider the simply supported beams carrying concentrated and uniformly distributed loads.

Fig. 8·35 shows a beam of length l, simply supported at the ends and carrying a concentrated load W at the mid span. Bending moment at any section $=\frac{Wx}{2}$. If the depth of the section is kept constant, the breadth of the beam will increase linearly upto the centre of the span and will then gradually decrease to zero at the other end. Breadth at any section $b_x=\frac{3Wx}{fd^2}$ and at the centre $B=\frac{3W}{fd^2}\times\frac{l}{2}=\frac{3Wl}{2fd^2}$. While keeping the breadth constant as B, depth at any section $d_x=\sqrt{\frac{3Wx}{fB}}$ and at the centre $D=\sqrt{\frac{3Wl}{2fB}}$. If we consider uniformly

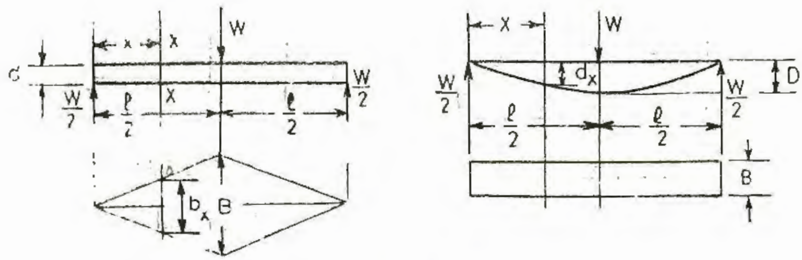


Fig. 8.35

distributed load on the beam, then bending moment at any section $=\frac{wl}{2}x-\frac{wx^2}{2}=\frac{w}{2}(lx-x^2)$.

Keeping the depth constant, breadth at any section, $b_x=\frac{3w}{fd^2}(lx-x^2)$ and at the centre $B=\frac{3wl^2}{4fd^2}$ as shown in Fig. 8·36. Again keeping the breadth constant as B, depth at any section, $d_x=\sqrt{\frac{3w(lx-x^2)}{fB}}$ and at the centre, $D=\sqrt{\frac{3w}{fB}}, \frac{l}{2}$.

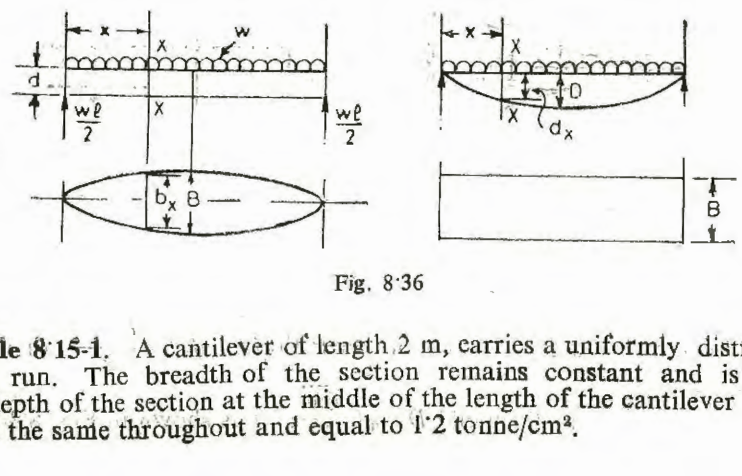


Fig. 8·36

Example 8·15-1. A cantilever of length 2 m, carries a uniformly distributed load of 2 tonnes/metre run. The breadth of the section remains constant and is equal to 10 cm. Determine the depth of the section at the middle of the length of the cantilever and at fixed end if stress remains the same throughout and equal to 1·2 tonne/cm².

Breadth, $B=10$ cm

l, length of the cantilever $=200$ cm

Uniform strength, $f=1{\cdot}2$ tonne/cm^2

Rate of loading, $w=2$ tonnes/metre run

$=0{\cdot}02$ tonne/cm

Depth at any section $d_x=\sqrt{\dfrac{6\,w}{fB}}\,.\,x$

$$=\sqrt{\frac{6\times0{\cdot}02}{1{\cdot}2\times10}}\;x=0{\cdot}1\,x$$

d at the middle of the length $=0{\cdot}1\times100=10$ cm

Depth at the fixed end $=0{\cdot}1\times200=20$ cm.

Example 8·15-2. A beam of uniform strength and varying rectangular section is simply supported over a span of 3 metres. It carries a uniformly distributed load of 10 kN per metre run. The uniform strength is 80 N/mm^2. (*a*) Determine the depth at a distance of 1 m from one end if the breadth is the same throughout and equal to 15 cm. (*b*) Find out the breadth at the centre of the span if the depth is constant throughout the length of the beam and is equal to 10 cm.

Solution. $l=3$ metres $=300$ cm

$w=10$ kN/m $=100$ N/cm

$f=80$ N/mm$^2=8000$ N/cm^2

(*a*) Breadth is constant,

$B=15$ cm

$$d_x=\sqrt{\frac{3\,w}{fB}\,(lx-x^2)}$$

$x=100$ cm

$$d_x=\sqrt{\frac{3\times100\times(300\times100-100^2)}{8000\times15}}$$

$$=\sqrt{\frac{3\times100\times10000\times2}{15\times8000}}=4{\cdot}08\text{ cm}$$

(*b*) Depth is constant, $d=10$ cm

Breadth at any section $b_x=\dfrac{3\,w}{fd^2}\;(lx-x^2)$

$x=150$ cm (*i.e.* at the centre)

$$B=\frac{3\times100}{8000\times10^2}\times(300\times150-150^2)$$

$=8{\cdot}4375$ cm

Exercise 8·15-1. A cantilever 250 cm long carries a load of 20 kN at the free end. The cantilever is of rectangular section with constant breadth $b=5$ cm but of variable depth. So as to have a cantilever of uniform strength. Determine the depth at intervals of 50 cm from the free end if the uniform strength is 100 N/mm^2.

[**Ans.** 0, 10·95, 15·49, 18·97, 21·90, 24·495 cm]

Exercise 8·15-2. A beam of span 4 metres carries a concentrated load 4 tonnes at its centre and its ends are simply supported. The beam is of rectangular section with uniform breadth 10 cm throughout. If the beam has uniform strength throughout and is equal to 1·2 tonnes/cm², determine the depth of the section at quarter spans from the ends and at the mid point.

[**Ans.** 10 cm, 14·14 cm]

8·16. BIMETALLIC STRIP

When two metal strips having different coefficients of thermal expansion are brazed together, a change in temperature will cause the assembly to bend. Fig. 8·37 shows a composite bar of rectangular strips of metal 1 and metal 2 permanently joined together. Say the coefficient of linear expansion of metal 1 is α_1 and that of metal 2 is α_2 and $\alpha_1 < \alpha_2$. Say the Young's modulus of metal 1 is E_1 and that of metal 2 is E_2. When this composite bar is heated through $T°$ it will bend because $\alpha_2 > \alpha_1$ and both the strips will deform together introducing compressive stress in metal 2 and tensile stress in metal 1, because $\alpha_1 < \alpha_2$, metal 1 will exert compressive force on metal 2 along the interface reducing its free expansion of $\alpha_2 lT$ and metal 2 will exert tensile force on metal 1 and further increasing its free expansion $\alpha_1 LT$. This we have already discussed in chapter 2.

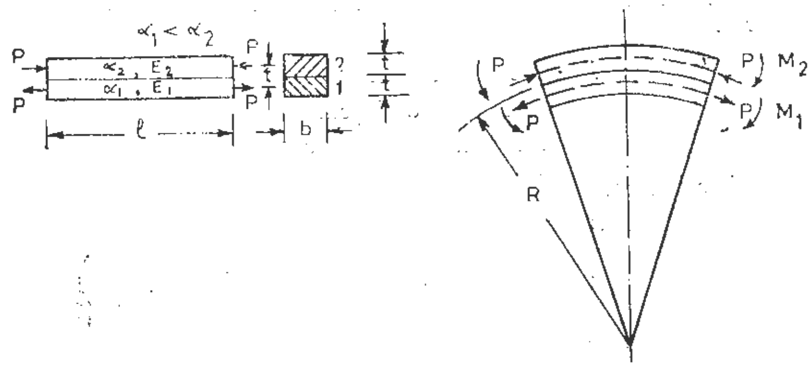


Fig. 8·37

For equilibrium, compressive force on strip of metal 2

$=$tensile force on strip of metal 1.

Say b is the breadth and t is the thickness of each strip.

$$f_1 bt = f_2\, bt \qquad \ldots(1)$$

where f_2 = compressive stress in strip 2

f_1 = tensile stress in strip 1.

The bending moment exerted on the bar,

$$M = Pt$$

$$M = M_1 + M_2$$

$=$Bending moment resisted by strip 1 + bending moment resisted by strip 2

$$Pt = \frac{bt^3}{12\,R} \times E_1 + \frac{bt^3}{12\,R}\, E_2 = \frac{bt^3}{12\,R}(E_1 + E_2) \qquad \ldots(1)$$

Let us assume that the composite bar has bent into the shape of an are of a circle. The radius of curvature upto the interface is R and Radius of curvature R is the same for both the strips because R is very large in comparison to t.

Resultant strain in strip 2,

$$\epsilon_2 = \frac{t}{2R} \qquad \text{(from flexure formula)}$$

Resultant strain in strip 1,

$$\epsilon_1 = -\frac{t}{2R}$$

Moreover resultant strain in strip 2,

$$= -\frac{P}{bt\,E_2} + \alpha_2 T$$

and resultant strain in strip 1,

$$= +\frac{P}{bt\,E_1} + \alpha_1 T$$

Difference of strains

$$\epsilon_2 - \epsilon_1 = \frac{t}{R} = -\frac{P}{bt\,E_2} + \alpha_2 T - \frac{P}{btE_1} - \alpha_1 T$$

$$\frac{t}{R} = (\alpha_2 - \alpha_1)T - \frac{P}{bt}\left(\frac{1}{E_1} + \frac{1}{E_2}\right)$$

or $$(\alpha_2 - \alpha_1)\,T = \frac{t}{R} + \frac{P}{bt}\left(\frac{1}{E_1} + \frac{1}{E_2}\right) \qquad \ldots(2)$$

Substituting the value of P from equation (1)

$$(\alpha_2 - \alpha_1)\,T = \frac{t}{R} + \frac{bt^2}{12\,R}(E_1 + E_2)\frac{1}{bt}\left(\frac{1}{E_1} + \frac{1}{E_2}\right)$$

$$= \frac{t}{R} + \frac{t}{12\,R}\,\frac{(E_1 + E_2)^2}{E_1 E_2}$$

$$= \frac{t}{R}\left[1 + \frac{E_1^2 + E_2^2 + 2E_1E_2}{12\,E_1E_2}\right]$$

$$= \frac{t}{12\,R\,E_1E_2}\,(E_1^2 + E_2^2 + 14\,E_1E_2)$$

or Radius of curvature, $$R = \frac{E_1^2 + E_2^2 + 14\,E_1\,E_2}{12\,E_1E_2(\alpha_2 - \alpha_1)} \times \frac{t}{T} \qquad \ldots(3)$$

Example 8·16-1. A bimetallic strip is made of brass and steel strips of width 6 mm and thickness 1·2 mm each. The composite strip is initially straight. Find the radius of bend if the temperature of the composite strip is raised by 80°C.

$\alpha_B = 19 \times 10^{-6}/°C, \quad \alpha_S = 11 \times 10^{-6}/°C,$
$E_B = 0·9 \times 10^5\ N/mm^2, E_S = 2 \times 10^5\ N/mm^2$

Solution. In this problem $\alpha_S < \alpha_B$, so the equation for the radius of curvature can be modified as

$$R=\frac{E_S^2+E_B^2+14\ E_S\ E_B}{12\ E_S\ E_B(\alpha_B-\alpha_S)}\times\frac{t}{T}$$

(taking $E_1=E_S$ and $E_2=E_B$)

$t=1{\cdot}2$ mm

$T=80°C$

If we take $\frac{E_s}{E_b}=m=\frac{2}{0{\cdot}9}=2{\cdot}22$

$$R=\frac{\left(m+\frac{1}{m}\right)+14}{12(\alpha_B-\alpha_S)}\times\frac{t}{T}$$

$$=\frac{(2{\cdot}22+0{\cdot}45)+14}{12(19-11)\times10^{-6}}\times\frac{1{\cdot}2}{80}$$

$$=\frac{16{\cdot}67}{96}\times\frac{1{\cdot}2\times10^6}{80}=2604{\cdot}7\text{ mm}$$

$=2{\cdot}60$ m.

Exercise 8·16-1. A bimetallic strip is made from copper and steel strips of width 50 mm and thickness 15 mm each. The composite strip is initially straight. Find the radius of the bend if the temperature of the composite strip is raised by 100°C.

Given : $\alpha_C=18\times10^{-6}/°C$ $\quad E_C=1\times10^5$ N/mm²

$\alpha_S=11\times10^{-6}/°C$ $\quad E_S=2\times10^5$ N/mm²

[**Ans.** 29·464 metre]

8·17. COMPOSITE BEAMS

In this chapter uptil now, we have studied beams of various sections but of single material subjected to bending moment. A beam having two or more than two materials rigidly fixed together is called a composite beam. A beam of two materials is most common, such as wooden beam reinforced by metal strips and concrete beams reinforced with steel rods. We will discuss the three cases as below—

1. Fig. 8·38 shows a beam of rectangular section $B\times D$ of material 1 strengthened by two strips of section $t\times D$ each of material 2.

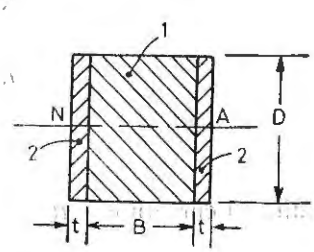


Fig. 8·38

Say the skin stress in material $1=f_1$

Skin stress in material $2=f_2$

Modular ratio $=\frac{E_2}{E_1}=m$

Moment of Resistance = Resisting moment offered by beam of material 1 + resisting moment offered by strips of material 2.

$$M=M_1+M_2$$

$$=f_1\cdot\frac{1}{6}BD^2+2\left(f_2\cdot\frac{1}{6}t\ D^2\right)$$

$$=\frac{f_1}{6}\cdot BD^2+\frac{f_2}{6}\cdot 2t\ D^2 \qquad ...(1)$$

Since the beam and strips are perfectly joined together, the deformation or the strain in the layers of both the materials at a particular distance from the Neutral axis is the same.

or $$\frac{f_1}{E_1}=\frac{f_2}{E_2}$$

or $$f_2=f_1\times\frac{E_2}{E_1}=mf_1 \quad ...(2)$$

Substituting in equation (1)

$$M=\frac{f_1\,BD^2}{6}+\frac{m\,f_1}{6}\,2t\,D^2=\frac{f_1}{6}\,D^2[B+2mt] \quad ...(3)$$

2. Fig. 8·39 show a plate of width t and depth d of material 2 sandwiched between two beams of rectangular section $B\times D$ of material 1. The neutral axis passes symmetrically through the section.

Say the skin stress developed in beam of material $1=f_1$

and modular ratio, $m=\dfrac{E_2}{E_1}$

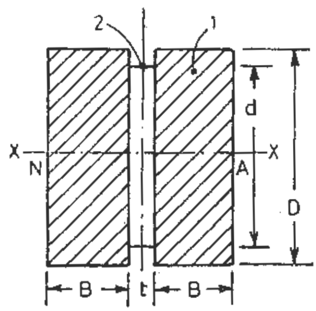


Fig. 8·39

Stress in material 1 at a distance of $d/2$ from neutral axis

$$=f_1\times\frac{d/2}{D/2}=f_1\,\frac{d}{D} \quad ...(1)$$

Skin stress developed in material 2,

$$f_2=f_1\,m\,\frac{d}{D} \quad ...(2)$$

Moment of resistance, $M=M_1+M_2$

=Resisting moment offered by two beams of rectangular section+resisting moment offered by the plate in between two beams

$$=f_1\times2\,.\left(\frac{BD^2}{6}\right)+f_2\times\left(\frac{td^2}{6}\right)$$

$$=f_1\times\frac{BD^2}{6}\times2+f_1\,m\,\frac{d}{D}\times\frac{td^2}{6}$$

$$=\frac{f_1}{6}\left[2BD^2+m\,\frac{td^3}{D}\right] \quad ...(3)$$

Fig. 8·40 (a) shows a rectangular beam of section $B\times D$ of material 1 strengthened by two plates at the top and bottom, of material 2. Say the modular ratio $\dfrac{E_2}{E_1}=m$.

In this case equivalent sections either of material 1 or of material 2 can be considered.

Fig. 8·40 (b) shows an equivalent section of material 1, the width of the plates is increased to mB. Neutral axis passes symmetrically through the equivalent I-section.

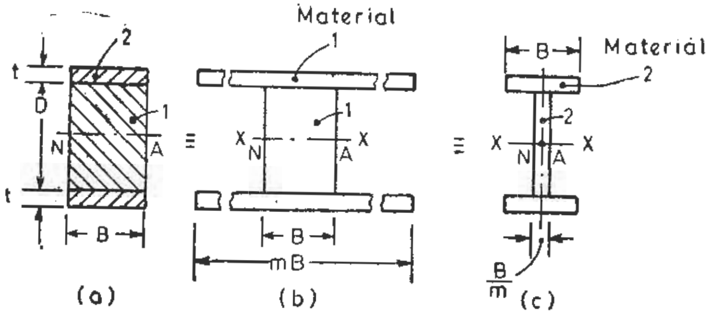


Fig. 8·40

Say skin stress in material 1 at a distance of

$\frac{D}{2}$ from the neutral axis$=f_1$

Then skin stress at a distance of $\left(t+\frac{D}{2}\right)$ from neutral axis

$$=\frac{f_1\left(t+\frac{D}{2}\right)}{D/2}=\frac{f_1(D+2t)}{D} \qquad \text{...(1)}$$

Moment of Inertia,

$$I_{NA} \text{ or } I_{xx}=\frac{mB(2t+D)^3}{12}-\frac{BD^3}{12}$$

Section modulus $\qquad Z=\frac{I_{xx}}{t+\frac{D}{2}}=\frac{2\,I_{xx}}{(2t+D)}$

Moment of resistance $\quad =f_1\,\frac{(D+2t)}{D}\times\frac{2\,I_{xx}}{(D+2t)}=f_1\times\frac{I_{xx}}{D/2} \qquad \text{...(2)}$

Fig. 8·40 (*c*) shows the equivalent section for material 2. The width of the beam, *B* of material 1 is reduced to *B*/*m* and the width of the plates remains unchanged. Neutral axis passes symmetrically through the equivalent I section.

Moment of inertia, $\quad I_{xx}=\frac{B(D+2t)^3}{12}-\frac{B}{m}\left(\frac{D^3}{12}\right) \qquad \text{...(1)}$

Say the skin stress developed at a distance of $t+\frac{D}{2}$ from neutral axis,

$$=f_2$$

Section modulus, $\qquad Z=\frac{I_{xx}}{\left(t+\frac{D}{2}\right)}$

Moment of resistance, $M=f_2\,.\,Z \qquad \text{...(2)}$

Example 8·17-1. A wooden beam 20×30 cm is strengthened by two steel plates 1×25 cm each as shown in Fig. 8·41. Determine the allowable bending moment if the allowable stress in steel is 1500 kg/cm² and in wood is 80 kg/cm². Given

$$E_{steel}=2\times10^6 \text{ kg/cm}^2$$

$$E_{wood}=8\times10^4 \text{ kg/cm}^2$$

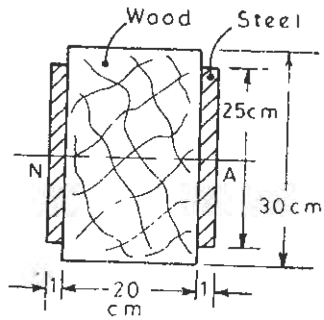


Fig. 8·41

Solution.

Modular ratio, $m=\frac{E_s}{E_w}=\frac{2\times10^6}{8\times10^4}=25$

The section is symmetrical, the neutral axis passes through the centre as shown.

Say the skin stress in wood,

$f_w=80$ kg/cm² (at a distance of 15 cm from NA)

Then skin stress in steel at a distance of 12·5 cm from NA,

$$f_s=80\times m\times\frac{12\cdot5}{15}$$

$$=80\times25\times\frac{12\cdot5}{15}=1666\cdot66 \text{ kg/cm}^2$$

But allowable stress in steel,

$f_s=1500$ kg/cm²

Then skin stress developed in wood (corresponding to allowable stress in steel),

$$f_w=\frac{f_s}{m}\times\frac{D}{d}=\frac{1500}{25}\times\frac{15}{12\cdot5}$$

$$=72 \text{ kg/cm}^2$$

So allowable bending moment,

$$M=M_s+M_w$$

$$=f_s\,.\,\frac{1}{6}\times2\ (1\times25^2)+f_w\,.\,\frac{1}{6}\ (20\times30^2)$$

$$=\frac{1500}{6}\times2\times25^2+72\times\frac{1}{6}\times20\times30^2$$

$$=312500+216000 \text{ kg-cm}=528500 \text{ kg-cm}$$

$$=5\cdot285 \text{ tonne-metre.}$$

Example 8·17-2. Determine the allowable bending moment about horizontal neutral axis for the composite beam of wood and steel shown in the Fig. 8·42. The allowable stress in wood=8 N/mm² and allowable stress in steel=150 N/mm².

$$E_{steel}=210\times10^3 \text{ N/mm}^2$$

$$E_{wood}=10\times10^3 \text{ N/mm}^2.$$

Solution.

Modulus ratio, $=\frac{E_s}{E_w}=21$

the equivalent steel section is shown in Fig. 8·43 (*a*) thickness of the web

$$=\frac{20}{21}\text{ cm}$$

CG will lie along *YY* axis due to symmetry.

Let us determine the position of CG along *Y-Y* axis.

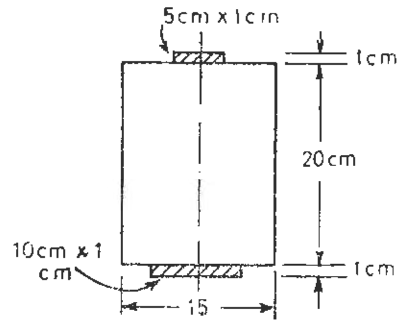


Fig. 8·42

$$y_1=10\times1\times0·5+20\times\frac{20}{21}\times(10+1)+\frac{5\times1\times(21+0·5)}{10+20\times\frac{20}{21}+5}$$

$$=\frac{5+209·52+107·5}{34·048}=9·46\text{ cm}$$

Distance, $y_2=22-9·46=12·54$ cm

Moment of inertia, $I_{xx}=\frac{10\times1^3}{21}+10\times(9·46-0·5)^2$

$$+\frac{20}{21}\times\frac{20^3}{12}+\frac{20}{21}\times20(11-9·46)^2$$

$$+\frac{5\times1^3}{12}+5\,(12·54-0·5)^2$$

$$=0·833+802·816+634·920+45·173+0·416+724·808$$

$$=2208·966\text{ cm}^4=2208·966\times10^4\text{ mm}^4.$$

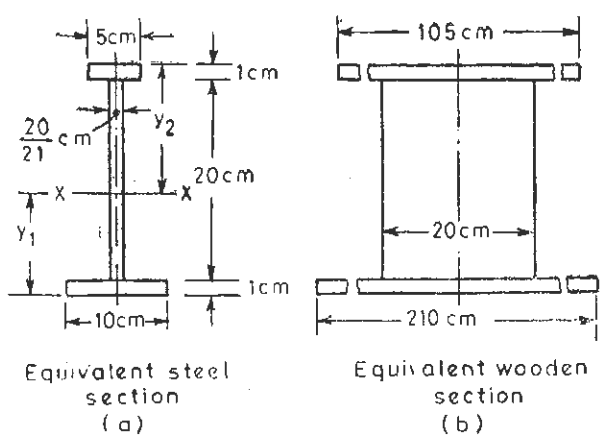


Fig. 8·43

Since $y_2 > y_1$, maximum stress will occur at the upper edge.

Allowable stress in steel, $f=150$ N/mm²

Section modulus, $Z_2=\frac{I_{xx}}{y_2}=\frac{2208 \cdot 966\times 10^4}{125 \cdot 4}=17 \cdot 61\times 10^4$ mm³

Bending moment, $M=f \, . \, Z_2$

$=150\times 17 \cdot 61\times 10^4$ N-mm

$=2641 \cdot 5\times 10^4$ N-mm$=26 \cdot 41$ kNm

The equivalent wooden section is shown in Fig. 8·43 (*b*). If we compare the two equivalent sections, we find that both are unsymmetrical I sections and in the case of wooden section, all the widths of flanges and web are 21 times the width of flanges and web in the equivalent steel section.

So $y_1=9 \cdot 46$ cm, $y_2=12 \cdot 54$ cm

$I_{xx}=21\ (2208 \cdot 966)$ cm$^4=21\times 2208 \cdot 966\times 10^4$ mm⁴

Section modulus, $Z_2=\frac{21\times 2208 \cdot 966\times 10^4}{125 \cdot 4}=21\times 17 \cdot 61\times 10^4$ mm³

Allowable stress $=8$ N/mm²

Bending moment, $M=f\,Z_2=8\times 21\times 17 \cdot 61\times 10^4$ Nmm

$=2958 \cdot 48\times 10^4$ Nmm$=29 \cdot 58$ kNm

So the allowable bending moment about the neutral axis

$=26 \cdot 41$ kNm.

Exercise 8·17-1. A wooden beam 15 cm×20 cm is strengthened by two steel plates of thickness 1 cm and depth 20 cm on both of its sides. Determine the allowable bending moment around the horizontal neutral axis if allowable stress in wood=10 N/mm² and allowable stress in steel=150 N/mm².

Given $E_{steel}=200\times 10^3$ N/mm²

$E_{wood}=10\times 10^3$ N/mm². [**Ans.** $M=55$ kNm]

Exercise 8·17-2. A wooden beam 20×30 cm is strengthened by two steel plates of thickness 12 cm and width 20 cm on its top and bottom. Determine the skin stresses developed in steel and wood if a bending moment of 4 tonne-metres is applied to the beam of this section.

Given

$E_{steel}=15\ E_{wood}=2100$ tonnes/cm².

[**Ans.** $f_w=42 \cdot 40$ kg/cm², $f_s=712 \cdot 476$ kg/cm²]

8·18. REINFORCED CEMENT CONCRETE (R.C.C.)

We have learnt that a portion of the beam comes under tension when the beam is subjected to a bending moment. Concrete is a common building material and when a column or a beam of concrete is subjected to a bending moment, a portion of the concrete comes in tension. Concrete has a very useful strength in compression but in tension it is very weak. Minute cracks are developed in concrete when subjected to even a small tensile stress. Therefore the tension side of a concrete beam is reinforced with steel bars. Concrete and steel make a very good composite because cement concrete contracts during setting and firmly grips the steel reinforcement. Moreover coefficient of thermal expansion of steel and most common mix of concrete *i.e.* 1 : 2 : 4 is more or less the same. Ratio 1 : 2 : 4 stands for 1 part of cement, 2 parts of sand and 4 parts of aggregate by volume. Therefore, the stresses developed in R.C.C. due to temperature changes are negligible.

In order to develop a theory for stresses developed in R.C.C. beam section, following assumptions are taken :

1. Concrete is effective only in compression and stress in concrete on the tension side of the beam is zero.
2. Sections which are plane before bending remain plane after bending.
3. Strain in a layer is proportional to its distance from the neutral axis.
4. Stress is proportional to strain in concrete.
5. Modulus of elasticity of concrete bears a constant ratio with the modulus of elasticity of steel.

The last two assumptions are not true since concrete does not obey Hooke's law, but it is possible to take a mean value of Young's modulus of concrete over the range of stress used. The allowable stresses for concrete and the value of the Young's modulus depend upon the type and mix of the concrete used.

Rectangular Section—R.C.C. Beam. In the Fig. 8·44, say B is breadth of the section and D is the depth of the reinforcement from the compression face.

Let H be the distance of neutral axis from the compression face and the maximum stresses developed in steel and concrete are f_s (tensile) and f_c (compressive) respectively.

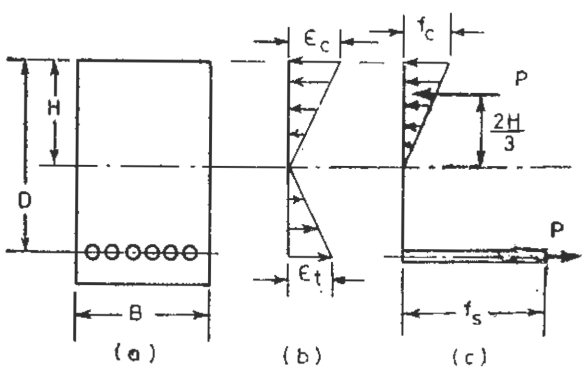


Fig. 8·44

Now the strain in any layer is proportional to its distance from the neutral axis.

Therefore,
$$\epsilon_c \propto H$$
$$\epsilon_t \propto (D-H)$$

or
$$\frac{\epsilon_c}{\epsilon_t} = \frac{H}{D-H}$$

or
$$\frac{f_c}{E_c} \times \frac{E_s}{f_s} = \frac{H}{D-H}$$

or
$$\frac{f_s}{f_c} = \frac{E_s}{E_c} \times \frac{(D-H)}{H}$$

where E_s = Young's modulus of steel
E_c = Young's modulus of concrete

$$= m\,\frac{(D-H)}{H} \qquad \text{where } m = \frac{E_s}{E_c}, \text{ modular ratio}$$

So
$$\frac{f_s}{f_c} = m\left(\frac{D-H}{H}\right) \qquad \text{...(1)}$$

If the beam is under the action of the pure bending, then the resultant force P in steel is the same as the resultant force P in concrete,

i.e. $$P = f_c\,A_c = f_s\,A_s$$

or $$\frac{f_s}{2}\,(BH) = f_s\,.\,A_s$$

where A_s = area of cross section of steel reinforcement

The stress in concrete linearly varies along the depth H. Therefore the mean stress in concrete is taken as $\frac{f_c}{2}$, where f_c is the maximum stress in concrete.

$$\therefore \qquad \frac{f_c}{2}\,(BH) = f_s\,.\,A_s \qquad \text{...(2)}$$

Then the resultant compressive force P in concrete and tensile force P in steel form a couple resisting the applied bending moment. The arm of the couple as shown in Fig. 8·44(c) is $\left(D-\frac{H}{3}\right)$.

$$\therefore \qquad M = P\left(D-\frac{H}{3}\right) = \frac{f_c}{2}\,BH\left(D-\frac{H}{3}\right)$$
$$= f_s\,.\,A_s\left(D-\frac{H}{3}\right) \qquad \text{...(3)}$$

If the maximum allowable stresses in steel and concrete are given, then knowing the ratio of f/f_c we can determine the value of H for the given dimensions of a beam, with the help of equation (1). After that area of steel reinforcement is found by using the equation (2). Finally we can determine the moment of resistance from equation (3). Thus is known as the "ECONOMIC SECTION" in which the allowable values of stresses in steel and concrete have been realised.

In case the dimensions of the beam *i.e.* B and D, area of steel reinforcement A_s are given, then H can be determined from equations (1) and (2) as follows :

$$\frac{f_s}{f_c} = m\,\frac{(D-H)}{H} \qquad \text{...(1)}$$

$$\frac{f_s}{f_c} = \frac{BH}{2A_s} \qquad \text{...(2)}$$

or $$\frac{m(D-H)}{H} = \frac{BH}{2A_s} \qquad \text{...(3)}$$

or $$2A_s\,(mD-mH) = BH^2$$

or $$BH^2+2\,m\,A_sH-2\,m\,A_sD = 0 \qquad \text{...(4)}$$

The value of H can be determined from equation (4) and the actual stresses in steel and concrete are then determined from the magnitude of applied moment M,

Example 8 18-1. A reinforced concrete beam is 20 cm wide and 40 cm deep. The maximum allowable stresses in steel and concrete are 1200 kg/cm² and 75 kg/cm² respectively. What area of steel reinforcement is required if both the stresses are developed and steel reinforcement is 6 cm above the tension face. Modular ratio = 16.

What uniformly distributed load can be carried over a span of 5 metres. The weight density of concrete is 2360 kg/m³. Neglect the weight of steel reinforcement.

Solution.

Allowable stress in steel,

$$f_s = 1200 \text{ kg/cm}^2$$

Allowable stress in concrete,

$$f_c = 75 \text{ kg/cm}^2$$

Width, $B = 20$ cm

Depth of the beam $= 40$ cm

Cover for steel $= 6$ cm

Distance of steel reinforcement from the compression face,

$$D = 40-6=34 \text{ cm}$$

Both the allowable stresses are to be realised, *i.e.* we are finding out the economic section of the beam.

From equation (1) $\dfrac{f_s}{f_c} = \dfrac{m(D-H)}{H}$

Modular ratio, $m = 16$

So
$$\frac{1200}{75} = 16\,\frac{(D-H)}{H}$$

$$16 = 16\,\frac{(D-H)}{H}$$

or
$$2H = D$$

$$H = \frac{34}{2} = 17 \text{ cm.}$$

From equation (2) $\dfrac{f_s}{f_c} = \dfrac{BH}{2A_s}$

Area of steel reinforcement,

$$A_s = \frac{20\times 17}{2\times 16} = 10{\cdot}625 \text{ cm}^2$$

Moment of resistance,

$$M = \frac{f_c}{2} BH\left(D-\frac{H}{3}\right)$$

$$= \frac{75}{2}\times 20\times 17\left(34-\frac{17}{3}\right)$$

$$= 750\times 17\times\frac{85}{3} = 361250 \text{ kg-cm}$$

$$= 3{\cdot}6125 \text{ tonne-metre.}$$

Say the uniformily distributed load per metre run $= w$ tonnes

Maximum bending moment

$$= \frac{wl^2}{8} = \frac{w \times 5^2}{8} = 3{\cdot}6125$$

$$w = 1{\cdot}156 \text{ tonnes/metre}$$

Weight of the beam per metre run

$$= 0{\cdot}2 \times 0{\cdot}4 \times 1 \times 2360 \text{ kg}$$
$$= 188{\cdot}8 \text{ kg}$$
$$= 0{\cdot}1888 \text{ tonne.}$$

Uniformly distributed load carried by the beam

$$= 1{\cdot}156 - 0{\cdot}1888$$
$$= 0{\cdot}9672 \text{ tonne/metre run}$$

Exercise 8·18-1. A reinforced concrete beam is of rectangular section 250 mm wide and 550 mm deep. Steel reinforcement of 1200 mm² is placed 50 mm above the tension face. The maximum stress in concrete is 5 N/mm². The modular ratio is 15. Calculate the stress in steel and moment of resistance. [**Ans.** 107·18 N/mm², 55·5 kNm]

Problem 8·1. A wooden joist of span 8 metres is to carry a brick wall 23 cm thick and 3·5 m high. The depth of the joist is 3 times its breadth and the maximum permissible stress is limited to 80 kg/cm². Find the dimensions of the joist.

Density of brick wall = 1800 kg/m³

Solution. Uniformly distributed load per metre length on the wooden joist,

$$w = 1 \times 0{\cdot}23 \times 3{\cdot}5 \times 1800 \text{ kg}$$
$$= 1449 \text{ kg.}$$

Span length, $l = 8$ metre

Maximum bending moment

$$= \frac{wl^2}{8} = \frac{1449 \times 8 \times 8}{8} = 11592 \text{ kg-m}$$
$$= 1159200 \text{ kg-cm}$$

Allowable stress, $f = 80$ kg/cm²

Required section modulus,

$$Z = \frac{M_{max}}{f} = \frac{1159200}{80}$$
$$= 14490 \text{ cm}^3$$
$$= \frac{1}{6} \times BD^2 \quad \text{but } D = 3\,B$$
$$= \frac{1}{6} \times B \times 9\,B^2 = 1{\cdot}5\,B^3$$

So $1{\cdot}5\,B^3 = 14490$

$$B^3 = \frac{14490}{1{\cdot}5} = 9660 \text{ cm}^3$$

Dimensions of the joist,

$$B = 21{\cdot}3 \text{ cm}$$
$$D = 63{\cdot}9 \text{ cm.}$$

Problem 8·2. A floor has to carry a load of 1000 kg per sq. metre (including its own weight). If the span of each joist is 5 metres, calculate the spacing centre to centre between the joists. The breadth of the joist is 10 cm and depth is 30 cm and the permissible stress due to bending is 8 N/mm².

Solution.

l, span of each joist $= 5$ m

Say the spacing between the joist

$= x$ m

Floor area per joist $= 5x$ m²

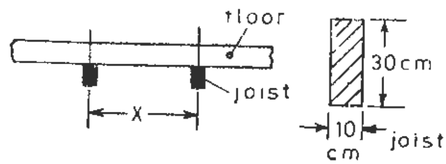


Fig. 8·45

w, uniformly distributed load per metre length of joist

$$= x \times 1 \times 1000 = 1000\ x \text{ kg/m}$$

$$= x \text{ tonne/metre run}$$

M_{max} Maximum bending moment

$$= \frac{wl^2}{8} = \frac{x \times 5 \times 5}{8} = \frac{25\ x}{8} \text{ tonne-metres}$$

Section modulus, $\quad Z = \dfrac{BD^2}{6} = \dfrac{10 \times 30 \times 30}{6} = 1500 \text{ cm}^3$

Permissible stress, $\quad f = 8 \text{ N/mm}^2 = 800 \text{ N/cm}^2 = 81{\cdot}63 \text{ kg/cm}^2$

$$M_{max} = f\,Z$$

$$= 81{\cdot}63 \times 1500$$

$$= 122445 \text{ kg cm} = \frac{25\ x}{8} \text{ tonne-metres}$$

$$= \frac{25 \times 10^5\ x}{8} \text{ kg-cm}$$

or $\quad x = \dfrac{122445 \times 8}{25 \times 10^5} = 0{\cdot}3918 \text{ m} = 39{\cdot}18 \text{ cm.}$

Problem 8·3. A beam subjected to bending moment M is of T-section as shown in Fig. 8·46. Determine the thickness of the flange and the web if the flange is 2 times as thick as web and the maximum tensile stress is 2 times the maximum compressive stress.

Solution. The stress in a layer due to bending is proportional to its distance from the neutral layer. With the type of bending moment shown, the flange will be in compression and lower portion of web will be in tension. Extreme fibres at top and bottom will have maximum compressive and maximum tensile stresses. Say the neutral layer lies at a distance of of y_1 from the bottom edge. As given in the problem.

$$y_t = 2y_c$$

or $$y_1=2y_2$$

But $$y_1+y_2=20 \text{ cm}$$

$$2y_2+y_2=20$$

$$y_2=6{\cdot}67 \text{ cm}$$

$$y_1=13{\cdot}33 \text{ cm}$$

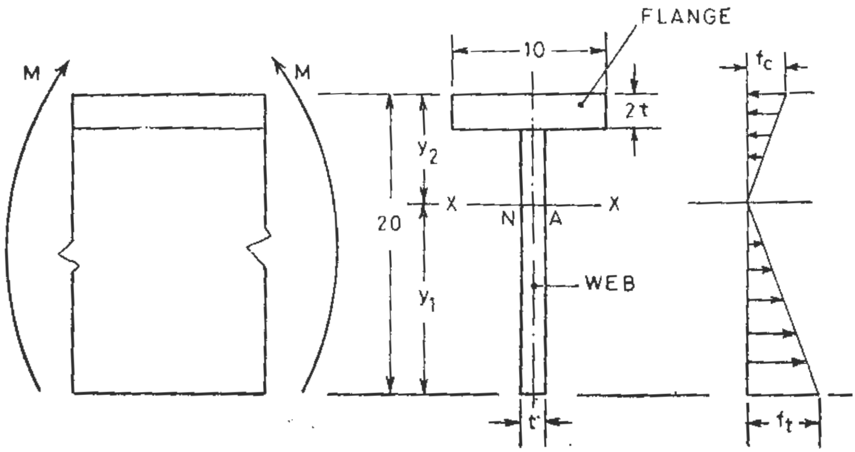


Fig. 8·46

Now flange area, $a_1=10\times 2t=20t$

CG of a_1 from bottom edge,

$$=20-t$$

Web area, $a_2=(20-2t)\ t=2t\ (10-t)$

CG of a_2 from the bottom edge

$$=\frac{(20-2t)}{2}=(10-t)$$

Now $$y_1=\frac{20t\ (20-t)+2t\ (10-t)(10-t)}{20t+2t\ (10-t)}$$

$$13{\cdot}33=\frac{10\ (20-t)+(10-t)^2}{10+(10-t)}$$

or $$13{\cdot}33\ (20-t)=200-10t+100-20t+t^2$$

$$266{\cdot}6-13{\cdot}33t=300-30t+t^2$$

$$t^2-16{\cdot}67t+33{\cdot}4=0$$

$$t=\frac{16{\cdot}67-\sqrt{(16{\cdot}67)^2-4\times 33{\cdot}4}}{2}$$

$$=\frac{16{\cdot}67-12{\cdot}012}{2}=2{\cdot}33 \text{ cm}$$

Thickness of web $=2{\cdot}33$ cm

Thickness of flange $=4{\cdot}66$ cm.

Problem 8·4. A *T*-beam of depth 12 cm is used as a beam with simply supported ends, so that the flange comes under tension. The material of the beam can be subjected to 900 kg/cm² in compression and 300 kg/cm² in tension. It is desired to achieve a balanced design so that the largest possible bending stresses are reached simultaneously. Determine the width of the flange. Find how much concentrated load can be applied to the beam at its centre if the span length is 4 metre.

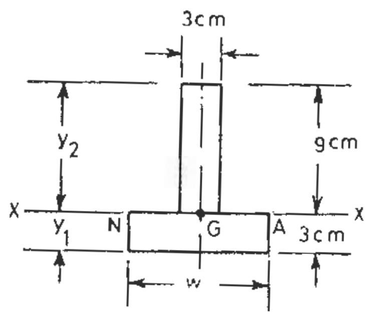


Fig. 8·47

Solution. Stress due to bending in a layer is proportional to its distance from the neutral layer

$$f_t = 300 \text{ kg/cm}^2 \text{ in flange } \propto y_1$$
$$f_c = 900 \text{ kg/cm}^2 \text{ in web } \propto y_2$$

where y_1 and y_2 are the distances of extreme layers from neutral layer

i.e. $300 \propto y_1,$

$900 \propto y_2$

But $y_1 + y_2 = 12$ cm

So $y_1 = 9$ cm, $y_2 = 3$ cm

This means that neutral axis is passing at the intersection of web and flange.

Area of web $= 27$ cm²

Area of flange $= 3w$ cm²

$$y_1 = \frac{3\,w \times 1{\cdot}5 + 27 \times (3 + 4{\cdot}5)}{3\,w + 27} = \frac{4{\cdot}5\,w + 81 + 121{\cdot}5}{3\,w + 27} = 3$$

or $4{\cdot}5\,w + 202{\cdot}5 = 9w + 81,$

$w = 27$ cm

Area of flange, $a_1 = 27 \times 3 = 81$ cm²

Area of web, $a_2 = 9 \times 3 = 27$ cm²

Moment of inertia, $I_{xx} = \dfrac{27 \times 3^3}{12} + 81 \times 1{\cdot}5^2 + \dfrac{3 \times 9^3}{12} + 27 \times 4{\cdot}5^2$

$= 60{\cdot}75 + 182{\cdot}25 + 182{\cdot}25 + 546{\cdot}75 = 972$ cm⁴

Section modulus, $Z_1 = \dfrac{I_{xx}}{y_1} = \dfrac{972}{3} = 324$ cm³

Section modulus, $Z_2 = \dfrac{I_{xx}}{y_2} = \dfrac{972}{2} = 108$ cm³

Permissible bending moment,

$M = f_t \,.\, Z_1 = f_c \,.\, Z_2$

$= 324 \times 300 = 97200$ kg-cm

Span length, $L = 4$ m $= 400$ cm

At the centre of the beam,

$$M_{max} = \frac{WL}{4} \quad \text{where } W \text{ is the central load}$$

$$=\frac{W\times 400}{4}=100\text{ W kg-cm}$$

$$=97200$$

or central load, $W=272$ kg

Problem 8·5. A cantilever has a free length of 3 metres. It is of T section with the flange 100 mm by 20 mm, web 200 mm by 10 mm; the flange being in tension. What load per metre run can be applied if the maximum tensile stress is 40 N/mm² ? What is the maximum compressive stress ?

Solution. The T-section is symmetrical about the vertical axis Y-Y so the C.G. of the section will lie on this axis To determine the position of X-X axis, the moments of areas can be taken about the top edge of the flange

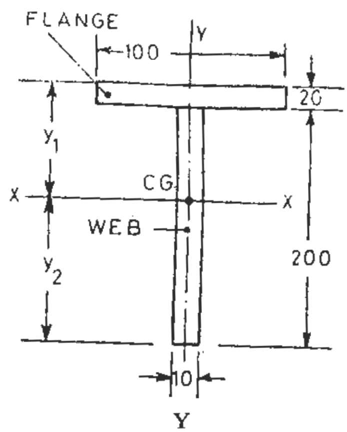


Fig. 8·48

Area, $a_1'=100\times 20\text{ mm}^2$

$y_1'=10$ mm

Area, $a_2'=200\times 10$

$y_2'=20+100=120$ mm

Now $(a_1'+a_2')\,y_1=a_1'\,y_1'+a_2'\,y_2'$

$(2000+2000)\,y_1=2000\times 10+2000\times 120$

$y_1=65$ mm

$y_2=220-65=155$ mm.

Maximum tensile stress is developed in the flange. So the maximum compressive stress will be developed in the web

$$\frac{f_{t\,max}}{f_{c\,max}}=\frac{y_1}{y_2}$$

$$f_{c\,max}=\frac{y_2}{y_1}\times f_{t\,max}=\frac{155}{65}\times 40$$

$$=7{\cdot}07\text{ N/mm}^2$$

Say the load per mm run $=w$ Newtons

Length of the cantilever $=3$ m$=3000$ mm

Maximum bending moment

$$=\frac{wl^2}{2}=\frac{w\times(3000)^2}{2}\text{ Nmm}$$

I_{xx} of T section

$$=\frac{(100)\,(20)^3}{12}+2000\,(65-10)^2+\frac{10\times(200)^3}{12}+2000\,(155-100)^2$$

$$=1983{\cdot}33\times 10^4\text{ mm}^4$$

Now $$\frac{M_{max}}{I_{xx}}=\frac{f_{t\,max}}{y_1}=\frac{f_{c\,max}}{y_2}$$

or $$M_{max}=I_{xx}\cdot\frac{f_{t\,max}}{y_1}$$

$$\frac{w}{2}\times(3000)^2=1983\cdot33\times10^4\times\frac{40}{65}$$

$$w=\frac{3966\cdot66}{325},\quad w=12\cdot205\ \text{N/mm}$$

Load per m run, $w=12205$ N/m

$=12\cdot205$ kN/m.

Problem 8·6. A compound beam for a crane runway is built up of a 150×50 mm rolled steel joist with a 100×50 rolled steel channel attached to the top flange. Calculate the position of the N.A. of the section and determine moment of inertia I_{xx} of composite section. For I section, Area=9·01 cm², $I_{x'x'}=322\cdot1$ cm⁴, $I_{y'y'}=9\cdot2$ cm⁴, and for the channel section, Area=10·02 cm², web thickness=4 mm, $I_{x'x'}=164\cdot7$ cm⁴ $I_{y'y'}=24\cdot8$ cm⁴. Distance of CG from outer edge of web=1·62 cm.

Solution. The composite section is shown in the Fig. 8·49. The section is symmetrical about *YY* axis but unsymmetrical about *XX* axis. Let us determine the position of neutral axis.

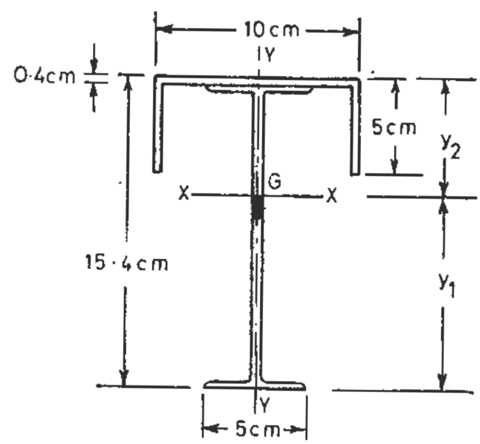


Fig. 8·49

$$y_1=\frac{9\cdot01\times7\cdot5+10\cdot02(15\cdot4-1\cdot62)}{9\cdot01+10\cdot02}$$

$$=\frac{67\cdot575+138\cdot076}{19\cdot03}=10\cdot80\ \text{cm}$$

$$y_2=15\cdot4-10\cdot8=4\cdot6\ \text{cm}$$

Moment of Inertia

$$I_{xx}=322\cdot1+9\cdot01(y_1-7\cdot5)^2+24\cdot8+10\cdot02(y_2-1\cdot62)^2$$

(322·1 cm⁴ is $I_{x'x'}$ of T section and 24·8 cm⁴ is the $I_{y'y'}$ of channel section)

$$I_{xx}=322\cdot1+9\cdot01(10\cdot8-7\cdot5)^2+24\cdot8+10\cdot02(4\cdot6-1\cdot62)^2$$

$$=322\cdot1+98\cdot119+24\cdot8+88\cdot981$$

$$=534\ \text{cm}^4.$$

Problem 8·7. Two channel sections 300 mm×100 mm are placed back to back and only the top flanges are joined by a plate 200 mm×10 mm. This compound section forms a simply supported beam 2 m long and carries a uniformly distributed load of 10 tonnes/metre run. Determine the maximum stress developed in the sections. Properties of a channel section.

Area=42·19 cm², $I_{x'x'}=6066$ cm⁴, $I_{y'y'}=346\cdot9$ cm⁴,

Solution. The compound section is shown in the Fig. 8·50. The section is symmetrical about YY axis but unsymmetrical about X-X axis. Let us determine the position of G from the bottom edge.

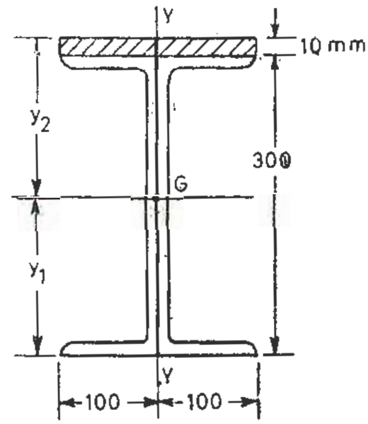


Fig. 8·50

$$y_1=\frac{2\times 42{\cdot}19\times 15+20\times 1{\cdot}0\times 30{\cdot}5}{2\times 42{\cdot}19+20\times 1}$$

(taking the dimensions in cm)

$$=\frac{1265{\cdot}7+610}{104{\cdot}38}$$

$$=\frac{1875{\cdot}7}{104{\cdot}38}=17{\cdot}97 \text{ cm}$$

$$y_2=31-17{\cdot}97=13{\cdot}03 \text{ cm}$$

Moment of Inertia, $I_{xx}=2\times I_{x'x'}+2\times 42{\cdot}19\,(y_1-15)^2$

$$+2\times\frac{20\times 1^3}{12}+2\times 20(y_2-0{\cdot}5)^2$$

$$=2\times 6066+84{\cdot}38\,(2{\cdot}97)^2+3{\cdot}333+40(12{\cdot}53)^2$$

(putting the values of y_1 and y_2)

$$=12132+744{\cdot}30+3{\cdot}333+6280{\cdot}0$$

$$=19159{\cdot}633 \text{ cm}^4$$

Length of the span, $l=2 \text{ m}=200 \text{ cm}$

Rate of loading, $w=10$ tonnes/metre$=0{\cdot}1$ tonnes/cm

Maximum bending moment

$$=\frac{wl^2}{8}=\frac{0{\cdot}1\times 200^2}{8}$$

$$=500 \text{ tonne-cm}$$

Maximum stress due to bending

$$=\frac{M_{max}}{I_{xx}}\times y_1$$

As, $y_1>y_2$, maximum stress will occur at the lower edge.

$$f_{max}=\frac{500\times 17{\cdot}97}{19159{\cdot}633}$$

$$=0{\cdot}469 \text{ tonne/cm}^2$$

Problem 8·8. A girder of I section is simply supported over a span of 8 metres. A uniformly distributed load of 1000 N/cm is carried throughout the span. The beam is strengthened wherever necessary by the addition of flangle plates 15 mm thick. Find the length and width of the flange plates such that the maximum stress due to bending does not exceed 120 N/mm².

Solution. The section is symmetrical, therefore C.G. of the section will lie at the centre of the web

$$y_1 = y_2 = 300 \text{ mm}$$

Moment of Inertia,

$$I_{xx} = \frac{240 \times 600^3}{12} - \frac{220 \times 550^3}{12}$$

$$= 10^6 \left[\frac{240}{12} \times 6^3 - \frac{220}{12} \times 5{\cdot}5^3 \right]$$

$$= 10^6 [4320 - 3050{\cdot}2]$$

$$= 1269{\cdot}8 \times 10^6 \text{ mm}^4$$

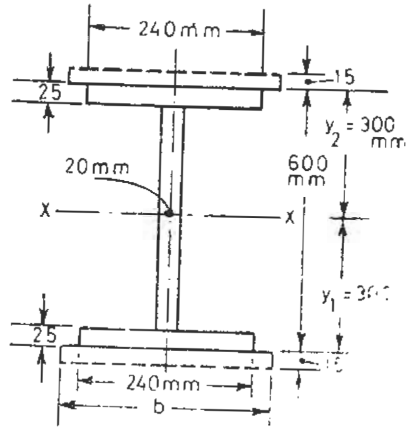


Fig. 8·51

The beam carries a uniformly distributed load

$$w = 1000 \text{ N/cm}$$
$$= 100 \text{ N/mm}$$

If $\quad l$ = length of span
$$= 8000 \text{ mm.}$$

The maximum bending moment will be at the centre

$$M_{max} = \frac{wl^2}{8} = \frac{100 \times 8000 \times 8000}{8}$$

$$= 8 \times 10^8 \text{ Nmm}$$

Maximum stress developed

$$= \frac{M_{max}}{I_{xx}} \cdot y_1$$

$$= \frac{8 \times 10^8}{1269{\cdot}8 \times 10^6} \times 300$$

$$= 189{\cdot}00 \text{ N/mm}^2.$$

This stress is more than the allowable stress of 120 N/mm². Therefore it is necessary to strengthen the section by the addition of flange plates as shown.

Thickness of the flange plates = 15 mm

Say the width of the flange plates = b mm

The moment of inertia with additional flange plates

$$I_{xx}' = I_{xx} + \frac{2 \times b \times 15^2}{12} + 2 \times b \times 15 \times (300 + 7{\cdot}5)^2$$

$$= I_{xx} + 37{\cdot}5\, b + 30\, b\, (307{\cdot}5)^2$$

$$= I_{xx} + 37{\cdot}5\, b + 2836687{\cdot}5\, b$$

$$= I_{xx} + 2836725\, b$$

$$y_1' = 300 + 15$$

$$= 315 \text{ mm}$$

Allowable stress, $120=\frac{M_{max}}{I_{xx}'} \times y_1'$

$$120=\frac{8\times 10^8\times 315}{(I_{xx}+2836725\ b)}$$

or $$I_{xx}+2836725\ b=\frac{8\times 10^8\times 315}{120}$$

$$2836725\ b=21\times 10^8-1269{\cdot}8\times 10^6$$

$$=830{\cdot}2\times 10^6$$

$$b=\frac{830{\cdot}2\times 10^6}{2836725}\text{ mm}$$

Width of additional flange plate

$$=292{\cdot}66\text{ mm}$$

Now it is not necessary to provide the strengthening of the beam throughout its length. For some central portion of the beam, additional flange plates can be provided.

The bending moment M_x, corresponding to allowable stress,

$$M_x=f_{allowable}\times\frac{I_{xx}}{y_1}$$

$$=\frac{125\times 1269{\cdot}8\times 10^6}{300}$$

$$=507{\cdot}92\times 10^6\text{ Nmm}$$

The beam is loaded as shown in the Fig. 8·52.

Reactions, $R_A=R_B$

$$=\frac{100\times 8000}{2}=40\times 10^4\text{ N}$$

$$M_x=R_A\,.\,x-\frac{wx^2}{2}$$

$$=40\times 10^4\ x-\frac{100\ x^2}{2}$$

$$=40\times 10^4\ x-50\ x^2\text{ Nmm}$$

$$=507{\cdot}92\times 10^6\text{ Nmm}$$

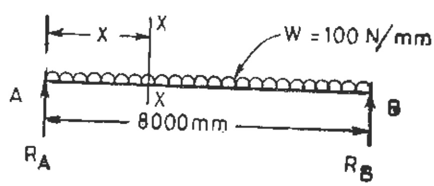


Fig. 8·52

or $$40\times 10^4\ x-50\ x^2=507{\cdot}92\times 10^6$$

or $$50\ x^2-40\times 10^4\ x+507{\cdot}92\times 10^6=0$$

or $$x^2-8000\ x+1015{\cdot}84\times 10^4=0$$

so $$x=\frac{8000\pm\sqrt{(8000)^2-4\times(1015{\cdot}84\times 10^4)}}{2}$$

$$=4000\pm\sqrt{(4000)^2-(1015{\cdot}84\times 10^4)}$$

$$=4000\pm 100\sqrt{1600-1015{\cdot}84}$$

$$=4000\pm 2417$$

$$=6417\quad\text{and}\quad 1583$$

This shows that upto a length of 1583 mm from both the sides, there is no necessity of additional flange plates because $M_x < 507.92 \times 10^6$ Nmm.

Therefore length of the additional flange plates

$$= 8000 - 1583 - 1583$$
$$= 4834 \text{ mm}$$

Problem 8·9. The section of a beam in shown in Fig. 8·53. X-X and YY are the axes of symmetry. Determine the ratio of the moment of resistance in the plane YY to that in the plane X-X, if the maximum stress due to bending is the same in both the cases.

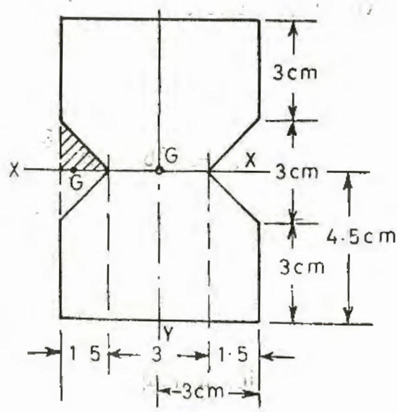


Fig. 8·53

Solution. The section is symmetrical about X-X and YY axis. Its CG lies at G as shown in Fig. 8·53.

Moment of inertia, $I_{xx} = \dfrac{6 \times 9^3}{12} - 4 \times$ (moment of inertia of triangle shown about its base)

$$= 364.5 - 4 \left(\frac{1.5 \times 1.5^3}{12} \right)$$

$$= 364.5 - 1.6875 = 362.8125 \text{ cm}^4$$

$$y_1 = y_2 = 4.5 \text{ cm}$$

Section modulus, $Z_x = \dfrac{362.8125}{4.5} = 80.625 \text{ cm}^3$

Moment of inertia, $I_y = \dfrac{9 \times 6^3}{12} - 2$ (moment of inertia of triangle 3×1.5 about YY)

$$= 169 - 2 \left[\frac{BH^3}{36} + \frac{BH}{2} (3 - 0.5)^2 \right]$$

where $B = 3$ cm

$H = 1.5$ cm

So $I_{yy} = 162 - 2 \left[\dfrac{3 \times 1.5^3}{36} + \dfrac{3 \times 1.5}{2} (2.5)^2 \right]$

$$= 162 - 2\,(0.28125 + 14.0625]$$

$$= 162 - 28.6875 = 133.3125 \text{ cm}^4$$

$$x_1 = x_2 = 3 \text{ cm}$$

Section modulus, $Z_y = \dfrac{I_{yy}}{3} = \dfrac{138.3125}{3} = 44.4375 \text{ cm}^3$

Moment of resistance,

$M_x = Z_x f$ where maximum stress due to bending is the same

Moment of resistance,

$$M_y = Z_y f$$

$$\frac{M_y}{M_x} = \frac{Z_y}{Z_x} = 0{\cdot}55$$

Problem 8·10. A beam of I section of moment of inertia 954 cm^4 and depth 14 cm is freely supported at its ends. Over what span can a uniform load of 500 kg/metre be carried if maximum stress is 60 N/mm^2.

What additional central load can be carried when maximum stress is 100 N/mm^2.

Solution. For a uniformly distributed load over a simply supported beam, the maximum bending moment occurs at the centre of the beam

$$M_{max} = \frac{wl^2}{8}$$

where w = rate of loading
$= 500$ kg/m
$= 5$ kg/cm $= 5 \times 9{\cdot}8$ N/cm

and l = length of the beam between the supports

Now $$M_{max} = f_{max} \times \frac{I}{y}$$

$$\frac{wl^2}{8} = f_{max} \times \frac{954 \text{ cm}^4}{7 \text{ cm}}$$

where y = half the depth of I section

and $f_{max} = 60$ $N/mm^2 = 6000$ N/cm^2

So $$5 \times 9{\cdot}8 \times \frac{l^2}{8} = \frac{6000 \times 954}{7}$$

$$l^2 = \frac{6000 \times 954}{7} \times \frac{8}{5 \times 9{\cdot}8} = 133504{\cdot}37 \text{ cm}^2$$

$$l = 365{\cdot}38 \text{ cm} = 3{\cdot}6538 \text{ metre}$$

Say the additional central load

$= W$

Additional maximum bending moment at the centre of the beam

$$\frac{Wl}{4} = M'_{max}$$

$$= \frac{W \times 365{\cdot}38}{4} \text{ Ncm}$$

Additional maximum stress due to central load

$$f_{max}' = 100 - 60 = 40 \text{ N/mm}^2 = 4000 \text{ N/cm}^2$$

So $$M_{max}' = f_{max} \times \frac{I}{y}$$

$$365{\cdot}38 \frac{W}{4} = 4000 \times \frac{954}{7}$$

Additional central load,

$$W=\frac{4\times4000\times954}{7\times365{\cdot}38}=5967{\cdot}95 \text{ N}$$

$$=5{\cdot}968 \text{ kN.}$$

Problem 8·11. A steel tube 4 cm outside diameter and 3 cm inside diameter safely carries a central load of 40 kg over a span of 6 metres.

Three of these tubes are firmly fixed together so that their centres make an equilateral triangle of the side 4 cm. Find the maximum central load which the beam can carry if the maximum stress is not to exceed to that of a single tube.

Solution.

Single tube. Moment of inertia,

$$I=\frac{\pi}{64}(4^3-3^3)=8{\cdot}590 \text{ cm}^4$$

Distance of extreme layer from the neutral layer = 2 cm

Section modulus, $=\frac{8{\cdot}59}{2}=4{\cdot}295 \text{ cm}^3$

l, span length $=6$ m

Central load, $W=40$ kg

Maximum BM, $M_{max}=\frac{WL}{4}=\frac{40\times600}{4}=600$ kg-cm

Safe stress developed in tube

$$=\frac{M_{max}}{Z}=\frac{600}{4{\cdot}295}=1396{\cdot}97 \text{ kg/cm}^2.$$

3 Tubes. The three tubes formly fixed together are shown in the Fig. 8·54. Their centres make an equilateral triangle of side 4 cm.

Vertical height $ad=ac \sin 60°$

$=4\times0{\cdot}866=3{\cdot}464$

C.G. will lie along ad but at a distance of $3{\cdot}464/3=1{\cdot}155$ cm from the base bc. So the neutral axis is parallel to bc and at a distance $1{\cdot}155$ from the base.

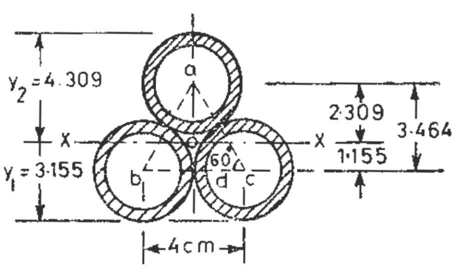


Fig. 8·54

Area of cross section of each tube

$$=\frac{\pi}{4}(4^2-3^2)=5{\cdot}4978 \text{ cm}^2$$

Moment of inertia, $I_{xx}=2\,[8{\cdot}590+5{\cdot}4978\,(y_1-2)^2]+8{\cdot}590+5{\cdot}4978\,(y_2-2)^2$

$=17{\cdot}18+14{\cdot}668+8{\cdot}590+29{\cdot}311=69{\cdot}749 \text{ cm}^4.$

Since $y_2>y_1$, maximum stress will occur at the upper edge

Section modulus, $Z_2 = \frac{I_{xx}}{y_2} = \frac{69{\cdot}749}{4{\cdot}309} = 16{\cdot}187 \text{ cm}^3$

Safe stress, $f = 1396{\cdot}97 \text{ kg/cm}^2$

Allowable BM $= f \cdot Z_2 = 16{\cdot}187 \times 1396{\cdot}97 = 22612{\cdot}5$ kg-cm

Span length, $l = 6 \text{ m} = 600$ cm

Say the central load $= W$ kg

$$M_{max} = \frac{WL}{4} = \frac{W \times 600}{4} = 150 \text{ kg-cm}$$

So $$150\,W = 22612{\cdot}5 \text{ kg-cm}$$

$$W = \frac{22612{\cdot}5}{150} = 150{\cdot}75 \text{ kg}$$

Problem 8·12. The 50×150 mm I section shown in Fig. 8·55 is simply supported at its ends over a span of 3·6 metres and carries a central load of 4 kN which acts through the centroid but inclined at an angle of 60° to the horizontal. Calculate the maximum stress.

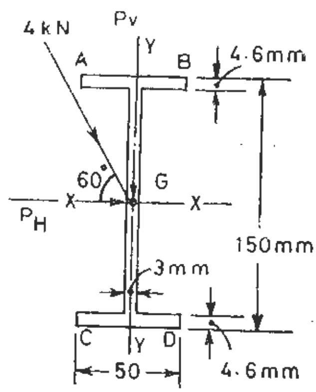


Fig. 8·55

Solution. Length of the beam,

$L = 3{\cdot}6$ m

Inclined load 4 kN can be resolved into two components

$$P_V = 4000 \cos 30° = 3464 \text{ N}$$

$$P_H = 4000 \cos 60° = 2000 \text{ N}$$

Moment of inertia, $I_{xx} = \frac{5 \times 15^3}{12} - \frac{4{\cdot}7 \times 14{\cdot}08^3}{12} = 313 \text{ cm}^4$ taking dimensions in cm

$$I_{yy} = 2 \times \frac{0{\cdot}46 \times 5^3}{12} + \frac{14{\cdot}08 \times 0{\cdot}3^3}{12} = 9{\cdot}55 \text{ cm}^4$$

Bending moment due to vertical component, P_V

$$M_x = \frac{P_V \times l}{4} = \frac{3464 \times 3{\cdot}60}{4} = 3117{\cdot}6 \text{ Nm}$$

Bending moment due to the horizontal component, P_H

$$M_y = \frac{P_H \cdot l}{4} = \frac{2000 \times 3{\cdot}6}{4} = 1800 \text{ Nm}$$

Stresses due to bending at extreme layers

Due to M_x, $$f_{AB} = -\frac{M_x}{I_x} 7{\cdot}5 = -\frac{3117{\cdot}6}{313} \times 7{\cdot}5 = -74{\cdot}7 \text{ N/cm}^2$$

$$f_{CD} = +\frac{M_x}{I_x} \times 7{\cdot}5 = +74{\cdot}7 \text{ N/cm}^2$$

Due to M_y $$f_{AC} = -\frac{M_y}{I_y} \times 2.5 = -\frac{1800 \times 2.5}{9.55} = -471.2 \text{ N/mm}^2$$

$$f_{BD} = +\frac{M_y}{I_y} \times 2.5 = +471.2 \text{ N/mm}^2$$

Maximum stress at A $= -74.7 - 471.2 = -545.9$ N/m²

Maximum stress at D $= +74.7 + 471.2 = 545.9$ N/m².

Problem 8.13. A vertical flag staff 10 m high is of circular section 200 mm diameter at the ground and 80 mm diameter at the top. A horizontal pull of 2 kN is applied at the top as shown in Fig. 8.56. Calculate the maximum stress due to bending.

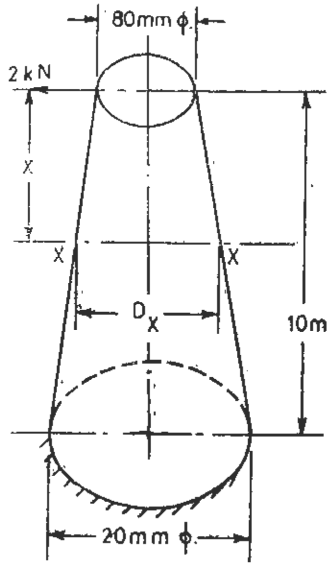


Fig. 8.56

Solution. Height of the flag staff

$= 10 \text{ m} = 10{,}000 \text{ mm}$

Diameter at top $= 80$ mm

Diameter at ground

$= 200$ mm

Consider a section at a distance of x mm from the top

Diameter, $$D_x = 80 + \frac{200-80}{10{,}000} x$$

$$= 80 + \frac{120}{10000} x$$

$$= (80 + 0.012x) \text{ mm}$$

Section modulus of the section

$$Z_x = \frac{\pi D_x^3}{32} = \frac{\pi}{32} (80 + 0.012\, x)^3$$

Bending moment at the section,

$$M_x = 2 \text{ kN} \times x = 2000\, x \text{ Nmm}$$

Stress in extreme layers at the cross-section considered

$$f = \frac{M_x}{Z_x} = \frac{2000\, x}{\frac{\pi}{32} (80 + 0.012\, x)^3}$$

$$= \frac{2000 \times 32}{\pi} x\, (80 + 0.012\, x)^{-3}$$

$$= kx\, (80 + 0.012x)^{-3}$$

where $$k = \frac{64000}{\pi} \text{ a constant}$$

For the stress to be maximum

$$\frac{df}{dx}=0$$

$$\frac{df}{dx}=k\ (80+0{\cdot}012x)^{-3}+kx\ (-3)\ (80+0{\cdot}012x)^{-4}\ (0{\cdot}012)=0$$

or $$k\ (80+0{\cdot}012x)^{-3}\ [1-3x\ (0{\cdot}012)\ (80+0{\cdot}012x)^{-1}]=0$$

or $$1-\frac{3x\times 0{\cdot}012}{(80+0{\cdot}012x)}=0$$

or $$80+0{\cdot}012x-0{\cdot}036x=0$$

$$80-{\cdot}024x=0$$

$$x=\frac{80}{0{\cdot}024}=3333{\cdot}33 \text{ mm}$$

$$x=3{\cdot}33 \text{ m}$$

Maximum stress due to bending,

$$f_{max}=\frac{64000}{\pi}\ (3333{\cdot}33)\ (80+0{\cdot}012\times 3333{\cdot}33)^{-3}$$

$$=\frac{20371{\cdot}8\ (3333{\cdot}33)}{(120)^3}=39{\cdot}29 \text{ N/mm}^2$$

Problem 8·14. The original dimensions of a tie bar of rectangular section are 80 mm × 30 mm. The dimensions are reduced by $\frac{1}{k}$ th of their original values by removing the material from two adjacent faces. If an axial load of 120 kN is applied through the centre of the original section, find the value of $\frac{1}{k}$ for a maximum tensile stress of 100 N/mm². Determine also the magnitude of the least stress.

Solution. The Fig. 8·57 shows the C.G. of the final section of the tie bar, but the tensile force is acting at the P *i.e.* at the C.G. of the original section of the tie bar.

eccentricity,

$$e_x=40-\frac{1}{2}\left(80-\frac{80}{k}\right)$$

$$=40-40+\frac{40}{k}=\frac{40}{k}\text{ mm}$$

eccentricity,

$$e_y=15-\frac{1}{2}\left(30-\frac{30}{k}\right)$$

$$=\frac{15}{k}\text{ mm}$$

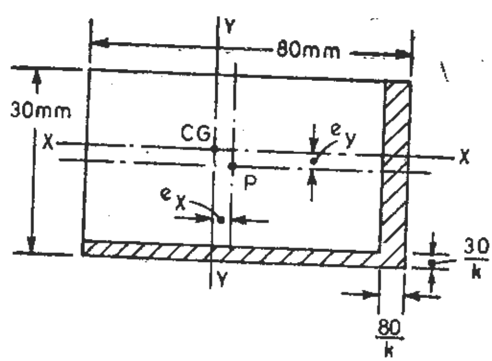


Fig. 8·57

Moment of inertia, $$I_{xx}=\frac{\left(80-\frac{80}{k}\right)\left(30-\frac{30}{k}\right)^3}{12}$$

$$=\frac{80\times30^3}{12}\times\left(1-\frac{1}{k}\right)^4$$

$$I_{yy}=\frac{\left(30-\frac{30}{k}\right)\left(80-\frac{80}{k}\right)^3}{12}$$

$$=\frac{30\times80^3}{12}\times\left(1-\frac{1}{k}\right)^4$$

Section modulus, $$Z_x=\frac{I_{xx}}{\left(15-\frac{15}{k}\right)}$$

$$=\frac{80\times30^3}{12}\times\frac{\left(1-\frac{1}{k}\right)^4}{15\left(1-\frac{1}{k}\right)}$$

$$=12000\left(1-\frac{1}{k}\right)^3$$

Section modulus, $$Z_y=\frac{I_{yy}}{\left(40-\frac{40}{k}\right)}$$

$$=\frac{30\times80^3}{12}\frac{\left(1-\frac{1}{k}\right)^4}{40\left(1-\frac{1}{k}\right)}$$

$$=32000\left(1-\frac{1}{k}\right)^3$$

Bending moment about plane $X-X$,

$$M_x=F\,.\,e_y$$

where F=tensile force

$$=120\text{ kN}$$

So $$M_x=120\times\frac{15}{k}=\frac{1800}{k}\text{ kNmm.}$$

Bending moment about plane Y-Y,

$$M_y=F\,.\,e_y$$

$$=120\times\frac{40}{k}$$

$$=\frac{4800}{k}\text{kNmm}$$

Bending stress due to M_x,

$$f_{bx}=\frac{M_x}{Z_x}$$

$$=\frac{1800}{k}\times\frac{1}{12000\left(1-\frac{1}{k}\right)^3}\ \text{kN/mm}^2$$

$$=\frac{0\cdot15}{k\left(1-\frac{1}{k}\right)^3}\ \text{kN/mm}^2$$

$$=\frac{150}{k\left(1-\frac{1}{k}\right)^3}\ \text{N/mm}^2$$

Bending stress due to M_y,

$$f_{by}=\frac{M_y}{Z_y}$$

$$=\frac{4800}{k}\times\frac{1}{32000\left(1-\frac{1}{k}\right)^3}\ \text{kN/mm}^2$$

$$=\frac{0\cdot15}{k\left(1-\frac{1}{k}\right)^3}\ \text{kN/mm}^2$$

$$=\frac{150}{k\left(1-\frac{1}{k}\right)^3}\ \text{N/mm}^3$$

Tensile stress due to direct force

$$=\frac{F}{80\left(1-\frac{1}{k}\right)30\left(1-\frac{1}{k}\right)}$$

$$=\frac{120}{80\times30\left(1-\frac{1}{k}\right)^2}\ \text{kN/mm}^2$$

$$=\frac{0\cdot05}{\left(1-\frac{1}{k}\right)^2}\ \text{kN/mm}^2$$

$$=\frac{50}{\left(1-\frac{1}{k}\right)^2}\ \text{N/mm}^2$$

Now maximum tensile stress

$$=\frac{50}{\left(1-\frac{1}{k}\right)^2}+\frac{150}{k\left(1-\frac{1}{k}\right)^3}+\frac{150}{k\left(1-\frac{1}{k}\right)^3}$$

$$=100\ \text{N/mm}^3$$

or
$$2=\frac{1}{\left(1-\frac{1}{k}\right)^2}+\frac{3}{k\left(1-\frac{1}{k}\right)^3}+\frac{3}{k\left(1-\frac{1}{k}\right)^3}$$

or
$$k\left(1-\frac{1}{k}\right)+3+3=2k\left(1-\frac{1}{k}\right)^3$$
$$k\left(1-\frac{1}{k}\right)+6=2k\left(1-\frac{1}{k}\right)^3$$
$$k-1+6=\frac{2k}{k^3}(k-1)^3$$

or
$$k^3+5k^2=2k^3-6k^2+6k-2$$
$$k^3-11k^2+6k-2=0$$

Solving the equation, $k \simeq 10{\cdot}4$

or
$$\frac{1}{k}=0{\cdot}096$$

Now $k\left(1-\frac{1}{k}\right)^3=10{\cdot}4\left(1-\frac{1}{10{\cdot}4}\right)^3=7{\cdot}679$
$$\left(1-\frac{1}{k}\right)^2=0{\cdot}817$$

Least stress
$$=\frac{50}{\left(1-\frac{1}{k}\right)^2}-\frac{150}{k\left(1-\frac{1}{k}\right)^3}-\frac{150}{k\left(1-\frac{1}{k}\right)^3}$$
$$=\frac{50}{0{\cdot}817}-\frac{150}{7{\cdot}679}-\frac{150}{7{\cdot}679}$$
$$=61{\cdot}199-19{\cdot}53-19{\cdot}53=22{\cdot}139 \text{ N/mm}^2.$$

Problem 8·15. A bimetallic strip is formed by using strips of copper and steel. Each strip is 60 mm wide and 12 mm thick. Both the strips are fastened together so that no relative movement can take place between them. This bimetallic strip is now heated through 100°C. Assuming that both the strips bend by the same radius and stresses are transmitted only through end connections, find radius of the bend, maximum tensile and compressive stresses in both.

Given : $E_s=2\times10^6$ kg/cm² $\quad E_c=1\times10^6$ kg/cm²

$\alpha_s=11\times10^{-6}/°C \quad \alpha_c=18\times10^{-6}/°C$

Solution. Thickness of each strip, $t=12$ mm

Temperature rise, $T=100$ °C

Radius of bend,
$$R=\frac{E_s^2+E_c^2+14\,E_s\,E_c}{12\,E_s\,E_c\,(\alpha_c-\alpha_s)}\times\frac{t}{T}$$

where
$$\frac{E_s}{E_c}=m=2$$

Then
$$R=\frac{m+\frac{1}{m}+14}{12(\alpha_c-\alpha_s)}\times\frac{t}{T}$$

$$=\frac{16{\cdot}5}{12\times7\times10^{-6}}\times\frac{12}{100}$$

$$=23571 \text{ mm}$$

$$=23{\cdot}571 \text{ m.}$$

There will be tensile stress developed in steel and compressive stress developed in copper because $\alpha_C>\alpha_S$.

Compressive force in copper

$$=\text{Tensile force in steel (due to temperature rise)}$$

$$P_c=\frac{bt^2}{12R}\ (E_s+E_c)$$

Direct compressive stress in copper

$$=\frac{P_c}{bt}=\frac{t}{12R}\ (E_s+E_c)$$

$$=\frac{12}{12\times23571}\times(E_s+E_c)$$

$E_s=2\times10^4$ kg/mm² $\qquad E_c=1\times10^4$ kg/mm²

$E_s+E_c=3\times10^4$ kg/mm²

Direct stress in copper $=\dfrac{12}{12\times23571}\times3\times10^4=1{\cdot}27$ kg/mm² (Compressive)

Therefore direct tensile stress in steel $=1{\cdot}27$ kg/mm²

Stresses due to bending. Bending moment shared by copper

$$M_c=\frac{E_c}{R}\times I_c=f_c\times\frac{bt^2}{6}$$

Stress due to bending, $f_c=\pm\dfrac{E_c}{R}\times\dfrac{bt^3}{12}\times\dfrac{6}{bt^2}$

$$=\pm\frac{E_ct}{2R}=\pm\frac{1\times10^4\times12}{2\times23571}$$

$$=\pm2{\cdot}54 \text{ kg/mm}^2$$

Maximum stress developed in copper strip

$=2{\cdot}54+1{\cdot}27=3{\cdot}81$ kg/mm² (compressive)

Minimum stress developed in copper strip

$=2{\cdot}54-1{\cdot}27=1{\cdot}27$ kg/mm² (tensile)

Similarly bending stress in steel, f_s

$$=\pm\frac{E_s\,t}{2R}=\frac{2\times10^4\times12}{2\times23571}$$

$$=\pm5{\cdot}08 \text{ kg/mm}^2$$

Maximum stress in steel strip

$=5{\cdot}08+1{\cdot}27=6{\cdot}35$ kg/mm² (tensile)

Minimum stress in steel strip

$=5{\cdot}08-1{\cdot}27=3{\cdot}81$ kg/mm² (compressive)

Problem 8·16. A steel bar 12 cm in diameter in completely encased in an aluminium tube of 18 cm outer diameter and 12 cm inner diameter so as to make a composite beam. The composite beam is subjected to a bending moment of 1·2 Tm. Determine the maximum stress due to bending in each material.

$$E_{steel}=3\ E_{aluminium}.$$

Solution. The composite section is shown in Fig. 8·58. The neutral axis will pass through the centre of the section as shown. Strain in any layer is proportional to its distance from the neutral layer.

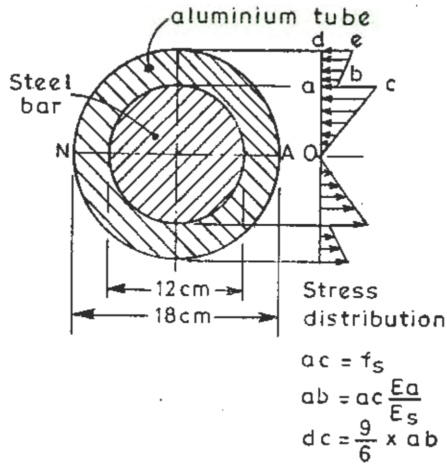


Fig. 8·58

Say maximum stress developed in steel$=f_s$

At a distance of 6 cm, from N.A., strain in steel$=\frac{f_s}{E_s}$

Strain in aluminium layer at a distance of 6 cm from N.A.$=\frac{f_s}{E_s}$

Strain in aluminium layer at a distance of 9 cm from N.A.$=\frac{f_s}{E_s}\times\frac{9}{6}=\frac{1·5 f_s}{E_s}$

Stress in aluminium layer at a distance of 9 cm N.A. (or the maximum stress),

$$f_a=\frac{1·5 f_s}{E_s}\times E_a$$

$$f_a=\frac{1·5 f_s}{3}=0·5 f_s \qquad \text{[putting } E_s=3\ E_a\text{]}$$

Bending moment, $M=M_s+M_a$

= Moment of resistance of steel bar

+ moment of resistance of aluminium tube.

Z_s, section modulus of steel section

$$=\frac{\pi\times 12^3}{32}=169·64 \text{ cm}^3$$

Z_a, section modulus of aluminium section

$$=\frac{\pi(18^4-12^4)}{32\times 18}=428·82 \text{ cm}^3$$

$$M_s=f_s\ .\ Z_s=f_s\times 169·64$$

$$M_a=f_a\ .\ Z_a=0·5\ f_s\times Z_a$$

$$=0·5\ f_s\times 428·82=214·41\ f_s$$

$$M=169·64\ f_s+214·41\ f_s=384.05\ f_s=1·2 \text{ Tm}$$

$$384{\cdot}05\, f_s = 1{\cdot}2 \times 10^5 \text{ kg-cm}$$
$$f_s = 312{\cdot}46 \text{ kg/cm}^2$$
$$f_a = 0{\cdot}5\ f_s = 156{\cdot}23 \text{ kg/cm}^2$$

Maximum stress in steel $= 312{\cdot}46$ kg/cm²

Maximum stress in aluminium

$$= 156{\cdot}23 \text{ kg/cm}^2.$$

Problem 8·17. A timber beam of breadth B and depth 32 cm is simply supported over a span of 8 metres. This beam is to be strengthened by the addition of steel flitches fixed on both the sides as shown in Fig. 8·59. With the original timber beam a load of 200 kg/m gave a maximum stress of 50 kg/cm². If the flitched beam is loaded with an extra load of 120 kg/m with the maximum stress in the steel of 600 kg/cm², the stress in timber remaining the same, determine the dimensions of timber beam and steel flitches.

Given : $\dfrac{E_s}{E_t} = 20$

Solution.

(a) Timber beam without steel flitches

Section modulus,

$$Z = \frac{B(32)^2}{6}$$

w = rate of loading
= 200 kg/m
= 2 kg/cm

l = length of the beam
= 8 m
= 800 cm

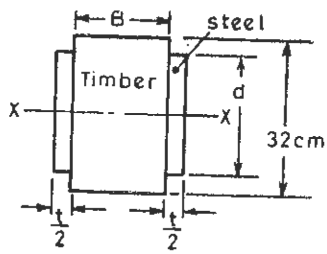


Fig. 8·59

Maximum bending moment at the centre of beam

$$M_{max} = \frac{wl^2}{8}$$

$$= \frac{2 \times 800 \times 800}{8} = 16 \times 10^4 \text{ kg-cm}$$

Now $\qquad M_{max} = f \times Z$

$$16 \times 10^4 = \frac{50 \times B \times 32^2}{6}$$

$$B = \frac{16 \times 10^4 \times 6}{50 \times 32^2} = 18{\cdot}75 \text{ cm}$$

Now considering steel flitches fixed on timber beam

M'_{max} = maximum bending moment at the centre of the beam considering additional load

Additional load $\qquad = 120$ kg/m $= 1{\cdot}2$ kg/cm

Total load $=2+1{\cdot}2=3{\cdot}2$ kg/cm

$$M_{max}'=\frac{3{\cdot}2\times 800^2}{8}=25{\cdot}6\times 10^4 \text{ kg-cm}$$

$$y_t=16 \text{ cm}$$

$$y_s=\frac{d}{2}$$

Stress in steel, $f_s=600$ kg/cm²

Stress in timber $f_t=50$ kg/cm²

Now $$\frac{M_{max}'}{I_t}=\frac{f_t}{y_t}=\frac{E_t}{R} \quad ...(1)$$

$$\frac{M_{max}'}{I_s}=\frac{f_s}{y_s}=\frac{E_s}{R} \quad ...(2)$$

or $$\frac{f_s}{y_s E_s}=\frac{f_t}{y_t E_t}$$

$$y_s=\frac{E_t}{E_s}\times y_t\times\frac{f_s}{f_t}=\frac{1}{20}\times 16\times\frac{600}{50}$$

$$y_s=\frac{1}{20}\times 16\times 12=9{\cdot}6 \text{ cm}$$

or depth of steel flitches $=2\,y_s=19{\cdot}2$ cm

Now I_t = Moment of inertia considering equivalent section in timber

$$=\frac{18{\cdot}75\times 32^3}{12}+\frac{t\times 19{\cdot}2^3}{12}\times\frac{E_s}{E_t}$$

where $\frac{t}{2}$ is thickness of each steel flitch

$$I_t=51200+11796{\cdot}48t \text{ cm}^4$$

From equation (1) above

$$I_t=M_{max}'\ \frac{y_t}{f_t}$$

$$51200+11796{\cdot}48\ t=25{\cdot}6\times 10^4\times\frac{16}{50}=81920$$

or $$11796{\cdot}48\ t=30720$$

$$t=2{\cdot}60 \text{ cm}$$

So thickness of each steel flitch $=1{\cdot}3$ cm

Depth of each steel flitch $=19{\cdot}2$ cm

Width of timber beam $=18{\cdot}75$ cm.

Problem 8·18. A composite beam consists of two wooden beams of breadth B and depth D each and a steel plate of width b and depth d sandwiched symmetrically between them. The allowable stress in steel is 160 N/mm² and in wood it is 10 N/mm². Determine (*i*) ratio of D/d if maximum stresses in steel and wood reach simultaneously (*ii*) ratio of B/b, if the moment of resistance of one wooden beam is equal to that of steel plate.

Given $E_{steel}=200\times 10^3$ N/mm²

$E_{wood}=10\times 10^3$ N/mm².

Solution. Maximum skin stress in wood,

$$f_w = 10 \text{ N/mm}^2$$

Modular ratio, $$m = \frac{E_s}{E_w} = \frac{200 \times 10^3}{10 \times 10^3} = 20$$

Distance of extreme layer of wood from neutral axis

$$= \frac{D}{2}$$

At a distance of $D/2$ from neutral axis,

Stress in steel, $f_s' = m\ f_w = 20 \times 10 = 200 \text{ N/mm}^2$

But the allowable stress in steel

$$f_s = 160 \text{ N/mm}^2$$

Therefore the depth of the steel plate has to be less than D.

Stress in a layer is proportional to its distance from the neutral layer.

So the value of $d/2$ for a stress of 160 N/mm²

$$= \frac{D}{2} \times \frac{160}{200} = 0{\cdot}4\ D$$

or $$d = 0{\cdot}8\ D$$

$$\frac{D}{d} = 1{\cdot}25$$

(*ii*) Moment of resistance of one wooden beam = moment of resistance of steel plate
is beam

$$\frac{BD^2}{6} \times f_w = \frac{bd^2}{6} \times f_s$$

$$BD^2 \times 10 = bd^2 \times 160$$

$$\frac{b}{B} = \frac{D^2}{16\,d^2} = \frac{D^2}{16 \times (0{\cdot}8\ D)^2} = \frac{D^2}{16 \times 0{\cdot}64\ D^2}$$

or $$\frac{B}{b} = 10{\cdot}24.$$

Problem 8·19. The reinforced concrete beam of T-section shown in the Fig. 8·60 as maximum stresses of 5/Nmm² in concrete and 100 N/mm² in steel. The modular ratio of eel and concrete is 16. Assume that the neutral axis lies within the full width of the section, nd the area of steel reinforcement and the moment of resistance.

Solution. Say the distance of the eutral axis from the compression face is H.

Breadth of concrete section in compression

$$= 600 \text{ mm}$$

Depth of steel reinforcement from the mpression face,

$$D = 300 \text{ mm}$$

Maximum stress in steel,

$$f_s = 100 \text{ N/mm}^2$$

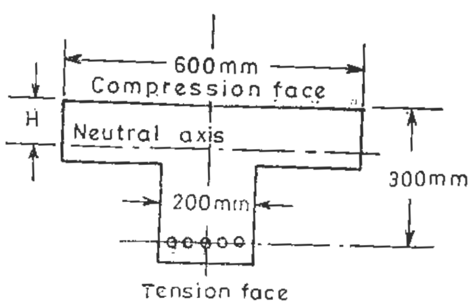


Fig. 8·60

Maximum stress in concrete,

$$f_c = 5 \text{ N/mm}^2$$

Modular ratio, $m = \dfrac{E_s}{E_c} = 16$

We know that $\dfrac{f_s}{f_c} = m\left(\dfrac{D-H}{H}\right)$ ...(1)

$$\frac{100}{5} = 16\left(\frac{300-H}{H}\right)$$

or $1{\cdot}25\,H = 300 - H$

$$H = \frac{300}{2{\cdot}25} = 133{\cdot}33 \text{ mm}$$

Now $\dfrac{f_c}{2}\cdot BH = f_s \cdot A_s$ ...(2)

$$\frac{5}{2}\times 600 \times 133{\cdot}33 = 100 \times A_s$$

Area of steel reinforcement,

$$A_s = 15 \times 133{\cdot}33 = 2000 \text{ mm}^2 = 20 \text{ cm}^2$$

Moment of resistanee, $M = f_s \cdot A_s\left(D - \dfrac{H}{3}\right) = 100 \times 2000\left(300 - \dfrac{133{\cdot}33}{3}\right)$

$$= 2\times 10^5 \times 255{\cdot}55 = 511{\cdot}1 \times 10^5 \text{ Nmm} = 51{\cdot}1 \text{ kNm}$$

SUMMARY

1. Flexure formula is,

$$\frac{M}{I_{xx}} = \frac{f}{y} = \frac{E}{R}$$

where M = bending moment at a particular section

I_{xx} = moment of inertia of the section

f = stress due to bending in a layer at a distance y from th neutral layer

E = young's modulus of elasticity

R = radius of curvature at the seetion.

2. Maximum stress in compression,

$$f_c = \frac{M}{Z_c}$$

where Z_c = section modulus in compression

Maximum stress in tension,

$$f_t = \frac{M}{Z_t}$$

where Z_t=section modulus in tension

$$Z_c=\frac{I_{xx}}{y_c}, \quad Z_t=\frac{I_{xx}}{y_t}$$

y_c=distance of extreme layer in compression from neutral axis

y_t=distance of extreme layer in tension from neutral axis.

3. Rectangular section,

$$Z_c=Z_t=\frac{BD^2}{6}$$

4. Circular section, $Z_c=Z_t=\frac{\pi d^3}{32}$

5. I section, $Z_c=Z_t=\frac{I_{xx}}{D/2}$

where D=depth of I section.

6. Modulus of rupture $=\frac{6M_{ult}}{bd^2}$

where M_{ult}=ultimate bending moment on the beam

b, d=breadth, depth of a rectangular section.

7. In beams, cantilevers of uniform strength, M/Z *i.e.*, ratio of bending moment and section modulus is kept constant throughout the length of beam/cantilever.

8. Bimettallic strip,

Radius of the bend, $R=\frac{E_1^2+E_2^2+14\,E_1\,E_2}{12\,E_1\,E_2\,(\alpha_2-\alpha_1)}\cdot\frac{t}{T}$

where t=thickness of each strip

T=temperature change

E_1, E_2=Young's modulus of strips 1 and 2 respectively

α_2, α_2=coefficient of linear expansion of materials 1 and 2.

9. In a flitched beam

Moment of resistance=Resisting moment offered by beam of material (1)+resisting moment offered by beams of strengthening material (2).

Equivalent section of the beam is made by considering the modular ratio E_2/E_1.

10. In reinforced cement concrete beam

$$\frac{f_s}{f_c}=m\left(\frac{D-H}{H}\right) \quad ...(1)$$

$$P=f_c\,.\,A_c=f_s\,A_s$$

$$\frac{f_c}{2}\,(BH)=f_s\,A_s \quad ...(2)$$

$$M=P\left(D-\frac{H}{3}\right)=\frac{f_c}{2}\,BH\left(D-\frac{H}{3}\right)=f_s\,A_s\left(D-\frac{H}{3}\right)$$

where f_s=maximum stress in steel

f_c=maximum stress in concrete

$m=E_s/E_c$

D=depth of steel reinforcement from compression face
H=distance of neutral axis from compression face
B=breadth of concrete section
A_s=area of steel reinforcement.

MULTIPLE CHOICE QUESTIONS

1. A beam of rectangular section of breadth 10 cm and depth 20 cm is subjected to a bending moment of 2 tonne-metres. The stress developed at a distance of 10 cm from the top face is

(a) 1200 kg/cm² (b) 600 kg/cm²
(c) 300 kg/cm² (d) 0·0.

2. A beam of T section is subjected to a bending moment M. The depth of the section is 12 cm. The moment of inertia of the section about plane of bending is 1200 cm⁴. The flange of the section is in compression. If the maximum tensile stress is two times the maximum compressive stress, then the value of section modulus in compression for the section is

(a) 300 cm³ (b) 200 cm³
(c) 150 cm³ (d) 100 cm³.

3. A mild steel beam is subjected to bending moment, a stress of 1 tonne/cm² is developed in a layer at a distance of 10 cm from the neutral layer. If E=2000 tonnes/cm², the radius of curvature is

(a) 400 m (b) 200 m
(c) 100 m (d) 50 m.

4. A beam of I section of depth 20 cm is subjected to a bending moment M. The flange thickness is 1 cm. If the maximum stress developed in I section is 100 N/mm², the stress developed at the inner edge of the flange is

(a) 95 N/mm² (b) 90 N/mm²
(c) 47·5 N/mm² (d) 45·0 N/mm².

5. A beam of rectangular section (breadth b, depth d) is tested under bending and M_{ult} is the ultimate bending moment recorded. The modulus of rupture of the beam is given by

(a) $6\ M_{ult}/bd^3$ (b) $6\ M_{ult}/bd^2$
(c) $12\ M_{ult}/bd^2$ (d) $12\ M_{ult}/bd^3$.

6. A beam of square section (with sides of the square horizontal and vertical) is subjected to a bending moment M and the maximum stress developed is 100 N/mm². If the diagonals of the section take vertical and horizontal directions, bending moment remaining the same, the maximum stress developed will now be

(a) $100\sqrt{2}$ N/mm² (b) $\frac{100}{\sqrt{2}}$ N/mm²
(c) 50 N/mm² (d) None of the above.

7. A cantilever of uniform strength f having rectangular section of constant depth d but variable breadth b, is subjected to a point load W at its free end. If the length of the cantilever is l, the breadth of the cantilever at the middle of its length is

(a) $\frac{6\,wl}{fd^2}$ (b) $\frac{3\,wl}{fd^2}$

(c) $\frac{2\,wl}{fd^2}$ (d) $\frac{wl}{fd^2}$

8. A cantilever of uniform strength f, having rectangular section of constant breadth b but variable depth d is subjected to a uniformly distributed load throughout its length. If the depth of the section at the fixed end is 16 cm, the depth at the middle of the length is

(a) 2 cm (b) 4 cm

(c) 8 cm (d) 12 cm.

9. A steel plate of breadth 1 cm and depth 15 cm is sandwiched between two wooden beams of breadth 10 cm and depth 20 cm each. The composite beam is subjected to a bending moment such that the maximum stress developed in steel plate is 1500 kg/cm². If $\frac{E_s}{E_w}=10$, the maximum stress developed in wooden beam is

(a) 50 kg/cm² (b) 100 kg/cm²

(c) 150 kg/cm² (d) 200 kg/cm².

10. In an R.C.C. beam, the depth of the steel reinforcement from compression face is 30 cm. The modular ratio $\frac{E_s}{E_c}=15$ and the ratio of maximum stress developed in steel and the maximum stress developed in concrete is also 15. The distance of the neutral axis from the compression face is

(a) 20 cm (b) 18 cm

(c) 15 cm (d) 12 cm.

ANSWERS

1. (d)	2. (a)	3. (b)	4. (b)	5. (b)
6. (a)	7. (b)	8. (c)	9. (d)	10. (c)

EXERCISES

8·1. A wooden joist of span 6 m is to carry a brick wall 23 cm thick and 3 m high. The depth of the joist is 2·5 times its breadth and the maximum permissible stress is limited to 70 kg/cm². Find the dimensions of the joist. Density of brick wall = 1800 kg/m³.
[**Ans.** $B=19\text{·}72$ cm, $D=49\text{·}30$ cm]

8·2. A floor has to carry a load of 8 kN/m² (including its own weight). If the span of each beam is 6 m. Calculate the spacing centre to centre between the joists. The breadth of each joist is 12 cm and depth is 30 cm and permissible stress due to bending is 5 N/mm².
[**Ans.** 25 cm]

8·3. A beam subjected to bending moment M is of T section, having flange 10 cm $\times$ $2t$ and web thickness t. The overall depth of T section is 20 cm. Determine the thicknesses of the flange and the web, if the maximum tensile stress is double the maximum compressive stress. (The flange being in compression).
[**Ans.** 4·648 cm, 2·324 cm]

8·4. A girder of T section of depth 25 cm is used as a cantilever with uniformly distributed load w throughout its length. The width of the flange is b and thickness 5 cm, while web is 20 cm×5 cm. The flange comes under tension. The material of cantilever can be subjected to 80 N/mm² in compression and 20 N/mm² in tension. It is desired to achieve a balanced design so that the maximum permissible bending stresses are reached simultaneously. Determine the width of the flange. Find the intensity of the load w on the cantilever if it is 2 m long. [**Ans.** 80 cm, 33·333 kN/m]

8·5. A cantilever has a free length of 2·4 m. It is of T section with the flange 12×2 cm and the web 24×1 cm. The flange being in tension. What load can be applied at the end of the cantilever if the maximum permissible stress in compression is 500 kg/cm². What is the maximum stress in tension ? [**Ans.** 359 kg, 202·7 kg/cm²]

8·6. A compound beam for a crane runway is built up of a 250×125 mm rolled steel joist with a 150×75 mm rolled steel channel attached to the top flange. Calculate the position of the neutral axis of the section and determine moment of inertia I_{xx}. For I section ; area=35·53 cm². $I_{x'x'}$=3717·8 cm⁴, $I_{y'y'}$=193·4 cm⁴. For channel section ; area=18·39 cm², web thickness=4·8 mm, $I_{x'x'}$=698·5 cm⁴, $I_{y'y'}$=103·1 cm⁴, distance of C.G. of channel section from outer edge of web=2·39 cm. [The over all depth of compound section is 254·8 mm]
[**Ans.** Neutral axis lies at a distance of 16·11 cm from lower edge ; 5179·9 cm⁴]

8·7. A 200×60 mm I section is strengthened by joining a plate 60 mm×10 mm at the bottom flange only. The compound section is used as a beam of span 4 m carrying a central load W. What is the maximum value of W if the stress due to bending in the section is not to exceed 80 N/mm². Properties of I section are. Area=12·64 cm², $I_{x'x'}$=780·7 cm⁴ $I_{y'y'}$=17·3 cm⁴. [**Ans.** 7·35 kN]

8·8. An I section, is to the used as a cantilever 2 m long. In I section, flanges are 100 mm×20 mm and the web is 210 mm×10 mm. If the permissible stress is 80 N/mm², What concentrated load can be carried at the end of the cantilever. If the cantilever is to be strengthened by steel plates 20 mm thick, welded on the top and bottom flanges, find the width of the plates required to withstand an increase of 40% in the load and the length over which the plate should extend, the maximum permissible stress remaining the same.
[**Ans.** 19·44 kN ; 52 mm ; 572 mm length]

8·9. The section of a beam is shown in the Fig. 8·61. X-X and Y-Y are the axes of symmetry. Determine the ratio of its moment of resistance in the plane YY to that in the plane XX for bending, if the maximum stress due to bending is the same in both the cases. [**Ans.** 0·366]

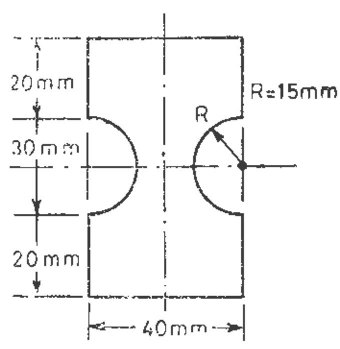


Fig. 8·61

8·10. A beam of I section of moment of inertia 1125 cm⁴ and depth 16 cm is freely supported at its ends. It carries a central load of 2 tonnes. Over what span can the beam be carried so that the maximum stress does not increase beyond 800 kg/cm².

(*b*) If the allowable stress is increased to 1200 kg/cm², what load uniformly distributed throughout its length can be applied on the beam. [**Ans.** 2·25 m, 888·8 kg/m run]

8·11. A steel tube 35 mm outside diameter and 30 mm inside diameter safely carries a load of 300 N over a span of 4 metres.

Four of these tubes are firmly fixed together with their centres forming a square of side 35 mm. Find the maximum central load which the beam can carry if the maximum stress is not to exceed to that in the single tube. [**Ans.** 1·98 kN]

8·12. An ISMB 250 I section is supported as a beam over a span of 2 metres, *A* load *W* acts at an angle of 45° to the vertical axis at the middle section of the beam. Determine *W* if the maximum stress in the section is not to exceed 560 kg/cm².

Specifications of the section are :

Depth=250 mm, Width=125 mm

Flange thickness=12·5 mm, Web thickness=6·9 mm

I_{xx}=5131·6 cm⁴, I_{yy}=334·5 cm⁴ [**Ans.** 750·7 kg]

8·13. A vertical flag staff 10 m high is of square section 160 mm×160 mm at the ground, uniformly tapering to 80 mm×80 mm at the top. A horizontal pull of 200 kg is applied at the top in the direction of a diagonal of the section. Calculate the maximum stress due to bending [**Ans.** 490·9 kg/cm²]

8·14. A bimetallic strip is formed by using strips of brass and steel, each of width 75 mm and thickness 20 mm. Both the strips are fastened together so that no relative movement can take place between them. The bimetallic strip is now heated through 120°C. Assuming that both the strips bend by the same radius and stresses are transmitted through end connections, find the radius of the bend and the maximum and minimum stresses in both the strips,

$E_s=2\times10^5$ N/mm², $E_b=0{\cdot}9\times10^5$ N/mm²

$\alpha_s=11\times10^{-6}$/°C, $\alpha_b=19\times10^{-6}$/°C

[**Ans.** 28·559 m ; +86·95, −53·11 N/mm² (in steel)
−51·93, +18·09 N/mm² (in brass)]

8·15. A steel bar 5 cm in diameter is completely encased in a brass tube of 10 cm diameter, so as to form a composite beam. This composite beam is subjected to a bending moment of 2 kNm. Determine the maximum bending stress in each material.

$$E_s=2E_b=210\times10^3 \text{ N/mm}^2$$

[**Ans.** $f_s=\pm19{\cdot}09$ N/mm², $f_b=\pm19{\cdot}09$ N/mm²]

8·16. A timber beam of breadth *B* and depth 30 cm is simply supported over a span of 6 m. The beam is to be strengthened by the addition of steel flitches fixed on both the sides as shown in Fig. 8·62. With the timber beam alone a load of 10 kN gave a maximum stress of 5 N/mm². If the flitched beam is loaded with an extra load of 5kN, with the maximum stress in steel of 60 N/mm², stress in the timber remaining the same, determine the dimensions of the timber beam and steel fitches.

$$\frac{E_s}{E_t}=15$$

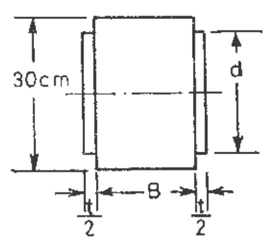


Fig. 8·62

[**Ans.** B=20 cm, d=24 cm, t=13·02 mm]

8·17. A composite beam consists of two wooden beams of breadth 10 cm and depth 30 cm each and a steel plate of width *b* and depth *d* is sandwiched between them. The allowable stress in steel is 1500 kg/cm² and in wood it is 90 kg/cm². Determine the dimensions of the steel plate if (*a*) maximum stresses in steel and wood reach simultaneously (*b*) moment of resistance of one wooden beam is equal to the moment of resistance of steel plate.

$$E_s = 2100 \text{ tonnes/cm}^2, \qquad \frac{E_s}{E_t} = 21$$

[**Ans.** $d=23\ 81$ cm, $b=9$·: mm]

8·18. The reinforced concrete beam of T section shown in Fig. 8·63 has maximum stress of 70 kg/cm² in concrete and 1540 kg/cm² in steel. The modular ratio of steel and concrete is 15. Assuming that the neutral axis lies within the full width of the section find : (*i*) distance of the neutral axis from the top face (*ii*) area of steel reinforcement (*iii*) moment of resistance.

[**Ans.** 10·135 cm, 11·52 cm², 3·835 Tm]

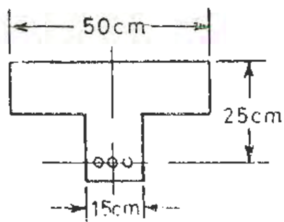


Fig. 8·63

9

Combined Bending and Direct Stresses

In the last chapter, we have studied about the variation of direct stresses due to bending moments along the depth of the beams and cantilevers of various sections. We have learnt that bending stress varies from maximum tensile stress in an extreme layer to the maximum compressive stress in other extreme layer on the other side of the neutral axis. Now, if in addition to the bending moment, the beam is subjected to axial pull or axial thrust, the direct stress due to pull or thrust will be superimposed on the bending stresses and for a certain value of thrust and bending moment, the section may have only one type of stress *i.e.*, either tensile stress or the compressive stress throughout the section. When a column carries a vertical load at a point not on its C.G. but away from C.G., the column will be subjected to a combination of a bending moment and a thrust.

9·1. BENDING MOMENT AND AXIAL THRUST

Fig. 9·1 shows a short column of rectangular cross section of breadth B and depth D. G is the centroid of the section *abcd* *i.e.*, the top edge of the column. A vertical load P acts at point G' along the X-X axis passing through the centroid. If the load acts on the C.G. of the section, there will be only direct compressive stress. But now the position of the application of the load is G', at a distance of e (eccentricity) from the centroid G. The effect of this will be to bend the column and as a result of bending, the edge *bc* will experience maximum compressive strain or stress and the edge *ad* will experience the maximum tensile strain or stress.

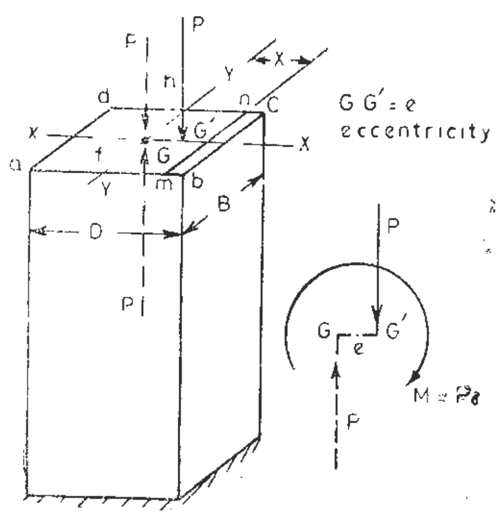


Fig. 9·1

Let us apply an equal and opposite vertical load of magnitude P at the centroid G of the section. A load P at G' can be replaced by a load P at G and a clockwise bending moment $\overset{\curvearrowright}{P . e}$ at G.

Bending moment, $M = Pe$

Direct load, $= P$ (compressive force)

Direct stress, $f_d = \dfrac{P}{BD}$ (which is constant throughout the section)

Bending stress on a layer *mn*, at a distance of x from the neutral layer *hf*

$$f_b = \frac{P.e}{I_{yy}} \cdot x$$

f_b is positive or negative depending upon the value of x.

Now
$$I_{yy} = \frac{BD^3}{12}$$

$$f_b = \frac{12\ Pe}{BD^3} \times x$$

$$= 0 \quad \text{at} \quad x = 0$$

$$= +\frac{6\ Pe}{BD^2} \text{ (compressive) at } x = \frac{D}{2}$$

$$= -\frac{6\ Pe}{BD^2} \text{ (tensile) at } x = -\frac{D}{2}$$

$$= \pm \frac{Pe}{Z}$$

where Z = section modulus.

Fig. 9·2 shows the stress distribution along the depth of the section, and is given by line $a''' b'''$. The stress on the edge bc is bb''' (compressive) and the stress on the edge ad is aa''' (tensile).

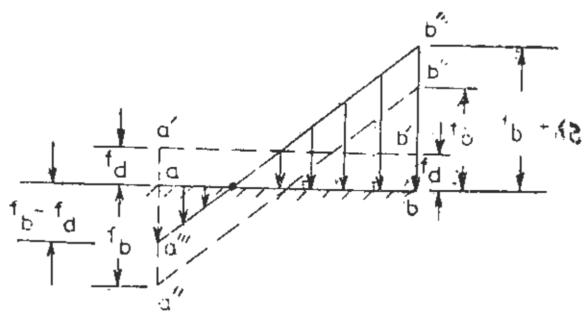


Fig. 9·2

Example 9·1-1. A cast iron column of 20 cm diameter carries a vertical load of 40 tonnes, at a distance of 4 cm from the centre. Determine the maximum and minimum stress developed in section, along the diameter passing through the point of loading.

Solution.

Vertical load, $P = 40$ tonnes $= 40{,}000$ kg

Diameter of section, $D = 20$ cm

Area of cross section, $A = \frac{\pi}{4}(20)^2 = 314{\cdot}16\ \text{cm}^2$

Direct stress, $f_d = \frac{P}{A} = \frac{40{,}000}{314{\cdot}16} = 127{\cdot}32\ \text{kg/cm}^2$ (compressive)

Eccentricity, $e=4$ cm

Bending moment, $M=P.e=40{,}000\times4=160{,}000$ kg-cm

Section modulus, $Z=\dfrac{\pi D^3}{32}=\dfrac{\pi\times20^3}{32}=785{\cdot}4$ cm³

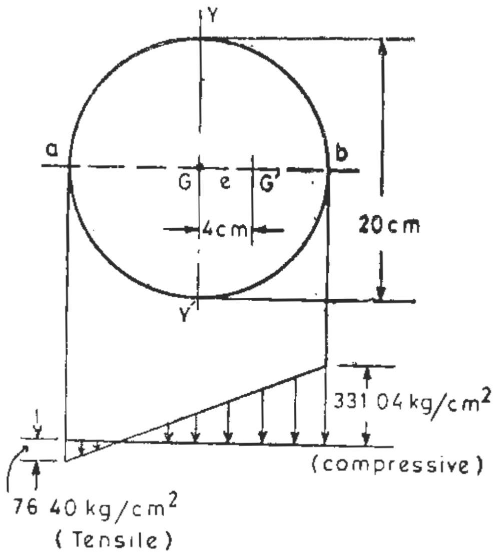


Fig. 9·3

Bending stress, $f_b=\pm\dfrac{Pe}{Z}=\dfrac{160{,}000}{785{\cdot}4}=\pm203{\cdot}72$ kg/cm²

Resultant stress at the edge b

$=127{\cdot}32+203{\cdot}72=331{\cdot}04$ kg/cm² (compressive)

Resultant stress at the edge a

$=127{\cdot}32-203{\cdot}72=-76{\cdot}40$ kg/cm² (tensile).

Exercise 9·1-1. A cast iron column of rectangular section 20 cm × 30 cm carries a vertical load of 25 tonnes at a point 3 cm away from the CG of the section on a line passing through the centroid and parallel to the larger side. Determine the maximum stress at the edges of the line passing through the centroid on which the point of application of load lies.

[**Ans.** 66·66 kg/cm² (compressive)
16·66 kg/cm² (compressive)]

9·2. LOAD ECCENTRIC TO BOTH THE AXES

Consider a rectangular section $B\times D$ of a column subjected to a vertical load P at the point G', at a distance of e from the centroid G (as shown in the Fig. 9·4 (a). The load is eccentric to both the symmetric axes XX and YY passing through the centroid. Component of eccentricity e along X-X axis is e_1 and along Y-Y axis is e_2.

Direct load on the column $=P$

Area of cross section, $A=BD$

Direct stress, $f_d=\dfrac{P}{BD}$ (compressive)

Bending moment about axis YY,

$$M_1=Pe_1$$

Section modulus, $Z_x=\dfrac{I_{yy}}{D/2}=\dfrac{BD^2}{6}$

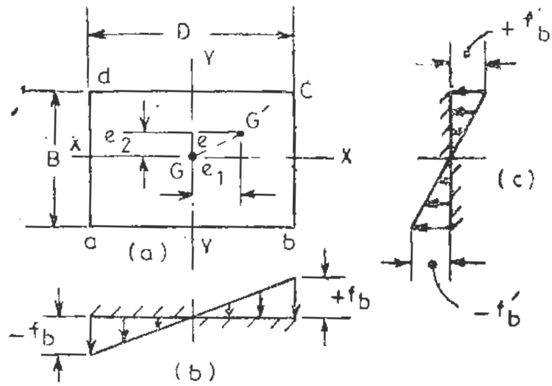


Fig. 9·4

Bending stress due to M_1 along the edge bc

$$=\frac{M_1}{Z_x}=\frac{6\ M_1}{BD^2}\text{ (compressive)}$$

Bending stress due to M_1 along the edge ad

$$=-\frac{6\ M_1}{BD^2}\text{ (tensile)}$$

Stress distribution is shown in Fig. 9·4 (*b*)

Bending moment about axis XX,

$$M_2=P.e_2$$

Section modulus $Z_y=\dfrac{I_{xx}}{B/2}=\dfrac{DB^2}{6}$

Bending stress due to M_2 along the edge cd

$$=\frac{M_2}{Z_y}=\frac{6\ M_2}{DB^2}\text{ (compressive)}$$

Bending stress due to M_2 along the edge ab

$$=-\frac{6\ M_2}{DB^2}\text{ (tensile)}$$

Stress distribution is shown in Fig. 9·4 (*c*).

Resultant stresses at the corners

$$f_a = f_d - f_b - f_b'$$
$$f_b = f_d + f_b - f_b'$$
$$f_c = f_d + f_b + f_b'$$
$$f_d = f_d - f_b + f_b'.$$

In this case, the compressive stress has been taken as positive and tensile stress is taken as negative, because the columns are subjected to compressive loads or thrusts and materials of columns like cast iron and concrete are strong in compression but weak in tension. In concrete columns it is desired that load should be placed at such an eccentricity that the resultant stress at any point in the section is only a compressive stress.

Example 9·2-1. A cast iron column of section 20 cm × 30 cm is subjected to a compressive load of 10 tonnes acting at a point 4 cm away from its CG and along a diagonal. Determine the resultant stresses at four corners of the top face of the column.

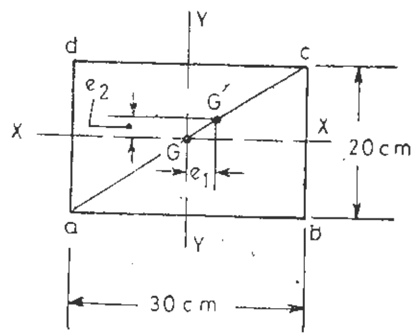


Fig. 9·5

Solution.

Vertical load, $P = 10$ tonnes $= 10{,}000$ kg

Area of cross section, $A = 20 \times 30 = 600\ \text{cm}^2$

Direct stress, $$f_d = \frac{P}{A} = \frac{10{,}000}{600} = 16{\cdot}66\ \text{kg/cm}^2$$

Eccentricity about the diagonal ac

$$= GG' = 4\ \text{cm}$$

Component of eccentricity along X-X axis,

$$e_1 = \frac{4 \times 15}{\sqrt{10^2 + 15^2}} = 3{\cdot}328\ \text{cm}$$

Component of eccentricity along Y-Y axis,

$$e_2 = \frac{4 \times 10}{\sqrt{10^2 + 15^2}} = 2{\cdot}218\ \text{cm}$$

Bending moment about YY axis,

$$M_1 = P.e_1 = 10000 \times 3{\cdot}328 = 33280\ \text{kg-cm}$$

Bending moment about X-X axis,

$$M_2 = P.e_2 = 10000 \times 2{\cdot}218 = 22180\ \text{kg-cm}$$

Section modulus, $$Z_x = \frac{I_{yy}}{15} = \frac{20 \times 30^2}{6} = 3000\ \text{cm}^3$$

Section modulus, $$Z_y = \frac{I_{xx}}{10} = \frac{30 \times 20^2}{6} = 2000\ \text{cm}^3$$

Maximum bending stress due to M_1,

$$f_b = \pm \frac{Pe_1}{Z_x} = \frac{33280}{3000} = \pm 11{\cdot}09 \text{ kg/cm}^2$$

Maximum bending stress due to M_2,

$$f_b' = \pm \frac{Pe_2}{Z_y} = \frac{22180}{2000} = \pm 11{\cdot}09 \text{ kg/cm}^2$$

Note that due to M_1 bending stress at b and c will be compressive and equal to 11·09 kg/cm² ; and the bending stress at a and d will be tensile and equal to 11·09 kg/cm².

Due to bending moment M_2, bending stress at c and d will be compressive and equal to 11·09 kg/cm² and bending stress at a and b will be tensile and equal to 11·09 kg/cm².

Resultant stresses

$f_a = 16{\cdot}66 - 11{\cdot}09 - 11{\cdot}09 = -5{\cdot}52$ kg/cm² (tensile)

$f_b = 16{\cdot}66 + 11{\cdot}09 - 11{\cdot}09 = +16{\cdot}66$ kg/cm² (compressive)

$f_c = 16{\cdot}66 + 11{\cdot}09 + 11{\cdot}09 = +38.84$ kg/cm² (compressive)

$f_d = 16{\cdot}66 - 11{\cdot}09 + 11{\cdot}09 = +16{\cdot}66$ kg/cm² (compressive)

Exercise 9·2-1. A cast iron column of square section 40×40 cm is subjected to a compressive load of 50 tonnes acting at a point which 6 cm from X-X axis and 8 cm from Y-Y axis, where X-X and Y-Y are the axes of symmetry passing through the C.G. of the top section of the column. Determine the resultant stresses at the extreme corners of the section.

[**Ans.** −34·375, +40·625, 96·875, +21·875 kg/cm²]

9·3. CORE OR THE KERNEL OF A RECTANGULAR SECTION

Fig. 9·6 shows a rectangular section of breadth B and depth D with C.G. located at its centre G. Say this is the section of a column and a load P is applied at G' at a distance e_1 from G along the axis X-X. Moment about the axis Y-Y.

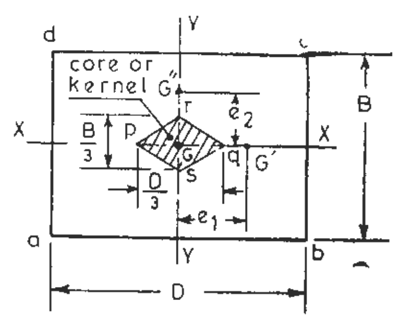


Fig. 9·6

$$M_1 = Pe_1$$

Maximum bending stress due to M_1

$$= \pm \frac{6P.e_1}{BD^2}$$

Resultant stress along the edge,

$$ad = \frac{P}{DB} - \frac{6Pe_1}{BD^2}$$

=direct compressive stress−tensile stress due to bending

Along the edge bc, resultant stress will be compressive throughout.

If the material of the column is brittle which is weak in tension, then it is desired that tensile stress should not be developed anywhere in the section, for that direct stress>bending stress

$$\frac{P}{BD} > \frac{6P.e_1}{BD^2}$$

or $$e_1 < \frac{D}{6}$$

Similarly we can consider the application of load on the other side of the neutral axis YY, for the resultant to stress be only the compressive stress,

$$e_1' < \frac{D}{6}$$

In other words, load can be applied anywhere along pq or on the middle third of the depth, the resultant stress in no part of the section will be tensile.

Again let us consider that load is applied at G'', at a distance of e_2 from the centroid G of the section.

Bending moment about axis XX,

$$M_2 = P.e_2$$

Maximum bending strese due to M_2

$$= \pm \frac{6P.e_2}{DB^2}$$

Resultant stress along the edge ab

$$= \frac{P}{BD} - \frac{6Pe_2}{DB^2}$$

The resultant stress along the edge cd will be compressive throughout.

If it is desired that tensile stress should not be developed anywhere in the section then direct compressive stress>bending stress

$$\frac{P}{BD} > \frac{6Pe_2}{DB^2}$$

or $$e_2 < \frac{B}{6}$$

Similarly we can consider the application of the load on the other side of neutral axis XX, and for the resultant stress to be only the compressive stress,

$$e_2' < \frac{B}{6}$$

or in other words, load can be applied anywhere along rs or on the middle third of the breadth, the resultant stress in no part of the section will be tensile.

Joining the points p, q, r and s gives a diamond shaped figure which is called *core-or kernel of the section.* If the vertical load is applied on the column on any point inside the area marked core or kernel, the resultant stress anywhere in the section of the column will be compressive.

9·4. CORE OF A CIRCULAR SECTION

Fig. 9·7 shows a circular section of diameter D, of a column carrying, the vertical load at G', at a distance of e from the centre G of the section.

Say X-X axis passes, through G and G' and axis YY is perpendicular to XX and passing through the centroid of the section, G.

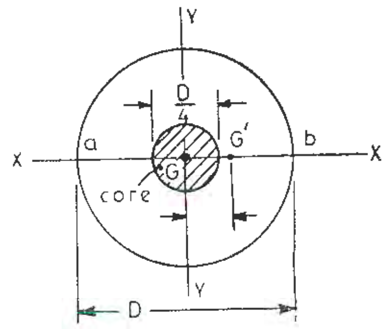


Fig. 9·7

Bending moment, $M = P.e$

Section modulus, $Z_x = \dfrac{\pi D^3}{32}$

Resultant stress at the extreme edge a

$$= \frac{4P}{\pi D^2} - \frac{32\,Pe}{\pi D^3}$$

While the resultant stress at edge b is wholly compressive.

If the resultant stress throughout the section has to be compressive, then

$$\frac{4P}{\pi D^2} - \frac{32\,Pe}{\pi D^3} > 0$$

or
$$\frac{4P}{\pi D^2} > \frac{32\,Pe}{\pi D^3}$$

or
$$e < \frac{D}{8}$$

Similarly eccentricity on the other side of the neutral axis YY can be considered and we get

$$e < \frac{D}{8} \text{ for no tension in the section anywhere.}$$

The area covered by a circle of diameter $D/4$ at the centre is called the core or kernel of circular section. If a load is applied on the column on any point within this core, the resultant stress at any point of the section will be only compressive.

Example 9·4-1. A short hollow cylindrical column carries a compressive force of 400 kN. The external diameter of the column is 200 mm and the internal diameter is 120 mm. Find the maximum permissible eccentricity of the load if the allowable stresses are 60 N/mm^2 in compression and 25 N/mm^2 in tension.

Solution.

External diameter, $D = 200$ mm

Internal diameter, $d = 120$ mm

Area of cross section, $A = \dfrac{\pi}{4}\,(D^2 - d^2)$

$$= \frac{\pi}{4}\,(200^2 - 120^2) = 2{\cdot}01 \times 10^4 \text{ mm}^2$$

Compressive load, $P=400 \text{ kN}=4\times10^5 \text{ N}$

Direct stress, $f_d=\frac{P}{A}=\frac{400\times10^3}{2\cdot01\times10^4}=19\cdot90 \text{ N/mm}^2$ (compressive)

Say the eccentricity of the load $=e$ mm

Bending moment, $M=P.e=4e\times10^5$ Nmm

Section modulus, $Z=\frac{\pi(D^4-d^4)}{32\,D}=\frac{\pi}{32}\times\frac{(200^4-120^4)}{200}$

$$=\frac{\pi}{32}\times\frac{13\cdot9264\times10^8}{200} \text{ mm}^3$$

$$=0\cdot6836\times10^6 \text{ mm}^3$$

Maximum bending stress,

$$f_b=\pm\frac{M}{Z}=\pm\frac{4e\times10^5}{0\cdot6836\times10^6}=\pm0\cdot585\,e$$

Resultant stress at extreme layers $=19\cdot90\pm0\cdot585\,e$

Allowable stress in compression $=60 \text{ N/mm}^2=19\cdot90+\cdot585\,e$

$$e=68\cdot55 \text{ mm}$$

Allowable stress in tension $=-25 \text{ N/mm}^2=19\cdot90-\cdot585\,e$

or $$e=\frac{25-19\cdot9}{\cdot585}=8\cdot72 \text{ mm}$$

Maximum permissible eccentricity is 8·72 mm from the centre of circular section.

Example 9·4-2. A short hollow pier 1·6 m×1·6 m outer sides and 1 m×1 m inner sides supports a vertical load of 2000 kN at a point located on a diagonal 0·5 m from the vertical axis of the pier. Neglecting the self weight of the pier, calculate the normal stresses at the four outside corners on a horizontal section of the pier.

Solution. Fig. 9·8 shows the section of the pier. Vertical axis passes through the centroid G of the section. ac is the diagonal and $GG'=0\cdot5$ m.

At G' the load 2000 kN is applied on the pier.

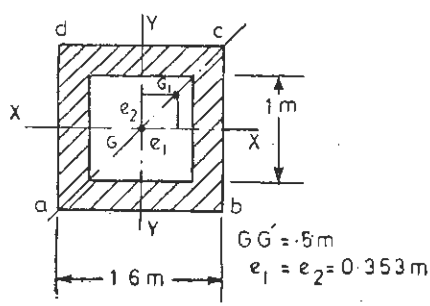


Fig. 9·8

Load applied $P=2000$ kN

Area of cross saction $=1\cdot6^2-1^2$
$=1\cdot56 \text{ m}^2$

Section modulus, $Z_x=Z_y=\left(\frac{1\cdot6^4-1^4}{12}\right)\frac{2}{1\cdot6}$

$$=\frac{5\cdot5536}{9\cdot6}=0\cdot5785 \text{ m}^3$$

Eccentricity about YY axis $=e_1=\cdot 5\times\cdot 707=0\cdot 353$ m
Eccentricity about XX axis $=e_2=\cdot 5\times\cdot 707=0\cdot 353$ m
Bending moment about YY axis $=M_1=P\times 0\cdot 353$
Bending moment about XX axis $=M_2=P\times 0\cdot 353$

Maximum bending stress due to M_1

$$=\pm\frac{M_1}{Z_x}=+\frac{2000\times 0\cdot 353}{0\cdot 5785}=\pm 1220\cdot 4\ \text{kN/m}^2$$

Maximum bending stress due to M_2

$$=\pm\frac{M_2}{Z_y}=\pm\frac{2000\times 0\cdot 353}{0\cdot 5785}=\pm 1220\cdot 4\ \text{kN/m}^2$$

Direct compressive stress,

$$f_d=\frac{P}{A}=\frac{2000\ \text{kN}}{1\cdot 56}=+1282\cdot 05\ \text{kN/m}^2$$

Resultant Stresses at Corners

$f_a=1282\cdot 05-1220\cdot 4-1220\cdot 4=-1158\cdot 75\ \text{kN/m}^2$
$f_b=1282\cdot 05+1220\cdot 4-1220\cdot 4=+1282\cdot 05\ \text{kN/m}^2$
$f_c=1282\cdot 05+1220\cdot 4+1220\cdot 4=+3722\cdot 85\ \text{kN/m}^2$
$f_d=1282\cdot 05-1220\cdot 4+1220\cdot 4=+1282\cdot 05\ \text{kN/m}^2$

Negative stress is a tensile stress.

9·5. WIND PRESSURE ON WALLS AND CHIMNEY SHAFTS

Many a times masonry walls and chimney shafts are subjected to strong wind pressures. The weight of the walls or the chimney produces compressive stress in the base while the wind pressure introduces bending moment producing tensile and compressive stresses in the base. Fig. 9·9 shows a masonry wall of height H and rectangular section $B\times D$. The horizontal wind pressure of intensity p is acting on the face of width B.

Say density of masonry structure $=\rho$

Weight of the masonry structure,

$W=\rho\ BDH$

Area of cross section at the base $=BD$

Compressive stress due to the weight of the structure on its base,

$$f_d=\frac{\rho BDH}{BD}=\rho H \quad \ldots(1)$$

Total wind force on the vertical face,

$P=p\ BH$

Distance of C.G. of the wind force from the base,

$$=\frac{H}{2}$$

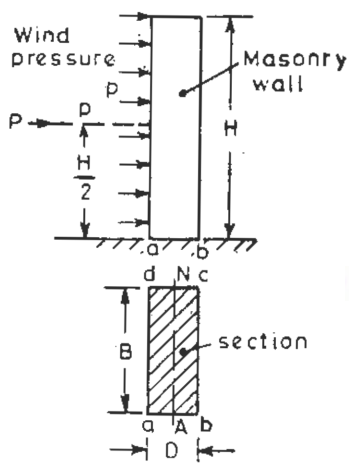


Fig. 9·9

Bending moment, $M=\frac{PH}{2}=\frac{pBH^2}{2}$

Section modulus, $Z=\frac{BD^2}{6}$

Bending stress, $f_b=\pm\frac{M}{Z}=\pm\frac{pBH^2}{2B}\times\frac{6}{D^2}$

$$=\pm\frac{3p\,H^2}{D^2}$$

Due to bending moment, there will be maximum tensile stress along edge *ad* and maximum compressive stress along edge *bc* of the base.

Resultant stresses, $f_R=f_d-\frac{3pH^2}{D^2}$ along edge *ad*

$$f_R=f_d+\frac{3pH^2}{D^2} \text{ along edge } bc$$

Fig. 9·10 shows a cylindrical chimney of height H, external diameter D and internal diameter d, subjected to horizontal wind pressure p as shown.

If ρ is the weight density of the masonry structure, direct stress due to the weight of the structure on its base $=\rho H$

Consider a small strip of width $R\,\delta\theta$, subtending an angle $\delta\theta$ at the centre and making an angle θ with the axis *ac* of the section.

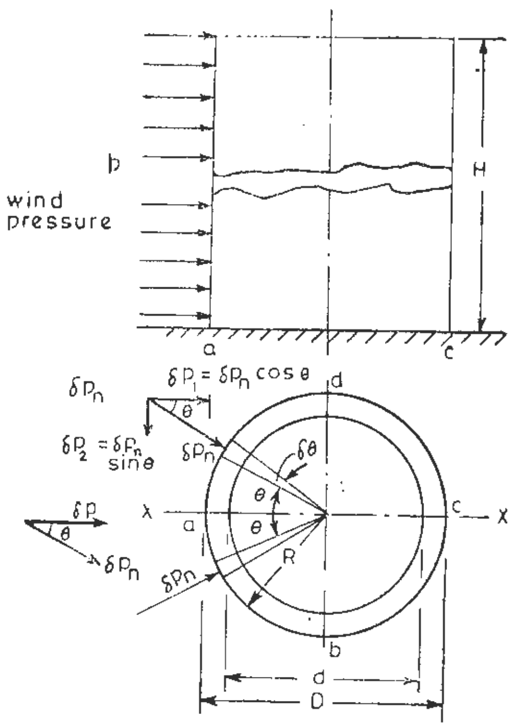


Fig. 9·10

δP=Wind force reaching the small strip

$$=p \times \vec{R}\, \delta\theta . H \cos\theta$$

$$=pH\, R\, \delta\theta \cos\theta$$

Cmponent of the force normal to the strip

$$=\delta P_n=\delta P \cos\theta$$

$$=pHR \cos\theta\,.\,\delta\theta \cos\theta=pHR \cos^2\theta\, \delta\theta$$

Horizontal component of δP_n,

$$\delta P_1=\delta P_n \cos\theta=pHR \cos^3\theta\,.\,\delta\theta$$

Another horizontal component of θP_{n1}, $\delta P_2=\delta P_n \sin\theta$. This component is cancelled out when we consider a strip in other quadrant (as shown), while the components $\delta P_n \cos\theta$ are added up.

Therefore total force in the direction X-X

$$=2\delta P_n \cos\theta=2pHR \cos^3\theta\,.\,\delta\theta$$

Integrating over the whole exposed surface from θ=0 to 90°.

Total wind force, $$P=\int_0^{\pi/2} 2pHR \cos^3 d\theta.$$

$$=pDH.\frac{2}{3}=kpDH$$

where k=coefficient of wind resistance

DH=projected area of the curved surface

CG of the force lies at a distance of $H/2$ from the base.

Bending moment due to wind force,

$$M=\frac{PH}{2}$$

$$=pDH\ \frac{2}{3}\times\frac{H}{2}=\frac{pDH^2}{3}$$

Section modulus, $$Z=\frac{\pi(D^4-d^4)}{32\,D}$$

Bending stress $$=\pm\frac{M}{Z}.$$

Generally the coefficient of wind resistanee is taken as 0·6 for cylindrical chimneys.

Example 9·5-1. A 10 m high masonry wall of rectangular section 4 m×1·5 m is subjected to horizontal wind pressure of 150 kg/m³ on the 4 m side. Find the maximum and minimum stress intensities induced on the base.

Density of masonry is 2200 kg/m³.

Solution. B, Breadth $=4$ m

Height, $H=10$ m

D, depth $=1{\cdot}5$ m

Area at the base $=4\times 1{\cdot}5$ m$^2=6$ m^2

Weight of the masonry structure

$$=pBDH=2200\times 4\times 1{\cdot}5\times 10=132000 \text{ kg}$$

Direct compressive stress at the base due to weight,

$$f_d=\frac{132000}{6}=22000 \text{ kg/m}^2$$

Wind pressure, $p=150$ kg/m^2

Wind force on the vertical face of side 4 m,

$$P=p\ BH$$
$$=150\times 4\times 10=6000 \text{ kg}$$

Distance of CG of P from base,

$$=\frac{H}{2}=5 \text{ m}$$

Bending moment, $M=\dfrac{PH}{2}=6000\times 5=30000$ kg-m

Section modulus, $Z=\dfrac{BD^2}{6}=\dfrac{4\times 1{\cdot}5^2}{6}=1{\cdot}5$ m^3

Bending stress due to bending moment,

$$f_b=\pm\frac{M}{Z}$$

$$=\frac{30{,}000}{1{\cdot}5}=20{,}000 \text{ kg/m}^2$$

Maximum stress $=22000+20000=42000$ kg/m^2 (compressive)

Minimum stress $=22000-20000=2000$ kg/m^2 (compressive).

Example 9·5-2. A masonry chimney 20 m high of uniform circular section, 5 m external diameter and 3 m internal diameter has to withstand a horizontal wind pressure of intensity 200 kg per square metre of the projected area. Find the maximum and minimum stress intensities at the base. Density of masonry structure=2100 kg/m^3.

Solution.

Height of the chimney, $H=20$ m

External diameter, $D=5$ m

Internal diameter, $d=3$ m

Density of masonry, $\rho=2100$ kg/m^3

Direct compressive stress due to self weight on the base of the chimney,

$$f_d=\rho H=20\times 2100=42000 \text{ kg/m}^2$$

Wind pressure, $p=200$ kg/m^2

Projected area, $A=DH=5\times 20=100\ m^2$

Wind force, $P=pA=200\times 100=20000$ kg

Distance of CG of the wind force from base

$$=\frac{H}{2}=10 \text{ m}$$

Bending moment, $M=\frac{PH}{2}=20{,}000\times 10=200{,}000$ kg-m

Section modulus, $Z=\frac{\pi}{32}\ \frac{(D^4-d^4)}{D}=\frac{\pi}{32}\ \frac{(5^4-3^4)}{5}=10\text{·}68\ m^3$

Bending stress, $f_b=\pm\frac{M}{Z}=\pm\frac{200{,}000}{10\text{·}68}$ kg/m²

$$=\pm 18626\text{·}6 \text{ kg/m}^2$$

Maximum stress intensity,

$$f_{max}=f_a+f_b=42000+18726\text{·}6=60726\text{·}6 \text{ kg/m}^2$$

Minimum stress intensity,

$$f_{min}=f_a-f_b=42000-18726\text{·}6=23273\text{·}4 \text{ kg/m}^2.$$

Exercise 9·5-1. The section of a masonry pier is a hollow rectangle, external dimensions 4 m × 1·2 m and internal dimensions 2·4 m × 0·6 m. A horizontal thrust of 3000 kg is exerted at the top of the pier in the vertical plane bisecting the length 4 m. The height of chimney is 5 m and density of the masonry is 2250 kg/m³. Calculate the maximum and minimum intensity of stress at the base. [**Ans.** 28142 kg/m², −5642 kg/m²]

Exercise 9·5-2. A cylindrical chimney shaft of a hollow circular section 2 m external diameter and 1 m internal diameter is 25 m high. Given that horizontal intensity of wind pressure is 100 kg/m²; determine the extreme intensities of stress at the base. Take the coefficient wind resistance as 0·6. Density of masonry=2280 kg/m³.

[**Ans.** 107930 kg/m², 6070 kg/m², both compressive]

Problem 9·1. A flat plate of section 20 mm × 60 mm placed in a testing machine is subjected to 60 kN of load along the line AB as shown in Fig. 9·11. An extensometer adjusted along the line of the load recorded an extension of 0·078 mm on a gauge length of 150 mm. Determine (*i*) maximum and minimum stresses set up, (*ii*) Young's modulus of the material of the plate.

Solution.

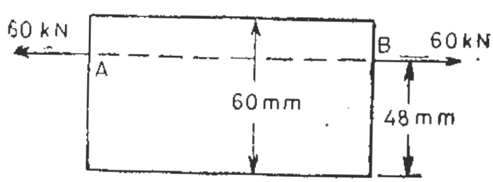


Fig. 9·11

Section of the plate, $=20\times 60$
$=1200\ mm^2$

Eccentricity of the load, $=48-30$
$=18$ mm

Bending moment,

$$M=60\times 1000\times 18 \text{ Nmm}=108\times 10^4 \text{ Nmm}$$

Section modulus, $Z=\frac{bd^2}{6}$ where $b=20$ mm, $d=60$ mm

$$=\frac{20\times 60\times 60}{6}=12000 \text{ mm}^3$$

Stress due to bending, $f_b=\pm\dfrac{M}{Z}$

$$=\pm\frac{108\times 10^4}{12000}=\pm 90 \text{ N/mm}^2$$

Direct stress, $f_d=+\dfrac{60\times 1000}{20\times 60}=-50 \text{ N/mm}^2$

Minimum stress $=f_d+f_b$

$=-50+90=40 \text{ N/mm}^2$ (compressive)

Maximum stress $=f_d+f_b$

$=-50-90=-140 \text{ N/mm}^2$ (tensile)

Now stress along the line of the load

$$=-50-f_b\times\frac{18}{30}$$

$$=-50-90\times\frac{18}{30}=-140 \text{ N/mm}^2 \text{ (tensile)}$$

Say Young's modulus $=E$

Then extension along the line of the load

$$=\frac{104}{E}\times\text{gauge length}$$

$$=\frac{104\times 150}{E}=0{\cdot}078 \text{ mm}$$

So $\dfrac{104\times 150}{E}=0{\cdot}078$

or Young's modulus, $E=\dfrac{104\times 150}{0{\cdot}078}=200{,}000 \text{ N/mm}^2$

Problem 9·2. A large C clamp is shown in the Fig. 9·12. As the screw is tightened down upon an object, the strain observed in the vertical direction at the point B is 800 micro-strain. What is the load on the screw?

$$E=2\times 10^5 \text{ N/mm}^2.$$

Solution. The section shown is symmetrical about the axis X-X. CG lies along X-X.

$$x_1=\frac{13\times 3\times 1{\cdot}5+2\times 15\times 8}{39+30}$$

$$=\frac{58{\cdot}5+240}{69}=4{\cdot}32$$

$x_1=4{\cdot}32$ cm

$x_2=13-4{\cdot}32=8{\cdot}68$ cm

Area of the section $=30+39=69 \text{ cm}^2$

Moment of inertia,

$$I_{yy}=\frac{13\times3^3}{12}+13\times3\,(4{\cdot}32-1{\cdot}5)^2+2\times1{\cdot}5\times\frac{10^3}{12}$$
$$+2\times1{\cdot}5\times10\,(8{\cdot}68-5{\cdot}0)^2$$
$$=29{\cdot}25+310{\cdot}1436+250{\cdot}00+406{\cdot}272$$
$$=995{\cdot}66\text{ cm}^4$$

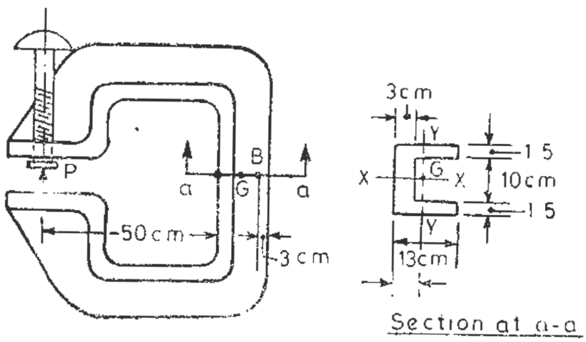


Fig. 9·12

Distance of the point B from neutral axis,

$$x=8{\cdot}68-3=5{\cdot}68\text{ cm}$$

Say the load on the screw $=P$ Newtons

Distance of CG of the section from the load line,

$$e=50+4{\cdot}32=54{\cdot}32\text{ cm}$$

Bending moment on the section,

$$=P.e\text{ Ncm}$$

Direct force on the section,

$$=P\text{ (compressive)}$$

On the point B $\quad f_d$ = direct stress is compressive

f_b = bending stress is also compressive

$$f_d=\frac{P}{69}\text{ N/cm}^2=0{\cdot}0145\,P$$

$$f_b=\frac{M\,.\,x}{I_{yy}}=\frac{54{\cdot}32\,P\times5{\cdot}68}{995{\cdot}66}=0{\cdot}3098\,P$$

Resultant stress, $\quad f_R=f_d+f_b=0{\cdot}3243\,P$

Strain at the point, $\quad \epsilon=\frac{f_R}{E}=800\times10^{-6}$

$$=\frac{0{\cdot}3243\,P}{E}=800\times10^{-6}\text{ where }E=2\times10^7\text{ N/cm}^2$$

So $$\frac{0{\cdot}3243\ P}{2\times10^7}=800\times10^{-6}$$

Load on the screw, $$P=\frac{800\times2\times10}{0{\cdot}3243}=49337\text{ N}=49{\cdot}337\text{ kN.}$$

Problem 9·3. A rectangular plate 1 cm thick with a hole of 5 cm diameter drilled in it as shown in the Fig. 9·13, is subjected to an axial pull $P=4500$ kg. Determine the greatest and the least intensities of stress at the critical cross section of the plate.

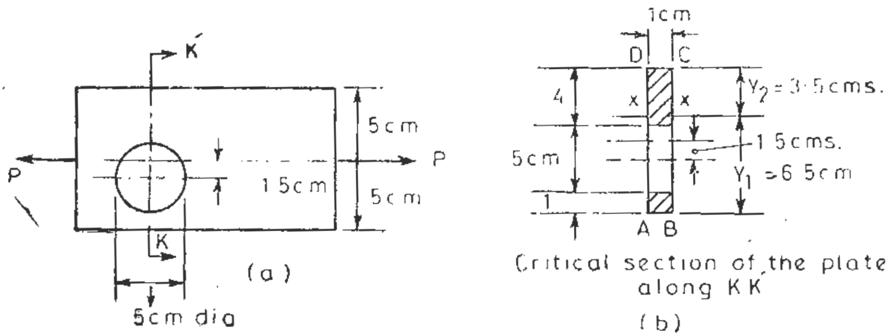


Fig. 9·13

Solution. For locating centroidal axis, take moments of areas about *AB*,

$$y_1=\frac{1\times1\times0{\cdot}5+4\times1\times8}{1+4}$$

$$=\frac{32{\cdot}5}{5}=6{\cdot}5\text{ cm}$$

$$y_2=10-6{\cdot}5=3{\cdot}5\text{ cm}$$

Moment of inertia, $$I_{xx}=\frac{1\times(4)^3}{12}+4\times(3{\cdot}5-2)^2+\frac{1\times(1)^3}{12}+1\times(6{\cdot}5-0{\cdot}5)^2$$

$$=\frac{65}{12}+4\times2{\cdot}25+36$$

$$=\frac{605}{12}\text{ cm}^4$$

Area of cross section, $A=1\times1+4\times1=5$ cm²

Axial load, $P=4500$ kg (tensile)

Eccentricity, $e=6{\cdot}5-5{\cdot}0=1{\cdot}5$

Maximum stress along the edge,

$$AB=-\frac{P}{A}-\frac{P.e.\,y_1}{I_{xx}}$$

$$=-\frac{4500}{5}-\frac{4500\times1{\cdot}5\times6{\cdot}5}{605}\times12$$

$$=-900-870$$

$$=-1770\text{ kg/cm}^2\text{ (tensile)}$$

Minimum stress along the edge,

$$CD=-\frac{P}{A}+\frac{P.e.\,y_2}{I_{xx}}$$

$$=-\frac{4500}{5}+\frac{4500\times1{\cdot}5\times3{\cdot}5}{605}\times12$$

$$=-900+468=-432 \text{ kg/cm}^2 \text{ (tensile).}$$

Problem 9·4. A short column of hollow circular section of internal diameter d and external diameter D is loaded with a compressive load W. Determine the maximum distance of the point of application of the load from the centre of the section such that the tensile stress does not exist at any point of the cross section if $D=1{\cdot}5\,d$.

Solution. Area of cross section,

$$A=\frac{\pi}{4}\,(D^2-d^2)$$

Moment of inertia,

$$I=\frac{\pi}{64}\,(D^4-d^4)$$

Say P is the point farthest from G (centre of the section) where the load acts ; eccentricity e=distance GP

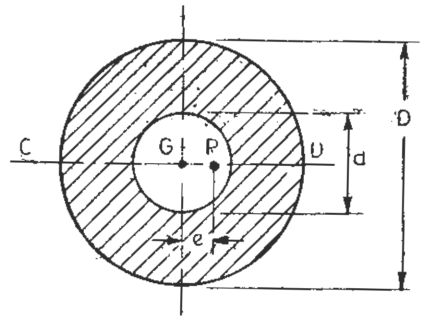


Fig. 9·14

Stress at the point D $=\frac{W}{A}+\frac{W.e.}{I}\times\frac{D}{2}$ (compressive)

Stress at the point C $=\frac{W}{A}-\frac{W.e}{I}\cdot\frac{D}{2}$

To satisfy the condition that tensile stress should not occur at any point of the section,

$$\frac{W}{A}-\frac{W.e.D}{2I}\geqslant0$$

or

$$\frac{1}{A}=\frac{e\,.\,D}{2I}$$

$$\frac{4}{\pi(D^2-d^2)}=\frac{e\,.\,D}{2\times\pi}\times\frac{64}{(D^4-d^4)}$$

or

$$e=\frac{D^2+d^2}{8D}=\frac{D^2+\frac{4}{9}\,D^2}{8D}=\frac{13}{72}\,D,$$

Problem 9·5. A short cast iron column has an external diameter of 20 cm. and internal diameter of 16 cm, the distance between the centres of the outer and inner circles due to the displacement of the core during casting is 6 mm. A load of 40 tonnes acts through a vertical centre line passing through the centre of the outer circle. Calculate the values of the greatest and least compressive stresses in a horizontal cross section of the column.

Solution.

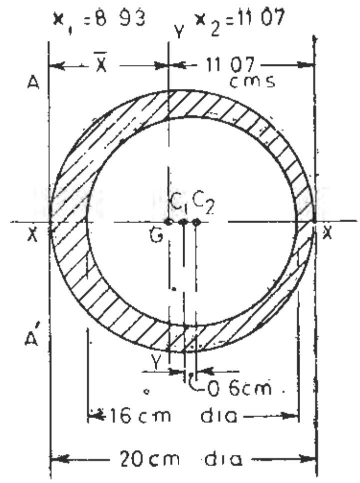


Fig. 9·15

C_1=Centre of outer circle

C_2=Centre of inner circle.

Dia. of outer circle=20 cm.

Dia. of inner circle=16 cm.

Taking moments of areas about AA', to locate C.G. of the section,

$$\bar{x}=\frac{\frac{\pi}{4}\times(20)^2\times 10-\frac{\pi}{4}(16)^2\times 10{\cdot}6}{\frac{\pi}{4}(20)^2-\frac{\pi}{4}(16)^2}$$

$$=\frac{4000-2713{\cdot}6}{400-256}=8{\cdot}93 \text{ cm.}$$

So $\qquad GC_1=10-8{\cdot}93=1{\cdot}07$ cm,

$\qquad GC_2=10{\cdot}6-8{\cdot}93=1{\cdot}67$ cm.

Moment of inertia about Y-Y axis,

$$I_{yy}=\left[\frac{\pi}{64}(20)^4+\frac{\pi}{4}(20)^2(G\,C_1)^2\right]-\left[\frac{\pi}{64}\times(16)^4+\frac{\pi}{4}(16)^2(G\,C_2)^2\right]$$

$$=\left[\frac{\pi}{4}\times 10^4+100\,\pi\,(1{\cdot}07)^2\right]-[1024\,\pi+64\,\pi\,(1{\cdot}67)^2]$$

$$=4436 \text{ cm}^4.$$

Area of cross section, $A=\frac{\pi}{4}\times(20)^2-\frac{\pi}{4}\times(16)^2=113{\cdot}2 \text{ cm}^2$

Vertical load at the point C_1, $W=40$ tonnes

Eccentricity, $\qquad e=G\,C_1=1{\cdot}07$ cm.

Greatest compressive stress,

$$=\frac{W}{A}+\frac{We}{I_{yy}}\times x_2$$

$$=\frac{40}{113{\cdot}2}+\frac{40\times 1{\cdot}07\times 11{\cdot}07}{4436}$$

$$=0{\cdot}353+0{\cdot}107=0{\cdot}460 \text{ tonne/cm}^2$$

Least compressive stress,

$$=\frac{W}{A}-\frac{W.e}{I_{yy}}.\ x_1$$

$$=\frac{40}{113 \cdot 2}-\frac{40\times 1 \cdot 07\times 8 \cdot 93}{4436}$$

$$=0 \cdot 353-0 \cdot 086=0 \cdot 267 \text{ tonne/cm}^2.$$

Problem 9·6. A steel rod 2 cm diameter passes through a copper tube 3 cm internal diameter and 4 cm external diameter. Rigid cover plates are provided at each end of the tube and steel rod passes through these cover plates also. Nuts are screwed on the projecting ends of the rod as so that the cover plates put pressure on the ends of the tube. Determine the maximum stress in the copper tube, if one of the nuts is tightened to produce a linear strain of $\frac{1}{1000}$ in the rod.

(*a*) if the rod is concentric with the tube.

(*b*) if the centre of the rod is 5 mm. out of the centre of the tube.

Given : $E_{steel}=2100$ tonnes/cm²

$E_{copper}=1050$ tonnes/cm²

Solution. (*a*) Strain in the steel rod, $\epsilon_s=\frac{1}{1000}$ tensile, by tightening the nuts the steel rod will be stretched.

Modulus of elasticity of steel, $E_s=2100$ tonnes/cm²

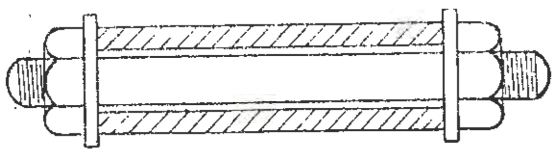


Fig. 9·16

Stress in steel, $f_s=\epsilon_s\times E_s$

$$=\frac{1}{1000}\times 2100=2 \cdot 1 \text{ tonnes/cm}^2 \quad \text{(tensile)}$$

Area of cross section of the steel rod,

$$A_s=\frac{\pi}{4}\times(2)^2=3 \cdot 14 \text{ cm}^3$$

Pull in steel rod, $P_s=f_s.A_s$

$=2 \cdot 1\times 3 \cdot 14=6 \cdot 6$ tonnes.

For equilibrium,

Pull in steel rod $=$Push in copper tube

$=f_c\times A_c$

Area of cross section of the copper tube,

$$A_c=\frac{\pi}{4}(16-9)=5 \cdot 5 \text{ cm}^2$$

Stress in copper tube, $f_c=\frac{6 \cdot 6}{5 \cdot 5}=1 \cdot 2$ tonnes/cm² (compressive)

(*b*) When the centre of the rod is 5 mm out of the centre of the tube.
Moment of inertia of the tube section,

$$I_{yy}=\frac{\pi}{64}(4^4-3^4)=8{\cdot}59 \text{ cm}^4$$

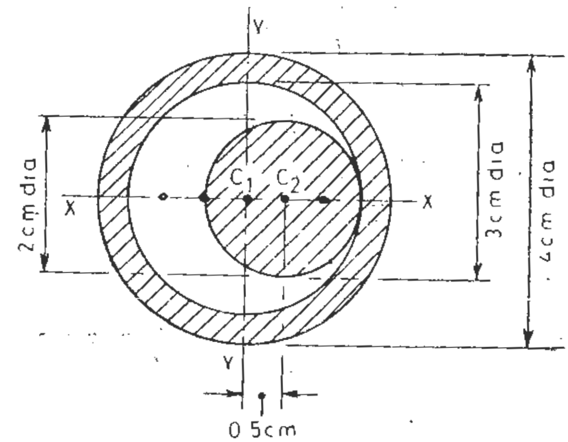


Fig. 9·17

Eccentricity of the load P_s, $e=0{\cdot}5$ cm.
Maximum bending stress in tube,

$$f_c'=\frac{P.e}{I_{yy}}\cdot\left(\frac{4}{2}\right)$$

$$=\pm\frac{6{\cdot}6\times0{\cdot}5\times2}{8{\cdot}59}=\pm0{\cdot}768 \text{ tonne/cm}^2$$

Therefore maximum stress in the copper tube

$$=1{\cdot}2+0{\cdot}768=1{\cdot}968 \text{ tonnes/cm}^2. \text{ (compressive)}$$

Problem 9·7. The cross section of a short column is as shown in the Fig. 9·18. A vertical load W tonnes acts at the point P. (*a*) Determine the value of W if the maximum stress set up in the cross section is not to exceed 750 kg/cm² (*b*) Draw the stress distribution diagram along the edge AD.

Solution. (*a*) The section is symmetrical about the X-X axis, the C.G. of the section lies at the point G, therefore,

$$I_{xx}=\frac{8\times8^3}{12}-\frac{\pi(4)^4}{64}$$

$$=328{\cdot}77 \text{ cm}^4$$

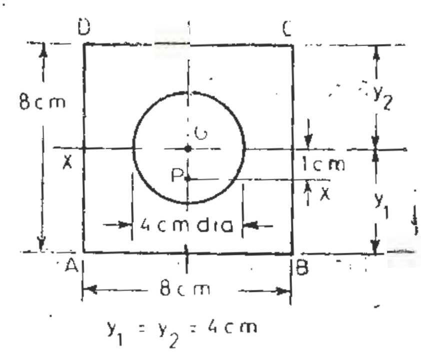


Fig. 9·18

Vertical load $=W$ tonnes

Eccentricity, $e=1$ cm.

Bending moment, $M=W.e$

$=W\times1$ tonne-cms.

Area of cross section, $A=8\times8-\dfrac{\pi\times4^2}{4}=51{\cdot}44\ \text{cm}^2$

Greatest stress along the edge

$$AB=\frac{W}{A}+\frac{W.e}{I_{xx}}.\,y_1$$

$$=\frac{W}{51{\cdot}44}+\frac{W\times1\times4}{328{\cdot}77}$$

$$=0{\cdot}75\ \text{tonne/cm}^2 \qquad \text{(compressive)}$$

$$W=23{\cdot}72\ \text{tonnes.}$$

(*b*) Stress distribution along AD

Considering origin at the centre and y to be +ve downwards.

Stress at $y=0$, $\quad f_0=\dfrac{23{\cdot}72}{A}+\dfrac{23{\cdot}72\times0}{I_{xx}}$

$$=\frac{23{\cdot}72}{51{\cdot}44}+0$$

$$=0{\cdot}462\ \text{tonne/cm}^2$$

Stress at $y=1$ cm, $\quad f_1=\dfrac{23{\cdot}72}{51{\cdot}44}+\dfrac{23{\cdot}72\times1}{328{\cdot}77}$

$$=0{\cdot}462+0{\cdot}072=0{\cdot}534\ \text{tonne/cm}^2$$

Similarly stress, $\quad f_{-1}=0{\cdot}462-\dfrac{23{\cdot}72\times1}{328{\cdot}77}$

$$=0{\cdot}462-0{\cdot}072=0{\cdot}39\ \text{tonne/cm}^2$$

Stress at $y=2$ cm, $\quad f_2=0{\cdot}462+\dfrac{23{\cdot}72\times2}{328{\cdot}77}$

$$=0{\cdot}462+0{\cdot}144=0{\cdot}606\ \text{tonne/cm}^2$$

$$f_{-2}=0{\cdot}462-0{\cdot}144=0{\cdot}318\ \text{tonne/cm}^2$$

Stress at $y=3$ cm, $\quad f_3=0{\cdot}462+\dfrac{23{\cdot}72\times3}{328{\cdot}77}$

$$=0{\cdot}462+0{\cdot}216=0{\cdot}678\ \text{tonne/cm}^2$$

$$f_{-3}=0{\cdot}462-0{\cdot}216=0{\cdot}246\ \text{tonne/cm}^2$$

Stress at $y=4$ cm, $\quad f_4=0{\cdot}462+\dfrac{23{\cdot}72\times4}{328{\cdot}77}$

$$=0{\cdot}462+0{\cdot}288=0{\cdot}75\ \text{tonne/cm}^2$$

$$f_{-4}=0{\cdot}462-{\cdot}288$$

$$=0{\cdot}174\ \text{tonne/cm}^2 \qquad \text{(compressive)}$$

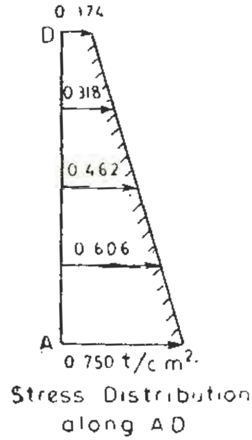


Fig. 9·19

Problem 9·8. The crosssection of a short column is as shown in the Fig. 9·20. A vertical load of 15 tonnes is applied at the point P. Determine the stresses at the corners A, B, C and D.

Solution. The section is symmetrical about X-X axis, therefore

$$y_1 = y_2 = \pm 6 \text{ cm}$$

To locate the centroid of the section, take moments of areas about the edge BC,

$$x_2 = \frac{12\times3\times1\cdot5+2\times10\times2\cdot5\times(5+3)+12\times2\times(13+1)}{36+50+24}$$

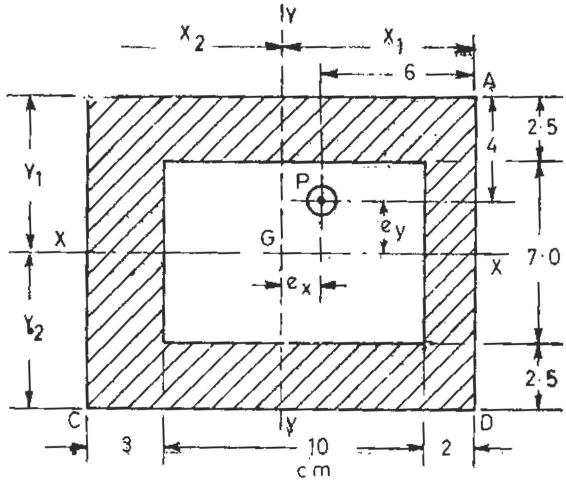


Fig. 9·20

$$=\frac{79}{11}\text{ cm}$$

$$x_1=15-\frac{79}{11}=\frac{86}{11}\text{ cm}$$

Moment of inertia, $I_{zz}=\dfrac{15\times 12^3}{12}-\dfrac{10\times 7^3}{12}$ (because of symmetry)

$$=1874{\cdot}17\text{ cm}^4$$

and $I_{yy}=\dfrac{2\times 2{\cdot}5\times 10^3}{12}+2\times 2{\cdot}5\times 10\left(8-\dfrac{79}{11}\right)^2+\dfrac{12\times 3^3}{12}$

$$+36\left(\frac{79}{11}-1{\cdot}5\right)^2+\frac{12\times 2^3}{12}+24\left(\frac{86}{11}-1\right)^2$$

$$=416{\cdot}67+40{\cdot}90+27+1162+8+1116$$

$$=2770{\cdot}57\text{ cm}^4$$

Eccentricity, $e_x=\dfrac{86}{11}-6=\dfrac{20}{11}$ cm

$$e_y=6-4=2\text{ cm}$$

Area of cross section, $A=110$ cm^2

Direct stresses

Compressive load, $W=15$ tonnes

Area, $A=110$ cm^2

Direct stress at all the points,

$$=\frac{15\times 1000}{110}=136{\cdot}4\text{ kg/cm}^2\text{ (compressive)}$$

Bending stresses

(*i*) Considering couple,

$$M_1=W_1\,e_x$$

Compressive stress at A and D

$$=\frac{W.e_x}{I_{yy}}\times x_1$$

$$=\frac{15000\times 1{\cdot}818}{2770{\cdot}57}\times\frac{86}{11}\text{ kg/cm}^2$$

$$=+76{\cdot}9\text{ kg/cm}^2\text{ (compressive)}$$

Tensile stresses at the points B and C,

$$=\frac{W\,.\,e_x\,.\,x_2}{I_{yy}}$$

$$=\frac{15000\times 1{\cdot}818\times 79}{2770{\cdot}57\times 11}=-70{\cdot}7\text{ kg/cm}^2\text{ (tensile)}$$

(*ii*) Considering couple,

$$M_2=W\,.\,e_y$$

Compressive stresses at A and B

$$=\frac{W\,.\,e_y}{I_{xx}}\,.\,y_1$$

$$=\frac{15000\times2\times6}{1874{\cdot}17}=+96{\cdot}0\ \text{kg/cm}^2$$

Tensile stresses at C and D

$$=\frac{W\,.\,e_y\,.\,y_2}{I_{xx}}$$

$$=\frac{15000\times2\times6}{1874{\cdot}17}=-96{\cdot}0\ \text{kg/cm}^2\ \text{(tensile)}$$

Resultant stresses at the corners,

$$f_A=136{\cdot}4+76{\cdot}9+96{\cdot}0=309{\cdot}3\ \text{kgf/cm}^2\ \text{(compressive)}$$
$$f_B=136{\cdot}4-70{\cdot}7+96{\cdot}0=161{\cdot}7\ \text{kgf/cm}^2\ \text{(compressive)}$$
$$f_C=136{\cdot}4+70{\cdot}7-96{\cdot}0=-30{\cdot}3\ \text{kgf/cm}^2\ \text{(tensile)}$$
$$f_D=136{\cdot}4+76{\cdot}9-96{\cdot}0=117{\cdot}3\ \text{kgf/cm}^2\ \text{(compressive).}$$

Problem 9·9. A rolled steel I section, flanges 15 cm wide and 2·5 cm thick, web 20 cm long and 1 cm thick is used as a short column, to carry a load of 80 tonne. The load line is eccentric, 5 cm above XX and 3 cm to the left of YY. Find the maximum and minimum stress intensities induced in the section.

Solution. Area of cross section of I section,

$$=2\times15\times2{\cdot}5+20\times1$$
$$=95\ \text{cm}^2$$

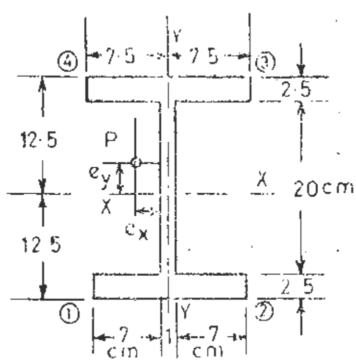


Fig. 9·21

Moment of inertia,

$$I_{xx}=\frac{15\times(25)^3}{12}-\frac{14\times(20)^3}{12}$$
$$=8565\ \text{cm}^4$$

Moment of inertia,

$$I_{yy}=\frac{2\times2{\cdot}5\times(15)^3}{12}+\frac{20\times(1)^3}{12}$$
$$=1407\ \text{cm}^4$$

Eccentricity, $e_x=3$ cm

and $e_y=5$ cm

Vertical load, $W=80$ tonnes

Direct stress, f_d at any point,

$$=\frac{80}{95}=+0{\cdot}842\ \text{tonne/cm}^2$$

Maximum bending stress (compressive) will occur at the edge (4) of the section,

$$f_4=\frac{P\,.\,e_x}{I_{yy}}\times7{\cdot}5+\frac{P\,.\,e_y}{I_{xx}}\times12{\cdot}5$$

$$=\frac{80\times3\times7.5}{1407}+\frac{80\times5\times12.5}{8565}=1.28+0.584$$

$$=1.864 \text{ tonnes/cm}^2$$

Maximum bending stress (tensile) will occur at the edge (2) of the section,

$$f_2=1.864 \text{ tonnes/cm}^2$$

Maximum resultant stress in the section,

$$=0.842+1.864=2.706 \text{ tonnes/cm}^2 \text{ (compressive)}$$

Minimum resultant stress in the section,

$$=0.842-1.864=-1.022 \text{ tonnes/cm}^2 \text{ tensile).}$$

Problem 9·10. A cylindrical chimney shaft 20 m high is of hollow circular section 2·4 m external diameter and 1 m internal diameter. The intensity of the horizontal wind pressure varies as $x^{2/3}$ where x is the height above the ground. Determine the maximum and minimum intensities of stress of the base. Given then

(1) Density of masonry structure is 2240 kg/cm³

(2) Coefficient of wind resistance is 0·6

(3) Wind pressure at a height of 27 m is 180 kg/m².

Solution. Say the intensity of pressure,

$$p=c\,x^{2/3}$$

where c is any constant

At $\quad x=27$ m, $p=180$ kg/m²

So $\quad 180=c\times27^{2/3}=c\times9$

or $\quad c=20$

So the pressure at any height

$$p=20\,x^{2/3}$$

Let us consider a small projected area of thickness δx.

Area $\quad \delta a=2.4\,\delta x$

Pressure $\quad p=20\,x^{2/3}$

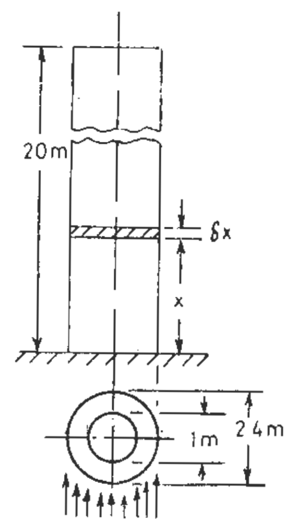


Fig. 9·22

Force on the area,

$$\delta P=kp\,\delta a=48\,kx^{2/3}\,dx$$

where $\quad k=0.6$

Moment of the force about the base,

$$\delta M=\delta P\,.\,x=28.8\,x^{5/3}\,dx$$

Total moment $$M=\int_0^{20}28.8\,x^{5/3}\,dx=\left[\frac{28.8}{8/3}\times x^{8/3}\right]_0^{20}$$

$$=\Big|\, 10{\cdot}8\ x^{8/8}\ \Big|_0^{20}=18\times 20^{8/3}=10{\cdot}8\times 2946 \text{ kg-m}$$

Section modulus, $Z=\frac{\pi}{32}\left(\frac{D^4-d^4}{D}\right)=\frac{\pi}{32}\left(\frac{2{\cdot}4^4-1^4}{2{\cdot}4}\right)=1{\cdot}316 \text{ m}^3$

Bending stress, $f_b=\pm\frac{M}{Z}=\frac{10{\cdot}8\times 2946}{1{\cdot}316}=24176{\cdot}9 \text{ kg/m}^2$

Density of masonry, $\rho=2240$ kg/m^3

Height of the chimney, $H=20$ m

Direct compressive stress on base due to self weight

$=\rho H=2240\times 20=44800$ kg/m^2

Maximum stress at base $=44800+24176{\cdot}9=68976{\cdot}9$ kg/m^2

$=6{\cdot}897$ kg/cm^2 (compressive)

Minimum stress at base $=44800-24176{\cdot}9=20623{\cdot}1$ kg/m^2

$=2{\cdot}062$ kg/cm^2 (compressive).

Problem 9·11. A tapering chimney of hollow circular section is 45 m high. Its external diameter at the base is 3·6 m and at the top it is 2·4 m. It is subjected to the wind pressure of 220 kg/m^2 of the projected area. Calculate the overturning moment at the base. If the weight of the chimney is 600 tonnes and the internal diameter at the base is 1·2 m, determine the maximum and minimum stress intensities at the base.

Solution.

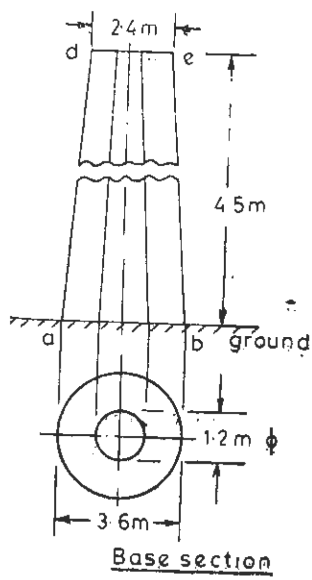


Fig. 9·3

Base. External diameter,

$=3{\cdot}6$ m

Internal diameter $=1{\cdot}2$ m

Area of cross section

$=\frac{\pi}{4}(3{\cdot}6^2-1{\cdot}2^2)$

$=\frac{\pi}{4}(11{\cdot}52)$

$=9{\cdot}048$ m^2

Weight of the chimney,

$=600$ tonnes

Direct compressive stress at base due to self weight,

$$f_d=\frac{600}{9{\cdot}048}=67{\cdot}31 \text{ T/m}^2$$

Wind pressure, $p=220$ kg/m^2

Projected area of the exposed surface,

$$=\left(\frac{3{\cdot}6+2{\cdot}4}{2}\right)\times 45=135 \text{ m}^2$$

Total force due to wind,

$$P=220\times 135=29700 \text{ kg}=29{\cdot}7 \text{ tonnes}$$

Distance of the centroid of the trapezoid *abcd* from the base,

$$=\frac{2{\cdot}4\times 45\times \frac{45}{2}+2\times\left(\frac{3{\cdot}6-2{\cdot}4}{2\times 2}\right)\times 45\times 15}{2{\cdot}4\times 45+2\times\frac{0{\cdot}6\times 45}{2}}$$

$$=\frac{2430+405}{135}=\frac{2835}{135}=21 \text{ m}$$

Bending moment, $M=29{\cdot}7\times 21=623{\cdot}7$ tonne-metres

Section modulus of base,

$$Z=\frac{\pi}{32}\left(\frac{D^4-d^4}{D}\right)=\frac{\pi}{32}\times\left(\frac{3{\cdot}6^4-1{\cdot}2^4}{3{\cdot}6}\right)=4{\cdot}524 \text{ m}^3$$

Bending stress, $f_b=\pm\frac{M}{Z}=\frac{623{\cdot}7}{4{\cdot}524}=137{\cdot}86$ tonne/metre²

Maximum stress at the base,

$$=f_d+f_b=67{\cdot}31+137{\cdot}86$$
$$=205{\cdot}17 \text{ tonne/metre}^2 \text{ (compressive)}$$

Minimum stress at the base,

$$=f_d-f_b=67{\cdot}31-137{\cdot}86=-70{\cdot}55 \text{ tonne/metre}^2 \text{ (tensile)}.$$

Problem 9·12. A masonry pillar D m in diameter is subjected to a horizontal wind pressure of intensity p kg/m². If the coefficient of wind resistance is k, prove that the maximum permissible height for the pillar so that no tension is induced at the base is given in metres by

$$H=\frac{\rho\pi D^2}{16\,kp} \quad \text{where } \rho=\text{density of masonry.}$$

Solution.

Density of masonry $=\rho$ kg/m³

Say permissible height $=H$ m

f_d, direct stress due to self weight

$$=\rho H \text{ kg/m}^2$$

Intensity of wind pressure $=p$ kg/m²

Coefficient of wind resistance $=k$

Diameter of exposed surface $=D$ metre

Wind force $P=kpHD$

Distance of C.G. of wind force from the base,

$$=\frac{H}{2}$$

W, Bending moment due to wind force,

$$=\frac{PH}{2}=\frac{kpDH^2}{2}$$

Z. section modulus at the base $= \frac{\pi D^3}{32}$

Bending stress, $f_b = \pm \frac{M}{Z} = \frac{kpDH^2}{2} \times \frac{32}{\pi D^3}$

$$= \pm \frac{16\ kpH^2}{\pi D^2}$$

For the condition that no tension is induced at the base,

$$f_d \geqslant f_b$$

$$\rho H \geqslant \frac{16\ kpH^2}{\pi D^2}$$

or $$H \leqslant \frac{\rho \pi D^2}{16\ kp}$$

or the maximum permissible height is equal to $\frac{\rho \pi D^2}{16\ kp}$.

SUMMARY

1. A short column of rectangular section with section modulus, $Z_x = BD^2/6$ and Z_y $DB^2/6$ carries the load P with eccentricity e_x or e_y, the resultant stresses in the extreme layers of the section will be

$$f_R = \frac{P}{BD} \pm \frac{P \cdot e_x}{Z_x}$$

where $$Z_x = \frac{I_{yy}}{D/2}, \quad Z_y = \frac{I_{xx}}{B/2}$$

$$f_R = \frac{B}{BD} \pm \frac{Pe_y}{Z_x}.$$

2. A short column of circular cross section of diameter D supports a load P at on eccentricity e from its axis. The resultant stresses developed in the extreme layers of the section will be

$$f_R = \frac{4P}{\pi D^2} \pm \frac{32\ Pe}{\pi D^3}.$$

3. A short column of rectangular section $B \times D$, supports a load P eccentric to both the axes XX and YY. If the eccentricities about XX and YY axes are e_x and e_y and section modulus $Z_x = BD^2/6$ and $Z_y = DB^2/6$ then stresses developed at the four corners of the section are (depending upon the location of the load P)

$$f_{1, 2, 3, 4} = \frac{P}{BD} \pm \frac{Pe_x}{Z_x} \pm \frac{Pe_y}{Z_y}$$

i.e., $$f_1 = \frac{P}{BD} + \frac{Pe_x}{Z_x} - \frac{Pe_y}{Z_y}$$

$$f_2 = \frac{P}{BD} + \frac{Pe_x}{Z_x} + \frac{Pe_y}{P_y}$$

$$f_3 = \frac{P}{BD} - \frac{Pe_x}{Z_x} + \frac{Pe_y}{Z_y}$$

$$f_4 = \frac{P}{BD} - \frac{Pe_x}{Z_x} - \frac{Pe_y}{Z_y}.$$

4. Core or the kernel of a section is a small area located around the centroid of the section of a column and if any vertical load is applied on the column within this area, there will not be any tensile stress developed any where in the section.

5. Core or kernel of a rectangular section $B \times D$ is a rhombous with its centre at the C.G. of the rectangular section and the two diagonals of the rhombous are

$B/3$ in the direction of length B

$D/3$ in the direction of depth D.

6. Corc or kernel of a circular section of diameter D is a circular area with its centre at the C.G. of the section and its diameter equal to $D/4$.

7. If a column is of any section ; square, hollow square, rectangular, hollow rectangular, circular solid and hollow and ρ is the density of the material of the column, then direct compressive stress due to self weight developed at the base of the column will be ρH, where H is the height of the column.

8. For a wall of rectangular section $B \times D$, wind pressure p acting on face of breadth B, stress due to bending moment created by the wind pressure will be $\pm 3pH^2/D^2$ at the base of the wall.

9. For a chimney of hollow circular section, outer diameter D, inner diameter d, height H.

Stresses at the base of the chimney due to hending moment created by wind pressure p,

$$f_b = \pm \frac{M}{Z}$$

where
$$Z = \frac{\pi (D^4 - d^4)}{32\, D}$$

$$M = \frac{pDH^2}{3} \quad \text{if coefficient of wind resistance} = \frac{2}{3}.$$

MULTIPLE-CHOICE QUESTIONS

1. A short cast iron column of 20 cm diameter is subjected to a vertical load P passing at a distance e from the CG of the section. What is the maximum value of e if no tensile stress is developed any where in the section

   (a) 4 0 cm (b) 3·0 cm
   (c) 2·5 cm (d) 2·0 cm.

2. A short column of square section of the side 20 cm carries a vertical load of 40 kN at a distance of 2 cm from its CG along one symmetric axis. The maximum stress developed in the section is

   (a) 240 N/mm² (b) 160 N/cm²
   (c) 80 N/cm² (d) None of the above.

3. A short cast iron column of circular section with area equal to 40 cm² subjected to a thrust 0·40 tonnes. Thrust is applied at a point 2 cm away from the centroid on the axis passing through the centroid. The section modulus of circular section is 80 cm³. The maximum stress at the extreme layer on the Y-Y axis passing through the centroid is

   (a) 20 kg/cm² (b) 15 kg/cm²
   (c) 13 kg/em² (d) None of the above.

4. A hollow cast iron column of circular section carrries a vertical load at a distance of e from the centroid of the section. The section modulus is 1·2 m³. The maximum and minimum intensities of the stress developed in the section are 42000 kg/m² and 12000 kg/m² (both compressive). The magnitude of the bending moment on the section is

   (*a*) 42000 kg-m (*b*) 36000 kg-m
   (*c*) 18000 kg-m (*d*) 14400 kg-m.

5. A short column is of hollow square section with outer side 2 a and inner side a. A load P acts at a distance of $a/4$ from the CG of the section, and along one diagonal. The maximum and minimum stresses at the corners of the section are 48 kN/m² and −12 kN/m². The bending stress introduced at the extreme corners of the section, by the eccentric load is

   (*a*) ±24 kN/m² (*b*) ±18 kN/m²
   (*c*) ±12 kN/m² (*d*) None of the above.

6. A short masonry square section 1 m side is 10 m high. Wind pressure of 200 kg/m² acts on one vertical face of the column. The weight density of masonry is 2000 kg/m³. The greatest stress acting at the base of the column is

   (*a*) 80 tonnes/m² (*b*) 40 tonne/m³
   (*c*) 20 tonnes/m² (*d*) None of the above.

7. A masonry chimney of hollow circular section is 10 m high. Outside diameter of chimney is 1 m and inside diameter is 0·5 m. The compressive stress on the base of the column due to its own weight is given by (if the weight density of masonry is 2200 kg/m³).

   (*a*) 44 kg/cm² (*b*) 2·75 kg/cm²
   (*c*) 2·2 kg/cm² (*d*) 1·65 kg/cm².

8. For a cylindrical chimney of hollow circular section subjected to wind pressure, the coefficient of wind resistance for calculating the total wind force on the chimney is generally taken

   (*a*) 0·3—0·5 (*b*) 0·45—0·6
   (*c*) 0·6—0·75 (*d*) 0·75—0·90

9. A short column of circular section of diameter D supports a load P at an eccentricity e from its axis. The maximum and minimum stresses developed in the section are 8 kg/cm² and 3 kg/cm². If the eccentricity is doubled and load remains. The same, the maximum stress developed in the section will be

   (*a*) 10·5 kg/cm² (*b*) 8·5 kg/cm²
   (*c*) 7·5 kg/cm² (*d*) 6·0 kg/cm².

10. A cylindrical chimney of hollow circular cross section is subjected to wind pressure p. The height of the chimney is H metres and density of masonry is 2000 kg/m³. The maximum and minimum stresses developed at the base of chimney are 65000 kg/cm² and 15000 kg/m². If the intensity of wind pressure increases by 50%, the maximum stress developed at the base will be

    (*a*) 1,30,000 kg/m² (*b*) 1,19,000 kg/m²
    (*c*) 77,500 kg/m² (*d*) 70,000 kg/m².

ANSWERS

1. (*c*).	2. (*b*).	3. (*a*).	4. (*c*).	5. (*d*).
6. (*a*).	7. (*c*).	8. (*c*).	9. (*a*).	10. (*c*).

EXERCISE

9·1. A flate plate of section 3 cm × 8 cm is placed in a testing machine and is subjected to 12 tonnes of force along a line as shown in Fig. 9·24. An extensometer adjusted along the line of force recorded an extension of 0·118 mm on a guage length marked of 200 mm. Determine

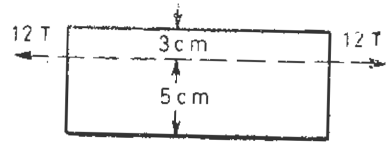


Fig. 9·24

(*a*) maximum and minimum stresses set up in plate

(*b*) Young's modulus of the material of the plate.

[**Ans.** (*a*) 875,125 kg/cm², (*b*) $1·006 \times 10^6$ kg/cm²]

9·2. A cantilever hydraulic crane is required to lift a load of 5 tonnes as shown in Fig. 9·25. The single rope supporting the load passes over two pulleys and then vertically down the axis of the crane to the hydraulic apparatus. The section of the crane at *CD* is also shown. Determine the maximum and minimum stress intensities in the section.

[**Ans.** 1·5235 tonne/cm² (compressive), −1·4865 tonne/cm² (tensile)]

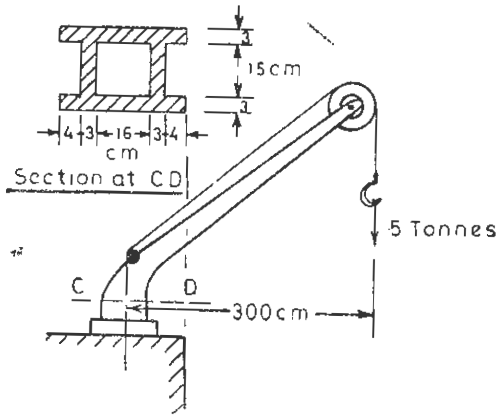


Fig. 9·25

9·3. A rectangular plate 2 cm thick containing a square hole of 2 cm side as shown in Fig. 9·26, is subjected to an axial pull of 3 tonnes as shown. Determine the greatest and least tensile stresses at the critical section of the plate.

[**Ans.** 388·06 kg/cm², 91·81 kg/cm²]

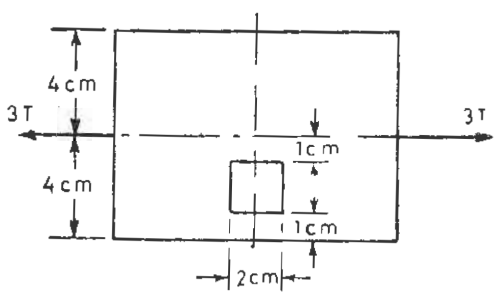


Fig. 9·26

9·4. A short column of hollow square section of inner side a and outer side A is loaded with a compressive load W. Determine the maximum distance of the point of application of the load from the CG of the section, along the diagonal so that the tensile stress does not exist at any point of the cross section if $A = 1{\cdot}4\ a$. [**Ans.** $0{\cdot}178\ A$]

9·5. A short cast iron column has an external diameter of 24 cm and internal diameter of 16 cm, the distance between the centres of the two circles due to the displacement of the core during casting is 1 cm. A load of 100 tonnes acts through the vertical line passing through the centre of the inner circle. Determine the greatest and least compressive stresses in a horizontal cross section of the column, neglecting the weight of the column.

[**Ans.** 579 kg/cm², (compressive) 239·5 kg/cm² (tensile)]

9·6. A steel rod 3 cm diameter passes through a cast iron tube 4 cm internal diameter and 6 cm external diameter, 200 cm long. Rigid cover plates are provided at each end of the tube and steel rod passes through these cover plates also. Nuts are tightened on the projecting ends of the rod, so that the cover plates bear on the ends of the tube. Determine the maximum stress in the cast iron tube, if one of the nuts is tightened to produce a stretch of 2·5 mm in the rod.

(*a*) if the rod is concentric with the tube.

(*b*) if the centre of the rod is 4 mm out of the centre of the tube.

Given $E_{steel} = 2100$ tonnes/cm²

$E_{C.I.} = 1080$ tonnes/cm²

[**Ans.** (*a*) 1·18 tonnes/cm², (*b*) 1·616 tonnes/cm²]

9·7. The cross section of a short column is as shown in the Fig. 9·27. A vertical load W tonnes acts at the point P. Determine the magnitude of the load W and the eccentricity e if the stresses at the points C and D are 1200 kg/cm² and 800 kg/cm² compressive respectively. [**Ans.** 48 tonnes ; 0·333 cm]

9·8. The cross section of a short vertical column is as shown in the Fig. 9·28. A vertical load of 12 tonnes is applied at the point P. Determine the stresses at the corners A, B, C, and D.

[**Ans.** 53·2 kg/cm² (tensile), 142·6 kg/cm² (compressive), 303·2 kg/cm² (compressive), 107·4 kg/cm² (compressive)]

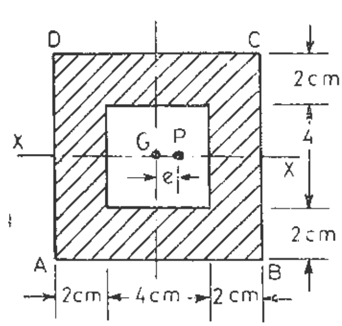


Fig. 9·27

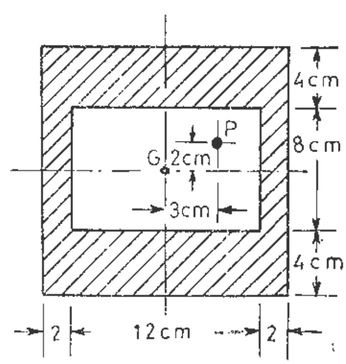


Fig. 9·28

9·9. A rolled steel I section, flanges 10 cm $\times$ 2 cm and web 26 cm $\times$ 1 cm is used as a short column to carry a load of 100 kN. The load acts eccentrically, 5 cm to the left side of axis *YY* passing through the centre of the web and 6 cm above the axis *X-X* passing through the centroid of the section. Find the maximum and minimum stress intensities induced in the section. [**Ans.** 112·6 N/mm^2 (compressive), $-82·3$ N/mm^2 (tensile)]

9·10. A cylindrical chimney 30 m high, external diameter 2 m and internal diameter 1 m is exposed to wind pressure whose intesity varies as the square root of the height above the ground. At a height of 9 m, the intensity of wind pressure is 18 kg/m^2. If the coefficient of wind resistance is 0·6, calculate the bending moment at the foot of the chimney. If the density of masonry structure is 2280 kg/m^3, what is the maximum stress developed at the base of the chimney. [**Ans.** 14195·52 kg-m, 81679·5 kg/m^2]

9·11. A tapering chimney of hollow circular section is 24 m high. Its external diameter at the base is 4 m and at the top it is 2·0 m. It is subjected to a wind pressure of 240 kg/m^2 of the projected area. If the weight of the chimney is 380 tonnes, and internal diameter at the base is 2 m, determine the maximum and minimum stress intensities at the base. [**Ans.** 71·61 T/m^2 (compressive), 9·028 T/m^2 (compressive)]

10

Distribution of Shear Stress in Beams

In the chapter 7, we have studied about the Shear Force and Bending Moment diagrams of beams and cantilevers and learnt that in a portion of the beam where B.M. is constant, there is no shear force and for an infinitesinal length of the beam where there is variation of BM, $\delta M = -F\delta x$ where δM is the change in BM along the length δx and F is the shear force transverse to the axis of the beam. Further in chapter 8 we have studied about the longitudinal stress f developed in the section due to bending moment M. When there is variation in M, there will be variation in f on both the sides of elementary length δx of the beam as shown in the Fig. 10·1. This figure shows a beam of reetangular section subjected to bending moment producing coneavity upwards in the beam. The neutral axis passes

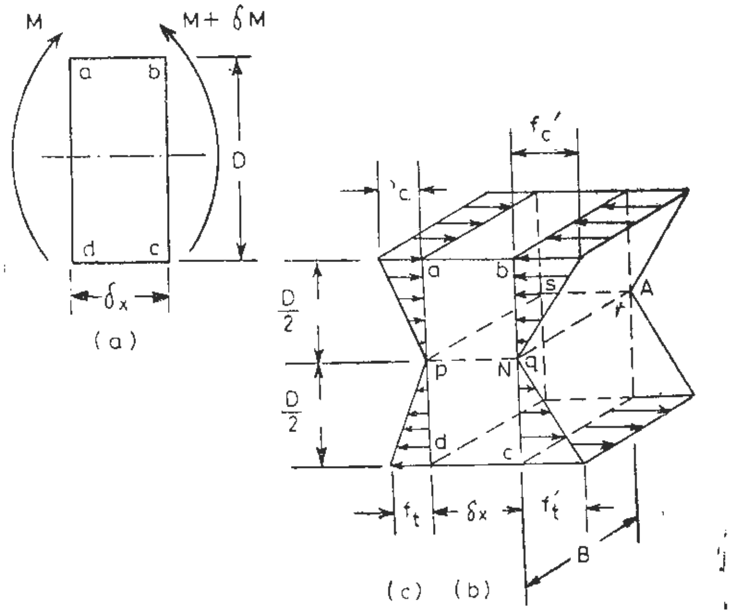


Fig. 10·1

symmetrically through the section. Upper half of the section will be under compression and lower half will be under tension.

The Fig 10·1 (*b*) shows the distribution of longitudinal stress across the thickness of the section. The compressive stress in extreme layer *ab* on one side *i.e.*, f_c' is greater than f_c on the other side. Similarly, the tensile stress in the extreme layer *cd* on the lower side *i.e.*, f_t' is greater than f_t on the other side. Due to the difference in longitudinal stresses on both the sides, there will be difference in resultant pull or push on the two sides which will be balanced by a horizontal shear force developed on the longitudinal plane of the beam or horizontal shear stress is developed on the horizontal section. Consider a rectangular block at a distance of y from the neutral axis and upto the extreme layer *ab*. The stress intensities on both the sides of layer *fe i.e.*, f_{ce} and f_{cf} are smaller than f_c' and f_c respectively as shown in Fig. 10·2. The resultant push F_c' on the right side of the section is greater than the resultant push on the left side of the section. For equilibrium, a shear force is developed on the horizontal plane *fegh*. Say the intensity of shear stress on this plane is q, then

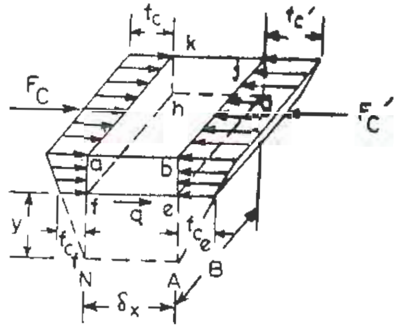


Fig. 10·2

$$F_c' - F_c = q \times \delta x \times B$$

or shear stress, $$q = \frac{F_c' - F_c}{B\,\delta x}.$$

With the help of flexure formula derived in chapter 8, we can determine f_c' and f_c or F_c' and F_c and the magnitude of shear stress q can be determined. It can be further observed that intensity of shear stress on plane *abjk* will be zero and the intensity of shear stress on the plane *pqrs* on the neutral axis Fig. 10·1 (*b*) will be maximum in this case of rectangular section.

10·1. SHEAR STRESS DISTRIBUTION

Consider a small length δx of a beam subjected to bending moment producing concavity upwards. Fig. 10·3 shows a trapezoidal section of a beam. Say NA is the neutral axis and M is the bending moment on the left side and $M+\delta M$ is the bending moment on the right side

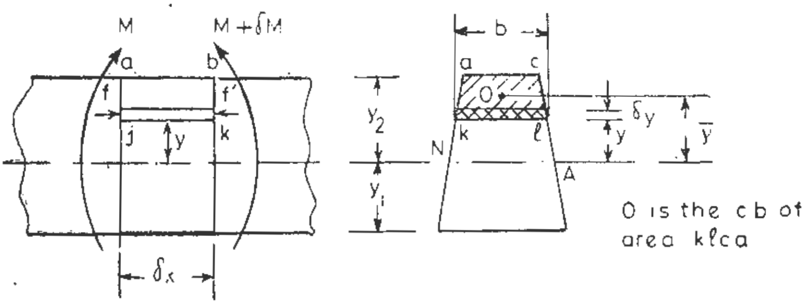


Fig. 10·3

of the element of the small length δx considered. Now consider a layer of thickness δy, at a distance of y from the neutral axis (as shown in Fig. 10·3.)

Due to the type of the bending moment shown, the upper portion of the section above NA will be in compression and the lower portion of the section below NA will be in tension.

Stress due to bending on the left side of the element,

$$f=\frac{M}{I_{NA}}\cdot y$$

Stress due to bending on the right side of the element,

$$f'=\frac{M+\delta M}{I_{NA}}\cdot y$$

where I_{NA}=moment of inertia of the section about the neutral axis

Area of the elementary layer,

$$\delta a=b\ \delta y$$

where b is the breadth of the layer under consideration.

Compressive force on the left side of the section,

$$\delta F=f\ .\ \delta a=f\ .\ b\ \delta y$$

Compressive force on the right side of the section,

$$\delta F'=f'\ \delta a=f'\ b\ \delta y$$

Unbalanced force on the elementary slice of length δx and thickness δy

$$=\delta F'-\delta F$$

$$=\left(\frac{M+\delta M}{I}\ y-\frac{M}{I}\ y\right)b\ \delta y$$

$$=\frac{\delta M}{I}\ y\ .\ \delta a$$

Summing up all these unbalanced forces on all elementary areas from y to y_2 we get

$$\delta F=\sum_{y}^{y_2}\frac{\delta M}{I}\ .\ y\ .\ \delta a$$

$$q\ .\ b\ \delta x=\frac{\delta M}{I}\sum_{y}^{y_2} y\ \delta a$$

where q is the shear stress developed on the horizontal plane $b\times\delta x$ to resist the unbalanced force δF for equilibrium.

So $$q\ .\ b\ .\ \delta x=\frac{\delta M}{I}\cdot\sum_{y}^{y_2} y\ \delta a$$

$$q=\frac{1}{Ib}\cdot\frac{\delta M}{\delta x}\cdot\sum_{y}^{y_2} y\ \delta a$$

where $\sum_{y}^{y_2} y\ .\ \delta a$ = first moment of area of the section above the layer kl and upto the extreme layer ca.

= first moment of the elementary area *klca* about the neutral axis

$= A\,.\,\bar{y}$

where A = area *klca*

$\bar{y}$ = distance of CG of *klca* from the neutral axis.

Shear stress or horizontal plane,

$$q = \frac{\delta M}{\delta x} \cdot \frac{A\bar{y}}{Ib}$$

$$= \frac{Fa\bar{y}}{Ib} \text{ because } \lim_{\delta x \to 0} \frac{\delta M}{\delta x} = \frac{dM}{dx} = F$$

where F is the transverse shear force on the section.

Example 10·1-1. A round beam of circular cross section of diameter D is simply supported at its ends and carries a load W at its centre. Determine the magnitude of the shear stress along the plane passing through the neutral axis, if the particular section lies at a distance of $l/4$ from one end, where l is the span length.

Solution. For a simply supported and centrally loaded beam, SF diagram is shown in Fig. 10·4. At the section X-X, at a distance of $l/4$ from one end, shear force,

$$F = \frac{W}{2}$$

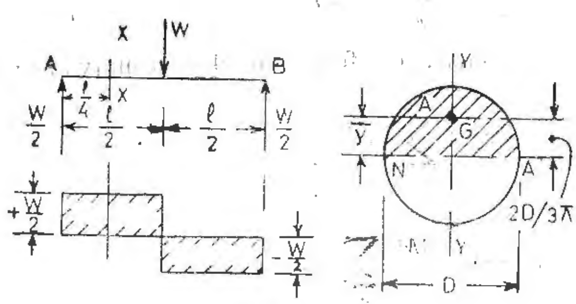


Fig. 10·4

Shear stress at NA, $q = \frac{Fa\bar{y}}{I_{NA}b}$

where $a = \frac{\pi D^2}{8}$, area *NYA*, above N.A. (neutral axis)

$\bar{y}$ = C.G. of the area from N.A. $= \frac{2D}{3\pi}$

$I_{NA} = \frac{\pi D^4}{64}$, moment of inertia about neutral axis

b = breadth of the section at the layer under consideration

$= D$ (in the present case)

Shear stress at neutral axis,

$$q=\frac{W}{2}\times\frac{\pi D^2}{8}\times\frac{2D}{3\pi}\times\frac{64}{\pi D^4}\times\frac{1}{D}$$

$$=\frac{8}{3\pi}\times\frac{W}{D^2}.$$

Exercise 10·1-1. A beam is of rectangular cross section of breadth B and depth D. At a particular section of the beam, the shear force is F. Determine the intensity of shear stress at the neutral axis.

Ans. $\left[\frac{3F}{2BD}\right]$

10·2. SHEAR STRESS DISTRIBUTION IN A CIRCULAR SECTION OF BEAM

Consider that a beam of circular section of diameter D has shear force F at a particular section. The neutral axis of the section will pass through the centre O as shown in Fig 10·5 shear stress at any layer,

$$q=\frac{Fa\bar{y}}{Ib}$$

Let us consider a layer at a distance of y from the neutral axis, subtending angle θ at the centre. Then

$$\sin\theta=\frac{y}{R} \quad \text{or} \quad y=R\sin\theta$$

and $$dy=R\cos\theta\, d\theta$$

$$R=\frac{D}{2}=\text{radius of the circle}$$

breadth $b=2R\cos\theta$

$$a\bar{y}=\int_y^R b\,\delta y.y$$

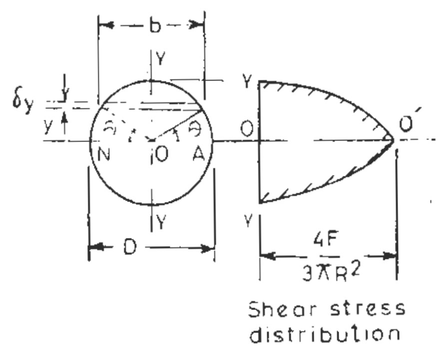


Fig. 10·5

$$=\int_y^R 2R\cos\theta.R\sin\theta.\delta y=\int_\theta^{\pi/2} 2R^2\cos\theta.\sin\theta(R\cos\theta)d\theta$$

$$=2R^3\int_0^{\pi/2}\cos^2\theta.\sin\theta\,d\theta=2R^3\left|\frac{\cos^3\theta}{3}\right|_{\theta}^{\pi/2}$$

$$=\frac{2R^3}{3}\left[\cos^3\frac{\pi}{2}-\cos^3\theta\right]=-\frac{2R^3\cos^3\theta}{3}$$

$$q=-\frac{F\times 2R^3\cos^3\theta}{3(2R\cos\theta)}\times\frac{4}{\pi R^4}=-\frac{4}{3}\frac{F\cos^2\theta}{\pi R^2}$$

Say $\theta=0°$, $\quad q=-\dfrac{4F}{3\pi R^2}$

$\theta=30°$, $\quad q=-\dfrac{F}{\pi R^2}$

$\theta=45°$, $\quad q=-\dfrac{2F}{3\pi R^2}$

$\theta=90°$ $\quad q=0$

Fig. 10·5 shows the shear stress distribution which is symmetrical about the X-X axis or the neutral axis.

Example 10·2-1. At a particular section of a beam carrying transverse loads, the shear force is 40kN. The section of th beam is circular of diameter 80 mm. Draw the shear stress distribution curve along the vertical axis passing through the centre of the section.

Solution. Diameter$=80$ mm

Radius, $R=40$ mm

Shear force, $F=40$ kN$=40000$ N

So $\dfrac{4}{3}\times\dfrac{F}{\pi R^2}=\dfrac{4}{3}\times\dfrac{40000}{\pi\times40\times40}=10{\cdot}61$ N/mm²

Shear stress at any layer sub-tending angle θ at the centre

$$=\frac{4F}{3\pi R^2}\cos^2\theta$$

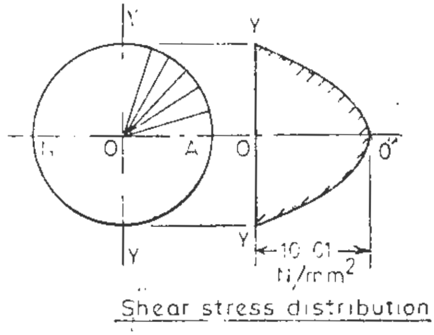


Fig. 10·6

At $\theta=0°$, $q=10{\cdot}61$ N/mm²

$\theta=30°$, $q=7{\cdot}957$ N/mm²

$\theta=45°$, $q=5{\cdot}305$ N/mm²

$\theta=60°$ $q=2{\cdot}6525$ N/mm²

$\theta=75°$ $q=0{\cdot}71063$ N/mm²

$\theta=90°$ $q=0$

The section is symmetrical about the neutral axis, therefore the shear stress distribution diagram is symmetrically repeated below the neutral axis as shown in Fig. 10·6.

Exercise 10·2-1. A beam of circular section 5 cm radius subjected to transverse loads, has the shear force 8 tonnes at a particular section. Draw the shear stress distribution diagram along a vertical diameter of the section.

Ans.

θ	0	30	60	90	
q	0·136	·102	·029	0	tonne/cm²

10·3. SHEAR STRESS DISTRIBUTION IN A BEAM OF RECTANGULAR SECTION

Fig. 10·7 shows a rectangular section of breadth B and depth D. Say at a particular section of the beam, shear force is F.

Shear stress at any layer,

$$q=\frac{Fa\bar{y}}{I_{NA}b}$$

a=area of the section above the layer

$\bar{y}$=distance of the CG of the area a from the neutral layer

$$I_{NA}=\frac{BD^3}{12}$$

$b=B$ (breadth at the section)

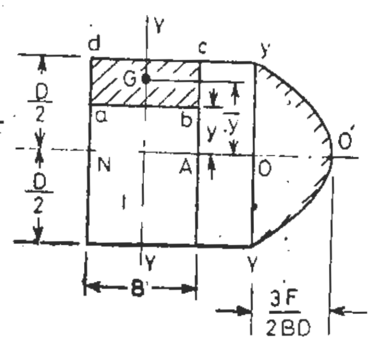


Fig. 10·7

Now $$\bar{y}=\left(y+\frac{\frac{D}{2}-y}{2}\right)=\left(\frac{D}{4}+y\right)$$

area, $$a=B\left(\frac{D}{2}-y\right)$$

$$a\bar{y}=\frac{1}{2}B\left(\frac{D}{2}-y\right)\left(\frac{D}{2}+y\right)=\frac{B}{2}\left(\frac{D^2}{4}-y^2\right)$$

or $$q=\frac{F\times 12}{BD^3\times B}\times\frac{B}{2}\left(\frac{D^2}{4}-y^2\right)=\frac{6F}{BD^3}\left(\frac{D^2}{4}-y^2\right)$$

at $$y=0, \qquad q=\frac{3}{2}\frac{F}{BD}$$

$$y=\frac{D}{4}, \qquad q=\frac{9}{8}\frac{F}{BD}$$

$$y=\frac{D}{2}, \qquad q=0.$$

The section is symmetrical about neutral axis, therefore shear stress distribution diagram is repeated symmetrically below the neutral axis as shown.

Maximum shear stress occurs at NA,

$$q_{max}=1{\cdot}5\,\frac{F}{BD}$$

but $$\frac{F}{BD}=q_{mean}, \text{ mean shear stress}$$

so $$q_{max}=1{\cdot}5\,q_{mean}$$

Example 10·3-1. A wooden beam of rectangular section 20 cm × 30 cm is used as a simply supported beam carrying uniformly distributed of ω tonnes/metre. What is the maximum value of ω if the maximum shear stress developed in the beam section is 50 kg/cm² and span length is 6 metres.

Solution. Maximum shear stress at neutral axis.

$$q_{max}=\frac{1{\cdot}5F}{BD}$$

so $$50=\frac{1{\cdot}5\times F}{20\times 30}$$

or $$F=\frac{50\times 600}{1{\cdot}5}=20{,}000 \text{ kg}$$

Now for a simply supported beam, carrying uniformly distributed load,

Maximum shear force

$$=\frac{\omega L}{2}$$

$$L=6 \text{ metres}$$

$$F_{max} = \frac{\omega \times 6}{2} = 3\ \omega \text{ tonnes}$$

$$= 3000\ w \text{ kg}$$

or $$3000\omega = 20{,}000$$

$$\omega = \frac{20{,}000}{3000} = \frac{20}{3} \text{ tonnes/metre run}$$

Permissible rate of loading = 6·667 tonnes/metre run.

Exercise 10·3-1. A wooden beam of square cross-section 20×20 cm is used as a cantilever of 3 metre length. How much load can be applied at the end of the cantilever if shear stress developed in the section is not to exceed 10 N/mm². What is the shear stress developed at a depth of 5 cm from the top.

Ans. [266·67 kN, 7·5 N/mm²]

10·4. SHEAR STRESS DISTRIBUTION IN A HOLLOW CIRCULAR SECTION

Fig. 10·8 shows a hollow circular section with outer radius R_2 and inner radius R_1. Say the section of a beam is a hollow circular section and subjected to a shear force F.

Shear stress at any layer

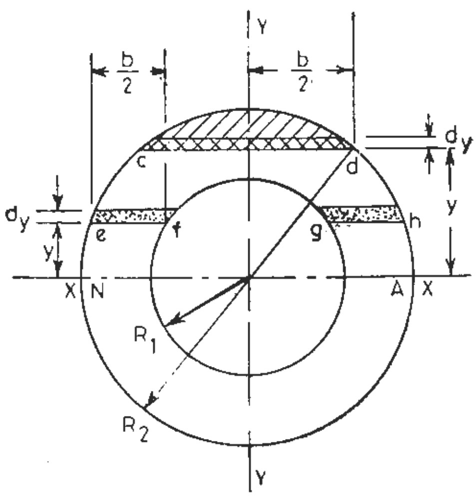


Fig. 10·8

$$q = \frac{Fa\bar{y}}{I_{NA}.b}$$

and $$I_{NA} = \frac{\pi}{4}(R_2^4 - R_1^4)$$

Case I, $y > R_1$

Consider a layer cd at a distance of y from the neutral axis.

Width of the layer, $b = 2(R_2^2 - y^2)^{1/2}$

Shear stress $$q=\frac{F}{I_{NA}b}\int_y^{R_2} by\,dy$$

$$=\frac{F}{I_{NA}\,2\sqrt{R_2^2-y^2}}\int_y^{R_2} 2[R_2^2-y^2)^{1/2}y\,dy$$

$$=\frac{F}{2I_{NA}\sqrt{R_2^2-y^2}}\left|-\frac{2}{3}(R_2^2-y^2)^{3/2}\right|_y^{R_2}$$

$$=\frac{F}{2I_{NA}\sqrt{R_2^2-y^2}}\left|0+\frac{2}{3}(R_2^2-y^2)^{3/2}\right|$$

$$=\frac{F}{3I_{NA}}[R_2^2-y^2]$$

$$=0 \text{ at } y=R_2$$

$$=\frac{F}{3I_{NA}}(R_2^2-R_1^2) \text{ at } y=R_1$$

Case II. $y<R_1$

$$\frac{b}{2}=\sqrt{R_2^2-y^2}-\sqrt{R_1^2-y^2}$$

Shear stress at any layer

$$q=\frac{F}{I_{NA}\,.\,b}\int_y^{R_2} by\,dy$$

$$=\frac{F}{I_{NA}\,.\,b}\left[\int_y^{R_1}(\sqrt{R_2^2-y^2}-\sqrt{R_1^2-y^2})\,2\,.\,y\,dy\right.$$

$$\left.+\int_{R_1}^{R_2} 2\sqrt{(R_2^2-y^2)}\,y\,dy\right]$$

$$=\frac{F}{I_{NA}\,.\,b}\left[\left|-\frac{2}{3}(R_2^2-y^2)^{3/2}+\frac{2}{3}(R_1^2-y^2)^{3/2}\right|_y^{R_1}\right.$$

$$\left.+\left|\frac{2}{3}-(R_2^2-y^2)^{3/2}\right|_{R_1}^{R_2}\right]$$

$$=\frac{F}{I_{NA}\,.\,b}\left[-\frac{2}{3}(R_2^2-R_1^2)^{3/2}+\frac{2}{3}(R_2^2-y^2)^{3/2}\right.$$

$$\left.-\frac{2}{3}(R_1^2-y^2)^{3/2}\right]+\frac{F}{I_{NA}\,.\,b}\left[+\frac{2}{3}(R_2^2-R_1^2)^{3/2}\right]$$

$$= \frac{F}{I_{NA} \cdot b}\left[\frac{2}{3}(R_2^2-y^2)^{3/2} - \frac{2}{3}(R_1^2-y^2)^{3/2}\right]$$

$$= \frac{2F}{3I_{NA}} \frac{[(R_2^2-y^2)^{3/2}-(R_1^2-y^2)^{3/2}]}{[2\sqrt{R_2^2-y^2}-2\sqrt{R_1^2-y^2}]}$$

$$= \frac{F}{3I_{NA}}\left[\frac{(R_2^2-y^2)^{3/2}-(R_1^2-y^2)^{3/2}}{(R_2^2-y^2)^{1/2}-(R_1^2-y^2)^{1/2}}\right]$$

$$q_{max} = \frac{F}{3I_{NA}} \times \left(\frac{R_2^3-R_1^3}{R_2-R_1}\right) \quad \text{at} \quad y=0$$

$$q = \frac{F}{3I_{NA}} \frac{(R_2^2-R_1^2)^{3/2}}{(R_2^2-R_1^2)^{1/2}} \quad \text{at} \quad y=R_1$$

$$= \frac{F}{3I_{NA}} \cdot (R_2^2-R_1^2) \quad \text{at} \quad y=R_1$$

Substituting the value of

$$I_{NA} = \frac{\pi}{4}(R_2^4-R_1^4)$$

$$q_{max} = \frac{F}{3} \times \frac{4}{\pi} \frac{(R_2^3-R_1^3)}{(R_2^4-R_1^4)(R_2-R_1)}$$

$$q_{mean} = \frac{F}{\text{area}} = \frac{F}{\pi(R_2^2-R_1^2)}$$

$$\frac{q_{max}}{q_{mean}} = \frac{4}{3} \times \frac{R_2^3-R_1^3}{(R_2^2+R_1^2)(R_2-R_1)} = \frac{4}{3} \times \frac{R_2^2+R_2R_1+R_1^2}{R_2^2+R_1^2}$$

if the section is very thin $R_1 \simeq R_2$

$$\frac{q_{max}}{q_{mean}} = 2$$

Example 10·4-1. A hollow circular section of a beam with inner radius 20 mm and outer radius 40 mm is subjected to a transverse shear force of 40000 N. Draw the shear stress distribution over the depth of the section.

Solution. Moment of inertia,

$$I_{NA} = \frac{\pi}{4}(40^4-20^4) = 60\pi \times 10^4 \text{ mm}^4$$

Shear Force, $F = 40000$ N

$$\frac{F}{I_{NA}} = \frac{40000}{60\pi \times 10^4} = 2{\cdot}12 \times 10^{-2} \text{ N/m}^4$$

Inner radius $R_1 = 20$ mm

Outer radius $R_2 = 40$ mm

Shear stress distribution

at $y = R_2 \quad q = 0$

$$y = 30 \text{ mm}, \; q = \frac{F}{3I_{NA}}[R_2^2-y^2] = \frac{2{\cdot}12 \times 10^{-2}}{3}[40^2-30^2]$$

$(y > R_1) \quad = 4{\cdot}95 \text{ N/mm}^2$

$$y=20 \text{ mm}, \quad q=\frac{F}{3I_{NA}}[40^2-20^2]=\frac{2{\cdot}12\times10^{-2}}{3}[1200]$$

$$=8{\cdot}96 \text{ N/mm}^2$$

$$y=10 \text{ mm}, \quad q=\frac{F}{3I_{NA}}\left[\frac{(R_2^2-y^2)^{3/2}-(R_1^2-y^2)^{3/2}}{(R_2^2-y^2)^{1/2}-(R_1^2-y^2)^{1/2}}\right]$$

$(y<R_1)$

$$=\frac{2{\cdot}12\times10^{-2}}{3}\left[\frac{(40^2-10^2)^{2/2}-(20^2-10^2)^{3/2}}{(40^2-10^2)^{1/2}-(20^2-10^2)^{1/2}}\right]$$

$$=\frac{2{\cdot}12\times10^{-2}}{3}\left[\frac{58094{\cdot}8-5196{\cdot}2}{38{\cdot}73-17{\cdot}32}\right]=17{\cdot}47 \text{ N/mm}^2$$

$$y=0, \qquad q=\frac{F}{3\,I_{NA}}\left[\frac{40^3-20^3}{40-20}\right]=\frac{2{\cdot}12\times10^{-2}}{3}\left[\frac{5600}{20}\right]$$

$$=19{\cdot}78 \text{ N/mm}^2$$

Fig. 10·9 shows the shear stress distribution across the depth of the section.

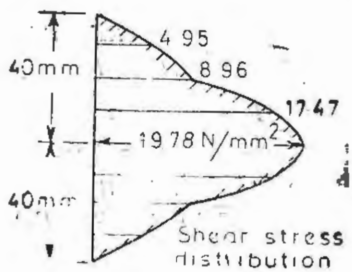


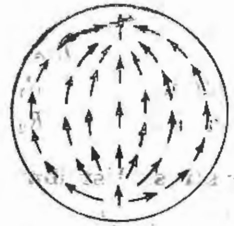
Fig. 10·9

Exercise 10·4-1. A cantilever is of hollow circular section, with outer diameter 10 cm and inner diameter 4 cm. At a particular section SF is 5 tonnes. Draw the shear stress distribution diagram across the depth of the section.

[**Ans.** $q_0=135{\cdot}97$ kg/cm², $q_1=123{\cdot}73$ kg/cm² $q_2=73{\cdot}20$ kg/cm²
$q_3=55{\cdot}770$ kg/cm², $q_4=31{\cdot}37$ kg/cm, $q_5=0$]

10·5. DIRECTIONAL DISTRIBUTION OF SHEAR STRESS

In chapter 1 we have learnt that every shear stress is accompanied by an equal complementary shear stress on planes at right angles. The directions of shear stresses on an element are either both towards the corner or both away from the corner to produce balancing couples. Moreover, near a free boundary, the shear stress on any section acts in a direction parallel to the boundary. This is due to the reason that if there were a shear stress in a direction perpendicular to the boundary, then it would require a complementary shear stress in the direction parallel to the boundary.



Directional distribution of shear stress

Fig. 10·10

The Fig. 10·10 shows a solid circular section subjected to shear force. The shear stress distribution along the boundary is parallel to the boundary and at the centre, shear stress direction is perpendicular to the boundary so as to provide complementary shear stress to to the shear stress along the boundary.

Fig. 10·11 shows the directional distribution of shear stress in I-section. Since the shear stress direction has to follow the boundary in flanges and in the web, the shear stress distribution must be of the form shown, *i.e.* horizontal in flanges and vertical in the web.

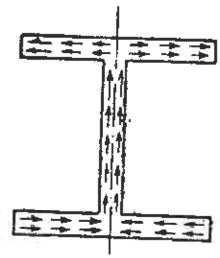


Fig. 10·11

Problem 10·1. A 12·5×30 cm RSJ of I section with flanges 12·5×1·2 cm and web 27·6×1 cm is subjected to a bending moment M and a shear force F. What percentage of M is carried by the flanges and what percentage of F is carried by the web.

Draw the shear stress distribution over the depth of the section.

Solution. The section is symmetrical about XX and YY axes and its G lies at the centre of the web as shown in Fig. 10·12(a).

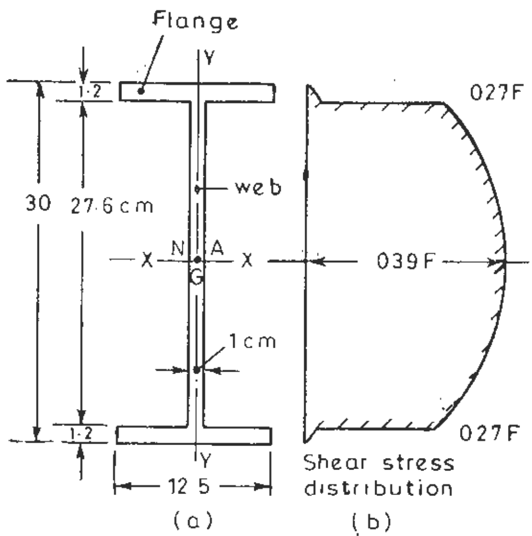


Fig. 10·12

$$\text{Moment of Inertia,} \quad I_{xx} = \frac{12·5 \times 30^3}{12} - \frac{11·5 \times 27·6^3}{12}$$

$$= 28125 - 20148·55 = 7976·45 \text{ cm}^4$$

Flanges. Direct stress due to M,

$$f=\frac{M.y}{I_{xx}}$$

where y is the distance of the layer from N.A.

Consider a layer in the flange,

$$f=\frac{My}{I_{xx}} \text{ where } y=13{\cdot}8 \text{ to } 15 \text{ cm.}$$

Say the thickness of the layer $=dy$

Force in layer, $\delta P=12{\cdot}5\ dy\ .\ f$

$$=\frac{My}{I_{xx}}\times 12{\cdot}5\ dy$$

Moment of δP about N.A.,

$$\delta M=\frac{My^2}{I_{xx}}\times 12{\cdot}5\ dy$$

Total moment shared by flanges,

$$M'=2\int_{13\cdot 8}^{15}\frac{My^2}{I_{xx}}\times 12{\cdot}5\ dy$$

$$=\frac{25}{I_{xx}}M\int_{13\cdot 8}^{15}y^2\,dy=\frac{25\ M}{I_{xx}}\left|\frac{y^3}{3}\right|_{13\cdot 8}^{15}$$

$$=\frac{25\ M}{7976{\cdot}45}\ [1125-876]=0{\cdot}78\ M$$

i.e. B.M. shared by flanges is 78%.

Web. Shear stress at any layer at a distance y from the NA,

$$q=\frac{Fa\bar{y}}{I_{xx}.b}=\frac{F}{7976{\cdot}45\times 1}\Big[12{\cdot}5\times 1{\cdot}2\times 14{\cdot}4 +(13{\cdot}8-y)\left(y+\frac{13{\cdot}8-y}{2}\right)\Big]$$

$$=\frac{F}{7976{\cdot}45}\left[311{\cdot}22-\frac{y^2}{2}\right] \text{ since } I_{xx}=I_{NA}$$

Shear force in the elementary layer of thickness δy and breadth 1 cm considered in web.

$$dF=q\times 1\times dy=\frac{F}{7976{\cdot}45}\ [311{\cdot}22-0{\cdot}5\ y^2]\ dy$$

Shear force shared by web

$$=\frac{F}{7976{\cdot}45}\times 2\int_{0}^{13\cdot 8}(311{\cdot}22-0{\cdot}5\ y^2)\ dy$$

$$=\frac{2F}{7976{\cdot}45}\left[311{\cdot}22\times13{\cdot}8-\frac{13{\cdot}8^2}{6}\right]$$

$$=0{\cdot}967\ F$$

Shear force shared by the web is 96·7%

Shear stress distribution

Flanges $(y>13{\cdot}8)$

$b=12{\cdot}5$ cm

$$q=\frac{Fa\bar{y}}{I_{xx}b}=\frac{F}{I_{xx}\times12{\cdot}5}\left[12{\cdot}5\times(15-4)\left(y+\frac{15-y}{2}\right)\right]$$

$$=\frac{F}{2I_{xx}}\ [225-y^2]$$

at $=15$ cm, $q=0$

$y=13{\cdot}8$ cm, $q=\frac{F}{2\times7976{\cdot}45}\ [225-13{\cdot}8^2]$

$$=0{\cdot}00216\ F$$

web. $b=1$ cm ; $y=13{\cdot}8$ cm or $y<13{\cdot}8$ cm

at $y=13{\cdot}8$ $q=\frac{F}{2I_{xx}\,.\,1}\ \times12{\cdot}5\ (225-y^2)$

$$=\frac{F}{2\times7976{\cdot}45}\times12{\cdot}5(225-13{\cdot}8^2)$$

$$=0{\cdot}027\ F$$

at $y=0$, (neutral axis), $q=\frac{F}{I_{xx}\,.\,1}\ [12{\cdot}5\times1{\cdot}2\times14{\cdot}4+13{\cdot}8\times1\times6{\cdot}9]$

$$=\frac{F}{7976{\cdot}45}\ [216+95{\cdot}22]$$

$$=0{\cdot}039\ F$$

Shear stress distribution is shown in the Fig. 10·12 (*b*).

Problem 10·2. Show that the difference between the maximum and the mean shear stress over the depth of the web of an I section is $Fd^2/24\ I_{NA}$ where I_{NA} is the moment of inertia of the section along an axis perpendicular to the shear force F and passing through the centroid of the section and d is depth of the web.

Solution. Fig. 10·13 shows an I section of flanges $B\times t$ and web $b\times d$. An approximate sketch of the shear stress distribution is also shown. Maximum shear stress occurs at the neutral axis, *NA*.

Say F is the shear force at the section. I_{NA} is the moment of inertia about NA. Then

$$q_{max}=\frac{F}{I_{NA}\,b}\left[Bt\left(\frac{d}{2}+\frac{t}{2}\right)+b\,\frac{d}{2}\times\frac{d}{4}\right] \qquad ...(1)$$

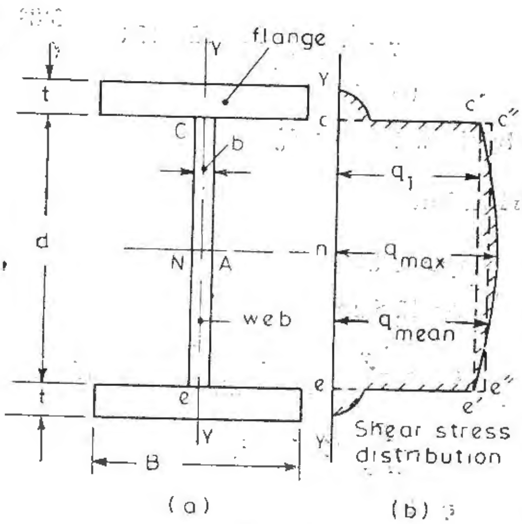


Fig. 10·13

Shear stress at the edge of the web

$$q_1=\frac{F}{I_{NA}\,.\,b}\left[\,Bt\left(\frac{d}{2}+\frac{t}{2}\right)\right] \quad ...(2)$$

$$q_{max}-q_1=\frac{F}{I_{NA}\,.\,b}\left[\,b\,\frac{d^2}{8}\right]=\frac{F}{8\,I_{NA}} \quad ...(3)$$

In the diagram for shear stress distribution (See Fig. 10·13 (*b*))

cc' or $ee'=q_1$, shear stress at edges of web

$nn'=q_{max}$, shear stress at centre of web, $c'n'e'$ is the parabolic curve.

So $\quad q_{mean}=$ mean shear stress $=cc''$ or ee''

$$=q_1+\frac{2}{3}\,(q_{max}-q_1)$$

$$=q_1+\frac{Fd^2}{12\,I_{NA}} \quad ...(4)$$

From equations (3) and (4), Difference between the maximum and mean shear stress in the web is

$$q_{max}-q_{mean}=\frac{Fd^2}{8\,I_{NA}}-\frac{Fd^2}{12\,I_{NA}}=\frac{Fd^2}{24\,I_{NA}}.$$

Problem 10·3. A beam of circular cross section, diameter d and span length l is supported at the ends. It carries a central load W. Show that the principal stresses developed at a layer $d/4$ from the edge at a section under the load are

$$\frac{2Wl}{\pi d^3}\left[1\pm\sqrt{1+\frac{d^2}{l^2}}\right]$$

Solution.

(*a*) **Direct Stress.** Span length $=l$

Central load $=W$

Bending moment at the centre

$$=\frac{Wl}{4}$$

Moment of inertia about neutral axis,

$$I_{NA}=\frac{\pi d^4}{64}$$

Direct stress at $d/4$ from the neutral axes (or $d/4$ from the edge)

$$f=\frac{M_{max}}{I_{NA}}\times\frac{d}{4}$$

$$=\frac{Wl}{4}\times\frac{64}{\pi d^4}\times\frac{d}{4}$$

$$=\frac{4Wl}{\pi d^3} \qquad \ldots(1)$$

(*b*) **Shear Stress**

$$q=\frac{Fa\bar{y}}{I_{NA}\,b}$$

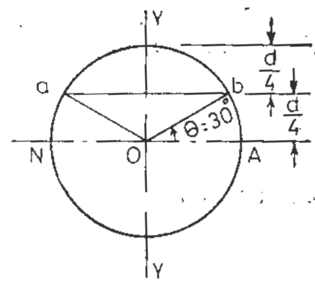


Fig. 10·14

Angle subtended by the layer *ab* Fig. 10·14.

$$\sin\theta=\frac{d/4}{d/2}=0\cdot 5$$

$$\theta=30°$$

$$\cos\theta=0\cdot 866$$

Breadth, $ab=d\cos 60°=d\times\frac{\sqrt{3}}{2}$ ...(1)

Shear stress at any layer subtending angle θ at the centre

$$q=\frac{4}{3}\times\frac{F\cos^2\theta}{\pi r^2}$$

In this case $\cos\theta=0\cdot 866$

$$\cos^2\theta=\frac{3}{4}$$

$$r=\frac{d}{2},\quad r^2=\frac{d^2}{4}$$

$$q=\frac{16\,F}{3\pi\,d^2}\times\frac{3}{4}=\frac{4\,F}{\pi d^2}$$

But shear force, $F=\frac{W}{2}$ at the centre

So $q=\frac{2W}{\pi d^2}$.

(*c*) **Principal Stresses**

$$p_1, p_2=\frac{f}{2}\pm\sqrt{\left(\frac{f}{2}\right)^2+q^2}$$

$$=\frac{2Wl}{\pi d^3}\pm\sqrt{\left(\frac{2Wl}{\pi d^3}\right)^2+\left(\frac{2W}{\pi d^2}\right)^2}$$

$$=\frac{2Wl}{\pi d^3}\pm\frac{2Wl}{\pi d^3}\sqrt{1+\frac{d^2}{l^2}}$$

$$=\frac{2Wl}{\pi d^3}\left[1\pm\sqrt{1+\frac{d^2}{l^2}}\right].$$

Problem 10·4. Show that for a beam section of triangular shape base *b*, **height *h*** subjected to shear force *F*, the maximum shear stress is $3F/bh$ and occurs at a **height of *h*/2** from the base.

Solution. The Fig. 10·15 shows a triangle *ABC* of base *b* and height *h*. Its neutral axes is parallel to the base and at a distance of *h*/3 from the base *BC*. Consider a layer *ef* at a distance of *y* from the neutral axis.

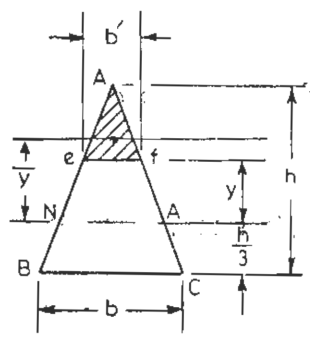


Fig. 10·15

Now $$I_{NA}=\frac{bh^3}{36}$$

Shear stress at the layer under consideration

$$=\frac{Fa\bar{y}}{I_{NA}\,.\,b'}$$

where a=area (shaded) above the layer

$\bar{y}$=CG of the shaded area from the neutral axis

b'=breadth of the layer

$$=\frac{\left(\frac{2h}{3}-y\right)}{h}\times b=\left(\frac{2}{3}-\frac{y}{h}\right)b$$

$$a=\frac{1}{2}\;b'\times\left(\frac{2}{3}\;h-y\right)=\frac{1}{2}\left(\frac{2}{3}-\frac{y}{h}\right)\left(\frac{2}{3}\;h-y\right)b$$

$$\bar{y}=y+\frac{1}{3}\left(\frac{2}{3}\;h-y\right)=\frac{2}{9}\;h+\frac{2}{3}\;y$$

$$a\bar{y}=\frac{b}{2}\left(\frac{2}{3}-\frac{y}{h}\right)\left(\frac{2}{3}\;h-y\right)\left(\frac{2}{9}\;h+\frac{2}{3}\;y\right)$$

Shear stress, $$q=\frac{F\,.\,\frac{b}{2}\left(\frac{2}{3}-\frac{y}{h}\right)\left(\frac{2}{3}\;h-y\right)\left(\frac{2}{9}\;h+\frac{2}{3}\;y\right)}{I_{NA}\,.\left(\frac{2}{3}-\frac{y}{h}\right)b}$$

$$=\frac{F}{I_{NA}}\left[\frac{1}{2}\left(\frac{2}{3}h-y\right)\left(\frac{2}{9}h+\frac{2}{3}y\right)\right]$$

$$=\frac{F}{I_{NA}}\left[\frac{2}{27}h^2+\frac{hy}{9}-\frac{y^2}{3}\right]$$

For the shear stress to be maximum

$$\frac{dq}{dy}=0 \quad \text{or} \quad \frac{h}{9}-\frac{2y}{3}=0$$

or $$y=\frac{h}{6}$$

Now $$\frac{h}{3}+y=\frac{h}{3}+\frac{h}{6}=\frac{h}{2}$$

i.e., maximum shear stress occurs at a height of $h/2$.

Now substituting the value of y

$$q_{max}=\frac{F}{I_{NA}}\left[\frac{2}{27}h^2+\frac{h}{9}\times\frac{h}{6}-\frac{h^2}{108}\right]=\frac{Fh^2}{12\, I_{NA}}$$

$$=\frac{Fh^2}{12}\times\frac{36}{bh^3}=\frac{3F}{bh}$$

$$q_{mean}=\text{mean shear stress } \frac{F}{bh/2}=\frac{2F}{bh}$$

$$\frac{q_{max}}{q_{mean}}=1\cdot 5.$$

Problem 10·5. A rolled steel joist of T-section shown in the Fig. 10·16 (*a*) is used as a beam. At a particular section, the transverse shear force is *F*. Plot the shear stress distribution over the depth of the section.

Solution. The section is symmetrical about *YY* axis. G lies on this axis.

Distance of G from the lower edge of the web

$$y_1=\frac{40\times 5\times 20+30\times 10\ (40+5)}{40\times 5+30\times 10}=35 \text{ mm}$$

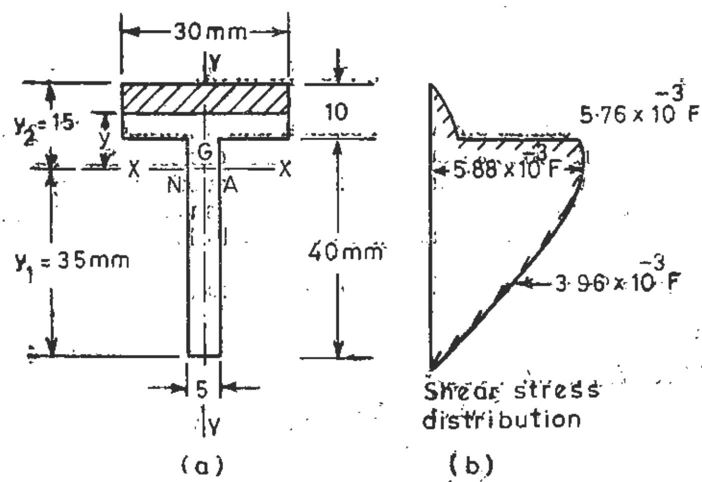


Fig. 10·16

then $y_2=50-35=15$ mm

Moment of inertia,

$$I_{NA} \text{ or } I_{xx}=\frac{5\times 40^3}{12}+200\,(35-20)^2+\frac{30\times 10^3}{12}+300\,(15-5)^2$$

$$=29166{\cdot}7+45000+30{,}000$$

$$=104{\cdot}1667\times 10^3 \text{ mm}^4$$

Shear stress at any layer (at a distance y from NA),

$$q=\frac{Fa\bar{y}}{I_{xx}\,b}$$

$a\bar{y}$=first moment of area above the layer

F=shear force

b=breadth of the layer

At $y=15$ mm, $q=0$

At $y=10$ mm $q=\dfrac{F\times(15-10)(30)(13+2{\cdot}5)}{104{\cdot}1667\times 10^3\times 30}$

$$=\frac{F\times 150\times 12{\cdot}5}{104{\cdot}1667\times 30\times 10^3}=0{\cdot}60\times 10^{-3}\,F$$

At $y=5$ mm $q=\dfrac{F\times 30\times 10\times 10}{104{\cdot}1667\times 10^3\times 30}=0{\cdot}96\times 10^{-3}\,F$

(when $b=30$ mm)

$q=5{\cdot}76\times 10^{-3}\,F$, (when $b=5$ mm)

At $y=0$, $q=\dfrac{F\times(30\times 10\times 10+5\times 5\times 2{\cdot}5)}{104{\cdot}1667\times 10^3\times 5}$ (Since $b=5$ mm)

$=5{\cdot}88\times 10^{-3}\,F$ (at the neutral axis)

At $y=20$ mm on the other side of neutral axis

$$q=\frac{F\,[15\times 5\times(20+7{\cdot}5)]}{104{\cdot}1667\times 10^3\times 5}=3{\cdot}96\times 10^{-3}\,F$$

At $y=35$ mm

on the other side of neutral axis, $q=0$.

Shear stress distribution is shown in the Fig. 10·16 (*b*).

Problem 10·6. Two beams one with I section and another with *T* section are simply supported at the ends and carry concentrated load at their centres. The span length for both is the same but the central loads applied are such that the maximum stress due to bending in both is the same. Determine the ratio of the maximum shear stresses developed in these sections. The dimensions of the sections are given in Fig. 10·17.

Solution. Let us first calculate I_{NA} for both the sections.

T-Section :

Distance of C.G. from lower edge of web,

$$y_1=\frac{10\times 5+12\times 10{\cdot}5}{10+12}$$

$$=8 \text{ cm}$$

then $y_2=11-8=3$ cm

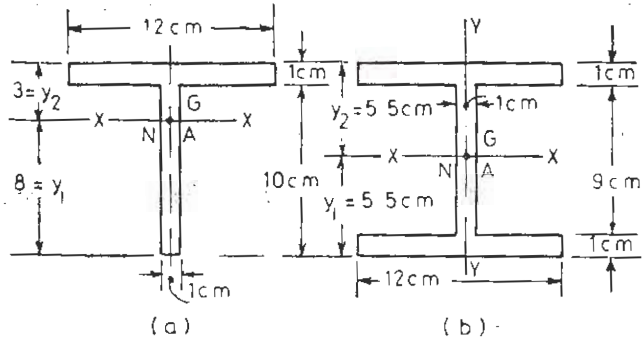


Fig. 10·17

Moment of Inertia, $I_{NA}=\frac{1\times10^3}{12}+10(8-5)^2+\frac{12\times1^3}{12}+12(2\cdot5)^2$

$$=83\cdot333+90+1+75=249\cdot333 \text{ cm}^4$$

I-Section. Section is symmetrical about YY and XX axis, therefore CG is located at the centre of the web.

$$y_1=y_2=\frac{9+1+1}{2}=5\cdot5 \text{ cm}$$

Moment of inertia, $I_{NA}=\frac{12\times11^3}{12}-\frac{11\times9^3}{12}=1331-668\cdot25$

$$=662\cdot75 \text{ cm}^4$$

Now say the load on the beam of T section $=W_1$

and load on the beam of I section $=W_2$

B.M. at the centre of T section beam $=\frac{W_1L}{4}$

B.M. at the centre of I section beam $=\frac{W_2L}{4}$

where L=span length.

In T-section, maximum bending stress will occur at the lower edge of the web, since $y_1>y_2$.

In I-section, maximum stress due to bending will occur at the extreme edge of the langes.

f_{max} in T-section beam$=\frac{W_1L}{4}\times\frac{8}{I_{NA}}=\frac{2W_1L}{249\cdot333}$ Since $y_1=8$ cm

f_{max} in I-section beam$=\frac{W_2L}{4}\times\frac{5\cdot5}{I_{NA}'}=\frac{W_2L\times5\cdot5}{4\times662\cdot75}$

But $f_{max}=f_{max}'$

$$\frac{2W_1L}{249\cdot333}=\frac{W_2\times L\times5\cdot5}{4\times662\cdot75}$$

or $$\frac{W_1}{W_2}=\frac{5\cdot5}{4\times662\cdot75}\times\frac{249\cdot333}{2}=0\cdot235 \quad ...(1)$$

Since the beams are simply supported at their ends and carry the central loads.

$$\text{Maximum SF in T section beam}=\frac{W_1}{2}$$

$$\text{Maximum SF in I section beam}=\frac{W_2}{2}$$

In both the cases, maximum shear stress across the section occurs at the neutral axis.

In T section,

$$q_{max}=\frac{W_1}{2}\times\frac{8\times1\times4}{249{\cdot}333\times1}=\frac{16W_1}{249{\cdot}333}$$

(Note that for convenience, $a\bar{y}$ is taken for the lower position of the web)

In I-section,

$$q'_{max}=\frac{W_2}{2}\times\frac{(12\times1\times5+4{\cdot}5\times1\times2{\cdot}25)}{662{\cdot}75\times1}$$

$$=\frac{W_2}{2}\times\frac{70{\cdot}125}{662{\cdot}75}$$

Now $$\frac{q_{max}}{q'_{max}}=\frac{16W_1}{249{\cdot}333}\times\frac{662{\cdot}75\times2}{W_2\times70{\cdot}125}=1{\cdot}213\,\frac{W_1}{W_2}$$

$$=1{\cdot}213\times0{\cdot}235$$

$$=0{\cdot}285$$

Problem 10·7. Determine the position of the layer at which transverse shear stress is maximum. The section of the beam is square of 10 cm side with one diagonal vertical, and the shear force at a particular section is F. Draw the shear stress distribution diagram.

Solution. Section symmetrical about YY and X-X axis as shown Fig. 10·18(a) with centroid at the centre of the section.

$$I_{xx}=\frac{10^4}{12}\text{ cm}^4=833{\cdot}33\text{ cm}^4$$

Consider a layer at a vertical height y from the neutral axis.

Height of the triangle yab

$$=\frac{d}{2}-y \quad \text{where } d \text{ is the diagonal of the square}$$

base of the triangle

$$=2\left(\frac{d}{2}-y\right)$$

area of the triangle yab,

$$=\frac{2}{2}\left(\frac{d}{2}-y\right)\left(\frac{d}{2}-y\right)$$

$$=\left(\frac{d}{2}-y\right)^2$$

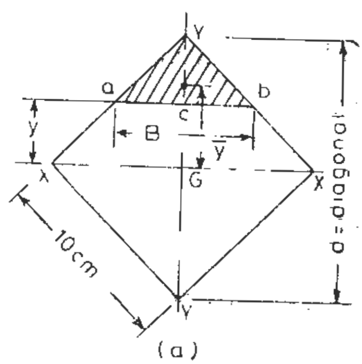


Fig. 10·18 (a)

Distance of the centroid of the triangle from the neutral layer,

$$\bar{y}=y+\frac{1}{3}\left(\frac{d}{2}-y\right)$$

$$=\frac{d}{6}+\frac{2y}{3}$$

$$=\frac{1}{3}\left(\frac{d}{2}+2y\right)$$

The shear stress at any layer,

$$q=\frac{Fa\bar{y}}{I_{xx}b}$$

$$=\frac{F\times\left(\frac{d}{2}-4\right)^2}{I_{xx}\times2\left(\frac{d}{2}-y\right)}\times\frac{1}{3}\left(\frac{d}{2}+2y\right)$$

$$=\frac{F}{6I_{xx}}\left(\frac{d}{2}-y\right)\left(\frac{d}{2}+2y\right) \qquad \ldots(1)$$

$$=\frac{F}{6I_{xx}}\left[\frac{d^2}{4}+\frac{yd}{2}-2y^2\right]$$

For maximum shear stress

$$\frac{dq}{dy}=0$$

or $$\frac{d}{2}-4y=0$$

or $$y=\frac{d}{8}$$

In the problem, diagonal,

$$d=10\sqrt{2}=14{\cdot}14 \text{ cm}$$

$\therefore$ $$y=\frac{10\sqrt{2}}{8}=1{\cdot}767 \text{ cms.}$$

At $y=\frac{d}{2}$, $q=0$

$y=\frac{d}{4}$, $q=\frac{Fd^2}{24I_{xx}}$ [Putting values in equation (1)]

$$=\frac{F\times(14{\cdot}14)^2}{24\times833{\cdot}33}=0{\cdot}01\ F$$

$y=\frac{d}{8}$, $\quad q=\frac{F\left(\frac{3d}{8}\right)\left(\frac{3d}{4}\right)}{6\times 833{\cdot}33}$

$=\frac{F\times 9d^2}{192\times 833{\cdot}33}$

$=\frac{9F\times 14{\cdot}14^2}{192\times 833{\cdot}33}$

$=0{\cdot}0112F$

$y=0,$

$q=\frac{Fd^2}{24\ I_{xx}}$

$=\frac{F\times 14{\cdot}14^2}{24\times 833{\cdot}33}$

$=0{\cdot}01\ F$

Fig. 10·18(*b*) shows the shear stress distribution. Maximum shear stress occurs at $d/8$ from N.A.

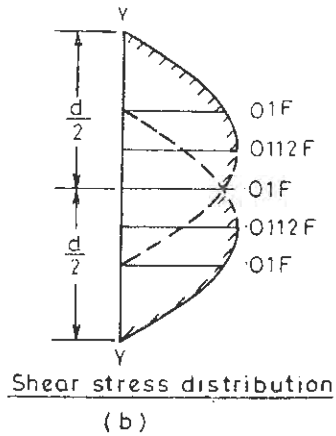


Fig. 10·18

Problem 10·8. A bar of hollow square section (as shown in the Fig. 10·19(*a*) is used as a cantilever of length 2 metres. What is the magnitude of the uniformly distributed load if the maximum shear stress in the section is not to exceed 150 kg/cm². Draw the shear stress distribution over the depth of the section. (for the maximum shear force on the cantilever).

Solution. The section is symmetrical about the *XX* and *YY* axis and its *G* lies at the centre as shown.

Semi diagonal $=\frac{8\sqrt{2}}{2}=5{\cdot}656$ cm

Moment of inertia

$$I_{NA} \text{ or } I_{xx}=\frac{8^4}{12}-\frac{4^4}{12}=320 \text{ cm}^4$$

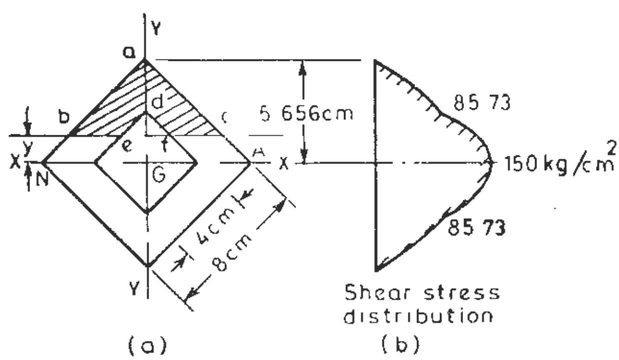


Fig. 10·19

Consider a layer bc at a distance of y from the neutral axis ($y < 2.828$ cm).

$$a\bar{y} \text{ about N.A.} = (5.656-y)^2\left(y+\frac{5.656-y}{3}\right)-(2.828-y)^2\left(y+\frac{2.828-y}{3}\right)$$

$$=\frac{(5.656-y)^2(5.656+2y)}{3}-\frac{(2.828-y)^2(2.828+2y)}{3}$$

Breadth of the layer

$$=2(5.656-y)-2(2.828-y)$$

$$=5.656 \text{ cm}$$

$$\frac{a\bar{y}}{b}=\frac{(5.656-y)^2(5.656+2y-2.828-y)^2(2.828+2y)}{3\times 5.656}$$

Shear stress,

$$q=\frac{Fa\bar{y}}{I_{xx}b}$$

For the shear stress to be maximum,

$$\frac{dq}{dy}=0 \quad \text{or} \quad \frac{d}{dy}\left(\frac{a\bar{y}}{b}\right)=0$$

or $\quad 2(5.656-y)(-1)(5.656+2y)+(5.656-y)^2(2)$

$$-2(2.828-y)(-1)(2.828+2y)-(2.828-y)^2.2=0$$

or $\quad (5.656-y)(5.656+2y)-(5.656-y)^2-(2.828-y)(2.828+2y)+(2.828-y)^2=0$

$$5.656-2y^2+32-32+2\times 5.656y-y^2-8+2y^2-2.828y+8-5.656y+y^2=0$$

or $\quad 8.484\, y=0$

$$y=0$$

In this case maximum shear stress occurs at the neutral axis.

$$\frac{a\bar{y}}{b} \text{ at N.A.}=\frac{5.656^3-2.828^3}{3\times 5.656}=\frac{180.937-22.617}{16.968}$$

$$=9.33$$

$$q_{max}=\frac{Fa\bar{y}}{I_{xx}b}=\frac{F\times 9.33}{320}=150 \text{ kg/cm}^2$$

or $\quad F=5144.7$ kg

$=\omega L$ (in the case of cantilever maximum SF occurs at fixed end)

$$=\omega\times 200$$

$$\omega=\frac{5144.7}{200}=25.7235 \text{ kg/cm}$$

$=2572.35$ kg/metre

$=2.572$ Tonne/metre mm.

Shear stress distribution

at $\quad y=5.656, \quad q=0$

$$y=2.828 \qquad q=\frac{Fa\bar{y}}{I_{xx}b}=\frac{5144.7}{320\times 5.656}\times\frac{(5.656)(2.828)}{2}\times\left(2.828+\frac{2.828}{3}\right)$$

$$=\frac{5144{\cdot}7}{320\times 5{\cdot}656}\times(2{\cdot}828)(2{\cdot}828)(3{\cdot}770)$$

$$=85{\cdot}73\ \text{kg/cm}^2$$

At $\qquad y=0, \quad q=150\ \text{kg/cm}^2$

The shear stress distribution is shown in the Fig. 10·19(*b*)

Problem 10·9. A beam section shown in the Fig. 10·20(*a*) is subjected to a shear force of 1 tonne. Plot a graph showing the variation of shear stress along the depth of the section. Determine also the ratio of the maximum shear stress and the mean shear stress.

Solution. The section is symmetrical about the *Y-Y* axis shown. *CG* will lie on *YY* axis.

Distance of *G* from the lower edge,

$$y_1=\frac{6\times 3\times 1{\cdot}5+2\times 3\times 2\times 4{\cdot}5}{6\times 3+2\times 3\times 2}=2{\cdot}7\ \text{cm}$$

Shear force, $\qquad F=1\ \ T=1000\ \text{kg}$

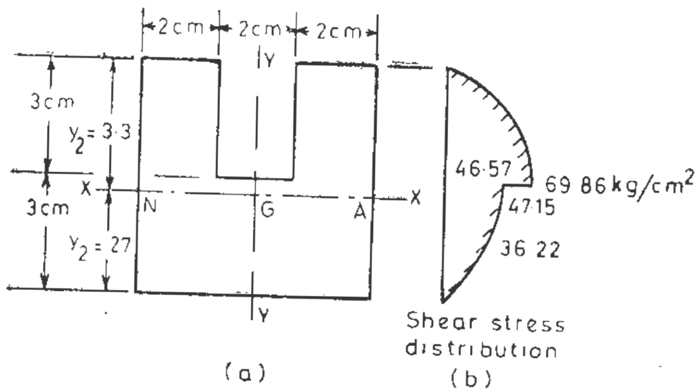


Fig. 10·20

Moment of Inertia

$$I_{NA}\text{ or }I_{xx}=\left[\frac{6\times 6^3}{12}+36(3-2{\cdot}7)^2\right]-\left[\frac{2\times 3^3}{12}+6(3{\cdot}3-1{\cdot}5)^2\right]$$

$$=(108+3{\cdot}24)-(4{\cdot}5+19{\cdot}44)$$

$$=101{\cdot}24-23{\cdot}94$$

$$=77{\cdot}30\ \text{cm}^4$$

Shear stress at any layer at a distance of *y* from N.A.

$$q=\frac{Fa\bar{y}}{I_{xx}b}$$

$y=3{\cdot}3 \qquad q=0$

$y=2{\cdot}0$ cm $\qquad q=\dfrac{1000[2\times 2\times 1{\cdot}3\times(2+0{\cdot}65)]}{2\times 2\times 77{\cdot}30}$

$$=\frac{250\times 5{\cdot}2\times 2{\cdot}65}{77{\cdot}30}$$

$$=44{\cdot}56\ \text{kg/cm}^2$$

$y=0{\cdot}3$ cm, $\quad q=\dfrac{1000[2\times3\times2(0{\cdot}3+1{\cdot}5)]}{77{\cdot}30\times2\times2}$ (breadth=4 cm)

$$=\frac{1000\times12\times1{\cdot}8}{4\times77{\cdot}30}=69{\cdot}86 \text{ kg/cm}^2$$

$q'=46{\cdot}57$ kg/cm² (breadth=6 cm)

$y=0$, $\quad q=\dfrac{1000\times6\times2{\cdot}7\times1{\cdot}35}{6\times77{\cdot}30}$ (we have taken first moment of the area $a\bar{y}$ on the other side of N.A. to avoid long calculations)

$$=47{\cdot}15 \text{ kg/cm}^2$$

At $y=1{\cdot}3$ cm on the other side of neutral axis

$$q=\frac{1000}{77{\cdot}3\times6}[6\times1{\cdot}4(1{\cdot}3+0{\cdot}7)]$$

$$=\frac{1000\times8{\cdot}4\times2}{77{\cdot}3\times6}=36{\cdot}22 \text{ kg/cm}^2$$

Fig. 10·20(*b*) shows the shear stress distribution over the depth. One can notice in this case, that maximum shear stress does not occur at the neutral axis.

Problem 10·10. A beam of hexagonal section with side *a*, is subjected to a transverse shear force *F* at a particular section. Shear force *F* is perpendicular to one of its diagonals. Derive an expression for the shear stress *q* at a distance *y* from the diagonal and plot the shear stress distribution over the depth.

Solution. The section is shown in Fig. 10·21(*a*). The depth of the section is $\sqrt{3}\,a$ and diagonal is $2a$. The section is symmetrical about *X-X* and *YY* axis and centroid lies at the centre *G* as shown.

Moment of inertia

$$I_{NA}=\frac{a}{12}(\sqrt{3}a)^3+4\,\frac{\left[\frac{a}{2}\times\left(\frac{\sqrt{3}}{2}a\right)^3\right]}{12}=\frac{5}{16}\sqrt{3}a^4$$

Now consider the layer *cd* at a distance of *y* from the neutral layer *X-X*.

$$a\bar{y}=a\left(\frac{\sqrt{3}a}{2}-y\right)\left(y+\frac{\frac{\sqrt{3}}{2}a-y}{2}\right)+2\,\frac{\left(\frac{\sqrt{3}}{2}a-y\right)}{\sqrt{3}}\,\frac{\left(\frac{\sqrt{3}}{2}a-y\right)}{2}\times\left\{y+\frac{\frac{\sqrt{3}}{2}a-y}{3}\right\}$$

Note that in $\triangle caa'$, $ca'=aa'\times\tan 30°$

and $\quad aa'=\dfrac{\sqrt{3}}{2}a-y$

$$a\bar{y}=\frac{a}{2}\left(\frac{3}{4}a^2-y^2\right)+\frac{1}{3\sqrt{3}}\left(\frac{\sqrt{3}}{2}a-y\right)^2\left(\frac{\sqrt{3}}{2}a+2y\right)$$

breadth at *cd*, $\quad b=a+\dfrac{2\left(\frac{\sqrt{3}}{2}a-y\right)}{\sqrt{3}}$

$$=\frac{2\sqrt{3}a-2y}{\sqrt{3}}$$

Simplifying the expression for $a\bar{y}$ we get

$$a\bar{y}=\frac{a^3}{2}-ay^2+\frac{2y^3}{3\sqrt{3}}=\frac{1}{6\sqrt{3}}\left[3\sqrt{3}a^3-6\sqrt{3}\,ay^2+4y^3\right]$$

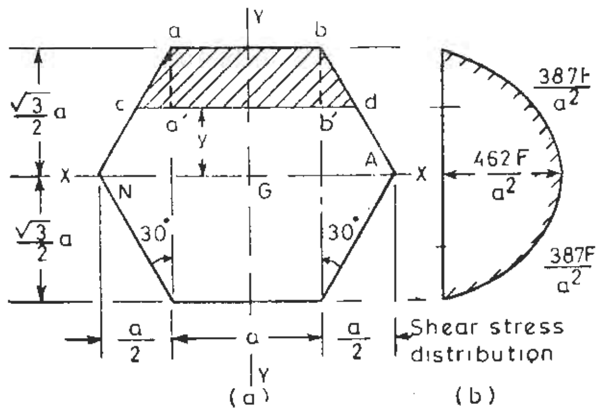


Fig. 10·21

So the shear stress,

$$q=\frac{Fa\bar{y}}{I_{NA}.b}=\frac{F[3\sqrt{3}a^3-6\sqrt{3}ay^2+4y^3]}{6\sqrt{3}\times5\sqrt{3}a^4[2\sqrt{3}a-2y]}$$

$$=\frac{16F}{5\sqrt{3}}\times\frac{\left[\frac{\sqrt{3}}{4}a^3-\frac{\sqrt{3}}{2}ay^2+\frac{y^3}{3}\right]}{a^4[\sqrt{3}a-y]}$$

when $\quad y=\frac{\sqrt{3}}{2}a,\ q=0$

$$y=\frac{a}{2},\quad q=\frac{16F}{5\sqrt{3}a^4}\ \frac{\left[\frac{\sqrt{3}}{4}\cdot a^3-\frac{\sqrt{3}}{8}a^3+\frac{a^3}{24}\right]}{\left[\sqrt{3}a-\frac{a}{2}\right.}$$

$$=0{\cdot}387\ \frac{F}{a^2}$$

$$y=0,\quad q=\frac{16F}{5\sqrt{3}a^4}\ \frac{\left[\frac{\sqrt{3}}{4}a^3\right]}{\sqrt{3}a}$$

$$=\frac{4F}{5\sqrt{3}}\times\frac{1}{a^2}$$

$$=0{\cdot}462\ \frac{F}{a^2}$$

Fig. 10·21(*b*) shows the shear stress distribution over the depth of the section.

Problem 10·11. A rolled steel section 60 mm × 40 mm is shown in the Fig. 10·22(*a*). A transverse shear force of 5 tonnes is acting on this section. Plot the shear stress distribution over the depth of the section.

Solution. Fig. 10·22 (*a*) shows the section, symmetrical about *X-X* and *YY* axis *i.e. G* lies at the centre as shown.

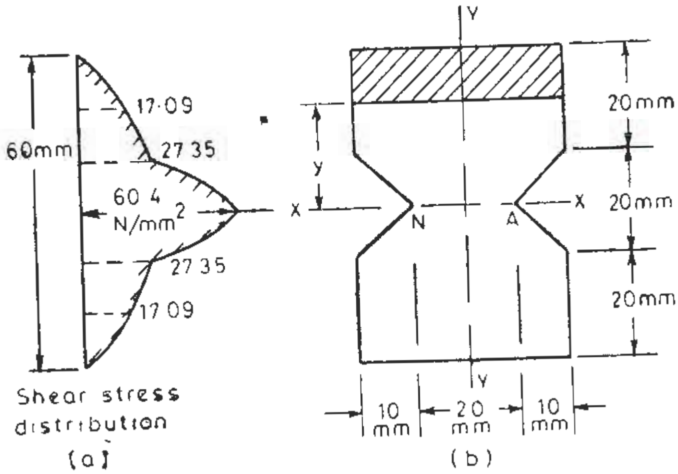


Fig. 10·22

Moment of Inertia

$$I_{NA}=\frac{40\times 60^3}{12}-4\left(\frac{10\times 10^3}{12}\right)$$

(Note that moment of inertia of 4 triangles of base 10 mm and height 10 mm is substracted from the moment of inertia of a rectangle of 40×60 mm)

$$I_{NA}= 0^4\left[\,72-\frac{1}{3}\right]=71{\cdot}667\times 10^4\ \text{mm}^4$$

Shear stress at any layer (at distance of y from the neutral layer)

$$q=\frac{Fa\bar{y}}{I_{NA}b}$$

$$F=5\times 9{\cdot}8\times 1000\ \text{N}=49\times 10^3\ \text{N}$$

$y=30$ mm, $\quad q=0$

$y=20$ mm, $$q=\frac{49\times 10^3[40\times 10\times(20+5)]}{I_{NA}\times 40}$$

$$=\frac{49\times 10^3\times 10^4}{71{\cdot}667\times 10^4\times 40}$$

$$=17{\cdot}09\ \text{N/mm}^2.$$

$y=10$ mm, $$q=\frac{49\times 10^3\times[20\times 40\times(10+10)]}{71{\cdot}667\times 10^4\times 40}$$

$$=\frac{49\times[16\times 10^3]}{71{\cdot}667\times 400}$$

$$=27{\cdot}35\ \text{N/mm}^2$$

$y=0$ mm, $$q=\frac{49\times 10^3\left[40\times 30\times 15-\frac{2\times(10\times 10)}{2}\times\frac{10}{3}\right]}{71{\cdot}667\times 10^4\times 20}$$

(at the neutral axis $b=20$ mm)

$$=\frac{49\times10^3\times17666\cdot667}{71\cdot667\times10^4\times20}$$

$$=60\cdot40\ \text{N/mm}^2$$

The shear stress distribution is shown in the Fig. 10·22 (*b*).

Problem 10·12. A beam is of T section, with flange 12 cm×1 cm and web 10 cm×1 cm. What percentage of shearing force at any section is shared by the web.

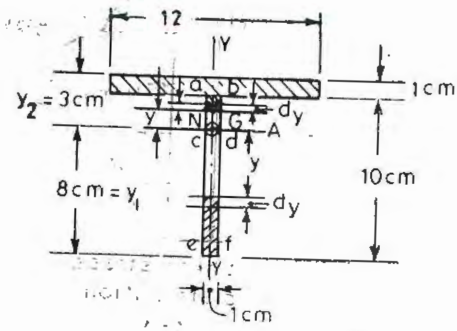


Fig 10·23

Solution. Fig. 10·23 shows the T section of given dimension. The section is symmetrical about the *Y-Y* axis. Let us find position of *G* along *YY* axis.

Taking first moments of the areas about the lower edge of the web

$$(10+12)\ y_1=10\times1\times5+12\times1\ (10+0\cdot5)$$

$$22\ y_1=50+126$$

or $$y_1=8\ \text{cm}$$

$$y_2=10+1-8=3\ \text{cm}$$

Moment of inertia about the neutral axis,

$$I_{NA}=\frac{12\times1^3}{12}+12\ (3-0\cdot5)^2+\frac{1\times10^3}{12}+10\ (8-5)^2$$

$$=1+75+83\cdot333+90$$

$$=249\cdot333\ \text{cm}^4.$$

Shear force shared by the portion abcd of the web

Shear stress at any layer

$$q=\frac{F\ a\bar{y}}{I_{NA}\ .\ b}$$

Say thickness of the layer$=\delta y$

Area of the layer $=b\ \delta y$

where $b=$breadth of the web

Shear force at the layer considered

$$=q\ bdy$$

$$=\frac{F\cdot a\bar{y}}{I_{NA}\ b}\times bdy$$

$$=\frac{F\ a\bar{y}}{I_{NA}}\ dy$$

$$a\bar{y}=12\times1\times2{\cdot}5+(2-y)\left(y+\frac{2-y}{2}\right)$$

$$=30+\frac{4-y^2}{2}=\frac{64-y^2}{2}$$

Total shear force shared by the portion *abcd*

$$=\int_0^2 \frac{F}{I_{NA}}\left[\frac{64-y^2}{2}\right]dy$$

$$=\frac{F}{2\,I_{NA}}\left[64\ y-\frac{y^3}{3}\right]_0^2$$

$$F_1=\frac{F}{2\,I_{NA}}\times\frac{376}{3}=62{\cdot}667\ \frac{F}{I_{NA}}$$

Shear force shared by the portion cdef

Shear stress at any layer at a distance y from N.A.,

$$q=\frac{F\,a\bar{y}}{I_{NA}\,b}$$

Shear force in a layer of thickness dy,

$$dF=q\ .\ bdy=\frac{Fa\bar{y}}{I_{NA}\ .\ b}\ bdy$$

$$=\frac{Fa\bar{y}}{I_{NA}}\ dy$$

where $$a\bar{y}=(8-y)\left(y+\frac{8-y}{2}\right)=\frac{1}{2}\ (8-y)(8+y)$$

$$=\frac{(64-y^2)}{2}$$

Total shear force shared by the portion *cdef*,

$$F_2=\int_0^8 \frac{F}{2\,I_{NA}}\ (64-y^2)\ dy$$

$$=\frac{F}{2\,I_{NA}}\left[64\ y-\frac{y^3}{3}\right]_0^8$$

$$=\frac{F}{2\,I_{NA}}\left[64\times8-\frac{8^3}{3}\right]=\frac{170{\cdot}667\ F}{I_{NA}}$$

Total shear force shared by the web

$$=F_1+F_2=(62{\cdot}667+170{\cdot}667)\ \frac{F}{I_{NA}}$$

$$=\frac{233{\cdot}334\ F}{249{\cdot}333}=0{\cdot}9358\ F$$

The shear force shared by the web is 93·58% of the shear force acting on the section.

Problem 10·13. A tee section with a flange $10\times1{\cdot}5$ cm and web 10×1 cm is subjected to a bending moment of 0·12 tonne-metre producing tension in flange and a shear force of ·6 tonnes. Determine the principal stresses at the following points.

(*a*) *A*, bottom edge of the flange

(*b*) *B*, at the neutral axis

(*c*) *C*, at the bottom edge of web.

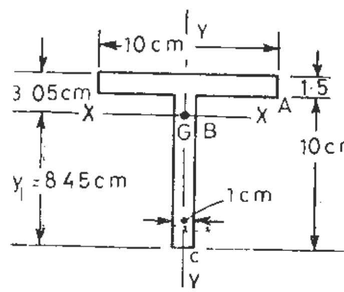


Fig. 10·24

Solution. The figure of the T section is shown in Fig. 10·24.

Let us calculate the moment of inertia I_{NA} or I_{xx}.

The section is symmetrical about *YY* axis and its *G* will lie on this axis,

$$y_1=\frac{10\times1\times5+10\times1{\cdot}5\ (10+0{\cdot}75)}{10+15}=\frac{50+161{\cdot}25}{25}=8{\cdot}45\text{ cm}$$

$$y_2=10+1{\cdot}5-8{\cdot}45=3{\cdot}05\text{ cm}$$

Momeat of inertia, $I_{NA}=I_{xx}=\frac{1\times10^3}{12}+10\ (8{\cdot}45-5)^2+\frac{10\times1{\cdot}5^3}{12}+15\ (3{\cdot}05-0{\cdot}75)^2$

$$=83{\cdot}333+119{\cdot}025+2{\cdot}8125+79{\cdot}35$$

$$=284{\cdot}52\text{ cm}^4$$

Stresses due to BM

$$M=0{\cdot}12\text{ T-m}=1{\cdot}2\times10^4\text{ kg-cm}$$

$$f_A=\frac{M}{I_{NA}}\times(y_A)$$

where $y_A=3{\cdot}05-1{\cdot}5=1{\cdot}55$

$$=\frac{1{\cdot}2\times10^4\times1{\cdot}55}{284{\cdot}52}=65{\cdot}77\text{ kg/cm}^2\text{ (tensile)}$$

$$f_B=0\text{ (at neutral axes)}$$

$$f_C=-\frac{1{\cdot}2\times10^4\times8{\cdot}45}{284{\cdot}52}=-356{\cdot}39\text{ kg/cm}^2\text{ (compressive)}$$

Stresses due to SF

$$F=1{\cdot}6\text{ tonne}=1600\text{ kg}$$

$$q_A=\frac{1600}{I_{xx}\,.\,(10)}\times10\times1{\cdot}5\ (3{\cdot}05-0{\cdot}75)$$

(The width of the flange at bottom is 10 cm)

$$=\frac{1600\times15\times2{\cdot}3}{284{\cdot}52\times10}=19{\cdot}40\ \text{kg/cm}^2$$

$$q_B=1600\times\frac{15\times2{\cdot}3+\left(1{\cdot}55\times1\times1\times\frac{1{\cdot}55}{2}\right)}{I_{xx}\times1}$$

(Width at neutral axis is 1 cm)

$$=\frac{35{\cdot}70\times1600}{284{\cdot}52}=200{\cdot}76\ \text{kg/cm}^2$$

$$q_C=0$$

Principal stresses

$$p_{1A},\ p_{2A}=\frac{f_A}{2}\pm\sqrt{\left(\frac{f_A}{2}\right)^2+q_A{}^2}$$

$$=\frac{65{\cdot}77}{2}\pm\sqrt{\left(\frac{65{\cdot}77}{2}\right)^2+19{\cdot}40)^2}=32{\cdot}885\pm38{\cdot}181$$

$$=71{\cdot}066,\ -5{\cdot}296\ \text{kg/cm}^2$$

$$p_{1B},\ p_{2B}=\pm\sqrt{q_B{}^2},\qquad \text{Since } f_B=0$$

$$=\pm200{\cdot}76\ \text{kg/cm}^2$$

$$p_{1C}=-356{\cdot}39\ \text{kg/cm}^2$$

$p_{2C}=0$ at this point.

Problem 10·14. The Fig. 10·25 shows a bracket of T-section supporting a shaft transmitting power. At a particular instant the thrust P on the bearing is 4000 N inclined at 45° to the vertical. Determine the principal stresses along the section a-a at the point b.

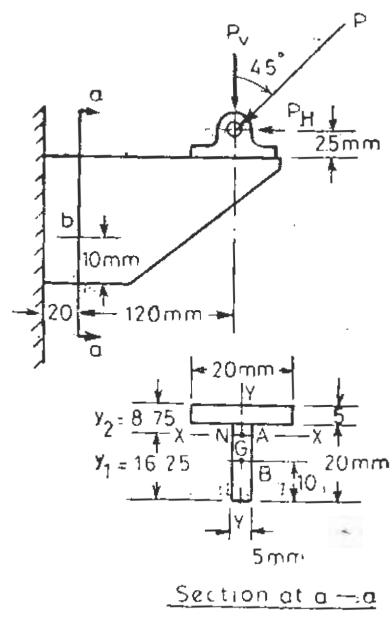


Fig. 10·25

Solution. Resolving the inclined load.

Vertical component,

$$P_V=4000\times0{\cdot}707=2828\ \text{N}$$

Horizontal component,

$$P_H=4000\times0{\cdot}707=2828\ \text{N}$$

Due to the vertical component P_V, there will be a bending moment and a shearing force on the section.

Bending moment at the section a-a

$$M=P_V\times120=2828\times120\ \text{Nmm}$$

Shear force at the section a-a

$$F=2828\ \text{N}.$$

T-Section

Section is symmetrical about the Y-Y axis. G lies on YY axis.

Distance of C.G. from lower edge of web,

$$y_1=\frac{20\times5\times10+20\times5\times22{\cdot}5}{100+100}=16{\cdot}25 \text{ mm}$$

then $$y_2=25-16{\cdot}25=8{\cdot}75 \text{ mm}$$

Moment of inertia,

$$I_{NA} \text{ or } I_{xx}=\frac{5\times20^3}{12}+100\ (16{\cdot}25-10)^2+\frac{20\times5^3}{12}+100\ (8{\cdot}75-2{\cdot}5)^2$$

$$=3333{\cdot}333+3906{\cdot}250+208{\cdot}333+3906{\cdot}250$$

$$=11354{\cdot}166 \text{ mm}^4$$

Due to the bending moment M, there will be compressive stress at the point B,

$$f'=\frac{M}{I_{xx}}\,y$$

where $$y=y_1-10=6{\cdot}25 \text{ mm}$$

$$=-\frac{2828\times120\times6{\cdot}25}{11354{\cdot}166}=-186{\cdot}80 \text{ N/mm}^2$$

Shear stress q (at $y=6{\cdot}25$ mm from neutral axis)

$$=\frac{Fa\bar{y}}{I_{xx}\,b}=\frac{2828\times10\times5\times5}{11354{\cdot}166\times5}=12{\cdot}453 \text{ N/mm}^2$$

Moreover the horizontal component P_H acts at an eccentricity

$$e=8{\cdot}75+25=33{\cdot}75 \text{ mm}$$

Bending moment due to P_H,

$$M'=2828\times33{\cdot}75 \text{ N mm}$$

Due to P_H there will be direct compressive stress on the section

$$f_d=-\frac{P_H}{\text{area}}=-\frac{2828}{100+100}=-14{\cdot}14 \text{ N/mm}^2$$

Due to M' there will be tensile stress on the point b

$$f''=\frac{M'\times y}{I_{xx}}=\frac{2828\times33{\cdot}75\times(16{\cdot}25-10)}{11354{\cdot}166}$$

$$=+52{\cdot}54 \text{ N/mm}^2 \text{ (tensile)}$$

Net direct stress at point B

$$=-186{\cdot}80+52{\cdot}54-14{\cdot}14$$

$$f=-148{\cdot}40 \text{ N/mm}^2 \text{ (compressive)}$$

Shear stress at the point B

$$q=12{\cdot}453 \text{ N/mm}^2$$

Principal stresses at the point B

$$p_1, p_2 = \frac{f}{2} \pm \sqrt{\left(\frac{f}{2}\right)^2 + q^2}$$
$$= -74{\cdot}2 \pm \sqrt{(-74{\cdot}2)^2 + (12{\cdot}453)^2}$$
$$= -74{\cdot}2 \pm 75{\cdot}23$$

or $\quad p_1 = -149{\cdot}43$ N/mm² (compressive)

$p_2 = +1{\cdot}03$ N/mm² (tensile)

Problem 10·15. The box section shown in Fig. 10·26 (*a*) is made up of four 18×3 cm wooden planks connected by screws. Each screw can safely transmit a load of 150 **kg.** Determine the minimum necessary spacing of screws along the length of the beam if the maximum shear force transmitted by section is 1000 kg.

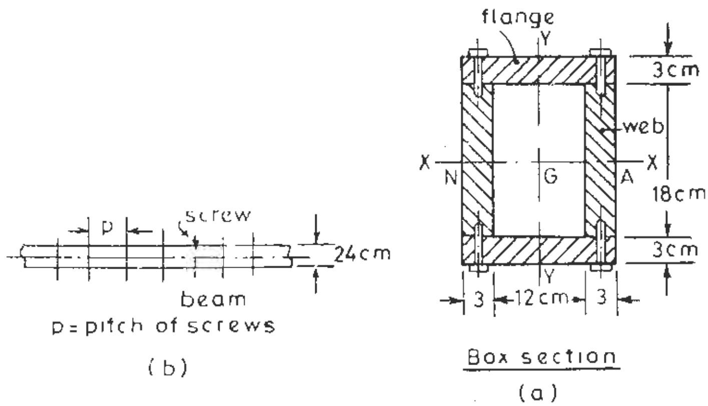


Fig. 10·26

Solution. Fig. 10·26 (*a*) shows the box section. The section is symmetrical about *YY* and *XX* axis and its *CG* lies at the centre as shown.

Moment of inertia,

$$I_{XX} = \frac{18 \times 24^3}{12} - \frac{12 \times 18^3}{12} = 20736 - 5832 = 14904 \text{ cm}^4$$

q at the edge of the web

$$= \frac{Fa\bar{y}}{I_{xx}b} = \frac{1000 \times (2 \times 9 \times 3 \times 10{\cdot}5)}{14904 \times 3 \times 2}$$
$$= 6{\cdot}34 \text{ kg/cm}^2 \qquad \text{(breadth is 6 cm)}$$

say *p* cm is the pitch of the screws

Then shear force per pitch length

$= p \times 3 \times 2 \times 6{\cdot}34$ kg

$= 38{\cdot}04\, p$ for two screws

$= 19{\cdot}02\, p$ for one screw

Load safely transmitted by a screw

$$=150 \text{ kg}=19{\cdot}02\, p$$

Pitch $\qquad p=7{\cdot}886 \simeq 8$ cm.

SUMMARY

1. Shear stress at any layer at a distance of y from the neutral axis,

$$q=\frac{Fa\bar{y}}{I_{NA}\, b}$$

where $\quad F$=shear force on the section

$a\bar{y}$=first moment about neutral axis of the area above the layer under consideration

I_{NA}=moment of inertia of the section about the neutral axis

b=breadth of the layer

2. In a circular section, maximum shear stress$=\frac{4}{3}\times$mean shear stress.
3. In a rectangular section, maximum shear stress$=1{\cdot}5$ mean shear stress.
4. In the case of a thin circular tube,

   Maximum shear stress$=2$ mean shear stress.
5. In the case of I section, most of the moment M is carried by the flanges and most of the shear force F is carried by the web.
6. In the case of rectangular, square, circular and I section maximum shear stress occurs at the neutral axis.
7. In the case of square section with diagonal lying in the plane of bending, maximum shear stress is 9/8 times the mean shear stress and occurs at a distance of $d/8$ from the neutral axis, where d is the diagonal of the section.
8. Near a free boundary, the shear stress on any section acts in a direction parallel to the boundary.

MULTIPLE CHOICE QUESTIONS

1. A rectangular section of a beam is subjected to a shearing force. The ratio of maximum shear stress to the mean shear stress developed in the section is

   (*a*) 2 $\qquad$ (*b*) 1·75

   (*c*) 1·50 $\qquad$ (*d*) 1·25

2. A square section with side a, of a beam is subjected to a shear force F, the magnitude of shear stress at the top edge of the square is

   (*a*) $1{\cdot}5\, F/a^5$ $\qquad$ (*b*) F/a^2

   (*c*) $0{\cdot}5\, F/a^2$ $\qquad$ (*d*) 0

3. In I-section of a beam subjected to transverse shear force F. The maximum shear stress is developed at

   (*a*) at the top edge of the top flange

   (*b*) at the bottom edge of the top flange

(c) at the centre of the web

(d) None of the above.

4. A circular section with area 100 mm^2 is subjected to a transverse shear force 6 kN. The magnitude of the maximum shear stress developed at the section is

(a) 120 N/mm^2 (b) 80 N/mm^2
(c) 60 N/mm^2 (d) 50 N/mm^2

5. A thin circular tube is used as a beam. At a particular section it is subjected to a transverse shear force F If the mean shear stress in the section is q, the maximum shear stress developed in the section is

(a) 2·5 q (b) 2·0 q
(c) 1·5 q (d) 1·25 q

6. A beam with a square cross-section 10 cm × 10 cm is simply supported at its ends and carries a central load W. If the maximum shear stress developed is not to exceed 6N/mm^2, the maximum value of W is

(a) 20 kN (b) 40 kN
(c) 60 kN (d) 80 kN

7. In an I section of a beam, subjected to a shear force F, the most of the shear force is shared by the web

(a) True (b) False

8. In an I-section of a beam, subjected to a bending moment M, the most of the moment M is shared by the web

(a) True (b) False

9. A beam with a square section of side a, is placed with one of its diagonals in the vertical plane. If the transverse shear force at a particular section is F, then the maximum shear stress developed in the section is

(a) 1·5 F/a^2 (b) 1·25 F/a^2
(c) 1·125 F/a^2 (d) None of the above.

10. A beam with square cross-section of side 100 mm, is placed with one of its diagonals in the horizontal plane. At a particular section shear force is 15 kN. The shear force developed at the neutral axis of the section is

(a) 3N/mm^2 (b) 2·25 N/mm^2
(c) 1·8 N/mm^2 (d) 1·5 N/mm^2

ANSWERS

1. (c)	2. (d)	3. (c)	4. (b)	5. (b)
6. (d)	7. (a)	8. (b)	9. (c)	10. (d)

EXERCISE

10·1. A 20×40 cm RSJ of I section with flanges 20×2 cm and web 36×1 cm is subjected to a bending moment M and a shear force F. What percentage of bending moment is carried by the flange and what percentage of shearing force is carried by the web ?

Ans. [88·14% of M, 95·28% of F]

10·2. A beam of rectangular section with breadth B and depth D is simply supported with a span length L. It carries a concentrated load P at its centre. Show that the principal stresses developed at a depth of $D/4$(in the central section of the beam) are

$$\frac{3PL}{8BD^2}\left[1 \pm \sqrt{1+\frac{9D^2}{4L^2}}\right]$$

10·3. Show that for a beam section of triangular shape base B, height H subjected to shear force F, the shear stress at the neutral axis is $\frac{8F}{3BH}$

10·4. A rolled steel section shown in the Fig. 10·27 is used as a beam. At a particular section the shear force is 5 kN. Plot the shear stress distribution over the depth of the section.

Ans. $[q_{NA}=48\cdot256 \text{ N/mm}^2]$

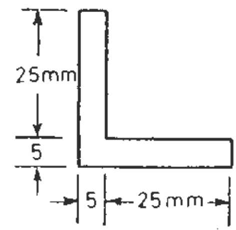


Fig. 10·27

10·5. Two beams, one with I section and the other with angle section are used as cantilevers with equal lengths and carry the uniformly distributed load on one w_1 and on the other w_2, such that the maximum shear stress in the web for both is the same. I section has flanges $12\times1\cdot5$ cm and web 15×1 cm. The angle section is 18 cm$\times$18 cm$\times$1·5 cm (thickness). Determine the ratio of the maximum stress due to bending developed in both.

Ans. $\left[\frac{w_1}{w_2}=0\cdot806, \quad \frac{f_1}{f_2}=0\cdot3296\right]$

10·6. A beam of square section is placed with its diagonal in the vertical plane. The shearing force at a certain cross-section is F. Show that the shear stress at the centroidal axis is equal to the mean shear stress in the section.

10·7. The section of a beam is a square with a small square cut-out from the centre as shown in the Fig. 10·28. Determine the shear stress at the neutral axis if the shear force at the section is 10 tonnes. **Ans.** $[404\cdot44 \text{ kg/cm}^2]$

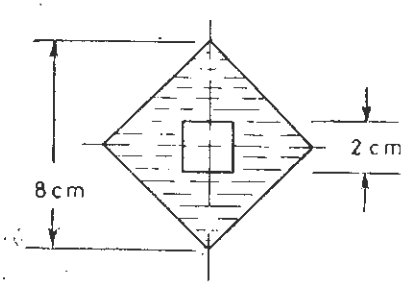


Fig. 10·28

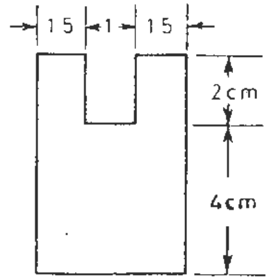


Fig. 10·29

10·8. A beam section subjected to shear force F is shown in the Fig. 10·29. Plot a graph showing the variation of shear stress over the thickness of the section. Determine ratio of maximum to mean shear stress.

Ans. $\left[q_A=0,\ q_B=\cdot0696\ F,\ q_B'=\cdot0522\ F,\ q_C=0,\ \frac{q_{max}}{q_{mean}}=1\cdot53\right]$

10·9. A timber beam of hexagonal section (side 10 cm) placed with one of its diagonals in a horizontal plane ; is simply supported at its ends and carries a central load W. Determine the distance between the supports and the load W if the maximum shearing stress is limited to 2 kg/cm² and the maximum direct stress due to bending is limited to 50 kg/cm².

[866 kg, 1·443 metre]

10·10. A rolled steel section shown in the Fig. 10·30 is subjected to a vertical shear force of 20 tonnes. Determine the shearing stresses at the corners A, B, C and D.

Ans. [$q_A=0$, $q_B=161{\cdot}4$ kg/cm², $q_C=695{\cdot}2$ kg/cm², $q_D=1108{\cdot}5$ kg/cm²]

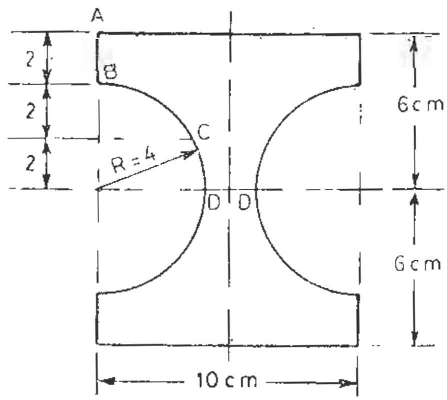


Fig. 10·30

10·11. A beam is of 10×30 cm I section, flanges 10×1·5 cm, web, 27×1 cm is subjected to a bending moment 20 k Nm (producing tension in top flange) and a shear force 120 kN. Determine principal stresses :

(*i*) at the bottom surface of top flange

(*ii*) at a distance of 5 cm from the top edge

(*iii*) at the neutral axis.

Ans. [(*i*) 54·896, −20·0 N/mm², (*ii*) +50·54, −24·70 N/mm², (*iii*) ±47·28 N/mm²]

10·12. A bracket of I section fixed in a wall supports a load of 3000 N as shown in the Fig. 10·31. Determine the principal stresses along the layer BB across the section aa.

Ans. [89·13 N/mm², −0·25 N/mm²]

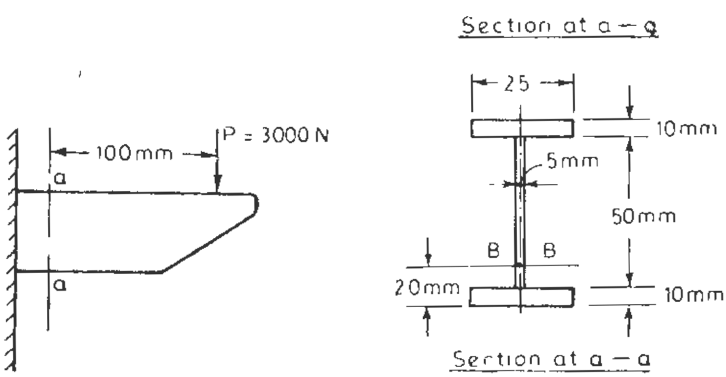


Fig. 10·31

10·13. The box section shown in Fig. 10·32 is made up of four 10×2 cm wooden planks connected by screws. Each screw can safely transmit a shear force of 1 kN. Estimate the maximum spacing between screws along the length of the beam if the maximum shear force transmitted by the section is 5 kN. **Ans.** [6 cm]

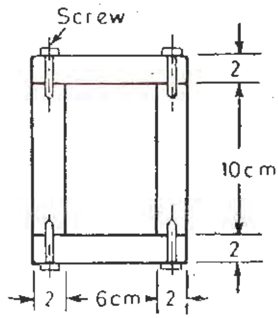


Fig. 10·32

11

Deflection of Beams and Cantilevers

In chapter 8, we have derived the flexure formula $M/I=E/R=f/y$ and studied about the maximum stresses developed in the extreme layers of the beam. The beam section is designed taking into consideration the allowable skin stress developed. When the beam carries the transverse points loads and distributed load over its length, the axis of the beam deflects. The deflection in the flexure elements of the machine must be with in the permissible limit so as to prevent misalignment and to maintain dimensional accuracy. Stiff flexural members are required in most engineering applications.

In this chapter basic differential equation relating slope and deflection with the bending moment will be developed. Only the deflection caused by bending will be discussed in this chapter and deflection due to shear will be discussed in the chapter on strain energy.

11·1. RELATION BETWEEN SLOPE, DEFLECTION, RADIUS OF CURVATURE AND BENDING MOMENT

Fig. 11·1 (*a*) shows a beam simply supported at its ends and carrying a point load W and a uniformly distributed load w as shown. Under the action of these transverse loads, the beam is deflected and its axis is bent as shown by *ACDB*. The radius of curvature of the beam at one section may be different than the radius of curvature at the other section.

As the bending moment along the length of the beam changes, its radius of curvature also changes as is obvious from the formula $M/I=E/R$. Consider a very small length δl

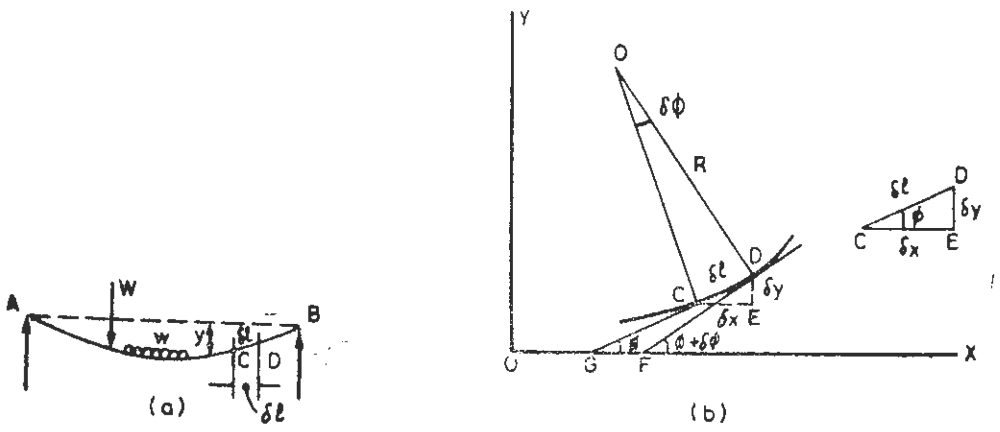


Fig. 11·1

marked by points C and D and for this very small length radius of curvature, R may be assumed as constant. Enlarged view of the length CD is shown in Fig. 11·1 (b), where R is the radius of curvature, O is the centre of curvature. CG is the tangent to the curve CD at the point C and DF is the tangent to the curve at the point D.

Slope of the curve at the point C $=\phi$

Slope of the curve at the point D $=\phi+\delta\phi$

then Angle subtended by length δl or CD at the centre of curvature

$$=\delta\phi$$

So
$$R\delta\phi=\delta l \qquad ...(1)$$

When $\delta l\to 0$, then segment CD can be approximated by a straight line CD having components δx and δy along x-axis and y-axis respectively.

$$\tan\phi=\frac{\delta y}{\delta x}$$

or in the limits
$$\tan\phi=\frac{dy}{dx} \qquad ...(2)$$

Differentiating both the sides of equation (2)

$$\sec^2\phi\,.\,d\phi=\frac{d^2y}{dx^2}\,dx$$

But
$$\sec^2\phi=1+\tan^2\phi=\left[1+\left(\frac{dy}{dx}\right)^2\right] \qquad ...(3)$$

So
$$d\phi=\frac{\dfrac{d^2y}{dx^2}\,dx}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}$$

From equation (1) $Rd\phi=dl=\sqrt{(dy)^2+(dx)^2}$

$$=dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Substituting above

$$dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}=R\,\frac{\dfrac{d^2y}{dx^2}\,dx}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}$$

or
$$\frac{1}{R}=\frac{\dfrac{d^2y}{dx^2}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}\times\frac{1}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}$$

$$=\frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \qquad ...(4)$$

Since dy/dx is the slope at a point in the beam is a very small quantity then $(dy/dx)^2$ will be much smaller than dy/dx and therefore can be neglected in comparison to 1.

So $$\frac{1}{R}=\frac{d^2y}{dx^2}$$

But from the flexure formula

$$\frac{1}{R}=\frac{M}{EI}$$

Therefore $$\frac{M}{EI}=\frac{d^2y}{dx^2}$$

or $$EI\frac{d^2y}{dx^2}=M \qquad \ldots(5)$$

This differential equation gives the relationship between the moment of resistance at a particular section and the cartesian co-ordinates of the point in the bent beam.

11·2. SIGN CONVENTIONS

In the chapter 7, we have taken the following conventions :

(*i*) Shear force tending to rotate the body in the clockwise direction is positive.

(*ii*) Bending moment producing concavity upwards in the beam is taken as positive.

(*iii*) If x-y is the cartesian co-ordinate system, then x is positive towards the right side of the origin O.

(*iv*) y is positive upwards.

Fig. 11·2 shows the bent shape of a beam loaded with transverse loads.

Slope at point A $=i_A$ is negative

Slope at point B $=i_B$ is positive

Deflection at point C $=y_C$ is negative

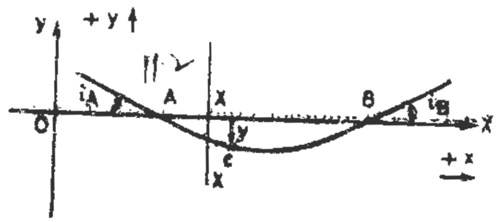


Fig. 11·2

When we follow these conventions, and consider a section X-X in the beam then (*i*) upward forces on the left side of the section are positive and (*ii*) clockwise moments of the forces on the left side of the section are positive.

11·3. A SIMPLY SUPPORTED BEAM WITH A CONCENTRATED LOAD

A beam AB of length l is simply supported at the ends A and B and carries a load W at its centre C, as shown in the Fig. 11·3. By symmetry the reactions at A and B will be equal *i.e.*, $R_A=R_B=W/2$. Now consider a section X-X at a distance of x from the end A.

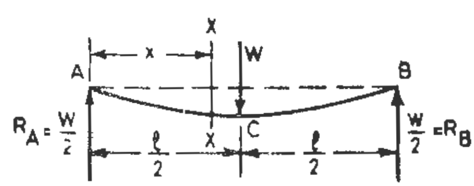


Fig. 11·3

Bending moment at X-X,

$$M=+R_A\,.\,x=\frac{W}{2}\,x$$

or $$EI\,\frac{d^2y}{dx^2}=\frac{W}{2}\,x \qquad ...(1)$$

Integrating the equation (1)

$$EI\,\frac{dy}{dx}=\frac{W}{2}\times\frac{x^2}{2}+C_1 \qquad ...(2)$$

where C_1 is the constant of integration.

As the beam is symmetrically loaded about its centre, slope at the centre of the beam will be zero.

i.e., $$\frac{dy}{dx}=0 \text{ at } x=\frac{l}{2}$$

Substituting the values in equation (2) we get

$$0=\frac{Wl^2}{2}+C_1$$

or $$C_1=-\frac{Wl^2}{16}$$

Equation (2) will now be

$$EI\,\frac{dy}{dx}=\frac{Wx^2}{4}-\frac{Wl^2}{16} \qquad ...(3)$$

Integrating the equation (3) we get

$$EI\,y=\frac{Wx^3}{12}-\frac{Wl^2}{16}\,x+C_2$$

where C_2 is another constant of integration.

At the end A, $x=0$, Deflection, $y=0$

Substituting in the equation above

$$0=0-0+C_2, \quad \text{or} \quad C_2=0$$

Therefore $$EI\,y=\frac{Wx^3}{12}-\frac{Wl^2x}{16} \qquad ...(4)$$

Maximum deflection occurs at the centre of the beam at $x=\frac{l}{2}$.

$$EI\,y_{max}=\frac{W}{12}\left(\frac{l}{2}\right)^3-\frac{Wl^2}{16}\left(\frac{l}{2}\right)=\frac{Wl^3}{96}-\frac{Wl^3}{32}$$

$$=-\frac{Wl^3}{48}$$

$$y_{max}=-\frac{Wl^3}{48\,EI} \text{ (indicating downward deflection)}$$

Slope at the end A, *i.e.*, at $x=0$

$$EI\, i_A = 0 - \frac{Wl^2}{16}$$

or
$$i_A = -\frac{Wl^2}{16\ EI}$$

By symmetry the slope at B,

$$i_B = \frac{Wl^2}{16\ EI}.$$

Note that bending moment is considered only in the portion AC. The equation (3) is valid only from portion AC. The slope at B can not be detemind by using equation (3).

Example 11·3-1. A girder or uniform section and constant depth is freely supported over a span of 2 metres. Calculate the central deflection under a central load of 2 tonnes, if $I_{xx}=780{\cdot}7\ \text{cm}^4$. Determine also the slopes at the ends of beam.

Given $E=2\times10^6\ \text{kg/cm}^2$.

Solution. Say a beam AB of length 2 m is simply supported at the ends and carries a central load 2 tonnes at C *i.e.*, the centre of the beam. Maximum deflection occurs at the point C.

$$y_{max} = \frac{Wl^3}{48\ EI}$$

$$W = 2\ \text{tonnes} = 2000\ \text{kg}$$

$$l = 2\ \text{m} = 200\ \text{cm}$$

$$E = 2\times10^6\ \text{kg/cm}^2,\quad I = 780{\cdot}7\ \text{cm}^4$$

$$y_{max} = \frac{2000\times(200)^3}{48\times2\times10^6\times780{\cdot}7} = 0{\cdot}213\ \text{cm} \quad \text{or} \quad 2{\cdot}13\ \text{mm}$$

In comparison to the span length of 2000 mm, deflection at the centre, 2·13 mm is very small.

Slope at the end A,
$$i_A = -\frac{Wl^2}{16\ EI} = -\frac{2000\times(200)^2}{16\times2\times10^6\times780{\cdot}7}$$

$$= -0{\cdot}0032\ \text{radian} \quad \text{or} \quad -0{\cdot}183^\circ$$

Slope at end B,
$$i_B = +0{\cdot}0032\ \text{radian} \quad \text{or} \quad +0{\cdot}183^\circ$$

Slope at the ends is also very small.

Exercise 11·3-1. A wooden beam of breadth 10 cm and depth 20 cm is used as a simply supported beam over a span length of 4 metres. If E for wood$=1\times10^5\ \text{kg/cm}^2$, what will be the magnitude of the load W at the centre of the beam to cause a deflection of 2 mm at the centre. What will then be the slope at the ends.

[**Ans.** 100 kg, $\pm1{\cdot}5\times10^{-3}$ radian]

11·4. A SIMPLY SUPPORTED BEAM WITH A UNIFORMLY DISTRIBUTED LOAD OVER ITS LENGTH

A beam AB of length l simply supported at the ends A and B carries a uniformly distributed load of w per unit length throughout its length as shown in the Fig. 11 4.

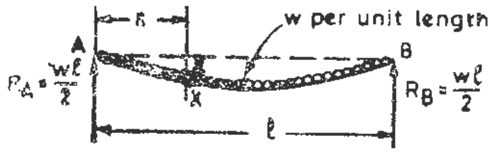


Fig. 11·4

Total vertical load on the beam $= wl$

By symmetry, reactions at A and B will be equal *i.e.*,

$$R_A = R_B = \frac{wl}{2}$$

Consider a section X-X at a distance of x from the end A

Bending moment on the section,

$$M = +\frac{wl}{2}x - \frac{wx^2}{2}$$

(note that C.G. of the load $w.x$ lies at a distance of $x/2$ from X-X)

So $$EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2} \quad \ldots(1)$$

Integrating the equation (1) we get

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1$$

where C_1 is the constant of integration

Since the beam is symmetrically loaded about its centre, slope at the centre of the beam will be zero.

i.e., $$\frac{dy}{dx} = 0 \quad \text{at} \quad x = \frac{l}{2}$$

Substituting in the equation above

$$0 = \frac{wl^3}{16} - \frac{wl^3}{48} + C_1$$

$$C_1 = -\frac{wl^3}{24}$$

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24} \quad \ldots(2)$$

Integrating the equation (2) again

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + C_2$$

where C_2 is another constant of integration at the end A, $x=0$, $y=0$.

$$0=0-0-C_2 \quad \text{or} \quad C_2=0$$

Then

$$EIy=\frac{wlx^3}{12}-\frac{wx^4}{24}-\frac{wl^3x}{24} \quad \ldots(3)$$

In this case, maximum deflection occurs at the centre of the beam *i.e.*, at $x=l/2$. Putting this value in equation (3)

$$EI\,y_{max}=\frac{wl^4}{96}-\frac{wl^4}{384}-\frac{wl^4}{48}$$

$$=-\frac{5}{384}wl^4$$

or

$$y_{max}=-\frac{5wl^4}{384EI} \quad \text{(indicating downward deflection)}$$

At the end A, $x=0$, $\frac{dy}{dx}=i_A$, slope at the end A

Putting $x=0$ in equation (2)

$$EI\,i_A=0-0-\frac{wl^3}{24}$$

or

$$i_A=-\frac{wl^3}{24EI}$$

By symmetry, slope at B,

$$i_B=+\frac{wl^3}{24\,EI}$$

Example 11·4-1. An I section steel girder of $I=2502$ cm^4 and depth 225 mm is used as a beam for a span length of 5 metres. The beam carries a uniformly distributed load w kg/cm run throughout its length. Determine the magnitude of w so that the maximum stress developed in the beam section does not exceed 600 kg/cm^2. Under this load determine slope and deflection in the beam at a distance of 1·5 m from one end.

Solution. $E=2\times10^6$ kg/cm^2

Span length, $l=5$ metres
$=500$ cm

Rate of loading, $w=?$

Allowable stress, $f=600$ kg/cm^2

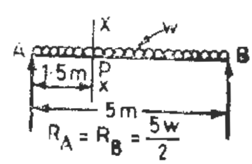


Fig. 11·5

Maximum bending moment occurs at the centre of the beam,

$$M_{max}=\frac{wl^2}{8}$$

$$=\frac{w\times500\times500}{8}=31250\,w \text{ kg cm}$$

Maximum stress will occur at the extreme layers at a distance of $\pm\, d/2$ from neutral axis, where d is the depth of the section and is equal to 22·5 cm.

$$f=\frac{M}{I}\times\frac{d}{2}=\frac{31250\,w}{2502}\times\frac{22\cdot5}{2}$$

or

$$600=\frac{31250\,w}{2502\times2}\times22\cdot5$$

w, rate of loading $=4\cdot27$ kg/cm run

At any section at a distance of x from the end A,

$$EI\frac{dy}{dx}=\frac{wlx^2}{4}-\frac{wx^3}{6}-\frac{wl^3}{24}$$

and

$$EI_y=\frac{wlx^3}{12}-\frac{wx^4}{24}-\frac{wl^3x}{24}$$

at $x=150$ cm *i.e.*, at point P

$$EI\,i_P=\frac{4\cdot27\times500\times150\times150}{4}-\frac{4\cdot27\times(150)^3}{6}-\frac{4\cdot27\times(500)^3}{24}$$

$$EI\,i_P=(1200\cdot9375\times10^4-2401\cdot875\times10^3-22\cdot2396\times10^6)$$

$$=(12\cdot009375-2\cdot401875-22\cdot2396)\times10^6$$

$$=-12\;63\times10^6$$

Slope at point P, $\quad i_P=-\dfrac{12\cdot63\times10^6}{2502\times2\times10^6}=-0\cdot0025$ radian

$$=-0\cdot145\text{ degree}$$

Then

$$EI\,y_P=\frac{wl}{12}(150)^3-\frac{w}{24}(150)^4-\frac{wl^3}{24}\,(150)$$

Putting $\quad w=4\cdot27$ kg/cm run, $\quad l=500$ cm

$$EI\,y_P=\frac{4\cdot27\times500}{12}\times(150)^3-\frac{4\cdot27}{24}\times(150)^4-\frac{4\cdot27\times(500)^3}{24}\times(150)$$

$$=6004\;68\times10^5-9007\cdot03\times10^4-333\cdot59\times10^7$$

$$=(60\cdot047-9\cdot00-333\cdot59)\times10^7$$

$$=-282\cdot543\times10^7$$

Deflection at point P, $\quad y_P=-\dfrac{282\cdot543\times10^7}{2502\times2\times10^6}$

$$=-0\cdot5646\text{ cm}=-5\cdot646\text{ mm}$$

Exercise 11·4-1. A rolled steel joist having $I=3600$ cm^4 is simply supported over a span of 6 metres. It carries a uniformly distributed load of 0·8 tonne/metre length. Determine slope and deflection at a distance of 2 metres from one end of the beam. $E=2000$ tonnes/cm^2.

[$\pm0\cdot2°$, $-1\cdot74$ cm]

11·5. A CANTILEVER WITH A CONCENTRATED LOAD

A cantilever AB of length l, free at end A and fixed at end B carries a concentrated load W at the free end as shown in the Fig. 11·6. At the end B, there will be a reaction, $R_B=W$ and a fixing couple $M_B=Wl$ (This we have already discussed in Chapter 7)

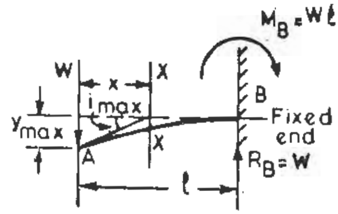


Fig. 11·6

Consider a section X-X at a distance of x from the end A.

Bending moment at X-X,

$$M=-Wx \quad \text{(a bending moment producing convexity upwards)}$$

or
$$EI\,\frac{d^2y}{dx^2}=-Wx \qquad ...(1)$$

Integrating the equation (1) we get

$$EI\,\frac{dy}{dx}=-\frac{Wx^2}{2}+C_1 \quad \text{(a constant of integration)}$$

at the end B, a fixed end, $\frac{dy}{dx}=0$

Therefore, $$EI\times 0=-\frac{Wl^2}{2}+C_1 \quad \text{or} \quad C_1=\frac{Wl^2}{2}$$

So $$EI\,\frac{dy}{dx}=-\frac{Wx^2}{2}+\frac{Wl^2}{2} \qquad ...(2)$$

Integrating the equation (2)

$$EIy=-\frac{Wx^3}{6}+\frac{Wl^2x}{2}+C_2 \quad \text{(another constant of integration)}$$

at the end B, a fixed end, $y=0$

Therefore $$EI\times 0=-\frac{Wl^3}{6}+\frac{Wl^3}{2}+C_2 \quad \text{or} \quad C_2=-\frac{Wl^3}{3}$$

$$EIy=-\frac{Wx^3}{6}+\frac{Wl^2x}{2}-\frac{Wl^3}{3} \qquad ...(3)$$

Maximum deflection takes place at the free end where $x=0$

$$EI\,y_{max}=-\frac{Wl^3}{3}$$

$$y_{max}=-\frac{Wl^3}{3EI} \quad \text{(showing downward deflection)}$$

Maximum slope also occurs at the free end, where $x=0$

$$EI\,i_{max}=0+\frac{Wl^2}{2}$$

$$i_{max}=\frac{Wl^2}{2EI}$$

Slope and deflection at any other point of the beam can be determined by using the equations (2) add (3).

Example 11·5-1. A cantilever 2 metres long is loaded with a point load of 50 kg at the free end. If the section is rectangular 8 cm × 16 cm deep, calculate slope and deflection at (*i*) free end of the cantilever, (*ii*) at a distance of 0·6 m from the free end. $E=10^5$ kg/cm².

Solution.

Width of the section, $B=8$ cm

Depth of the section, $D=16$ cm

Moment of inertia, $I=\frac{BD^3}{12}=\frac{8\times 16^3}{12}=2730{\cdot}67 \text{ cm}^4$

Load at the free end, $W=50$ kg

Length of the cantilever,

$$l=2 \text{ metre}=200 \text{ cm}$$

Young's modulus, $E=10^5$ kg/cm²

(*i*) At the free end

$$\text{deflection, } y_{max}=-\frac{Wl^3}{3EI}=-\frac{50\times(200)^3}{3\times 2730{\cdot}7\times 10^5}=-0{\cdot}488 \text{ cm}$$

Slope,
$$i_{max}=+\frac{Wl^2}{2EI}=\frac{50\times(200)^2}{2\times 2730{\cdot}7\times 10^5}$$
$$=0{\cdot}003662 \text{ radian}=0{\cdot}21°$$

(*ii*) **At a distance of 0·6 m from free end**

Taking $x=60$ cm from free end

$$EI\frac{dy}{dx}=-\frac{W}{2}x^2+\frac{Wl^2}{2}$$

$$EI\,i=-\frac{50}{2}\times 60^2+\frac{50\times 200^2}{2}=910000$$

$$i=\frac{910000}{2730{\cdot}67\times 10^5}=0{\cdot}0033 \text{ radian}=0{\cdot}19°$$

And
$$EI\,y=-\frac{Wx^3}{6}+\frac{Wl^2x}{2}-\frac{Wl^3}{3}$$

Putting $x=60$ cm

$$EI\,y=-\frac{50\times 60^3}{6}+\frac{50\times 200^2\times 60}{2}-\frac{50\times 200^3}{3}$$
$$=-1800\times 10^3+6000\times 10^4-133{\cdot}33\times 10^6$$
$$=(-1{\cdot}8+60-133{\cdot}33)\times 10^6=-75{\cdot}133\times 10^6$$
$$y=-\frac{75{\cdot}133\times 10^6}{2730{\cdot}67\times 10^5}=-0{\cdot}275 \text{ cm}$$

Exercise 11·5-1. A rolled steel joist of I section, depth 15 cm and moment of inertia 1455·6 cm⁴ is fixed at one end and a load W acts on the other end at a distance of 6 m from the fixed end. What is the maximum value of W such that the deflection at the centre of the cantilever does not exceed 2·5 mm. $E=2\times 10^6$ kg/cm². **[Ans. 32·346 kg]**

11·6. A CANTILEVER WITH A UNIFORMLY DISRIBUTED LOAD

A cantilever AB of length l, fixed at end B and free at the end A carries a uniformly distributed load w per unit length, as shown in the Fig. 11·7. There is reaction, $R_B=wl$ and fixing couple, $M_B=\frac{wl^2}{2}$ to maintain equilibrium.

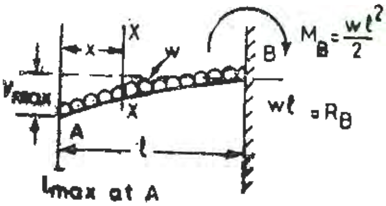


Fig. 11·7

Consider a section X-X at a distance of x from the end A.

B.M. at the section XX, $M=-wx.\frac{x}{2}=-\frac{wx^2}{2}$

or $$EI\frac{d^2y}{dx^2}=-\frac{wx^2}{2} \quad ...(1)$$

Integrating the equation (1) we get

$$EI\frac{dy}{dx}=-\frac{wx^3}{6}+C_1 \quad \text{(a constant of integration)}$$

At $x=l$; fixed end ; $\frac{dy}{dx}=0$

So $$0=-\frac{wl^3}{6}+C_1 \quad \text{or} \quad C_1=+\frac{wl^3}{6}$$

$$EI\frac{dy}{dx}=-\frac{wx^3}{6}+\frac{wl^3}{6} \quad ...(2)$$

Integrating the equation (2) further we get

$$EI\,y=-\frac{wx^4}{24}+\frac{wl^3x}{6}+C_2 \quad \text{(another constant of integration)}$$

At the fixed end B ; $x=l$; $y=0$

So $$0=-\frac{wl^4}{24}+\frac{wl^4}{6}+C_2 \quad \text{or} \quad C_2=-\frac{wl^4}{8}$$

Therefore, $$EI\,y=-\frac{wx^4}{24}+\frac{wl^3x}{6}-\frac{wl^4}{8} \quad ...(3)$$

Maximum deflection occurs at the free end, *i.e.*, at $x=0$.

Substituting this in equation (3)

$$EI\,y_{max}=-0+0-\frac{wl^4}{8}$$

Maximum deflection, $y_{max}=-\frac{wl^4}{8EI}$ (indicating downward deflection)

Moreover maximum slope in the cantilever also takes place at the free end as is obvious from the diagram *i.e.*, at $x=0$, substituting this value in equation (2)

$$EI\,i_{max}=-0+\frac{wl^3}{6}$$

Maximum slope, $i_{max}=\dfrac{wl^3}{6EI}$

The slope and deflection at any other point of the beam can be found out by substituting the value of x in equations (2) and (3) respectively.

Example 11·6-1. An aluminium cantilever of rectangular section 48 mm wide and 36 mm deep, of length 250 mm carries a uniformly distributed load. What is the maximum value of w if the maximum deflection in the cantilever is not to exceed 1 mm.

$$E \text{ for aluminium}=70\times 10^3 \text{ N/mm}^2$$

Solution.

Length of the beam, $l=250$ mm

Breadth, $b=48$ mm

Depth, $d=36$ mm

Moment of inertia, $I=\dfrac{bd^3}{12}=\dfrac{48\times 36^3}{12}=186624 \text{ mm}^3$

Young's modulus, $E=70\times 10^3 \text{ N/mm}^2$

Maximum deflection,

$$y_{max}=\frac{wl^4}{8EI}=1 \text{ mm}$$

or $$\frac{w\times 250^4}{70\times 10^3\times 186624}=1$$

or $$w=\frac{70\times 10^3\times 186624}{(250)^4}=3\cdot 344 \text{ N/mm}$$

$$=3\ 344 \text{ kN/metre run}$$

Exercise 11·6-1. A cantilever of circular section of diameter d and length 1 metre carries a uniformly distributed load of 500 kg/metre run. What is the minimum diameter of the section if the deflection at the free end is not to exceed 2 mm. $E=1\times 10^6$ kg/cm².

[**Ans.** 15·022 cm]

11·7 MACAULAY'S METHOD

The method followed so far for the determination of slopes and deflections in a beam is laborious when we consider each portion of the beam (between two adjacent loads) separately, making equation for the bending moment for a particular portion and integrating the expressions and finding out the constants of integration for each portion and then finding out slope deflection at a particular section lying in that portion of the beam. The method devised by Macaulay gives one continuous expression for the bending moment which applies for all the portions of the beam and the constants of integration determined by using boundary conditions, are also applicable for all portions of the beam. By using this method slope or deflection at any section throughout the length of the beam is determined by a single expression. This method can be best explained by taking an example.

(*a*) **Concentrated Loads.** Fig. 11·8 shows a beam *ABCDE* of length *l*, supported at *A* and at *D*, at a distance of x_3 from *A*. There are concentrated loads on the beam *i.e.*, W_1 at *B*, at a distance of x_1 from the end *A*, load W_2 at *C*, at a distance of x_2 from the end *A* and load W_3 at the end *E* of the beam. R_1 and R_2 are the support reactions at *A* and *D*.

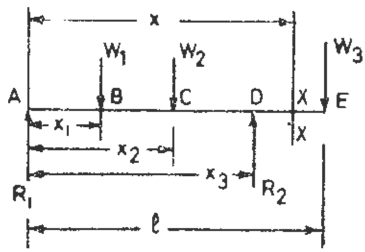


Fig. 11·8

Macaulay's method can be briefly outlined as follows :

1. Consider a section *X-X*, in the last portion of the beam starting from one end, and at a distance of *x* from the starting end. In the example shown, last portion is *DE*.

2. Make an equation for the bending moment at the section *X-X* in the last portion of the beam. In the example shown,

B.M. at the section *X-X*,

$$M=R_1x-W_1(x-x_1)-W_2(x-x_2)+R_2(x-x_3)$$

or

$$EI\frac{d^2y}{dx^2}=R_1x-W_1(x-x_1)-W_2(x-x_2)+R_2(x-x_3) \quad ...(1)$$

3. Integrate the expression for the bending moment and the brackets as shown above will be integrated as a whole, such as

$$EI\frac{dy}{dx}=\frac{R_1x^2}{2}-\frac{W_1}{2}(x-x_1)^2-\frac{W_2}{2}(x-x_2)^2+\frac{R_2}{2}(x-x_3)^2+C_1$$

and

$$EIy=\frac{R_1x^3}{6}-\frac{W_1}{6}(x-x_1)^3-\frac{W_2}{6}(x-x_2)^3+\frac{R_2}{6}(x-x_3)^3+C_1x+C_2$$

where C_1 and C_2 are the constants of integration.

4. Boundary conditions are used to determine the constants C_1 and C_2, subject to the conditions that all terms for which the quantity inside a brackets is negative is omitted. As an example at the end *A*, $x=0$, deflection $y=0$. This boundary condition can be used to determine one of the two constants. The terms $(x-x_1)$, $(x-x_2)$, $(x-x_3)$ all become negative and are to be omitted. In other words, when $x=0$, the point lies only in the portion *AB* expressions other than R_1x are not valid for the portion *AB*.

5. Similarly the value of other constant is determined. Say in the example at $x=x_3$, $y=0$ *i.e.*, we have to consider the portion *CD*.

6. Once the constants C_1 and C_2 are determined, they are applicable for all the portions of the beam.

7. Say slope and deflection are to be determined in portion *BC*. then *x* will be taken in this portion and $(x-x_2)$ is either zero or negative and $(x-x_3)$ a negative term are to be omitted. Similarly for the calculation of slope and deflection in the portion *AB*, the terms $(x-x_2)$ and $(x-x_3)$ become negative and are to be omitted.

(*b*) **Uniformly distributed loads.** Consider a beam *ABCD* of length l supported at *A* and *D* and carrying uniformly distributed load of intensity w per unit length over *BC*, where *B* is at a distance of x_1 from *A*, and *C* is at a distance of x_2 from *A*. There are 3 portions of the beam *i.e.*, *AB*, *BC* and *CD*. Consider a section *X-X* at a distance of x from the end *A*, in the portion *CD* of the beam. Now in order to obtain an expression for the bending moment, which will apply for all values of x (*i.e.*, in the portion *AB*, *BC* also) it is necessary to continue the loading upto the section x and applying equal and opposite load of intensity w from x_2 to x as shown in the Fig. 11·9.

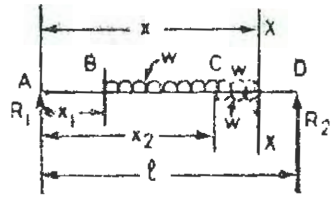


Fig. 11·9

BM at the section *X-X*,

$$M = \underset{\text{I}}{R_1 x} - \underset{\text{II}}{\frac{w}{2}(x-x_1)^2} + \underset{\text{III}}{\frac{w}{2}(x-x_2)^2}$$

First term is applicable for portion *AB*, upto II term for *BC* and upto III term expression is applicable for the portion *CD*.

$$EI\frac{d^2y}{dx^2} = R_1\ x - \frac{w}{2}(x-x_1)^2 + \frac{w}{2}(x-x_2)^2$$

After integrating $EI\,y\dfrac{dy}{dx} = \dfrac{R_1 x^2}{2} - \dfrac{w}{6}(x-x_1)^3 + \dfrac{w}{6}(x-x_2)^3 + C_1$

and $$EI\,y = \frac{R_1 x^3}{6} - \frac{w}{24}(x-x_1)^4 + \frac{w}{24}(x-x_2)^4 + C_1x + C_2$$

The constants of integration are C_1 and C_2 which are determined using boundary conditions. The values of slope and deflection at any section in any portion of the beam can be determined as per the procedure explained in part (*a*).

Example 11·7-1. A beam *ABCD*, 6 m long carries a concentrated load of 20 kN at end *A* and 40 kN at point *C*, at a distance of 4 m from *A*. The beam is supported over a span of 4 metres at points *B* and *D* as shown in the Fig. 11·10. Determine maximum deflection and state where it occurs.

$$E = 2\times10^5\ \text{N/mm}^2$$

$$I = 8000\ \text{cm}^4.$$

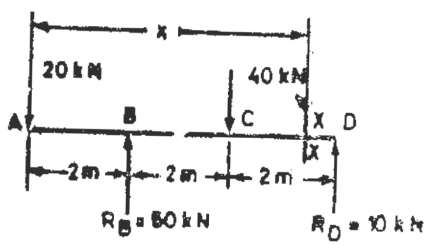


Fig. 11·10

Solution. For support reactions take moments of the forces about the point *D*.

$$20\times6+40\times2 = R_B\times4$$

$$R_B = 50\ \text{kN}$$

then $$R_D = 20+40-R_B = 10\ \text{kN}$$

Using Macaulay's method consider a section X-X in the last position CD of the beam at a distance of x from end A.

BM at the section, $\quad M=-20\,(x)+50\,(x-2)-40\,(x-4)$

or
$$EI\,\frac{d^2y}{dx^2}=-20x+50\,(x-2)-40\,(x-4) \quad \ldots(1)$$

Integrating equation (1) we get
$$EI\,\frac{dy}{dx}=-\frac{20\,x^2}{2}+\frac{50}{2}\,(x-2)^2-\frac{40}{2}\,(x-4)^2+C_1 \quad \ldots(2)$$

and
$$EI\,y=-\frac{10\,x^3}{3}+\frac{25}{3}\,(x-2)^3-\frac{20}{3}\,(x-4)^3+C_1x+C_2 \quad \ldots(3)$$

Boundary conditions are at $x=2$ m, $y=0$, putting this value of x, in this case term $(x-4)$ will become negative and, therefore, is to be omitted.

So
$$0=-10\times\frac{2^3}{3}+0-\text{omitted term}+2\,C_1+C_2$$

or
$$2\,C_1+C_2=+26{\cdot}667 \quad \ldots(4)$$

Another boundary condition is that at $x=6$ m, $y=0$, putting this value of x,
$$0=-10\times\frac{6^3}{3}+\frac{25}{3}\,(6-2)^3-\frac{20}{3}\,(6-4)^3+6\,C_1+C_2$$
$$0=-720+533{\cdot}33-53{\cdot}33+6\,C_1+C_2$$
$$=-240+6\,C_1+C_2$$

or
$$6\,C_1+C_2=240 \quad \ldots(5)$$

From equations (4) and (5), the values of constants are
$$C_1=52{\cdot}333,\quad C_2=-80$$

Equation for deflection will now be
$$EI\,y=-10\,\frac{x^3}{3}+\frac{25}{3}\,(x-2)^3-\frac{20}{3}\,(x-4)^3+53{\cdot}333\,x-80 \quad \ldots(6)$$

At $x=0$ $\quad y=y_A$
$$EI\,y_A=-80$$

Now
$$E=200\text{ kN/mm}^2=200\times10^6\text{ kN/m}^2$$
$$I=8000\text{ cm}^4=8000\times10^{-8}\text{ m}^4$$
$$EI=200\times10^6\times8000\times10^{-8}=16000\text{ kNm}^2$$

So
$$y_A=-\frac{80}{16000}=-0{\cdot}005\text{ m}$$
$$=-0{\cdot}5\text{ cm or }-5\text{ mm}$$

At $x=4$ m, $\quad y=y_C$, deflection at the point C
$$EI\,y_C=-10\times\frac{4^3}{3}+\frac{25}{3}\,(4-2)^3-0+53{\cdot}333\times4-80$$

$$=-213{\cdot}333+66{\cdot}667+213{\cdot}332-80$$

$$=-13{\cdot}333$$

$$y_C=-\frac{13{\cdot}343}{16000}=-0{\cdot}0008 \text{ m}=-0{\cdot}08 \text{ cm}$$

In this case maximum deflection occurs at the free end where the load 20 kN is applied.

Example 11·7-2. A beam 6 m long simply supported on ends carries a uniformly distributed load of 2·4 tonnes per metre length over 3 metres length starting from the point B, at a distance of 1 m from the end A as shown in Fig. 11·11.

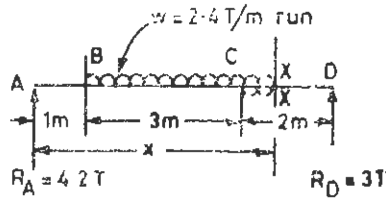


Fig. 11·11

Determine the magnitude of slope at the ends A and D. Determine also the maximum deflection in the beam.

$$E=2000 \text{ tonnes/cm}^2$$

$$I=4800 \text{ cm}^4.$$

Solution. For support reactions take moments about the point A

$$2{\cdot}4\times3\times(1+1{\cdot}5)=6\ R_D,$$

and $$R_D=3 \text{ tonnes},\ R_A=3\times2{\cdot}4-3=4{\cdot}2 \text{ tonnes}$$

Now consider a section X-X at a distance of x from the end A, in the portion CD of the beam. Continue the uniformly distributed load upto the section and apply equal and opposite uniformly distributed load from C to x as shown in the figure.

BM at any section X-X,

$$M=+4{\cdot}2x-\frac{w}{2}(x-1)^2+\frac{w}{2}(x-4)^2$$

or $$EI\frac{d^2y}{dx^2}=4{\cdot}2\,x-\frac{2{\cdot}4}{2}(x-1)^2+\frac{2{\cdot}4}{2}(x-4)^2$$

Integrating $$EI\frac{dy}{dx}=\frac{4{\cdot}2\,x^2}{2}-\frac{2{\cdot}4\,(x-1)^3}{3\times2}+\frac{2{\cdot}4}{2\times3}(x-4)^3+C_1$$

$$=2.1\,x^2-0{\cdot}4\,(x-1)^3+0{\cdot}4\,(x-4)^3+C_1$$

Integrating further $$EI\,y=\frac{2{\cdot}1\,x^3}{3}-\frac{0{\cdot}4}{4}(x-1)^4+\frac{0{\cdot}4}{4}(x-4)^4+C_1x+C_2$$

or $$EI\,y=0{\cdot}7\,x^3-0{\cdot}1\,(x-1)^4+0{\cdot}1\,(x-4)^4+C_1x+C_2$$

Taking the boundary conditions that at $x=0$, $y=0$ terms $(x-1)$ and $(x-4)$ are to be omitted

$$0=0-\text{omitted term}+\text{omitted term}+C_1\times0+C_2$$

or $$C_2=0$$

Taking another boundary condition that is at

$$x=6 \text{ m},\ y=0$$

we get $$0=0{\cdot}7\times6^3-0{\cdot}1\,(6-1)^4+0{\cdot}1\,(6-4)^4+6\,C_1$$

or $$0=151{\cdot}2-62{\cdot}5+1{\cdot}6+6\,C_1$$

$$0=90{\cdot}3+6\,C_1$$

or $$C_1=-15{\cdot}05$$

The expression for slope will now be

$$EI\,\frac{dy}{dx}=2{\cdot}1\,x^2-0{\cdot}4\,(x-1)^3+0{\cdot}4\,(x-4)^3-15{\cdot}05$$

At A, $$x=0,\ \frac{dy}{dx}=i_A$$

$$EI\,i_A=-15{\cdot}05$$

$$E=2000\times10^4\ \text{tonne/metre}^2$$

$$I=4800\times10^{-8}\ \text{m}^4$$

$$EI=960\ \text{tonne metre}^2$$

Therefore $$i_A=-\frac{15{\cdot}05}{960}=-0\ 0157\ \text{radian}$$

Slope at A $$=-0{\cdot}9^\circ$$

At B, $$x=6\ \text{m},\ \frac{dy}{dx}=i_B$$

So $$EI\,i_B=2\ 1\,(6)^2-0{\cdot}4\,(6-1)^3+0{\cdot}4\,(6-4)^3-15{\cdot}05$$

$$=75{\cdot}6-50+3{\cdot}2-15{\cdot}05$$

$$=+13{\cdot}75$$

$$i_B=\frac{13{\cdot}75}{960}=+0{\cdot}01433\ \text{radian}$$

Slope at B $$=0{\cdot}82^\circ.$$

Maximum deflection. Maximum deflection in the beam may occur at a section in the portion BC. The term $(x-4)$ will be negative and is to be omitted. At this section dy/dx will be zero.

For the portion BC,

$$EI\,\frac{dy}{dx}=2{\cdot}1\,x^2-0{\cdot}4\,(x-1)^3+C_1$$

$$=2{\cdot}1\,x^2-0{\cdot}4\,(x-1)^3-15{\cdot}05$$

$$=2{\cdot}1\,x^2-0{\cdot}4\,(x^3-3x^2+3x-1)-15{\cdot}05$$

$$=2{\cdot}1\,x^2-0{\cdot}4x^3+1{\cdot}2x^2-1{\cdot}2x+0{\cdot}4-15{\cdot}05$$

or $$=-0{\cdot}4x^3+3{\cdot}3x^2-1{\cdot}2x-14{\cdot}65$$

or $$x^3-8{\cdot}25x^2+3x+36{\cdot}625=0$$

$x\simeq2{\cdot}92$ m from end A. (by trial)

$$EI\,y_{max}=0{\cdot}7\times2{\cdot}92^3-0{\cdot}1\,(2{\cdot}92-1)^4-15{\cdot}05\times2{\cdot}92$$

$$=17{\cdot}428-1{\cdot}359-43{\cdot}946$$

$$=-27{\cdot}877$$

$$y_{max}=-\frac{27{\cdot}877}{960}=-0{\cdot}029\ \text{m}$$

Maximum deflection $$=-2{\cdot}9\ \text{cm}.$$

Exercise 11·7-1. A beam 4 m long simply supported at the ends carries loads of 2 tonnes each at a distance of 1 m from each end. Determine the slope at the ends and the maximum deflection.

Given $E=2\times10^6$ tonnes/cm²

$I=5000$ cm⁴. [**Ans.** $\pm0{\cdot}171°$, 0·366 cm]

Exercise 11·7-2. A beam 7 m long carries a uniformly distributed load of 2 tonnes/metre run throughout its length. The beam is supported over a span of 5 metres with overhang of 2 m on one side. Determine the slope and deflection at the cantilevered end.

$E=2000$ tonnes/cm²

$I=802$ cm⁴. [**Ans.** 0·506°, 9·35 mm]

11·8. ECCENTRIC LOAD ON A SIMPLY SUPPORTED BEAM

Fig. 11·12 shows a beam AB of length l, simply supported at the ends and carrying a load W at a point C, at a distance of a from the end A or at a distance of b from the end B. Say $a<b$. For support reactions, take moments of the forces about the end A

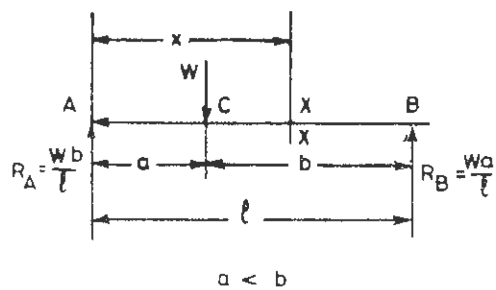


Fig. 11·12

$$W.a=R_B.l$$

$$R_B=\frac{Wa}{l}$$

$$R_A=W-R_B=\frac{Wb}{l}$$

Macaulay's method can be used to determine the slope and deflection at any point of the beam

B.M. at the section. $M=R_A\,.\,x-W(x-a)$

or
$$EI\frac{d^2y}{dx^2}=\frac{Wb}{l}x-W(x-a) \qquad \ldots(1)$$

Integrating equation (1),

$$EI\frac{dy}{dx}=\frac{Wbx^2}{2l}-\frac{W(x-a)^2}{2}+C_1 \qquad \ldots(2)$$

$$EIy=\frac{Wbx^3}{6l}-\frac{W(x-a)^3}{6}+C_1x+C_2 \qquad \ldots(3)$$

where C_1 and C_2 are constants of integration

Boundary Conditions

at $x=0,\ y=0$

and at $x=l,\ y=0$

Substituting the first condition we get

$0=0+C_1\times0+C_2$ or $C_2=0$ (term $(x-a)$ to be omitted), Substituting the second boundary condition

$$0=\frac{Wbl^3}{6l}-\frac{W(l-a)^3}{6}+C_1l$$

or
$$C_1=\frac{W(l-a)^3}{6l}-\frac{Wbl}{6}$$

$$=\frac{Wb^3}{6l}-\frac{Wbl}{6}=\frac{Wb}{6l}(b^2-l^2)$$

$$C_1=\frac{Wb}{6l}(b+l)(b-l)$$

$$=-\frac{Wab}{6l}(l+b)=-\frac{Wab}{6l}(a+2b) \qquad \text{because } l=a+b$$

The expressions for slope and deflection will now be

$$EI\frac{dy}{dx}=\frac{Wbx^2}{2l}-\frac{W}{2}(x-a)^2-\frac{Wab}{6l}(a+2b)$$

$$EIy=\frac{Wbx^3}{6l}-\frac{W}{6}(x-a)^3-\frac{Wab}{6l}(a+2b)x$$

The constants determined above are valid for both the portions AC and CB.

Example 11·8-1. A beam 6 m long, simply supported at both the ends carries a load 4 kN at a distance of 2 m from one end. Determine the slope at the ends and maximum deflection. Given $E=2\times10^5$ N/mm^2 I=4800 cm^4

Solution.

Load $W=6$ kN

Distance $a=2$ m, $b=4$ m

Length $l=6$ m

$$E=2\times10^5 \text{ N/mm}^2=2\times10^{11} \text{ N/m}^2=2\times10^8 \text{ kN/m}^2$$

$$I=4800 \text{ cm}^4=4800\times10^{-8} \text{ m}^4$$

$$EI=2\times10^8\times4800\times10^{-8}=9600 \text{ kN m}^2$$

Expressions for slope and deflection will now be

$$EI\frac{dy}{dx}=\frac{6\times4x^2}{2\times6}-\frac{6}{2}(x-2)^2-\frac{6\times2\times4}{6\times6}(2+8)$$

$$=2x^2-3(x-2)^2-\frac{40}{3} \qquad \ldots(1)$$

$$EIy=\frac{6\times4\times x^3}{6\times6}-\frac{6}{6}(x-2)^3-\frac{6\times2\times4}{6\times6}(2+8)x$$

$$=\frac{2}{3}x^3-(x-2)^3-\frac{40}{3}x \qquad \ldots(2)$$

Slope at the end A, where $x=0$, (term $(x-2)$ is negative and so omitted)

$$EI\, i_A = 2\times 0 - \frac{40}{3}$$

$$i_A = \frac{-40}{3}\times\frac{1}{9600} = -0.00139 \text{ radian} = -0.08^\circ$$

Slope at the end B, where $x=6$ m

$$EI\, i_B = 2\times 6^2 - 3(6-2)^2 - \frac{40}{3} = 72 - 48 - \frac{40}{3} = +10.667$$

$$i_B = \frac{10.667}{9600} = +0.0011 \text{ radian} = 0.063^\circ$$

Maximum Deflection. Maximum deflection will occur at the section where the slope is zero. This section may lie in the portion CB. (Fig. 11·12)

So,
$$2x^2 - 3(x-2)^2 - 13.333 = 0$$
$$2x^2 - 3x^2 + 6\times 2x - 12 - 13.333 = 0$$
$$-x^2 + 12x - 25.333 = 0$$
or
$$x^2 - 12x + 25.333 = 0$$
$$x = \frac{12 - \sqrt{144 - 101.332}}{2} = \frac{12 - 6.532}{2}$$
$$= 2.734 \text{ m}$$

Substituting $x = 2.734$ m for the maximum value of deflection

$$EIy_{max} = \frac{2}{3}(2.734)^3 - (2.734-2)^3 - \frac{40}{3}\times 2.734$$
$$= 13.624 - 5.213 - 36.453 = -28.042$$
$$y_{max} = -\frac{28.042}{9600} = -0.00292 \text{ m} = -0.29 \text{ cm}$$

Exercise 11.8-1. A beam of length l simply supported at both the ends carries a load W at a distance of $l/3$ from one end. Determine :

(*i*) Slope at the ends

(*ii*) Deflection under the load.

EI is the flexural rigidity of the beam.

[**Ans.** (*i*) $-\dfrac{5Wl^2}{81EI}$, $+\dfrac{4Wl^3}{81EI}$, (*ii*) $-\dfrac{2}{243}\dfrac{Wl^3}{EI}$]

11·9. IMPACT LOADING OF BEAMS

If a load W is dropped from a height h onto a beam supported at the ends, the kinetic energy of the load is converted into the strain energy for the beam. An instantaneous deflection δ_i is produced in the beam at the point where the falling load strikes and an instantaneous stress f is developed in the beam. The beam starts vibrating and finally settles down to a deflection $\delta < \delta_i$, which would have occurred if the load W had been applied gradually.

Potential energy loss by the falling weight $= W(h+\delta_i)$

Strain energy absorbed by the beam $= \frac{1}{2} P\, \delta_i$

Where δ_t is the instantaneous deflection and P is the equivalent gradually applied load *i.e.*, if P is applied gradually on the beam at the point where load W strikes, the deflection in the beam will be δ_t.

Consider a beam of length l, simply supported at the ends and the load W falling from a height h strikes the beam in its middle, say under the falling load, the instantaneous defleetion is δ_t.

Then $$W(h+\delta_t)=\tfrac{1}{2}P\delta_t \qquad ...(i)$$

where equivalent gradually applied load is

$$P=\frac{48EI\ \delta_t}{l^3}=K\delta_t \qquad ...(ii)$$

where K is a constant and equal to $\dfrac{48EI}{l^3}$.

So $$W(h+\delta_t)=\tfrac{1}{2}\ K\delta_t.\delta_t=\tfrac{1}{2}\ K\delta_t^2$$

$$\tfrac{1}{2}\ K\delta_t^2-W\delta_t-Wh=0 \qquad ...(3)$$

If W and h are given, then δ_t ean be found out and the maximum instantaneous stress developed in the beam section is determined.

Example 11·9·1. An ISJB 150 rolled steel joist is simply supported over a span of 4 metres A weight of 40 kg is dropped onto the middle of the beam, producing an instantaneous maximum stress of 800 kg/cm². Calculate the height from which the weight was dropped and the maximum instantaneous deflection in the beam.

$$I=322{\cdot}1\ \text{cm}^4\ ; \quad E=2000\ \text{tonnes/cm}^2$$

Solution. Say the equivalent gradually applied load $=P$ kg

Maximum bending moment at the centre of the beam

$$M_{max}=\frac{Pl}{4}$$

where l=span length=400 cm

So $$M_{max}=\frac{P\times 400}{4}=100\ P\ \text{kg-cm}$$

f, Maximum stress produced $=800$ kg/cm²

I for the section $=322{\cdot}1$ cm⁴

d, depth of the section $=15$ cm

Now $$f=\frac{M}{I}\times\frac{d}{2}$$

$$800=\frac{100\ P}{322{\cdot}1}\times\frac{7{\cdot}5}{}$$

$$P=\frac{800\times 322{\cdot}1}{750}=343{\cdot}57\ \text{kg}$$

Instantaneous defleetion, $\delta_t=\dfrac{Pl^3}{48EI}$

where $$E=2000\times 1000\ \text{kg/cm}^2$$

So $$\delta_t=\frac{343{\cdot}57\times(400)^3}{48\times2\times10^6\times322{\cdot}1}=\frac{343{\cdot}57\times64}{96\times322{\cdot}1}=0{\cdot}711 \text{ cm}$$

Now $$W(h+\delta_t)=\frac{1}{2}P\delta_t \text{ where } W=40 \text{ kg}$$

So $$40(h+0{\cdot}711)=\frac{1}{2}\times343{\cdot}57\times0{\cdot}711=122{\cdot}139$$

$$h=\frac{122{\cdot}139}{40}-0{\cdot}711=3{\cdot}053-0{\cdot}711=2{\cdot}342 \text{ cm}$$

Exercise 11·9-1. A gradually applied load of 250 kg at the middle of a beam simply supported at the ends, produces a deflection of 0·5 mm. What will be the maximum instantaneous deflection produced by a weight of 50 kg dropped onto the middle of the beam from a height of 20 cm. [**Ans.** 0·645 cm]

11·10. PROPPED CANTILEVERS AND BEAMS

A cantilever of length l carring uniformly distributed load w per unit length is propped at the end, so that the level at the free end is the same as the level of fixed end, then the reaction of the prop will be equivalent to a load producing deflection in the opposite direction so that the deflection produced by the uniformly distributed load is nullified.

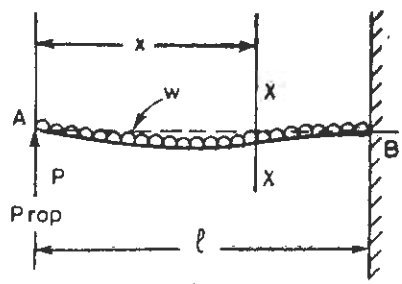


Fig. 11·13

If the prop is not provided then deflection at the free end due to uniformly distributed load w (throughout the length of the cantilever)

$$=-\frac{wl^4}{8\,EI}$$

Deflection due to prop at the free end

$$=+\frac{Pl^3}{3\,EI}$$

So $$\frac{Pl^3}{3\,EI}-\frac{wl^4}{8\,EI}=0$$

or Reaction at prop, $$P=\frac{3\,wl}{8} \quad ...(1)$$

Consider a section X-X at a distance of x from the end A as shown in the Fig. 11·13.

BM at the section $$=+Px-\frac{wx^2}{2}$$

$$EI\,\frac{d^2y}{dx^2}=Px-\frac{wx^2}{2}=\frac{3\,wl}{8}\,x-\frac{wx^2}{2} \quad ..(2)$$

Integrating equation (2)

$$EI\frac{dy}{dx}=\frac{3\,wlx^2}{16}-\frac{wx^3}{6}+C_1$$

Slope is zero at $x=l$

Therefore, $0=\frac{3\,wl^3}{16}-\frac{wl^3}{6}+C_1$

or $C_1=\frac{wl^3}{6}-\frac{3\,wl^3}{16}=-\frac{wl^3}{48}$

So $$EI\frac{dy}{dx}=\frac{3\,wlx^2}{16}-\frac{wx^3}{6}-\frac{wl^3}{48} \qquad ...(3)$$

Integrating again equation (3)

$$EI\,y=\frac{3\,wlx^3}{48}-\frac{wx^4}{24}-\frac{wl^3x}{48}+C_2$$

At $x=l$, at the fixed end, deflection is zero

So $0=\frac{wl^4}{16}-\frac{wl^4}{24}-\frac{wl^4}{48}+C_2$

or $C_2=0$

So $$EI\,y=\frac{wlx^3}{16}-\frac{wx^4}{24}-\frac{wl^3x}{48} \qquad ...(4)$$

Equations (3) and (4) can be used to determine slope and deflection at any section of the cantilever. For the maximum deflection the slope is zero on the section where maximum deflection occurs. Therefore to find the position of the section where maximum deflection occurs

$$EI\frac{dy}{dx}=0=\frac{3wlx^2}{16}-\frac{wx^3}{6}-\frac{wl^3}{48}$$

or $\frac{3lx^2}{16}-\frac{x^3}{6}-\frac{l^3}{48}=0$

or $9lx^2-8x^3-l^3=0$

After solving this equation, $x=0{\cdot}425\,l$

At $\mathbf{x=0{\cdot}425}\,l$

$$EI\,y_{max}=\frac{wl^4}{16}(0{\cdot}425)^3-\frac{w}{24}(0{\cdot}425)^4\,l^4-\frac{wl^4}{48}(0{\cdot}425)$$

$$=[0{\cdot}0048-0{\cdot}00136-0{\cdot}00885]\,wl^4$$

$$y_{max}=\frac{0{\cdot}0054\,wl^4}{EI}.$$

Example 11·10-1. A beam of length 6 metres, simply supported at the ends carries a uniformly distributed load of 6 kN per metre run throughout its length. The beam is propped at its centre so that centre of the beam is brought back to the level of the supports. Determine (*i*) reaction of the prop (*ii*) slope at the ends (*iii*) maximum deflection in the beam.

$$I=760 \text{ cm}^4,\ E=200 \text{ kN/mm}^2.$$

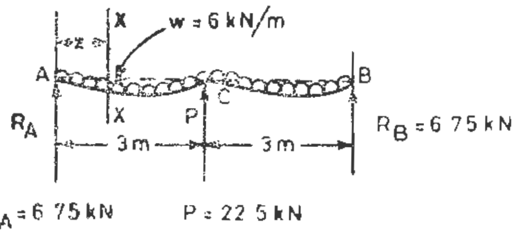


Fig. 11·14

Solution.

$$E=200 \text{ kN/mm}^2=200\times10^6 \text{ kN/m}^2$$

$$I=760 \text{ cm}^4=760\times10^{-8} \text{ m}^4$$

$$EI=200\times10^6\times760\times10^{-8}=1520 \text{ kNm}^2$$

Deflection at the centre of a beam having uniformly distributed load w throughout its length,

$$=-\frac{5wl^4}{384\ EI}$$

Say Reaction of the prop$=P$

Upward deflection due to P at the centre of the beam

$$=\frac{Pl^3}{48\ EI}$$

So $$\frac{Pl^3}{48\ EI}-\frac{5wl^4}{384\ EI}=0$$

$$P=\frac{5\ wl}{8}$$

$$w=6 \text{ kN/m},\ l=6 \text{ m}$$

Therefore $$P=\frac{5}{8}\times6\times6=22{\cdot}5 \text{ kN}$$

Total load on the beam $=6\times6=36$ kN

Reactions, $$R_A=R_B=\frac{36-22{\cdot}5}{2}=6{\cdot}75 \text{ kN}$$

(Reactions R_A and R_B are equal because of symmetrical loading).

Take the portion AC only of the beam and consider a section X-X at a distance of x from the end A.

BM at the section $$M==+6{\cdot}75\ x-\frac{wx^2}{2}$$

$$=6{\cdot}75x-\frac{6x^2}{2}=6{\cdot}75x-3x^2$$

Therefore $$EI\frac{d^2y}{dx^2}=6{\cdot}75x-3x^2 \qquad \ldots(1)$$

Integrating equation (1) we get

$$EI\frac{dy}{dx}=6{\cdot}75\frac{x^2}{2}-x^3+C_1 \text{ (constant of integration)}$$

Slope is zero at $x=3$ m *i.e.*, at the centre because of symmetrical loading on both the sides of the centre of the beam

Therefore $$0=6{\cdot}75\times\frac{3^2}{2}-3^3+C_1$$

$$C_1=27-30{\cdot}375=-3{\cdot}375$$

$$EI\frac{dy}{dx}=6{\cdot}75\frac{x^2}{2}-x^3-3{\cdot}375 \qquad \ldots(2)$$

Integrating equation (2) we get

$$EI\ y=6{\cdot}75\frac{x^3}{6}-\frac{x^4}{4}-3{\cdot}375x+C_2 \text{ (constant of integration)}$$

At the end $x=0,\ y=0$

So $0=0-0-0+C_2$ or $C_2=0$

Therefore $$EI\ y=6{\cdot}75\frac{x^3}{6}-\frac{x^4}{4}-3{\cdot}375\ x.$$

Slope at the end A

$$x=0$$

$$EI\ i_A=0-0-3{\cdot}375$$

$$i_A=-\frac{3{\cdot}375}{1520}=-0{\cdot}0022 \text{ radian}=-0{\cdot}127^\circ$$

Slope at B, $i_B=+0{\cdot}127^\circ$ (due to symmetry).

Maximum Deflection. For the maximum deflection, slope is zero

So $$EI\frac{dy}{dx}=0=6{\cdot}75\frac{x^2}{2}-x^3-3{\cdot}375$$

will give section at which slope is zero

$$3{\cdot}375\ x^2-x^3-3{\cdot}375=0$$

Solving this equation gives

$$x=1{\cdot}264 \text{ m (by trial)}$$

Now $$EI\ y_{max}=\frac{6{\cdot}75}{6}(1{\cdot}264)^3-\frac{(1{\cdot}264)^4}{4}-3{\cdot}375\times1{\cdot}264$$

$$=2{\cdot}272-0{\cdot}638-4{\cdot}266$$

$$=-2{\cdot}632$$

$$y_{max}=-\frac{2{\cdot}632}{1520}=-0{\cdot}0017 \text{ m}=-1{\cdot}7 \text{ mm.}$$

Exercise 11·10-1. A cantilever 3 metres long carries a load of 1·5 tonnes/metre run throughout its length. The free end of the cantilever is propped so that this end is brought to the level of fixed end. Determine (*i*) reaction at the prop (*ii*) slope at the free end (*iii*) maximum deflection and where it occurs.

$$E=2000 \text{ tonne/cm}^2,\ I=448 \text{ cm}^4.$$

[**Ans.** (*i*) 1·6875 tonne, (*ii*) $-0{\cdot}495°$, (*iii*) $-6{\cdot}72$ mm at 1·275 m from free end]

11·11. SLOPE AND DEFECTION BY THE USE OF BENDING MOMENT DIAGRAM

Slope and deflection at a section of a beam carrying transverse loads can be obtained by the use of BM diagram. Fig. 11·15 shows the BM diagram for a portion *AB* of a beam subjected to transverse loads. Let *M* be the bending moment at a distance of *x* from the origin *A*.

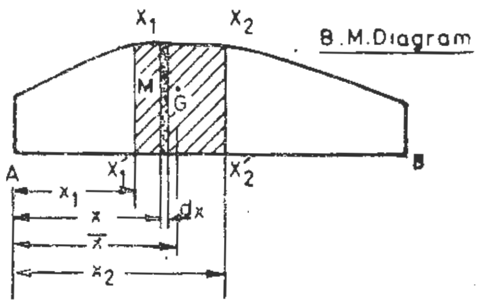


Fig. 11·15

$$EI\,\frac{d^2y}{dx^2}=M$$

(bending moment at any section)

or $$EI\,\frac{d^2y}{dx^2}\,.\,dx=M\,.\,dx$$

(area of the BM diagram for a very small length *dx*).

Integrating both the sides

$$\int_{x_1}^{x_2} EI\,\frac{d^2y}{dx^2}\,dx=\int_{x_1}^{x_2} M\,dx$$

$$EI\left|\frac{dy}{dx}\right|_{x_1}^{x_2}=\text{area of BM diagram between } x_2 \text{ and } x_1$$

Say $X_1\,X_1'\,X_2'\,X_2$ is the area of the BM diagram for the values of x between x_2 and x_1.

If $$i_2=\text{slope at section } X_2\,X_2'$$

$$i_1=\text{slope at section } X_1\,X_1'$$

then $$(i_2-i_1)=\frac{\text{area of the BM diagram between the sections } X_2 \text{ and } X_1}{EI} \quad \ldots(1)$$

Let us consider the differential equation again

$$EI\,\frac{d^2y}{dx^2}=M$$

Multiplying both the sides by $x\,.\,dx$ and integrating between the limits x_2 to x_1 we get

$$\int_{x_1}^{x_2} EI\,\frac{d^2y}{dx^2}\,.\,x\,dx=\int_{x_1}^{x_2} Mx\,dx$$

$$EI \left| x \frac{dy}{dx} - y \right|_{x_1}^{x_2} = \text{moment of the BM diagram between the section } X_1 \text{ and } X_2 \text{ about the origin } A$$

or $$EI\,[(x_2\, i_2 - y_2) - (x_1\, i_1 - y_1)] = a\bar{x} \text{ where}$$

a=area of the B.M. diagram between X_2 and X_1

$\bar{x}$=distance of the centroid of the area a from the origin A

Consider a cantilever of length l, fixed at end 2 and free at end 1, carrying a concentrated load W at the free end as shown in Fig. 11·16 (a). The cantilever will bend showing convexity upwards throughout its length. So negative bending moment acts on every section of the cantilever. Fig. 11·16 (b) shows the B.M. diagram for the cantilever with

B.M.$=-Wl$ at the fixed end

End 1 is the free end

End 2 is fixed end,

where slope and deflection are zero

EI(i_2-i_1)=area of the B M. diagram 2 and 1

$$= -Wl \times \frac{l}{2} = -\frac{Wl^2}{2}$$

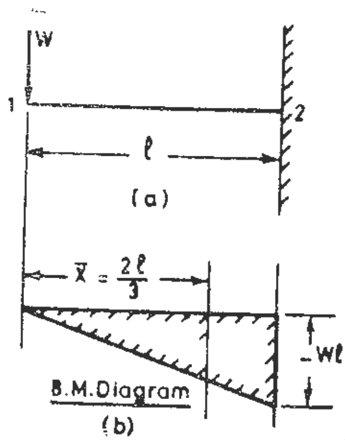


Fig. 11·16

Slope, $i_2=0$, at the fixed end

So $$EI\,(-i_1) = -\frac{Wl^2}{2}$$

Slope, $$i_1 = +\frac{Wl^2}{2EI}$$ *i.e.*, the slope at the free end of the cantilever

Again,

$EI\,[(x_2\, i_2 - y_2) - (x_1 i_1 - y_1)]$=moment of the B.M. diagram about the origin 1

Now $$i_1 = +\frac{Wl^2}{2EI}$$

$i_2=0$ fixed end

$x_1=0,\quad x_2=l$

$y_1=?\quad y_2=0$ (at the fixed end)

Area of B.M. diagram, $a = -\dfrac{Wl^2}{2}$

C.G. of the area from end 1,

$$\bar{x} = \frac{2l}{3}$$

Substituting the values in the equation above

$$EI[(l\times 0-0)-(0\times i_1-y_1)]=-\frac{Wl^2}{2}\times\frac{2l}{3}=-\frac{Wl^3}{3}$$

$$EI\,y_1=-\frac{Wl^3}{3}$$

Deflection, $y_1=-\dfrac{Wl^3}{3EI}$ (indicating downward deflection at the free end)

Consider now a beam AB of length l simply supported at the ends and carrying uniformly distributed load throughout its length. Say the intensity of loading is w per unit length. B.M. diagram is a parabolic curve as shown in the diagram 11·17 (*b*). Since the beam is symmetrically loaded about its centre, slope at the centre C is zero. Using the relationship for slopes between C and A.

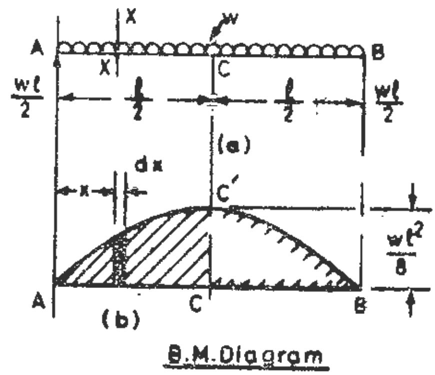


Fig. 11·17

$EI\,(i_C-i_A)$=area of the B.M. diagram between C and A

$$=+\frac{wl^2}{8}\times\frac{2}{3}\times\frac{l}{2}=\frac{wl^3}{24}$$

$i_C=0$ at the centre of the beam

So $\quad i_A=-\dfrac{wl^3}{24EI}$

Deflection at C. Reactions at A and C are equal,

$$R_A=R_B=\frac{wl}{2}$$

B.M. at a distance x from the end A,

$$=\frac{wl}{2}x-\frac{wx^2}{2}$$

(Clockwise B.M. on the left side of the section is positive)

Now $\quad EI\dfrac{d^2y}{dx^2}=\left(\dfrac{wlx}{2}-\dfrac{wx^2}{2}\right)$

or $\quad EI\dfrac{d^2y}{dx^2}x\,dx=\left(\dfrac{wlx}{2}-\dfrac{wx^2}{2}\right)xdx=\left(\dfrac{wlx^2}{2}-\dfrac{wx^3}{2}\right)dx$

Integrating both the sides between the points C and A

$$EI\int_A^C\frac{d^2y}{dx^2}\,xdx=\int_0^{l/2}\left(\frac{wlx^2}{2}-\frac{wx^3}{2}\right)dx$$

$$EI[(i_C x_C - y_C) - (i_A x_A - y_A)] = \left| \frac{wlx^3}{6} - \frac{wx^4}{8} \right|_0^{l/2}$$

$$EI\left[\left(0 \times \frac{l}{2} - y_C\right) - (i_A \times 0 - 0)\right] = \frac{wl^4}{48} - \frac{wl^4}{128}$$

$$-EIy_C = \frac{5}{384} \times wl^4$$

$$y_C = -\frac{5}{384} \times \frac{wl^4}{EI} \text{ (indicating downward deflection)}$$

Similarly any beam or cantilever with any type of loading can be considered and deflection and slope at any section can be determined by using the area and moment of the area of the B.M. diagram.

Example 11·11-1. A cantilever of length l fixed at one end and free at the other end carries a load W_1 at the free end and a load W_2 at its centre. If EI is the flexural rigidity of the cantilever determine the slope and deflection at the free end of the cantilever using the B.M. diagram.

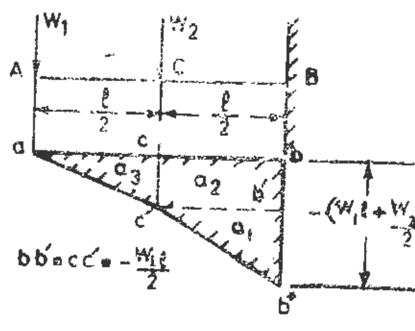
Solution. Fig. 11·8 shows the B.M. diagram of the cantilever of length l and carrying load W_1 at free end and load W_2 at the centre.

For the fixed end B

Slope, $i_B = 0$

Deflection, $y_B = 0$

Let us divide the B.M. diagram into 3 parts, a_1, a_2 and a_3 as shown.



(B.M. Diagram)

Fig. 11·18

Area, $a_1 = -(W_1 + W_2)\frac{l}{2} \times \frac{l}{2} \times \frac{1}{2} = -(W_1 + W_2)\frac{l^2}{8}$

x_1 = Distance of C.G. of a_1 from $A = \frac{5l}{6}$

Area, $a_2 = -\frac{W_1 l}{2} \times \frac{l}{2} = -\frac{W_1 l^2}{4}$

x_2 = Distance of C.G. of a_2 from $A = \frac{3l}{4}$

Area, $a_3 = -\frac{W_1 l}{2} \times \frac{l}{2} \times \frac{l}{2} = -\frac{W_1 l^2}{8}$

$x_3 = \frac{l}{3}$ (distance of C.G of a_3 from A)

Now $$a_1+a_2+a_3=-\left[\frac{W_1l^2}{8}+\frac{W_2l^2}{8}+\frac{W_1l^2}{4}+\frac{W_1l^2}{8}\right]$$

$$=-\left[\frac{W_1l^2}{2}+\frac{W_2l^2}{8}\right]$$

Now $$EI(i_B-i_A)=-\left[\frac{W_1l^2}{2}+\frac{W_2l^2}{8}\right]$$

But slope, $$i_B=0$$

So, slope, $$i_A=\frac{1}{EI}\left[\frac{W_1l^2}{2}+\frac{W_2l^2}{8}\right]$$

Moreover $$a_1x_1=-(W_1+W_2)\ \frac{5l^3}{48}$$

$$a_2x_2=-(W_1)\ \frac{3l^3}{16}$$

$$a_3x_3=-\frac{W_1l^3}{24}$$

$$a_1x_1+a_2x_2+a_3x_3=-\frac{l^3}{8}\left[\frac{5}{6}W_1+\frac{5}{6}W_2+\frac{3}{2}W_1+\frac{W_1}{3}\right]$$

$$=-\frac{l^3}{8}\left[\frac{16}{6}W_1+\frac{5}{6}W_2\right]$$

$$=-\left[\frac{W_1l^3}{3}+\frac{5}{48}\ W_2l^3\right]$$

Now $$EI[(x_B\ .\ i_B-y_B)-(x_A\ .\ i_A-y_A)]=a_1x_1+a_2x_2+a_3x_3$$

$$=-\left[\frac{W_1l^3}{3}+\frac{5}{48}W_2l^3\right]$$

But $$i_B=0,\ y_B=0,\ x_A=0$$

Therefore, $$EI(+y_A)=-\frac{l^3}{3}\left[W_1+\frac{5}{16}W_2\right]$$

$$y_A=-\frac{l^3}{3EI}\left[W_1+\frac{5}{16}W_2\right]$$

Exercise 11 11-1. A beam AB of length l carries concentrated loads W each at a distance of a from both the ends. The beam is simply supported on ends A and B. If EI is the flexural rigidity of the beam, determine (i) slope at ends (ii) maximum deflection.

$$\left[\textbf{Ans.}\ \pm\frac{Wa}{2EI}(l-a),\ -\frac{Wa}{EI}\left(\frac{l^2}{8}-\frac{a^2}{6}\right)\right]$$

11·12. SLOPE AND DEFLECTION OF BEAMS BY A GRAPHICAL METHOD

In the Chapter 7, we derived relationship between rate of loading, shear force and bending moment at any section of a beam carrying transverse loads producing bending in the beam. Say w is the rate of loading, F is the shear force and M is the bending moment at a particular section of beam.

Then $\dfrac{dF}{dx}=-w$ (rate of loading)

Let us consider a case of variable loading as shown in the Fig. 11·19.

$$dF=-wdx \qquad \ldots(1)$$

Integrating both the sides between X and A [A is the simply supported end of the beam and X is the section under consideration] we get

$$\int_A^x dF=\int_0^x -wdx$$

$$F_x-F_A=-\int_0^x wdx$$

$F_x=F_A-$Area of load diagram between A and X

$F_x=R_A-$Area of the diagram $AXXA'$

$=R_A-a_1 \qquad \ldots(2)$

(as shear force $F_A=R_A$)

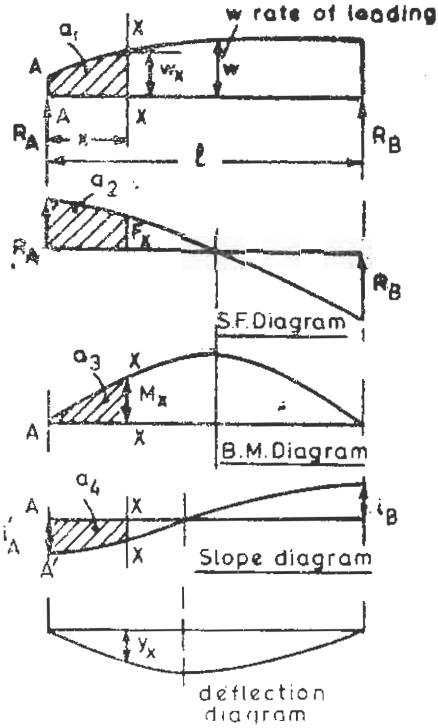


Fig. 11·19

Similarly we know that $\dfrac{dM}{dx}=F$

or $$\int_A^x dM=\int_0^x F\,dx$$

or $M_x-M_A=$area of SF diagram between A to X

Bending moment, $M_A=0$ as the beam is simply supported

Therefore, $M_x=a_2$ (area of SF diagram between A to X) $\ldots(3)$

Moreover we know that

$$EI\,\frac{d^2y}{dx^2}=M \text{ (bending moment)}$$

Integrating both the sides of this equation

$$EI\int_A^x \frac{d^2y}{dx^2}\,.dx=\int_0^x Mdx$$

$$EI\left|\frac{dy}{dx}\right|_A^x=\text{area of the BM diagram between } A \text{ and } X$$

$$EI\,[i_X - i_A] = a_3 \text{ [area of BM diagram } (A\ XX)]$$

or
$$i_X = i_A + \frac{a_3}{EI} \quad \ldots(4)$$

Slope at X $= \text{slope at } A + \frac{1}{EI}$ [area of BM diagram between A and X]

Now $\frac{dy}{dx} = i$ (slope)

or $dy = i\,dx$

Integrating both the sides

$$\int_A^x dy = \int_0^x i\,dx$$

$y_X - y_A$ = area of slope diagram between A and X ($AA'XX$)

$= a_4$

But deflection, $y_A = 0$, as the end is simply supported

So $y_X = a_4$ (area of the slope diagram between A and X)

Like this any problem can be solved.

Example 11·12-1. A beam of length l, simply supported at ends carries a concentrated load W at its centre. Determine the slope at the ends and deflection at the centre. EI is the flexural rigidity of the beam. Use graphical method for solution.

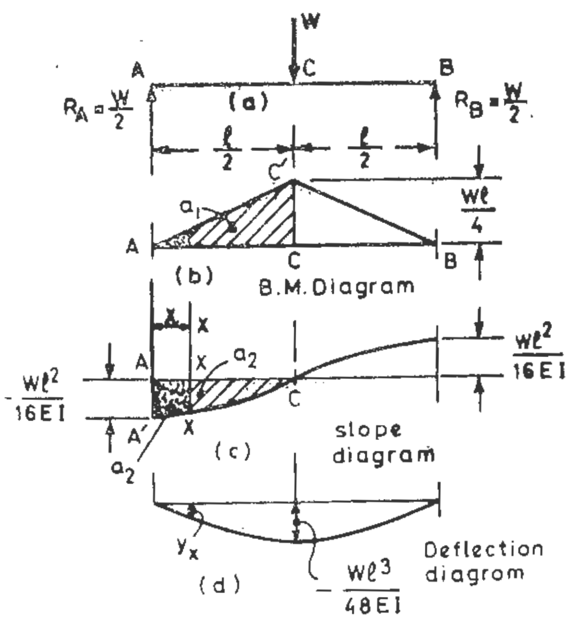


Fig. 11·20

Solution. Fig. 11·20 (*a*) shows the load diagram of a simply supported beam carrying a concentrated load W at its centre.

BM at centre $=R_A \times \frac{l}{2}=\frac{Wl}{4}$

$\left(\text{Since} \quad R_A=R_B=\frac{W}{2}\right)$

We know that slope at the centre of the beam is zero, because the beam is symmetrically loaded about its centre

$$EI\,(i_C-i_A)=\text{slope at } C-\text{slope at } A$$
$$=\text{area of BM diagram between } A \text{ and } C$$
$$=\frac{Wl}{4}\times\frac{l}{2}\times\frac{1}{2}=\frac{Wl^2}{16}$$

But $\quad i_C=0$

So $\quad i_A=-\frac{Wl^2}{16\,EI}$

Starting from $\quad AA'=-\frac{Wl^2}{16\,EI}$

further slope diagram can be made by integrating the area of BM diagram.

To deiermine i_X

Area of BM diagram upto X

$$=\frac{W}{2}\,x\times\frac{x}{2}=\frac{Wx^2}{4}$$

$$i_X=i_A+\frac{Wx^2}{4\,EI} \text{ (a parabolic equation)}$$

At C, $\quad i_C=-\frac{Wl^2}{16\,EI}+\frac{W}{4\,EI}\left(\frac{l}{2}\right)^2=0$

We know that deflection at the freely supported end A is zero

$$y_C-y_A=\text{deflection at } C-\text{deflection at } A$$
$$=\text{area of slope diagram between } C \text{ and } A$$
$$=-\frac{Wl^2}{16\,EI}\times\frac{l}{2}\times\frac{2}{3}=-\frac{Wl^3}{48\,EI}$$

or $\quad y_C=-\frac{Wl^3}{48\,EI}$, deflection at the centre.

Deflection at any section X-X, *i.e.*, y_X is equal to the area of the slope diagram between X to A, as shown by shaded area a_2 in the diagram for slopes.

Exercise 11·12-1. A cantilever of length l carries a load W at the free end. If EI is the flexural rigidity of the cantilever determine slope and deflection at the free end by using the graphical method.

$$\left[\textbf{Ans.}\quad +\frac{Wl^2}{2\,EI},\ -\frac{Wl^3}{3\,EI}\right]$$

11·13. SLOPE AND DEFLECTION OF BEAMS BY CONJUGATE BEAM METHOD

If a beam is supported at its ends and carries any type of transverse loads, its SF and BM diagrams can be plotted. Now if the same beam supported at its ends is shown to carry the variable transverse loads as (a variable load diagram) shown by the BM diagram then it is said to be the conjugate beam. Fig. 11·21 shows a beam AB of length, simply supported at ends A and B and carrying a concentrated load W at the point C at a distance of a from the end A. Then reaction

$$R_A=\frac{Wb}{l},\qquad R_B=\frac{Wa}{l}$$

BM at C $=\dfrac{Wab}{l}.$

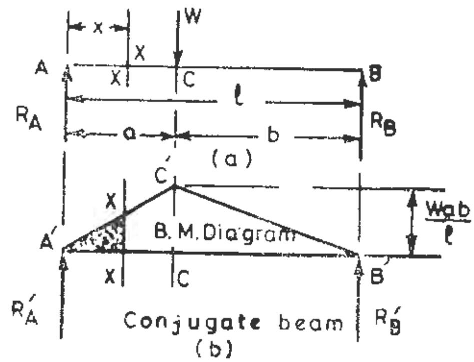


Fig. 11·21

Fig. 11·21 (*b*) shows the BM diagram with maximum bending moment Wab/l at the point C. A beam supporting this BM diagram as a variable load is called a conjugate beam, reactions R_A' and R_B' can be obtained for this conjugate beam. Then R_A'/EI and R_B'/EI give the slopes at the ends A and B.

In the example given above, taking moments about the point A' of the conjugate beam, we get

$$\frac{Wab}{l}\times\frac{a}{2}\times\frac{2a}{3}+\frac{Wab}{l}\times\frac{b}{2}\left(a+\frac{b}{3}\right)=R_B'\times l$$

$$\frac{Wab}{l}\left[\frac{a^2}{3}+\frac{ab}{2}+\frac{b^2}{6}\right]=R_B'\times l$$

$$\frac{Wab}{6l}\,[2a^2+3ab+b^2]=R_B'\times l$$

or $$R_B'=\frac{Wab}{6l^2}\,(2a+b)(a+b)=\frac{Wab\,(2a+b)}{6l}\quad\text{as } l=a+b$$

Similarly $$R_A'=\frac{Wab}{6l^2}\,(a+2b)(a+b)=\frac{Wab\,(a+2b)}{6l}$$

Slope at the end A, $$i_A=-\frac{R_A'}{EI}=-\frac{Wab\,(a+2b)}{6l\,EI}$$

$$i_B=+\frac{R_B'}{EI}=+\frac{Wab\,(2a+b)}{6l\,EI}.$$

To determine slope at any section X-X, at a distance of x from the end A, we have to take into account the area of the loading diagram of the conjugate beam as shown by the shaded portion.

$EI\cdot(i_X-i_A)=$area of the loading diagram of conjugate beam from A to X.

Again let us consider the conjugate beam and determine BM at any section say M_x'. Then M_x'/EI gives the deflection at the point X.

Conjugate beam

BM at C, $$M_x' = +R_A' \cdot a - \frac{Wab}{l} \times \frac{a}{2} \times \frac{a}{3}$$

$$= \frac{Wab\,(a+2b)}{6l} \times a - \frac{Wa^3b}{6l}$$

$$= \frac{Wab}{6l}\,[a^2+2ab-a^2] = \frac{Wa^2b^2}{3l}$$

Deflection at C, $$y_C = \frac{M_x}{EI} = \frac{Wa^2b^2}{3\,EIl}.$$

Example 11·13-1. A beam 6 m long, simply supported at its ends carries a load of 6 tonnes at a distance of 2 metres from one end and another load of 3 tonnes at a distance of 2 metres from the other end. Determine slope at ends and deflection under the loads using conjugate beam method.

Given $EI = 3600$ tonne-metre^2

Solution. The load-diagram for the beam of 6 m length is shown in the Fig. 11·22 (*a*). For support reactions, let us take moments about the point A

$$6\times2+3\times4 = 6\,R_B$$

$$R_B = 4 \text{ tonnes}$$

$$R_A = 6+3-4 = 5 \text{ tonnes}$$

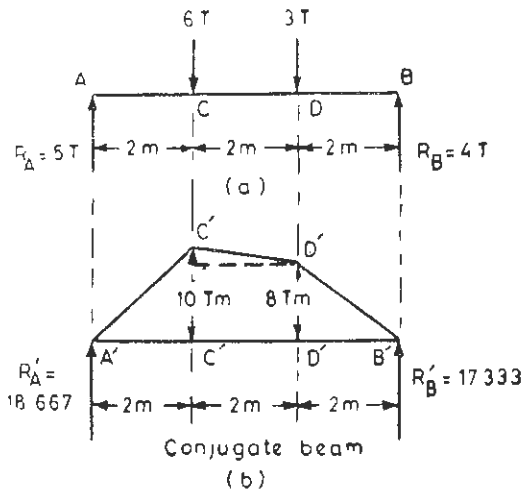


Fig. 11·22

Bending moments

At A, $M_A = 0$

At C, $M_C = +5\times2 = 10$ tonne metres

At D, $M_D = +4\times2 = 8$ tonne metres

At B, $M_B = 0$

The conjugate beam is shown in the Fig. 11·22 (*b*).

Let us determine support reactions, R_A' and R_B' taking moments of the forces (in this case moment area) about the point A'

$$10\times2\times\frac{1}{2}\times\left(\frac{4}{3}\right)+8\times2\times3+(10-8)\times\frac{2}{2}\left(2+\frac{2}{3}\right)$$

$$+8\times2\times\frac{1}{2}\left(4+\frac{2}{3}\right) = R_B'\times l$$

$$\frac{40}{3}+48+\frac{16}{3}+\frac{112}{3} = R_B'\times6$$

$$104 = R_B' \times l$$

$$R_B' = \frac{104}{6} = \frac{52}{3} = 17.333 \text{ Tm}^2$$

Similarly taking moments about the point B'

$$8 \times 2 \times \frac{1}{2}\left(\frac{4}{3}\right) + 8 \times 2 \times 3 + (10-8)\left(2 \times \frac{1}{2}\right)\left(2 + \frac{4}{3}\right)$$

$$+10 \times 2 \times \frac{1}{2}\left(4 + \frac{2}{3}\right) = R_A' \times 6$$

$$\frac{32}{3} + 48 + \frac{20}{3} + \frac{140}{3} = R_A' \times 6$$

$$R_A' = \frac{112}{6} = 18.667 \text{ T-m}^2$$

Slope at A $\quad = -\frac{R_A'}{EI} = -\frac{18.667}{3600} = -5.185 \times 10^{-3}$ radian

Slope at B $\quad = +\frac{R_B'}{EI} = +\frac{17.333}{3600} = +4.815 \times 10^{-3}$ radian

Considering the conjugate beam again, let us find moments under the loads.

$$M_C' = +18.667 \times 2 - \frac{10 \times 2}{2}\left(\frac{2}{3}\right) = 37.334 - 6.606$$

$$= 30.668 \text{ Tm}^3$$

$$M_D' = +17.333 \times 2 - 8 \times 2 \times \frac{1}{2}\left(\frac{2}{3}\right) = 34.666 - 5.333$$

$$= 29.333 \text{ Tm}^3$$

Deflection under the loads

$$y_C = \frac{M_C'}{EI} = \frac{30.668}{3600} = 8.52 \times 10^{-3} \text{ m} = 8.52 \text{ mm}$$

$$y_D = \frac{M_D'}{EI} = \frac{29.333}{3600} = 8.15 \times 10^{-3} \text{ m} = 8.15 \text{ mm}$$

Exercise 11·13-1. A beam 8 m long simply supported at its ends carries a load of 40 kN at a distance of 2 m from one end and another load of 40 kN at a distance of 2 m from the other end. Determine slope at ends and deflection at the centre of the beam using conjugate beam method.

EI for the beam $= 50{,}000$ kNm² [**Ans.** $\pm 4.8 \times 10^{-3}$ radian, 1·117 cm]

11·14. SLOPE AND DEFLECTION OF CANTILEVER WITH STEPPED SECTIONS

Uptil now we have considered cantilever and beams of uniform sections throughout their length, or cantilevers beams of continuously varying section. Now we will determine the slope and deflection of cantilever with stepped sections as shown in the Fig. 11·23. In such cases conjugate beam method is very useful for finding out the slope and deflection at any section. In the conjugate beam method, B.M. diagram is plotted for the beam or the cantilever with B.M. diagram as the load, reactions at the ends are obtained. Then the ratio of Reaction/EI gives the slope at the end. Then bending moment at any section obtained from the conjugate beam divided by EI gives deflection at any section. In the case of cantilever, maximum slope and deflection occur at the free end, while the slope and deflection are zero at the fixed end. Therefore in the conjugate cantilever, free end becomes the fixed end and fixed end becomes the free end, so that the reaction and bending moment obtained at this end of conjugate cantilever give slope and deflection at the free end of the original cantilever.

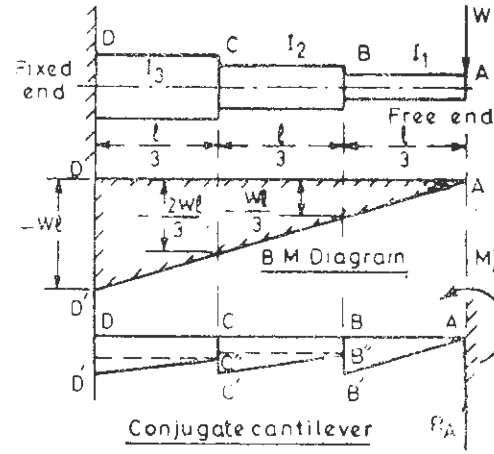


Fig. 11·23

In the figure, there is a cantilever of length l, fixed at end D and free at end A. At the free end a concentrated load W is applied. The section is in steps and the moment of inertia of portion DC is I_3, of portion CB is I_2 and of portion BA is I_1. Such that $I_3=3I_1$ and $I_2=2I_1$. ADD' is the bending moment diagram of the cantilever. Let us draw $\frac{M}{EI}$ diagram for the cantilever, or the conjugate cantilever. Since I is variable, so in place of M we have taken $\frac{M}{EI}$ for the conjugate cantilever.

$$BB'=-\frac{Wl}{3EI_1},\quad BB''=-\frac{Wl}{3EI_2}=-\frac{Wl}{6EI_1}$$

$$CC'=-\frac{2Wl}{3EI_2}=-\frac{Wl}{3EI_1},\quad CC''=-\frac{2Wl}{3EI_3}=-\frac{2Wl}{9EI_1}$$

$$DD'=-\frac{Wl}{EI_3}=-\frac{Wl}{3EI_1}$$

Reaction, R_A' at the fixed end of conjugate cantilever

$$=\frac{BB'}{2}\times\frac{l}{3}+\left(\frac{BB''+CC'}{2}\right)\times\frac{l}{3}+\left(\frac{CC''+DD'}{2}\right)\frac{l}{3}$$

$$=-\frac{Wl^2}{18EI_1}+\left(-\frac{Wl}{6EI_1}-\frac{Wl}{3EI_1}\right)\frac{l}{6}+\left(-\frac{2Wl}{9EI_1}-\frac{Wl}{3EI_1}\right)\frac{l}{6}$$

$$=-\frac{Wl^2}{18EI_1}-\frac{Wl^2}{12EI_1}-\frac{5\,Wl^2}{54EI_1}=-\frac{25\,Wl^2}{108\,EI_1}$$

Slope at the free end of the cantilever $=-\dfrac{25\,Wl^2}{108\,EI_1}$

Bending moment at the fixed end of the conjugate cantilever

$$M_A' = +\left[\frac{BB'}{2}\times\frac{l}{3}\times\frac{2l}{9}+BB''\times\frac{l}{3}\left(\frac{l}{2}\right)\right.$$
$$+\left(\frac{CC'-BB''}{2}\right)\times\frac{l}{3}\left(\frac{l}{3}+\frac{2l}{9}\right)$$
$$\left.+CC''\times\frac{l}{3}\times\left(\frac{5l}{6}\right)+\left(\frac{DD'-CC''}{2}\right)\times\frac{l}{3}\left(\frac{2l}{3}+\frac{2l}{9}\right)\right]$$
$$=-\left[\frac{Wl}{3EI_1}\times\frac{l^2}{27}+\frac{Wl}{6EI_1}\times\frac{l^2}{6}+\frac{Wl}{2\times6EI_1}\times\frac{l}{3}\times\frac{5l}{9}\right.$$
$$\left.+\frac{2Wl}{9}\times\frac{1}{EI_1}\times\frac{5l^2}{18}+\frac{1}{2\times9EI_1}\frac{Wl}{}\times\frac{l}{3}\times\frac{8l}{9}\right]$$
$$=-\frac{Wl^3}{EI_1}\left[\frac{1}{81}+\frac{1}{36}+\frac{5}{324}+\frac{1}{81}+\frac{8}{486}\right]$$
$$=-\frac{65\,Wl^3}{486\,EI_1}$$

Example 11·14-1. A cantilever 4 m long, has moment of inertia 1200 cm⁴ for 2 m starting from free end and a moment of inertia 1600 cm⁴ for the rest of the length. What load applied at the free end would cause a maximum deflection of 2 mm What will be the slope at the free end ?

$$E=2000 \text{ tonnes/cm}^2$$

Solution. Fig. 11·24 shows a cantilever of length l carrying a load W at its free end. The moment of inertia for half of its length is I_1 and for another half of its length is I_2. A bending moment diagram is shown below the loading diagram. In the problem

ratio of $\frac{I_2}{I_1}=\frac{1600}{1200}=\frac{4}{3}$ or $I_2=\frac{4}{3}I_1$

Let us first draw the conjugate cantilever diagram with ordinates $\frac{M}{EI}$ as I is variable.

$$BB'=-\frac{Wl}{2EI_1}$$
$$BB''=-\frac{Wl}{2EI_2}=-\frac{3Wl}{8EI_1}$$
$$CC'=-\frac{Wl}{EI_2}=-\frac{3Wl}{4EI_1}$$

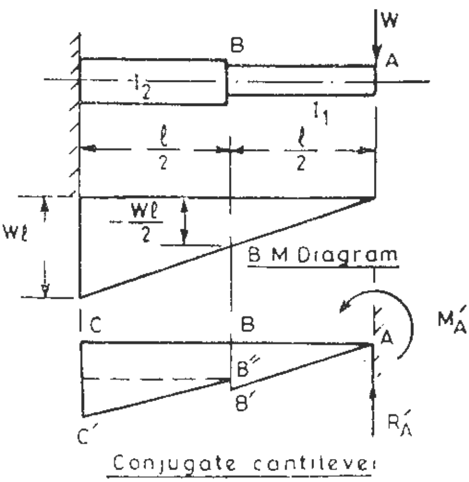


Fig. 11·24

Reaction, R_A' at the fixed end of the conjugate cantilever

$$=+\frac{BB'}{2}\times\frac{l}{2}+\left(\frac{BB''+CC'}{2}\right)\frac{l}{2}$$

$$R_A' = -\frac{Wl}{2EI_1} \times \frac{l}{4} + \left(-\frac{3Wl}{8EI_1} - \frac{3}{4}\frac{Wl}{EI_1}\right)\frac{l}{4}$$

$$= -\frac{Wl^2}{8EI_1} - \frac{9\,Wl^2}{8 \times 4EI_1} = -\frac{13\,Wl^2}{32\,EI_1}$$

or the slope at the end A of the cantilever $= -\frac{13}{32}\frac{Wl^2}{EI_1}$

Bending moment M_A' at the fixed end of the conjugate cantilever

$$= -\frac{Wl}{2EI_1} \times \frac{l}{4} \times \frac{l}{3} - \frac{3}{8}\frac{Wl}{EI_1} \times \frac{l}{2} \times \frac{3l}{4}$$

$$-\left(\frac{3Wl}{4EI_1} - \frac{3}{8}\frac{Wl}{EI_1}\right) \times \frac{l}{2} \times \frac{1}{2} \times \frac{5}{6}\,l$$

$$= -\frac{Wl^3}{24EI_1} - \frac{9Wl^3}{64EI_1} - \frac{3 \times Wl}{8EI_1} \times \frac{5l^2}{24}$$

$$= -\frac{Wl^3}{EI_1}\left[\frac{1}{24} + \frac{9}{64} + \frac{5}{64}\right]$$

$$= -\frac{Wl^3}{EI_1} \times \frac{25}{96}$$

Deflection at the free end of the cantilever

$$= -\frac{Wl^3}{EI_1} \times \frac{25}{96} = -0{\cdot}002 \text{ m}$$

$$l = 4 \text{ m}$$

$$E = 2000 \times 10^4 \text{ T/m}^2$$

$$I_1 = 1200 \text{ cm}^4 = 1200 \times 10^{-8} \text{ m}^4$$

$$EI_1 = 2000 \times 10^4 \times 1200 \times 10^{-8} = 240 \text{ Tm}^2$$

Therefore $\quad 0{\cdot}002 = \frac{W \times 4^3}{240} \times \frac{25}{96}$

Load at free end, $\quad W = \frac{{\cdot}002 \times 240 \times 96}{64 \times 25} = 0{\cdot}0288$ Tonne

$$= 28{\cdot}8 \text{ kg}$$

Slope at the free end $\quad = -\frac{13}{32}\frac{Wl^2}{EI_1}$

$$= -\frac{13}{32} \times \frac{{\cdot}0288 \times 4^2}{240} = -0{\cdot}00078 \text{ radian}$$

Exercise 11·14-1. A cantilever 3 metres long, has moment of inertia 800 cm⁴ for 1 m length from the free end, 1600 cm⁴ for next 1 m length and 2400 cm⁴ for the last 1 m length. At the free end a load of 1 kN acts on the cantilever. Determine the slope and deflection at the free end of the cantilever. $E = 210$ GN/m². [**Ans.** $-1{\cdot}24 \times 10^{-3}$ radian, $-2{\cdot}15$ mm]

11·15. SLOPE AND DEFLECTION OF BEAMS OF STEPPED SECTIONS

In the article 11·13 we have studied about the conjugate beam method for the determination of slope and deflection in beams and we considered B.M. diagram as the load diagram over the span as the conjugate. beam. Now as the section is varying in steps we will make $\frac{M}{EI}$ diagram which will be the conjugate beam diagram. Reactions at the ends of the conjugate beam give the values of slope at the ends and the bending moment at any section of the conjugate beam gives the deflection at the section of original beam. Fig. 12·25 shows a beam AB of length l, having moment of inertia I_1 for quarter length from the ends and I_2 for the middle half length The beam carries a central load W. Diagram AC_1B shows the B.M. diagram with $CC_1=\frac{Wl}{4}$.

Let us construct the $\frac{M}{EI}$ conjugate beam diagram.

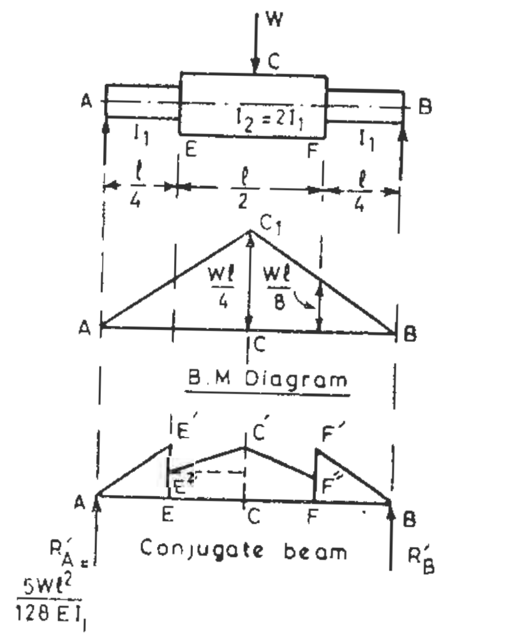


Fig. 11.25

$$EE'=\frac{Wl}{8}\times\frac{1}{EI_1}$$

$$EE''=\frac{Wl}{8}\times\frac{1}{EI_2}=\frac{Wl}{16EI_1}$$

$$CC'=\frac{Wl}{4}\times\frac{1}{EI_2}=\frac{Wl}{8EI_1}$$

The beam is symmetrically loaded therefore reactions,

$$R_A'=R_B'$$

or R_A'=area of conjugate beam diagram upto the centre

$$=EE'\times\frac{1}{2}\times\frac{l}{4}+\frac{EE''+CC'}{2}\times\frac{l}{4}$$

$$=\frac{Wl}{8EI_1}\times\frac{l}{8}+\left(\frac{Wl}{16EI_1}+\frac{Wl}{8EI_1}\right)\frac{l}{8}$$

$$=\frac{Wl^2}{64EI_1}+\frac{3Wl}{16EI_1}\times\frac{l}{8}$$

$$=\frac{Wl^2}{64EI}+\frac{3Wl^2}{128EI_1}=\frac{5Wl^2}{128EI_1}$$

Since the beam is symmetrically loaded about its centre, slope at the centre of the beap is zero.

$$i_C - i_A = \text{area of conjugate beam diagram from } A \text{ to } C$$

$$0 - i_A = \frac{5\,Wl^2}{128\,EI_1}$$

or

$$-i_A = \frac{5\,Wl^2}{128\,EI_1} \quad ...(1)$$

To determine deflection at the centre, let us calculate the bending moment at the centre of conjugate beam

$$= R_A' \times \frac{l}{2} - \frac{EE'}{2} \times \frac{l}{4}\left(\frac{l}{4} + \frac{l}{12}\right) + EE'' \times \frac{l}{4} \times \frac{l}{8}$$

$$+ \left(\frac{CC' - EE''}{2}\right)\left(\frac{l}{4}\right)\frac{l}{12}$$

$$= \frac{5\,Wl^2}{128\,EI_1} \times \frac{l}{2} - \frac{Wl}{8\,EI_1} \times \frac{l}{8} \times \frac{l}{3} - \frac{Wl}{16\,EI_1} \times \frac{l^2}{32}$$

$$-\left(\frac{Wl^3}{8\,EI_1} - \frac{Wl}{16\,EI_1}\right)\frac{l}{8} \times \frac{l}{12}$$

$$= \frac{5\,Wl^3}{256\,EI_1} - \frac{Wl^3}{192\,EI_1} - \frac{Wl^3}{512\,EI_1} - \frac{Wl^3}{128 \times 12}$$

$$= \frac{5\,Wl^3}{256\,EI_1} - \frac{Wl^3}{192\,EI_1} - \frac{Wl^2}{512\,EI_1} - \frac{Wl^3}{1536\,EI_1}$$

$$= \frac{Wl^3}{1536\,EI_1}\,[30 - 8 - 3 - 1]$$

$$= \frac{18\,Wl^3}{1536\,EI_1} = \frac{3\,Wl^3}{256\,EI}$$

Deflection at the centre of the beam

$$= \frac{3\,Wl^3}{256\,EI_1}.$$

Example 11·15-1. A beam 5 m long is simply supported at the ends. The moment of inertia for half the length from one end to the centre is 1256 cm^4 and that for the rest of the beam is 1884 cm^4. The beam carries a load of 10 kN at the centre. Determine the slope at the ends and deflection at the centre of the beam.

$$E = 210 \text{ kN/mm}^2.$$

Solution. Fig. 11·26 shows a beam of length l, simply supported at the ends carrying a load W at the centre. The moment of inertia for half the length is I_1 and for another half length is I_2. Below the loading diagram is the BM diagram drawn for the beam.

Let us first draw the M/EI conjugate beam because I is variable in steps

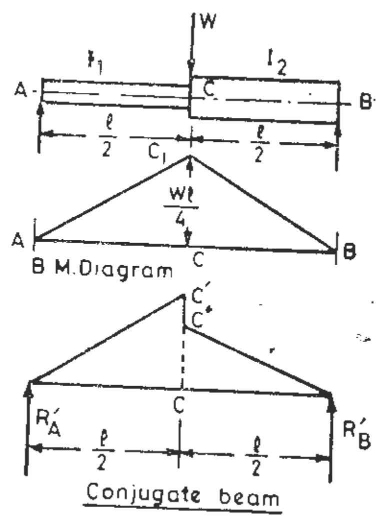


Fig. 11·26

$$CC' = \frac{Wl}{4} \times \frac{1}{EI_1}$$

$$CC'' = \frac{Wl}{4} \times \frac{1}{EI_2} \cdot$$

Reactions of conjugate beam. Taking moments about A

$$R_B' \times l = \frac{CC'}{2} \times \frac{l}{2} \times \frac{l}{3} + \frac{CC''}{2} \times \frac{l}{2} \times \left(\frac{l}{2} + \frac{l}{6}\right)$$

$$= \frac{Wl}{4\,EI_1} \times \frac{l^2}{12} + \frac{Wl}{4\,EI_2} \times \frac{l}{4} \times \frac{2l}{3}$$

$$= \frac{Wl^3}{48\,EI_1} + \frac{Wl^3}{24\,EI_2}$$

$$R_B' = \frac{Wl^2}{48\,EI_1} + \frac{Wl^2}{24\,EI_2}$$

Taking moments about B

$$R_A' \times l = \frac{CC''}{2} \times \frac{l}{2} \times \frac{l}{3} + \frac{CC'}{2} \times \frac{l}{2} \times \frac{2l}{3}$$

$$= CC'' \times \frac{l^2}{12} + \frac{CC' \times l^2}{6}$$

$$= \frac{Wl}{4\,EI_2} \times \frac{l^2}{12} + \frac{Wl}{4\,EI_1} \times \frac{l^2}{6} = \frac{Wl^3}{48\,EI_2} + \frac{Wl^3}{24\,EI_1}$$

$$R_A' = \frac{Wl^2}{48\,EI_2} + \frac{Wl^2}{24\,EI_1}$$

Slope at the end A $\quad = -R_A' = -\dfrac{Wl^2}{24\,E}\left(\dfrac{1}{I_1} + \dfrac{1}{2I_2}\right)$

Slope at the end B $\quad = R_B' = \dfrac{Wl^2}{24\,E}\left(\dfrac{1}{I_2} + \dfrac{1}{2I_1}\right)$

Obviously the slope at A is negative and slope at B is positive.

Bending moment at the centre

$$= R_A' \times \frac{l}{2} - \frac{CC'}{2} \times \frac{l}{2} \times \frac{l}{6}$$

$$= \frac{Wl^3}{96\,EI_2} + \frac{Wl^3}{48\,EI_1} - \frac{Wl}{4\,EI_1} \times \frac{l^2}{24}$$

or deflection at the centre, $\quad y_C = \dfrac{Wl^3}{96\,EI_2} + \dfrac{Wl^3}{96\,EI_1}$

where $W = 10$ kN, $l = 5$ m

$E = 210 \times 10^6$ kN/m^2

$I_1 = 1256$ cm^4, $I_2 = 1884$ cm^4

$EI_1 = 210 \times 10^6 \times 1256 \times 10^{-8} = 2637{\cdot}6$ kNm^2

$EI_2 = 210 \times 10^6 \times 1884 \times 10^{-8} = 3956{\cdot}4$ kNm^2

Slope at the end A, $\quad i_A = -\dfrac{Wl^2}{48\ EI_2} - \dfrac{Wl^2}{24\ EI_1}$

$$= -\frac{10\times5^2}{48\times3956{\cdot}4} - \frac{10\times5^2}{24\times2637{\cdot}6}$$

$$= -0{\cdot}0013164 - 0{\cdot}0039492$$

$$= -0{\cdot}0052656 \text{ radian.}$$

Slope at the end B, $\quad i_B = +\dfrac{Wl^2}{48\ EJ_1} + \dfrac{Wl^2}{24\ EI_2}$

$$= \frac{10\times5^2}{48\times2637{\cdot}6} + \frac{10\times5^2}{24\times3956{\cdot}4}$$

$$= 0{\cdot}0019746 + 0{\cdot}0026328$$

$$= 0{\cdot}0046074 \text{ radian.}$$

Deflection at the centre $= \dfrac{Wl^3}{96}\left[\dfrac{1}{EI_1} + \dfrac{1}{EI_2}\right]$

$$= \frac{10\times5^3}{96}\left[\frac{1}{2637{\cdot}6} + \frac{1}{3956{\cdot}4}\right]$$

$$= 0{\cdot}0082276 \text{ m} = 8{\cdot}2 \text{ mm.}$$

Exercise 11·15-1. A beam 6 m long, simply supported at the ends carries a concentrated load of 1·5 tonnes at its centre. The moment of inertia for the middle half length is 2400 cm^4 and that for the outer quarter lengths is 1200 cm^4. Determine (*i*) slope at the ends, (*ii*) deflection at the centre.

E=2000 tonnes/cm^2. [**Ans.** $\pm$0·0088 radian, 15·82 mm]

Problem 11·1. A beam $ABCD$, 10 m long carries concentrated loads of 2 tonnes and 4 tonnes at points B and C. The beam is simply supported at ends A and D. Point B is at a distance of 3 metres from end A and point C is at a distance of 3 meters from the end D. Determine (*i*) deflection under the loads of 2 tonnes and 4 tonnes, (*ii*) maximum deflection and the section of beam where it occurs.

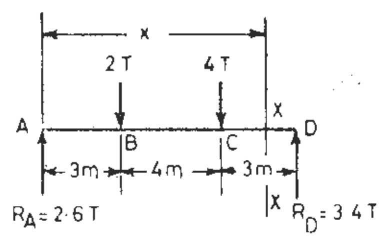


Fig. 11·47

Solution. For support reactions take moments of the forces about the point A.

$$3\times2+7\times4=R_D$$

$$R_D=\frac{34}{10}=3{\cdot}4 \text{ tonnes}$$

$$R_A+R_D=2+4=6 \text{ tonnes}$$

$$R_A=6-3{\cdot}4=2{\cdot}6 \text{ tonnes}$$

Consider a section X-X in the portion CD of the beam, at a distance of x from the end A.

BM at the section X-X,

$$M=2{\cdot}6x-2(x-3)-4\ (x-7)$$

or
$$EI\ \frac{d^2y}{dx^2}=2{\cdot}6x-2\ (x-3)-4\ (x-7) \qquad \ldots(1)$$

Integrating equation (1) we get

$$EI\ \frac{dy}{dx}=2{\cdot}6\ \frac{x^2}{2}-\frac{2\ (x-3)^2}{2}-\frac{4\ (x-7)^2}{2}+C_1 \text{ (constant of integration)}$$

$$=1{\cdot}3x^2-(x-3)^2-2\ (x-7)^2+C_1 \qquad \ldots(2)$$

Integrating equation (2) also we get

$$EI\ y=\frac{1{\cdot}3\ x^3}{3}-\frac{(x-3)^3}{3}-\frac{2\ (x-7)^3}{3}+C_1\ x+C_2 \text{ (constant of integration)} \quad \ldots(3)$$

at $x=0$, $y=0$

Substituting in equation (3) and omitting the negative terms in bracket

$$0=C_2$$

then at $x=10$ m, $y=0$, *i.e.*, at the other end of the beam

So
$$0=\frac{1{\cdot}3\times 10^3}{3}-\frac{(10-3)^3}{3}-\frac{2\ (10-7)^3}{3}+C_1\times 10$$

$$-10\ C_1=433{\cdot}33-114{\cdot}33-18=+301$$

constant, $C_1=-30{\cdot}1$

Equation for deflection becomes

$$EI\ y=\frac{1{\cdot}3\ x^3}{3}-\frac{(x-3)^3}{3}-\frac{2\ (x-7)^3}{3}-30{\cdot}1\ x$$

Deflection under the loads

at $x=3$ m, $y=y_B$

$$EIy_B=\frac{1{\cdot}3\times 3^3}{3}-0-0-30{\cdot}1\times 3$$

(omitting the negative terms in the bracket)

$$EIy_B=11{\cdot}7-90{\cdot}3=-78{\cdot}6$$

where $E=2000$ tonnes/cm^2$=2000\times 10^4$ tonnes/m^2

$I=9800\times 10^4$ mm^4$=9800\times 10^4\times 10^{-12}$ m^4

$=9800\times 10^{-8}$ m^4

$EI=2000\times 10^4\times 9800\times 10^{-8}=1960$ Tm^2

$$y_B=-\frac{78{\cdot}6}{1960}=-0{\cdot}040 \text{ m}=-4{\cdot}0 \text{ cm}$$

at $x=7$ m, $y=y_C$

$$EIy_C = \frac{1\cdot3\times7^3}{3} - \frac{(7-3)^3}{3} - 0 - 30\cdot1\times7$$

$$= 148\cdot633 - 21\cdot333 - 210\cdot7 = -83\cdot4$$

$$y_C = -\frac{83\cdot4}{1960} = -0\cdot0425 = -4\cdot25 \text{ cm.}$$

Maximum deflection. Let us assume that maximum deflection occurs in the portion *BC*. At the position $dy/dx=0$, deflection will be maximum. Considering the equation (2) and omitting the term $(x-7)^2$

$$EI\frac{dy}{dx} = 1\cdot3\,x^2 - (x-3)^2 - 30\cdot1 = 0$$

or $\quad 1\cdot3\,x^2 - x^2 - 9 + 6x - 30\cdot1 = 0$

or $\quad 0\cdot3x^2 + 6x - 39\cdot1 = 0$

or $\quad x^2 + 20x - 130\cdot33 = 0$

$$x = \frac{-20+\sqrt{400+4\times130\cdot33}}{2}$$

$$= \frac{-20+30\cdot35}{2} = 5\cdot175$$

At a distance of 5·175 m from *A*, slope is zero.

$$EI\,y_{max} = \frac{1\cdot3\times5\cdot175^3}{3} - \frac{(5\cdot175-3)^3}{3} - \text{omitted term} - 30\cdot1\times5\cdot175$$

$$= 60\cdot05 - 3\cdot43 - 155\cdot77 = -99\cdot15$$

$$y_{max} = -\frac{99\cdot15}{1960} = -0\cdot0506 \text{ m} = -5\cdot06 \text{ cm} \quad \textbf{Ans.}$$

Problem 11·2. A cantilever of symmetrical cross section 3 metre long and 40 cm deep carries a uniformly distributed load of 2·4 tonnes/metre run throughout its length. If $I=50{,}000$ cm^4 and $E=2100$ tonnes/cm^2, calculate the deflection at the free end.

What is the maximum concentrated load which the cantilever can carry at a distance of 2 m from the fixed end (in addition to the distributed load) if

(*a*) stress due to bending is not to exceed 800 kg/cm^2 anywhere in the cantilever

(*b*) deflection at the free end of the cantilever is not to exceed 4 mm.

Solution. Taking length in metres and load in tonnes, let us convert the units of *E* and *I*

$$E = 2100 \text{ tonnes/cm}^2 = 2100\times10^4 \text{ tonnes/m}^2$$

$$I = 50{,}000 \text{ cm}^4 = 50{,}000\times10^{-8} \text{ m}^4$$

$$EI = 2100\times10^4\times50000\times10^{-8} = 10500 \text{ T-m}^2$$

Uniformly distributed load,

$$w = 2\cdot4 \text{ tonnes/metre}$$

Length of the cantilever, $l=3$ m

Deflection at the free end,

$$\delta=\frac{wl^4}{8\ EI}$$

$$=\frac{2{\cdot}4\times 3^4}{8\times 10500}=2{\cdot}314\times 10^{-3}\text{ m}$$

$$=2{\cdot}314\text{ mm}$$

Maximum bending moment occurs at the fixed end

$$M_{max}=\frac{wl^2}{2}=\frac{2{\cdot}4\times 3^2}{2}=10{\cdot}8\text{ tonne-metres}$$

Depth of the section, $d=40\text{ cm}=0{\cdot}4\text{ m}$

Maximum stress due to bending

$$=\frac{M}{I}\ \frac{d}{2}=\frac{10{\cdot}8\times 0{\cdot}2}{50000\times 10^{-8}}$$

$$=\frac{10{\cdot}8\times 0{\cdot}2}{5}\times 10^4=0{\cdot}432\times 10^4\text{ tonnes/m}^2$$

$$=0{\cdot}432\text{ tonnes/cm}^2=432\text{ kg/cm}^2$$

(*a*) Due to additional concentrated load W

f', stress due to bending

$=800-432=368\text{ kg/cm}^2$

$=0{\cdot}368\times 10^4\text{ tonnes/m}^2.$

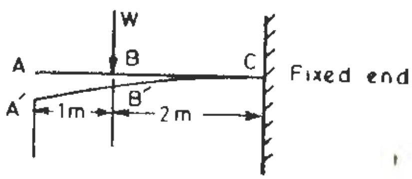


Fig. 11·28

Maximum bending moment occurs at the fixed end due to the concentrated load W, at a distance of 2 m from the fixed end. Say, load is W tonnes

$$M_{max}=W\times 2=2\ W\text{ tonne-metre}$$

$$f'=\frac{M_{max}}{I}\cdot\frac{d}{2}=\frac{2\ W\times 0{\cdot}2}{50{,}000\times 10^{-8}}=0{\cdot}08\times 10^{+4}\ W$$

$$0{\cdot}368\times 10^4=0{\cdot}08\times 10^4\ W$$

$$W=4{\cdot}6\text{ tonnes.}$$

(*b*) Due to concentrated load W, additional deflection AA' at the free end (See Fig. 11·28).

$$AA'=4\text{ mm}-2{\cdot}314\text{ mm}=1{\cdot}786\text{ mm}$$

$$=1{\cdot}786\times 10^{-3}\text{ m}$$

Deflection $BB'=\dfrac{Wl'^3}{3\ EI}$

where $l'=2$ m

$$=\frac{W\times 2^3}{3\times 10500}=0{\cdot}254\times 10^{-3}\ W$$

i_B, slope at B $=\frac{Wl'^2}{2\ EI}=\frac{W\times 2^2}{2\times 10{,}500}=0{\cdot}190\times 10^{-3}\ W$

Slope at A will be the same as slope at B, because portion AB is not subjected to bending due to the additional load W.

Deflection at A $=AA'=BB'+i_B\times AB=(0{\cdot}254\ W+0{\cdot}190\times 1\ W)\times 10^{-3}$

$=0{\cdot}444\ W\times 10^{-3}=1{\cdot}786\times 10^{-3}$

or $W=3{\cdot}98$ tonnes.

So the allowable load W is 3·98 tonnes.

Problem 11·3. A uniform beam of length L is supported symmetrically over a span l; $l<L$. Determine the ratio of l/L if the upward deflection at the ends is equal to the downward deflection at the centre due to a concentrated load at the mid span.

Solution. Fig. 11·29 shows a beam of length L supported symmetrically over a span l.

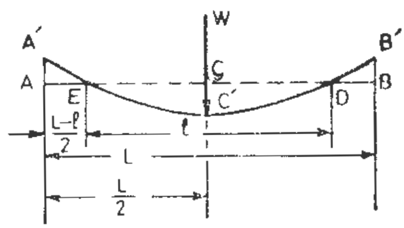


Fig. 11·29

Overhang $AE=\frac{L-l}{2}$

Say the central load$=W$

Deflection at the centre

$$=\frac{Wl^3}{48EI}=CC'$$

where EI is the flexural rigidity of the beam.

Slope at the ends $=\frac{Wl^2}{16\ EI}=i_e$

Portion EA' remains straight since this portion is not subjected to bending. Portion EA' follows the slope of the point E.

So the upward deflection,

$$AA'=i_e\times AE$$

$$=i_e\times\left(\frac{L-l}{2}\right)=\frac{Wl^2}{16\ EI}\left(\frac{L-l}{2}\right)$$

But $CC'=AA'$

$$\frac{Wl^3}{48\ EI}=\frac{Wl^2}{16EI}\left(\frac{L-l}{2}\right)$$

or $$l^3=3l^2\left(\frac{L}{2}-\frac{l}{2}\right)$$

$$l=\frac{3L}{2}-\frac{3l}{2}$$

or $$\frac{5}{2}l=\frac{3}{2}L$$

or $$\frac{l}{L}=\frac{3}{5}=0·6$$

Problem 11·4. A girder 6 m long is supported at one end and at 1·5 m from the other end. It carries a uniformly distributed load of 10 tonnes/metre over the supported length of 4·5 m and a concentrated load of 6 tonnes at the overhang end. Calculate the maximum downward deflection and state where it occurs. Given $EI=20\times10^{10}$ kg cm²

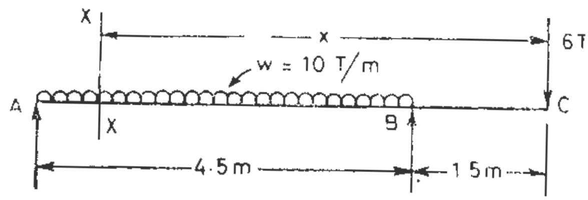


Fig. 11·30

Solution. Taking moments about A for reactions,

$$R_B\times4·5=4·5\times10\times\frac{4·5}{2}+6\times6$$

Reaction, $$R_B=\frac{4·5\times10\times4·5}{2\times4·5}+\frac{36}{4·5}$$
$$=30·5 \text{ Tonnes}$$

Reaction, $$R_A=4·5\times10+6-30·5$$
$$=20·5 \text{ Tonnes}$$

Consider a section X-X at a distance of x from the end C

Bending moment at the section,

$$M_x=6x+\frac{w}{2}(x-1·5)^2-R_B(x-1·5)$$

or $$EI\frac{d^2y}{dx^2}=6x+\frac{10}{2}(x-1·5)^2-30·5(x-1·5)$$

$$EI\frac{d^2y}{dx^2}=6x+5(x-1·5)^2-30·5(x-1·5) \qquad ...(1)$$

Integrating equation (1)

$$EI\frac{dy}{dx}=\frac{6x^2}{2}+\frac{5}{3}(x-1·5)^3-\frac{30·5}{2}(x-1·5)^2+C_1 \qquad ...(2)$$

where C_1 is the constant of integration.

Integrating further the equation (2),

$$EIy=\frac{6x^3}{6}+\frac{5}{12}(x-1·5)^4-\frac{30·5}{6}(x-1·5)^3+C_1x+C_2$$

$$=x^3 + \frac{5}{12}(x-1{\cdot}5)^4 - \frac{30{\cdot}5}{6}(x-1{\cdot}5)^3 + C_1x + C_2 \qquad \text{...(3)}$$

where C_2 is another constant of integration.

Now deflection $y=0$ at $x=1{\cdot}5$ m and $x=6$ m

Substituting in equation (3) above

$$0=1{\cdot}5^3+0-0+C_1\times 1{\cdot}5+C_2 \qquad \text{...(4)}$$

$$0=6^3+\frac{5}{12}(6-1{\cdot}5)^4-\frac{30{\cdot}5}{6}(6-1{\cdot}5)^3+6C_1+C_2 \qquad \text{...(5)}$$

or $$1{\cdot}5\,C_1+C_2=-3{\cdot}375 \qquad \text{...(4)}$$

$$6C_1+C_2=-216-170{\cdot}859+463{\cdot}218 \qquad \text{...(5)}$$

or $$1{\cdot}5C_1+C_2=-3{\cdot}375 \qquad \text{...(4)}$$

$$6C_1+C_2=+76{\cdot}359 \qquad \text{...(5)}$$

From equations (4) and (5)

$$C_1=17{\cdot}718$$

$$C_2=-29{\cdot}953$$

Deflection at the overhang end C *i.e.*, at $x=0$

$$EI\,y_c=0+0-0+0+C_2$$

$$=-29{\cdot}953$$

$$y_c=-\frac{29{\cdot}953}{EI}$$

$$EI=20\times 10^{10}\text{ kg-cm}^2$$

$$=20\times 10^6\text{ kg-m}^2$$

$$=20\times 10^3\text{ tonne-m}^2$$

So $$y_c=-\frac{29{\cdot}953}{20\times 10^3}\text{ m}$$

$$=-0{\cdot}00149\text{ m}$$

$$=-0{\cdot}149\text{ cm}$$

Deflection can be more than y_c in the span AB *i.e.* $x>1{\cdot}5$. For this, let us find out where slope $\frac{dy}{dx}=0$. Putting equation (2) equal to zero.

$$0=3x^2+\frac{5}{3}(x-1{\cdot}5)^3-\frac{30{\cdot}5}{2}(x-1{\cdot}5)^2+17{\cdot}718 \qquad \text{...(6)}$$

or $$18x^2+10(x-1{\cdot}5)^3-30{\cdot}5\times 3(x-1{\cdot}5)^2+6\times 17{\cdot}718=0$$

or $$18x^2+10(x^2-3x+2{\cdot}25)(x-1{\cdot}5)-91{\cdot}5(x^2-3x+2{\cdot}25)+106{\cdot}308=0$$

or $$10x^3-118{\cdot}5\,x^2+342\,x-136{\cdot}317=0$$

$$x=3{\cdot}87\text{ metres (By trial and error)}$$

Now deflection at $x=3\cdot87$ m

$$EI\,y_{3\cdot87}=3\cdot87^3+\frac{5}{12}(3\cdot87-1\cdot5)^4-\frac{30\cdot5}{6}(3\cdot87-1\cdot5)^3$$
$$+17\cdot718\times3\cdot87-29\cdot953$$
$$=57\cdot960+13\cdot146-67\cdot669+68\cdot568-29\cdot953=42\cdot052$$

$$y_{3\cdot87}=\frac{42\cdot052}{20\times10^3}=0\cdot0021 \text{ m}$$

Maximum deflection $=0\cdot21$ cm

This shows that deflection $y_{3\cdot87}$ is downward and deflection y_o is upwards and $y_{3\cdot87}$ is maximum deflection.

Problem 11·5. A beam of length l is hinged at one end and supported at a distance of $\frac{2l}{3}$ from the hinged end. It carries a load W at the free end and a load W distributed over a length $\frac{l}{3}$, starting from a distance of $\frac{l}{3}$ from the hinged end. If EI is flexural rigidity show that the deflection under the concentrated load W at the free end is $\frac{13}{432}\ \frac{Wl^3}{EI}$.

Solution. The loads on the beam are as shown in the Fig. 11·31. For support reactions let us take moments of the forces about the point D.

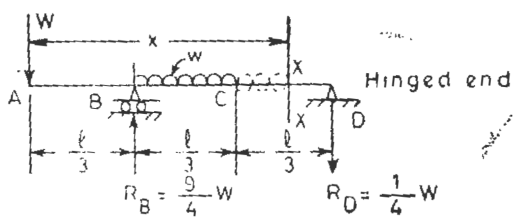


Fig. 11·31

$$Wl+\frac{wl}{3}\left(\frac{l}{3}+\frac{l}{6}\right)=\frac{2l}{3}R_B$$

$$Wl+\frac{wl}{3}\times\frac{l}{2}=\frac{2l}{3}R_B$$

where $\frac{wl}{3}=W$ or $w=\frac{3W}{l}$

So $$Wl+\frac{Wl}{2}=\frac{2}{3}l.\ R_B$$

$$\frac{3}{2}W=\frac{2}{3}R_B$$

or $$R_B=\frac{9}{4}W$$

$$R_A=W+W-\frac{9}{4}W=-\frac{1}{4}W$$

...ction X-X at a distance of x from the end A and extend the unif

B.M. at the section, $M=-Wx+\frac{9}{4}W\left(x-\frac{l}{3}\right)$

$$-\frac{w}{2}\left(x-\frac{l}{3}\right)^2+\frac{w}{2}\left(x-\frac{2l}{3}\right)^2$$

or $$EI\frac{d^2y}{dx^2}=-Wx+\frac{9}{4}W\left(x-\frac{l}{3}\right)$$

$$-\frac{w}{2}\left(x-\frac{l}{3}\right)^2+\frac{w}{2}\left(x-\frac{2l}{3}\right)^2 \qquad ...(1)$$

Integrating equation (1) we get

$$EI\frac{dy}{dx}=-\frac{Wx^2}{2}+\frac{9}{8}W\left(x-\frac{l}{3}\right)^2$$

$$-\frac{w}{6}\left(x-\frac{l}{3}\right)^3+\frac{w}{6}\left(x-\frac{2l}{3}\right)^3+C_1 \qquad ...(2)$$

and $$EIy=-\frac{Wx^3}{6}+\frac{9}{24}W\left(x-\frac{l}{3}\right)^3$$

$$-\frac{w}{24}\left(x-\frac{l}{3}\right)^4+\frac{w}{24}\left(x-\frac{2l}{3}\right)^3+C_1x+C_2 \qquad ...(3)$$

Now $y=0$ at $x=\frac{l}{3}$, and omitting $\left(x-\frac{2l}{3}\right)$ term we get

$$0=-\frac{W}{6}\left(\frac{l}{3}\right)^3+0-0+C_1\times\frac{l}{3}+C_2$$

or $$C_1\frac{l}{3}+C_2=+\frac{Wl^3}{162}$$ where C_1 and C_2 are constants of integration ...(4)

Moreover $y=0$ at $x=l$

$$\therefore \qquad 0=-\frac{Wl^3}{6}+\frac{9}{24}W\left(l-\frac{l}{3}\right)^3$$

$$-\frac{w}{24}\left(l-\frac{l}{3}\right)^4+\frac{w}{24}\left(l-\frac{2l}{3}\right)^4+C_1l+C_2$$

$$0=-\frac{Wl^3}{6}+\frac{Wl^3}{9}-\frac{w}{24}\times\frac{16\ l^4}{81}+\frac{w}{24}\times\frac{l^3}{81}+C_1l+C_2$$

or $$0=-\frac{Wl^3}{18}-\frac{5wl^4}{648}+C_1l+C_2$$

or $$C_1l+C_2=\frac{Wl^3}{18}+\frac{5}{648}\times wl^4 \quad \text{but} \quad w=\frac{3W}{l}$$

$$=\frac{Wl^3}{18}+\frac{5}{648}\times\frac{3W}{l}\times l^4=\frac{Wl^3}{18}+\frac{5}{216}Wl^3$$

$$=\frac{17}{216}Wl^3 \qquad ...(5)$$

From equations (5) and (4),

$$\frac{2l}{3}\,C_1=Wl^3\left[\frac{17}{216}-\frac{1}{162}\right]=\frac{Wl^3}{18}\left[\frac{17}{12}-\frac{1}{9}\right]$$

$$=\frac{Wl^3}{18}\left[\frac{47}{36}\right]$$

$$C_1=\frac{47\,Wl^2}{12\times 36}$$

$$C_2=\frac{17}{216}\,Wl^3-\frac{47}{12\times 36}\times Wl^3=-\frac{13}{432}\,Wl^3$$

So

$$EIy=-\frac{Wx^3}{6}+\frac{9}{24}\,W\left(x-\frac{l}{3}\right)^3-\frac{w}{24}\left(x-\frac{l}{3}\right)^4$$

$$+\frac{w}{24}\left(x-\frac{2l}{3}\right)^4+\frac{47\,Wl^2}{432}\,x-\frac{13}{432}\,Wl^3$$

At the free end $x=0$, and omitting $\left(x-\frac{l}{3}\right)$ and $\left(x-\frac{2l}{3}\right)$ terms we get

$$EI\,y_A=-\frac{13}{432}\,Wl^3$$

$$y_A=\frac{13}{432}\times\frac{Wl^3}{EI}$$ is the deflection at the free end.

Problem 11·6. A beam AB of length l is hinged at both the ends. An anticlockwise turning moment M is applied at the point C. Point C is at a distance of $l/4$ from end A. Determine slope and deflection at the point C. Given that EI is the flexural rigidity of the beam. Indicate the slope of the deflected beam.

Solution. The beam AB with a turning moment M is shown in Fig. 11·32 (*a*).

For support reactions, taking moments of the forces about the point A,

$$M=R_B\times l$$

or $$R_B=\frac{M}{l}\downarrow$$

then for equilibrium $$R_A=\frac{M}{l}\uparrow$$

If a bending moment diagram is drawn for the beam, it will be of the shape given in Fig. 11·32 (*b*).

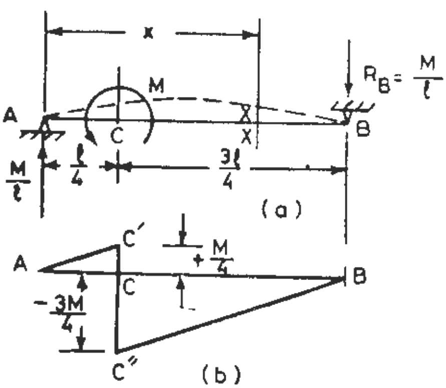


Fig. 11·32

Consider a section X-X at a distance of x from the end A.

BM at the section, $M_x=\frac{M}{l}\cdot x-M$

To take into account the distribution of bending moment, the term M can be written as $M\ (x-l/4)^{\circ}$, because any quantity raised to the power zero is equal to one

So $$M_x=\frac{Mx}{l}-M\left(x-\frac{l}{4}\right)^{\circ}$$

or $$EI\ \frac{d^2y}{dx^2}=\frac{Mx}{l}-M\left(x-\frac{l}{4}\right)^{\circ} \qquad ...(1)$$

Integrating the equation (1)

$$EI\ \frac{dy}{dx}=\frac{Mx^2}{2l}-M\left(x-\frac{l}{4}\right)+C_1 \text{ (a constant of integration)} \qquad ...(2)$$

and $$EIy=\frac{Mx^3}{6l}-\frac{M}{2}\left(x-\frac{l}{4}\right)^2+C_1x+C_2$$

(another constant of integration) ...(3)

At $x=0$, $y=0$, term $(x-l/4)$ will become negative and so it will be omitted

So $0=0+0+C_2$

$C_2=0$

At $x=l$, $y=0$; another boundary condition

So $$0=\frac{Ml^3}{6l}-\frac{M}{2}\left(l-\frac{l}{4}\right)^2+C_1\ l$$

$$=\frac{Ml^2}{6}-\frac{9\ Ml^2}{32}+C_1\ l$$

or $$C_1=\left(\frac{9}{32}-\frac{1}{6}\right)Ml=\frac{11}{96}\ Ml$$

Expression (2) and (3) will now be

$$EI\ \frac{dy}{dx}=\frac{Mx^2}{2l}-M\left(x-\frac{l}{4}\right)+\frac{11}{96}\ Ml \qquad ...(2)$$

$$EI\ y=\frac{Mx^3}{6l}-\frac{M}{2}\left(x-\frac{l}{4}\right)^2+\frac{11}{96}\ Mlx \qquad ...(3)$$

At the point C, $x=\frac{l}{4}$, $\frac{dy}{dx}=ic$ and $y=yc$

$$EI\ ic=\frac{M}{2l}\left(\frac{l^2}{16}\right)-0+\frac{11}{96}\ Ml$$

$$=\frac{Ml}{32}+\frac{11}{96}\ Ml=\frac{14}{96}\ Ml$$

$$EI\ yc=\frac{M}{6l}\left(\frac{l}{4}\right)^3+\frac{11}{96}\times Ml\ .\frac{l}{4}$$

$$=\frac{Ml^2}{384}+\frac{11}{384}\ Ml^2=\frac{12}{384}\ Ml^2$$

$$i_C = \frac{14\ Ml}{96\ EI}$$

$$y_C = \frac{12\ Ml^2}{384\ EI} = \frac{Ml^2}{32\ EI}$$

Shape of the deflected beam

Slope at the end B, at $x=l$

$$EI\ i_B = \frac{Ml}{2} - \frac{3\ Ml}{4} + \frac{11}{16}\ Ml = -\frac{23}{96}\ Ml$$

At a position where deflection is maximum, slope is zero. Substituting in expression (2) $dy/dx=0$ we get

$$\frac{x^2}{2l} - \left(x - \frac{l}{4}\right) + \frac{11}{96}\ l = 0$$

$$\frac{x^2}{2l} - x + \frac{l}{4} + \frac{11}{96}\ l = 0$$

or $$x = 0{\cdot}48\ l$$

$$EI\ y_{max} = \frac{M}{6l}\ (0{\cdot}48\ l)^3 - \frac{M}{2}\ (0{\cdot}48\ l - 0{\cdot}25\ l)^2 + \frac{11}{96}\ Ml\ .\ (0{\cdot}48\ l)$$

$$= 0{\cdot}018432\ Ml^2 - 0{\cdot}02645\ Ml^2 + 0{\cdot}055\ Ml^2$$

$$= 0{\cdot}047\ Ml^2$$

The deflected shape of the beam is shown in diagram 11·32 (*a*) showing thereby that the beam is entirely lifted up from its original axis.

Problem 11·7. A propped cantilever of length l is fixed at one end and freely supported at the other end. The cantilever is subjected to a couple M in the vertical plane about an axis $3l/4$ from the fixed end. Determine the reaction at the prop and fixing moment at the fixed end.

Solution. Fig. 11·33 shows the cantilever of length l fixed at end B and simply supported at end A. A moment M in the anticlockwise direction is applied at C, at a distance of $3l/4$ from the fixed end.

Say the reaction at the prop $=P$

Consider a section X-X at a distance of x from the end A.

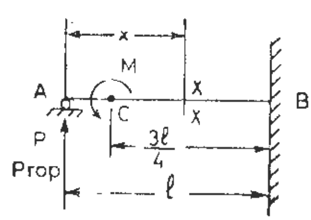


Fig. 11·33

BM at the section, $$M = Px - M = Px - M\left(x - \frac{l}{4}\right)^{\circ}$$

or $$EI\ \frac{d^2y}{dx^2} = Px - M\ \left(x - \frac{l}{4}\right)^{\circ} \qquad \ldots(1)$$

Integrating equation 1,

$$EI\ \frac{dy}{dx}=\frac{Px^2}{2}-M\left(x-\frac{l}{4}\right)+C_1$$

where C_1 is the constant of integration

$$\text{at} \quad x=l\ ;\ \text{fixed end}\ \ \frac{dy}{dx}=0$$

So $$0=\frac{Pl^2}{2}-\frac{3\ Ml}{4}+C_1$$

or $$C_1=\frac{3Ml}{4}-\frac{Pl^2}{2}$$

So $$EI\ \frac{dy}{dx}=\frac{Px^2}{2}-M\left(x-\frac{l}{4}\right)+\frac{3\ Ml}{4}-\frac{Pl^2}{2} \qquad \ldots(2)$$

Integrating further $$EIy=\frac{Px^3}{6}-\frac{M}{2}\left(x-\frac{l}{4}\right)^2+\frac{3\ Mlx}{4}-\frac{Pl^2}{2}\ x+C_2$$

(another constant of integration)

But at $x=l$, fixed end, $y=0$

So $$0=\frac{Pl^3}{6}-\frac{M}{2}\times\frac{9l^2}{16}+\frac{3\ Ml^3}{4}-\frac{Pl^3}{2}+C_2$$

or $$C_2=\frac{Pl^3}{3}-\frac{15}{32}\ Ml^2$$

Therefore $$EIy=\frac{Px^3}{6}-\frac{M}{2}\left(x-\frac{l}{4}\right)^2+\frac{3\ Mlx}{4}-\frac{Pl^2x}{2}+\frac{Pl^3}{3}-\frac{15}{32}\ Ml^2$$

But at $x=0$, $y=0$, neglecting $\left(x-\frac{l}{4}\right)$ a negative term

$$0=0+0-0+\frac{Pl^3}{3}-\frac{15\ Ml^2}{32}$$

or $$P=\frac{45\ Ml^2}{32l^3}=\frac{45\ M}{32l} \text{ (reaction at the prop)}$$

BM at fixed end, $$M_B=\frac{45\ M}{32l}\times l-M=\frac{13M}{32}$$

Problem 11·8. A vertical pole 6 m high carries a concentrated load of 500 kg inclined at an angle of 30° to its vertical axis. The pole is of uniform round section throughout. A pull P is applied at an angle of 45° to the axis of the pole, at a distance of 3 m from its base. Determine the magnitude of P such that the deflection at the top of the pole is zero. Neglect the effect of axial forces.

Solution. Fig. 11·34 shows a vertical pole, 6 m high subjected to an inclined load 500 kg as shown.

Horizontal component of load applied

$$=500\times\sin 30°$$
$$=250 \text{ kg}$$

Say the pull applied at the point C is P kg, inclined at an angle of 45° to the axis of the pole.

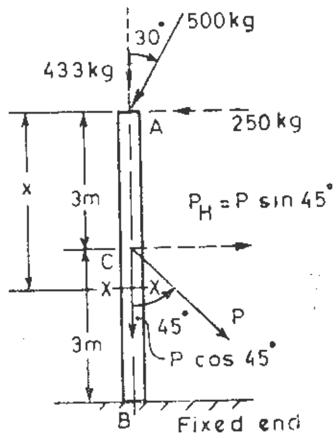


Fig. 11·34

Horizontal component of pull,

$$P_H = P \sin 45° = 0{\cdot}707\ P$$

Consider a section X-X of the pole, at a distance of x from the end A

BM at the section $= -250\ x + P_H\ (x-3)$

or
$$EI\frac{d^2y}{dx^2} = -250\ x + P_H\ (x-3) \quad \ldots(1)$$

Integrating equation (1)

$$EI\frac{dy}{dx} = -250\ \frac{x^2}{2} + \frac{P_H}{2}\ (x-3)^2 + C_1$$

where C_1 is the constant of integration

At $x=6$ m, fixed end of the pole, slope is zero

i.e.,
$$\frac{dy}{dx} = 0$$

$\therefore$
$$0 = -125\ (6)^2 + \frac{P_H}{2}\ (3)^2 + C_1$$

So
$$C_1 = 4500 - 4{\cdot}5\ P_H$$

Therefore
$$EI\frac{dy}{dx} = -125\ x^2 + 0{\cdot}5\ P_H\ (x-3)^2 + 4500 - 4{\cdot}5\ P_H \quad \ldots(2)$$

Integrating equation (2)

$$EIy = -\frac{125\ x^3}{3} + \frac{P_H}{6}\ (x-3)^3 + 4500\ x - 4{\cdot}5\ P_H\ .\ x + C_2$$

where C_2 is another constant of integration

At $\quad x=6$ m, fixed end, deflection, $y=0$

$$\therefore \quad 0=-\frac{125\times6^3}{3}+\frac{P_H}{6}(3)^3+4500\times6-4{\cdot}5\,P_H\,.\,6+C_2$$

$$=-9000+4{\cdot}5\,P_H+27000-27\,P_H+C_2$$

or $\quad C_2=22{\cdot}5\,P_H-18000$

Therefore $\quad EIy=-\frac{125\,x^3}{3}+\frac{P_H}{6}(x-3)^3+4500x-4{\cdot}5\,P_H.x+22{\cdot}5\,P_H-18000$

At $\quad x=0$ free end, deflection is zero as per the condition given in the problem,

So $\quad 0=0+0-0+22{\cdot}5\,P_H-1800$ omitting the term $(x-3)$

$P_H=800$ kg

Pull, $\quad P=\frac{800}{0{\cdot}707}=1131{\cdot}54$ kg

Problem 11·9. A beam of length l simply supported at the ends carries a uniformly varying distributed load throughout its length. The load intensity varies from zero at one end to w per unit length at the other end. If EI is the flexural rigidity of the beam, determine the maximum deflection in the beam.

Solution. The load on the beam is as shown in the Fig. 11·35. For support reactions, take moments of the forces about the point A.

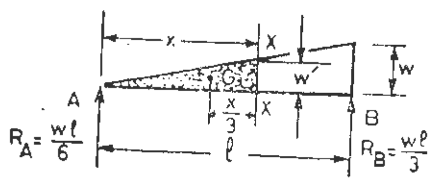


Fig. 11·35

Total load on beam $=\frac{wl}{2}$

C.G. of the loads acts at a distance of $\frac{2l}{3}$ from the end A. Therefore,

$$\frac{wl}{2}\times\frac{2l}{3}=R_B\times l$$

or $\quad R_B=\frac{wl}{3}$

and $\quad R_A=\frac{wl}{2}-R_B=\frac{wl}{6}$

Consider a section X-X at a distance of x from the end A.

Rate of loading at X, $w'=\frac{w.x}{l}$

Total load upto X, $\quad =\frac{wx}{l}\times\frac{x}{2}=\frac{wx^2}{2l}$

C.G. of the load AXX, acts at a distance of $\frac{x}{3}$ from the section X-X

Bending moment at the section,

$$M=+R_A.x-\frac{wx^2}{2l}\times\left(\frac{x}{3}\right)$$

$$=\frac{wl}{6}.\ x-\frac{wx^3}{6l}$$

or
$$EI\frac{d^2y}{dx^2}=\frac{wlx}{6}-\frac{wx^3}{6l}\qquad ...(1)$$

Integrating the equation (1)

$$EI\frac{dy}{dx}=\frac{wlx^2}{12}-\frac{wx^4}{24\ l}+C_1\qquad ...(2)$$

and
$$EIy=\frac{wlx^3}{36}-\frac{wx^5}{120\ l}+C_1x+C_2\qquad ...(3)$$

At $x=0,\quad y=0$

$$0=0-0+0+C_2\quad\text{or}\quad C_2=0$$

Moreover, at $x=l$, $y=0$

$$0=\frac{wl^4}{36}-\frac{wl^4}{120}+C_1\ l$$

or
$$C_1=-\frac{wl^3}{36}+\frac{wl^3}{120}=-\frac{7}{360}\ wl^3$$

The expressions will now be

$$EI\frac{dy}{dx}=\frac{wlx^2}{12}-\frac{wx^4}{24\ l}-\frac{7}{360}\ wl^3\qquad ...(2)$$

$$EIy=\frac{wlx^3}{36}-\frac{wx^5}{120\ l}-\frac{7}{360}\ wl^3x\qquad ...(3)$$

Now the maximum deflection occurs in the beam at the section where slope is zero. Therefore to determine the position of maximum bending moment let us put equation (2) equal to zero.

or
$$\frac{wlx^2}{12}-\frac{w}{24l}x^4-\frac{7}{360}wl^3=0$$

or
$$lx^2-\frac{x^4}{2l}-\frac{7}{30}\ l^3=0$$

or
$$x^4-2x^2l^2+\frac{7}{15}\ l^4=0$$

$$x^2=\frac{2l^2\pm\sqrt{4l^4-\dfrac{28}{15}\,l^4}}{2}=\frac{2l^2\pm\sqrt{\dfrac{32}{15}\,l^4}}{2}$$

$$=l^2-l^2\sqrt{\frac{8}{15}}=l^2(1-{\cdot}730)$$

$$=l^2\times 0{\cdot}27$$

$$x=0{\cdot}5196\ l$$

Therefore $$EIy_{max}=\frac{wl}{36}\,({\cdot}5196\ l)^3-\frac{w}{120l}\,({\cdot}5196)^5-\frac{7}{360}\,wl^2\times{\cdot}5196l$$

$$=wl^4[0{\cdot}003897-0{\cdot}000315-{\cdot}010103]$$

Maximum deflection, $y_{max}=-\dfrac{wl^4}{EI}\times\dfrac{0{\cdot}00652}{}=-\dfrac{0{\cdot}00652\ wl^4}{EI}$

occurs at a distance of $0{\cdot}5196\ l$ from the end where load intensity is zero.

Problem 11·10 A horizontal steel beam 25 cm dia and 5 m long carries a uniformly distributed load of 1 tonne/metre run throughout its length. The beam is supported by 3 vertical steel tie rods each 2·4 m long, one at each end and one in the middle. The diameter of the outer rods is 15 mm and that of the middle rod is 20 mm. Calculate the deflection at the centre of the beam below its end points. $E=2000$ tonnes/cm².

Solution. Fig. 11·36 shows a horizontal beam 5 m long supported by 3 rods. Say the reaction at the middle rod is P, then reaction at the outer rods will be

$$\left(\frac{1\times 5-P}{2}\right)=\left(\frac{5-P}{2}\right)$$

as the total load on the beam is 5 tonnes.

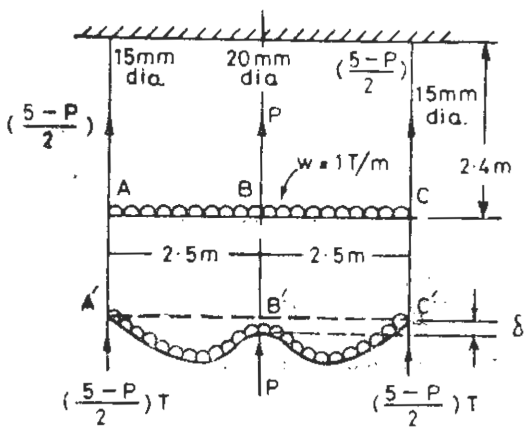


Fig. 11·36

Area of cross section of outer rods

$$=\frac{\pi}{4}\ (1{\cdot}5)^2=1{\cdot}767\ \text{cm}^2$$

Area of cross section of middle rod

$$=\frac{\pi}{4}(2^2)=3{\cdot}1416\ \text{cm}^2$$

Length of each rod $=2{\cdot}4$ m $=240$ cm

Extension in length of outer rods,

$$\delta l_1=\left(\frac{5-P}{2}\right)\times\frac{340}{1{\cdot}767\times E}=\left(\frac{5-P}{2}\right)\times\frac{240}{1{\cdot}767\times E}$$

$$=\frac{67{\cdot}912}{E}\ (5-P) \qquad \ldots(1)$$

Extension in the length of middle rod,

$$\delta l_2 = \frac{P}{\pi} \times \frac{240}{E} = \frac{P}{3{\cdot}1416} \times \frac{240}{E} = \frac{76{\cdot}394\ P}{E} \qquad \ldots(2)$$

Deflection at the centre of the beam,

$$\delta = \delta l_2 - \delta l_1$$

$$= \frac{76{\cdot}394\ P}{E} - \frac{67{\cdot}912}{E}(5-P) \qquad \ldots(3)$$

Moreover, deflection at the centre of the beam, δ (as per the formulae already derived)

$$= \frac{5}{384}\frac{wl^4}{EI} - \frac{Pl^3}{48\ EI}$$

where l=length of the beam=500 cm

w=rate of loading=1 T/m='01 T/cm

$$\delta = \frac{5}{384} \times \frac{{\cdot}01 \times 500^4}{EI} - \frac{P \times 500^3}{48\ EI}$$

$$= \frac{500^2}{EI}[32{\cdot}552 - 10{\cdot}416\ P]$$

I of the beam $= \frac{\pi}{64}(25)^4 = 19174{\cdot}8\ \text{cm}^4$ where diameter of the beam is 25 cm

$$\delta = \frac{500^2}{E \times 19174{\cdot}8}[32{\cdot}552 - 10{\cdot}416\ P]$$

$$= \frac{13{\cdot}038}{E}[32{\cdot}552 - 10{\cdot}416\ P] \qquad \ldots(4)$$

From equations (3) and (4)

$$\frac{13{\cdot}038}{E}[32{\cdot}552 - 10{\cdot}416\ P] = \frac{76{\cdot}394\ P}{E} - \frac{67{\cdot}912}{E}(5-P)$$

or $$13{\cdot}038 \times 32{\cdot}552 - 10{\cdot}416 \times 13{\cdot}038 = 76{\cdot}394\ P - 67{\cdot}912 \times 5 + 67{\cdot}912\ P$$

$$424{\cdot}413 - 135{\cdot}804 = 76{\cdot}394\ P - 339{\cdot}56 + 67{\cdot}912\ P$$

or $$(67{\cdot}912 + 76{\cdot}394)\ P = 424{\cdot}413 - 135{\cdot}804 + 339{\cdot}56$$

$$= 628{\cdot}169$$

$$P = \frac{628{\cdot}169}{144{\cdot}306} = 4{\cdot}353 \text{ Tonnes}$$

$$\frac{5-P}{2} = 0{\cdot}3235 \text{ Tonne}$$

$$\delta l_1 = \frac{67.912}{E}\ (5-4.353) = \frac{.647 \times 67.912}{2000} = 0.022 \text{ cm}$$

$$\delta l_2 = \frac{76.394\ P}{E} = \frac{76.394 \times 4.353}{2000} = 0.166 \text{ cm}$$

Deflection, $\delta = \delta l_2 - \delta l_1 = 0.166 - 0.022 = 0.144$ cm

$= 1.44$ mm

Stress in outer rods $= \left(\frac{5-P}{2}\right)\frac{1}{1.767} = \frac{.3235}{1.767}$

$= 0.183$ tonne/cm²

Stress in middle rod $= \frac{P}{3.1416} = \frac{4.353}{3.1416} = 1.386$ tonnes/cm²

Problem 11·11. A cantilever of length l carries a total load P distributed over its length l It is supported over a prop at a distance of kl from the fixed end. Determine the ratio of the deflection of the cantilever at the free end with the deflection of the cantilever if it is unpropped.

Solution. Say EI is the flexural rigidity of the cantilever. Fig. 11·37 shows a cantilever of length l carrying uniformly distributed load, $w = \frac{P}{l}$ per unit length. A prop is at a distance of kl from the end

Reaction at the prop $= P$

Consider a section X-X at a distance of x from the end A

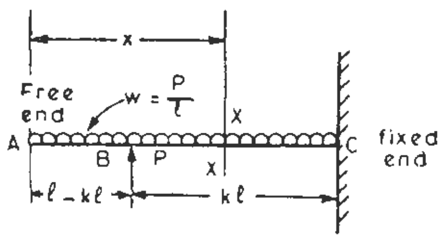


Fig. 11·37

B.M. at the section is $M = -\frac{wx^2}{2} + P(x-l+kl)$

$$EI\frac{d^2y}{dx^2} = -\frac{P}{2l} \times x^2 + P(x-l+kl) \quad \text{as } w = \frac{P}{l}$$

Integrating equation (1)

$$EI\frac{dy}{dx} = -\frac{Px^3}{6l} + \frac{P}{2}(x-l+kl)^2 + C_1 \qquad \ldots(1)$$

where C_1 is a constant of integration

at $x = l$, $\frac{dy}{dx} = 0$

Therefore $0 = -\frac{Pl^3}{6l} + \frac{P}{2}(l-l+kl)^2 + C_1$

or $C_1 = \frac{Pl^2}{6} - \frac{Pk^2l^2}{2} = \frac{Pl^2}{6}(1-3k^2)$

So $$EI\,\frac{dy}{dx}=-\frac{Px^3}{6l}+\frac{P}{2}\,(x-l+kl)^2+\frac{Pl^2}{6}\,(1-3k^2) \qquad ...(2)$$

Integrating equation (2),

$$EIy=-\frac{Px^4}{24\,l}+\frac{P}{6}\,(x-l+kl)^3+\frac{Pl^2}{6}\,(1-3k^2)x+C_2$$

Again at $x=l$, $y=0$

Therefore, $$0=-\frac{Pl^3}{24}+\frac{P}{6}\,(k^3l^3)+\frac{Pl^3}{6}\,(1-3k^2)+C_2$$

or $$C_2=\frac{Pl^3}{24}-\frac{P}{6}\,k^3l^3-\frac{Pl^3}{6}+\frac{Pl^3k^2}{2}, \text{ constant of integration}$$

$$=-\frac{Pl^3}{8}-\frac{P}{6}k^3l^3+\frac{P}{2}\,k^2l^3$$

$$=Pl^3\left[\frac{k^2}{2}-\frac{k^3}{6}-\frac{1}{8}\right]=\frac{Pl^3}{24}\,(12k^2-4k^3-3)$$

$$=-\frac{Pl^3}{24}\,(4k^3-12k^2+3)$$

Therefore $$EIy=-\frac{Px^4}{24\,l}+\frac{P}{6}\,(x-l+kl)^3+\frac{Pl^2}{6}\,(1-3k^2)x$$

$$-\frac{Pl^3}{24}\,(4k^3-12k^2+3)$$

Deflection at the free end at $x=0$, neglecting the term $(x-l+kl)$

$$EIy_A=-\frac{Pl^3}{24}\,(4k^3-12k^2+3)$$

or $$y_A=-\frac{Pl^3}{8EI}\left[\frac{4}{3}\,k^3-4k^2+1\right]$$

When the cantilever is unpropped deflection at the free end,

$$y_A'=-\frac{Pl^3}{8EI}$$

So $$\frac{y_A}{y_A'}=\frac{4}{3}\,k^3-4k^2+1$$

Problem 11·12. A circular steel pipe 400 mm bore and 10 mm wall thickness is supported at its ends and at the centre. When the pipe is full of water the central support sinks by 2 mm below the ends. Find the load on each support. Draw also the BM diagram.

Given $\rho_{steel}=7{\cdot}8\ g/cc$, $\rho_{water}=1\ g/cc$

$E_{steel}=205\ \text{kN/mm}^2$.

Solution.

Inside diameter of pipe $=40$ cm

Outside diameter of pipe $=42$ cm

$$\rho_{steel}=9{\cdot}8\ g/cc,\ \rho_{water}=1\ g/cc.$$

Let us first determine the weight of water and pipe per unit length.

Area of cross section of water pipe

$$=\frac{\pi}{4}\,(40)^2=400\ \pi\ \text{cm}^2$$

Area of cross section of steel pipe

$$=\frac{\pi}{4}\,(42^2-40^2)=41\ \pi\ \text{cm}^2$$

Weight per cm of pipe $=400\ \pi\times 1+41\times\pi\times 7{\cdot}8$

$=2261{\cdot}3\ \text{g}=2{\cdot}261\ \text{kg/cm}$

Weight of the pipe per metre length

$$=2{\cdot}261\times 100\times 9{\cdot}8\ \text{N/m}$$

or $$w=2{\cdot}216\ \text{kN/m}$$

Length of the span $=10$ m

$$E=205\ \text{kN/mm}^2=205\times 10^6\ \text{kN/m}^2$$

Moment of inertia, $$I=\frac{\pi}{64}\,(42^4-40^4)=\frac{\pi}{64}\,(3364)\,(164)$$

$$=27081\ \text{cm}^4=27081\times 10^{-8}\ \text{m}^4$$

If the beam (pipe) is not supported at the centre, the central deflection would have been

$$y_C=\frac{5}{384}\times\frac{wl^4}{EI}=\frac{5}{384}\times\frac{2{\cdot}216\times 10^4}{205\times 10^6\times 27081\times 10^{-8}}\ \text{m}$$

$$=\frac{5}{384}\times\frac{2{\cdot}216\times 10^4}{205\times 270{\cdot}81}=5{\cdot}197\times 10^{-3}\ \text{m}=5{\cdot}197\ \text{mm}$$

But the central support sinks by 2 mm only. This means that the upward reaction provided by the central support produces a deflection of $5{\cdot}197-2=3{\cdot}197$ mm upward.

Say the support reaction at central support

$$=P\ \text{kN}$$

then $$3{\cdot}197\ \text{mm}=3{\cdot}197\times 10^{-3}\ \text{m}=\frac{Pl^3}{48\ EI}$$

$$\frac{P\times 10^3}{48\times 205\times 10^6\times 27081\times 10^{-8}}=3{\cdot}197\times 10^{-3}$$

$$\frac{P}{4{\cdot}8\times 2{\cdot}05\times 270{\cdot}81}=3{\cdot}197\times 10^{-3}$$

$$P=3{\cdot}197\times 10^{-3}\times 4{\cdot}8\times 2{\cdot}05\times 270{\cdot}81=8{\cdot}52\ \text{kN}$$

Reaction at outer supports

$$= \frac{\text{Total load on supports} - P}{2}$$

$$= \frac{2{\cdot}216 \times 10 - 8{\cdot}52}{2} = 6{\cdot}82 \text{ kN}$$

BM diagram

$$w = 2{\cdot}216 \text{ kN/m}$$

Reactions at the ends

$$R_A = R_B = 6{\cdot}82 \text{ kN}$$

Reaction at centre $= 8{\cdot}52$ kN

Consider a section at a distance of x from the end A.

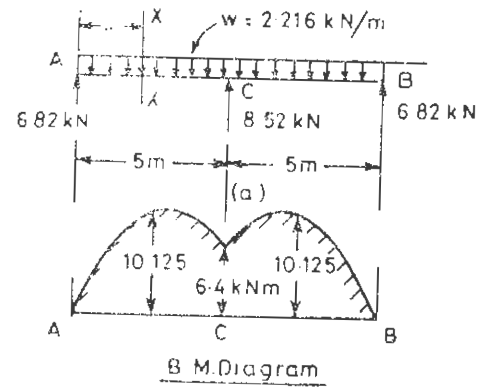


Fig. 11·38

BM at section, $M = 6{\cdot}82 \times x - \dfrac{wx^2}{2}$

$$= 6{\cdot}82 \times x - \frac{2{\cdot}216}{2} \times x^2$$

$$= 6{\cdot}82x - 1{\cdot}108\, x^2$$

$$M_A = 0$$

$$M_2 = 6{\cdot}82 \times 2 - 1{\cdot}108 \times 4 = 9{\cdot}208 \text{ kNm}$$

(This shows BM at a section 2 m away from end A)

$$M_{2\cdot5} = 6{\cdot}82 \times 2{\cdot}5 - 1{\cdot}108 \times 2{\cdot}5^2 = 10{\cdot}125 \text{ kNm}$$

$$M_4 = 6{\cdot}82 \times 4 - 1{\cdot}108 \times 16 = 9{\cdot}552 \text{ kNm}$$

$$M_5 = 6{\cdot}82 \times 5 - 1{\cdot}108 \times 25 = +6{\cdot}4 \text{ kNm}$$

Fig. 11·38 (*b*) shows the bending moment diagram.

Problem 11·13. A long steel strip of uniform width and thickness 2·5 mm is lying on a level floor. Its one end is passing over a roller of 4 cm diameter lying on the floor at one point. For what distance on either side of the roller will the strip be clear of the ground. What is the maximum stress induced in steel strip ?

$$\rho_{steel} = 7{\cdot}8 \text{ g/cm}^3$$

$$E = 2 \times 10^6 \text{ kg/cm}^2.$$

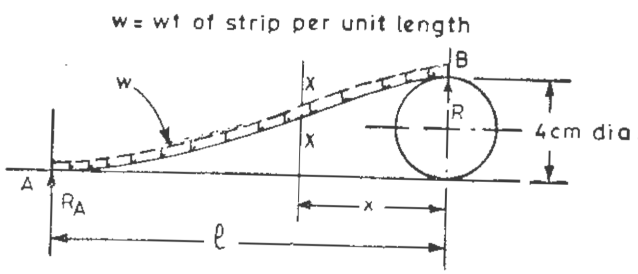


Fig. 11·39

Solution. Say the strip is clear of the ground for a distance l from the point where the roller is lying.

Reaction R at the roller is the upward force at the end B and w is the weight of the strip per unit length acting downwards on the strip.

Slope of the strip at A where the strip just leaves the ground is zero and slope of the strip at the point B where the strip smoothly passes over the roller, is also zero.

Consider a section X-X at a distance of x from the end B.

BM at the section, $M = Rx - \dfrac{wx^2}{2}$

$$EI\frac{d^2y}{dx^2} = Rx - \frac{wx^2}{2} \qquad \ldots(1)$$

Integrating equation (1)

$$EI\frac{dy}{dx} = \frac{Rx^2}{2} - \frac{wx^3}{6} + C_1$$

where C_1 is the constant of integration

At the point B, $x=0,\ \dfrac{dy}{dx}=0$

or $0 = 0 - 0 + C_1$ or $C_1 = 0$

So $$EI\frac{dy}{dx} = \frac{Rx^2}{2} - \frac{wx^3}{6} \qquad \ldots(2)$$

Integrating the equation (2) we get

$$EIy = \frac{Rx^3}{6} - \frac{wx^4}{24} + C_2 \text{ (another constant of integration)}$$

Now $y=0$ at $x=l$

or $$0 = \frac{Rl^3}{6} - \frac{wx^4}{24} + C_2$$

or $$C_2 = \frac{wl^4}{24} - \frac{Rl^3}{6}$$

Therefore $$EIy = \frac{Rx^3}{6} - \frac{wx^4}{24} + \frac{wl^4}{24} - \frac{Rl^3}{6} \qquad \ldots(3)$$

But $y = 4$ cm (roller dia) at $x=0$

$\therefore$ $$-EI \times 4 = \frac{wl^4}{24} - \frac{Rl^3}{6} \qquad \ldots(4)$$

But at $x=l,\ \dfrac{dy}{dx}=0$

So $$0 = \frac{Rl^2}{2} - \frac{wl^3}{6}$$

or $$R = \frac{wl}{3}$$

Substituting in equation (4)

$$-4\ EI=\frac{wl^4}{24}-\frac{wl}{3}\times\frac{l^3}{6}=-\frac{wl^4}{72}$$

or

$$l^4=\frac{288\ EI}{w}$$

w=weight of the strip per unit length

$$=\frac{7{\cdot}8\times b\times 0{\cdot}25}{1000}\ \text{kg/cm}$$

where b is breadth of the strip

$$I=\frac{bt^3}{12}=\frac{b\times 0{\cdot}25^3}{12}=\frac{b}{768}$$

$$E=2\times 10^6\ \text{kg/cm}^2$$

$$\therefore \qquad l^4=\frac{288\times 2\times 10^6\times b}{768\times 7{\cdot}8\times 0{\cdot}25\ b}\times 1000=3{\cdot}846\times 10^8$$

Length, $l=140$ cm

Maximum bending moment occurs at the point A,

$$M_{max}=Rl-\frac{wl^2}{2}=\frac{wl^2}{3}-\frac{wl^2}{2}=-\frac{wl^2}{6}$$

$$=\frac{7{\cdot}8\times 0{\cdot}25\ b}{1000}\times\frac{140\times 140}{6}$$

Section modulus, $Z=\dfrac{bt^2}{6}=\dfrac{b\times 0{\cdot}25^2}{6}=\dfrac{b}{96}$

Maximum stress induced in strip

$$=\frac{M_{max}}{Z}=\frac{7{\cdot}8\times 0{\cdot}25\ b\times 140\times 140\times 96}{6000\times b}$$

$$=611{\cdot}52\ \text{kg/cm}^2.$$

Problem 11·14. A long flat strip 4 cm wide and 2·5 mm thick is lying on a flat horizontal plane. One end of the strip is lifted by 20 mm from the plane by a vertical force applied at the end. The strip is so long that the other ends remains undisturbed. Calculate (*a*) the force required to lift the end (*b*) the maximum stress in the strip. Weight density of steel $=7{\cdot}8$ g/cm^3.

$$E=2\times 10^6\ \text{kg/cm}^2.$$

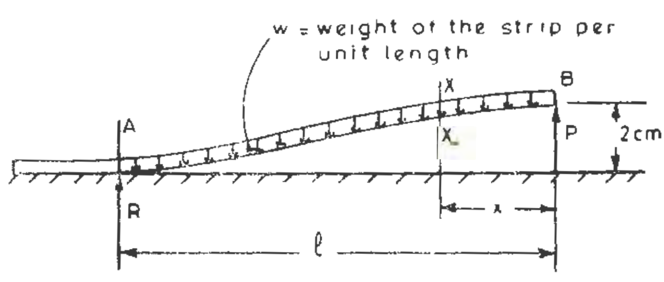


Fig. 11·40

Solution The strip is lifted from the ground by 2 cm by a force as shown in the Fig. 11·40. The slope at the point A where the strip just leaves the ground is zero.

Say the reaction at $A = R$

Taking moments of the forces about the point A

$$wl \times \frac{l}{2} = P \times l$$

or
$$P = \frac{wl}{2} \qquad \text{...(1)}$$

Consider a section X-X at a distance of x from the end B.

BM at the section, $M = -Px + \dfrac{wx^2}{2}$

or
$$EI \frac{d^2y}{dx^2} = -Px + \frac{wx^2}{2} \qquad \text{...(2)}$$

Integrating equation (2)

$$EI \frac{dy}{dx} = -\frac{Px^2}{2} + \frac{wx^3}{6} + C_1$$

But
$$\frac{dy}{dx} = 0 \quad \text{at} \quad x = l$$

$\therefore$
$$0 = -\frac{Pl^2}{2} + \frac{wl^3}{6} + C_1$$

or
$$C_1 = -\frac{wl^3}{6} + \frac{Pl^2}{2} \text{ (constant of integration)}$$

So
$$EI \frac{dy}{dx} = -\frac{Px^2}{2} + \frac{wx^3}{6} + \frac{wl^3}{6} - \frac{Pl^2}{2} \qquad \text{...(3)}$$

Integrating the equation (2) further

$$EIy = -\frac{Px^3}{6} + \frac{wx^3}{24} - \frac{wl^3}{6} x + \frac{Pl^2}{2} x + C_2$$

But $y = 0$ at $x = l$

$\therefore$
$$0 = -\frac{Pl^3}{6} + \frac{wl^4}{24} - \frac{wl^4}{6} + \frac{Pl^3}{2} + C_2 \text{ (constant of integration)}$$

or
$$C_2 = -\frac{Pl^3}{3} + \frac{wl^4}{8}$$

Therefore
$$EIy = -\frac{Px^3}{6} + \frac{wx^4}{24} - \frac{wl^3}{6} x + \frac{Pl^2}{2} x - \frac{Pl^3}{3} + \frac{wl^4}{8}$$

at $x = 0$, deflection, $y = 2$ cm

So
$$-EI \times 2 = -\frac{Pl^3}{3} + \frac{wl^4}{8} \quad \text{but } P = \frac{wl}{2}$$

$$-2EI=-\frac{wl^4}{6}+\frac{wl^4}{8}=-\frac{wl^4}{24}$$

or $$l^4=\frac{48EI}{w}$$

Strip. $b=4$ cm, $t=\cdot25$ cm $\quad I=\frac{bt^3}{12}=\frac{0\cdot0625}{12}$ cm^4

$E=2\times10^6$ kg/cm^2

$$w=\text{wt. of strip per cm length}=\frac{4\times\cdot25\times1\times7\cdot8}{1000}\text{ kg}$$

$$=\frac{7\cdot8}{1000}\text{ kg/cm run}$$

$\therefore$ $$l^4=\frac{48\times2\times10^6}{7\cdot8}\times\frac{0\cdot0625}{12}\times10^3=0\cdot641\times10^8$$

Length, $l=90$ cm

Force required to lift the load,

$$P=\frac{wl}{2}=\frac{0\cdot0078\times90}{2}=0\cdot35\text{ kg}$$

Maximum bending moment,

$$M_{max}=\frac{wl^2}{8}\text{ since the beam is simply supported at the ends}$$

$$M_{max}=0\cdot0078\times\frac{90^2}{8}=7\cdot8975\text{ kg-cm}$$

Section modulus, $$Z=\frac{bt^2}{6}=\frac{4\times0\cdot25^2}{6}=\frac{0\cdot25}{6}\text{ cm}^3$$

Maximum stress in strip$=\frac{M_{max}}{Z}=\frac{7\cdot8975}{0\cdot25}\times6$

$$=189\cdot54\text{ kg/cm}^2$$

Problem 11·15. A cantilever 3 m long carries a uniformly distributed of 20 kN/m for 1·5 m length starting from the free end. Its free end is attached to a vertical tie rod 2·4 m long and 16 mm in diameter. This tie rod is initially straight. Determine the load taken by the rod and the deflection of the cantilever.

$$E=208\text{ kN/mm}^2,\qquad I=800\text{ cm}^4$$

Solution. The Fig. 11·41 shows a cantilever 3 m long, carrying *udl* of 20 kN/m over 1·5 m length. The free end of the cantilever is attached to a tie rod.

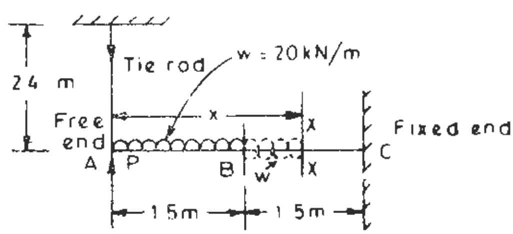


Fig. 11.41

Say reaction offered by tie rod$=P$ kN

A, Area of tie rod

$$=\frac{\pi}{4}(16)^2=201\cdot06\text{ mm}^2$$

Stress in tie rod$=\frac{P}{A}=\frac{P}{201\cdot06}$

$$E=208 \text{ kN/mm}^2$$

Length of the rod $=2{\cdot}4$ m

Extension in tie rod, $\delta l=\dfrac{P}{201{\cdot}06}\times\dfrac{2{\cdot}4}{208}=\dfrac{0{\cdot}057\,P}{1000}$ metre

Consider a section X-X at a distance of x from end A and continue the uniformly distributed load upto the section XX as shown and compensate this extra load by applying the load in the opposite direction as shown in the Fig.

B.M. at the section, $M=Px-\dfrac{wx^2}{2}+\dfrac{w}{2}(x-1{\cdot}5)^2$

where $w=20$ kN/m

or $$EI\frac{d^2y}{dx^2}=Px-10x^2+10(x-1{\cdot}5)^2 \qquad \ldots(1)$$

Integrating equation (1) we get

$$EI\frac{dy}{dx}=\frac{Px^2}{2}-\frac{10x^3}{3}+\frac{10}{3}(x-1{\cdot}5)^3+C_1$$

(constant of integration)

But $\dfrac{dy}{dx}=0$ at $x=3$ m, at fixed end

or $$0=\frac{P\times3^2}{2}-\frac{10\times27}{3}+\frac{10}{3}(1{\cdot}5)^3+C_1$$

or $$0=4{\cdot}5\,P-90+11{\cdot}25+C_1$$

$$C_1=78{\cdot}75-4{\cdot}5\,P$$

$$EI\frac{dy}{dx}=\frac{Px^2}{2}-\frac{10x^3}{3}+\frac{10}{3}(x-1{\cdot}5)^3+78{\cdot}75-4{\cdot}5\,P \qquad \ldots(2)$$

Integrating equation (2)

$$EIy=\frac{Px^3}{6}-\frac{10x^4}{12}+\frac{10}{12}(x-1{\cdot}5)^4+78{\cdot}75x-4{\cdot}5\,Px+C_2$$

where C_2 is another constant of integration

At $x=3$ m, $y=0$, at the fixed

$\therefore$ $$0=\frac{P\times3^3}{6}-\frac{10\times81}{12}+\frac{10}{12}(1{\cdot}5)^4+78{\cdot}75\times3-4{\cdot}5\,P\times3+C_2$$

$$=4{\cdot}5\,P-67{\cdot}5+4{\cdot}2188+236{\cdot}25-13{\cdot}5\,P+C_2$$

$$=-9P+172{\cdot}97+C_2$$

or $$C_2=9P-172{\cdot}97$$

Therefore $$EIy=\frac{Px^3}{6}-\frac{10x^4}{12}+\frac{10}{12}(x-1{\cdot}5)^4+78{\cdot}75x-4{\cdot}5Px+9P-172{\cdot}97$$

at the free end $x=0$, neglecting $(x-1{\cdot}5)$ term we get

$$EIy_A=9P-172{\cdot}97$$

But deflection at free end $=-\dfrac{0{\cdot}057\ P}{1000}$ (indicating downward moment)

Therefore $$-\frac{0{\cdot}057P}{1000}=\frac{9P-172{\cdot}97}{EI}$$

$$EI=208\times10^6\times800\times10^{-8}\ \text{kNm}^2=1664\ \text{kNm}^2$$

$$-\frac{0{\cdot}057P}{1000}=\frac{9P-172{\cdot}97}{1664}$$

$$(-0{\cdot}09485-9)\ P=-172{\cdot}97$$

Load taken by the rod, $P=\dfrac{172{\cdot}97}{9{\cdot}09485}=19{\cdot}0$ kN

Deflection of the cantilever

$$=-\frac{0{\cdot}057\times19{\cdot}0}{1000}\text{m} \quad \text{(indicating downward deflection)}$$

$$=-1{\cdot}083\ \text{mm.}$$

Problem 11·16. A cantilever of circular section of length l, carries a load W at its free end. The diameter of the cantilever for 2/3 rd of its length starting the free end is D while the diameter for the rest of the length is $2D$. Determine the slope and deflection at the free end. E is the Young's modulus of the material.

Solution. A cantilever of length l and diameter D for $2l/3$ and $2D$ for $l/3$ is shown in the Fig. 11·42. Load applied at the free end is W.

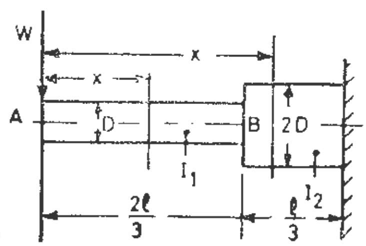


Fig. 11·42

Let us consider the cantilever in two parts AB and BC.

B.M. at any section in the portion AB

$$=-Wx$$

$$\left(x \text{ varies from 0 to } \frac{2l}{3}\right)$$

B.M. at any section in the portion BC

$$=-Wx$$

$$\left(x \text{ varies from } \frac{2l}{3} \text{ to } l\right)$$

Now $$E\frac{d^2y}{dx^2}=\frac{M}{I} \quad \text{(since } I \text{ changes from one portion to another portion)}$$

$$=-\frac{Wx}{I} \qquad ...(1)$$

Moment of inertia, $I_1=\dfrac{\pi D^4}{64}$

$$I_2 = \frac{\pi (2D)^4}{64} = \frac{\pi D^4}{4}$$

or $$I_2 = 16\ I_1 \qquad ...(2)$$

From equation (1),

$$\left| E\frac{dy}{dx} \right|_0^l = \int_0^{2l/3} -\frac{Wx}{I_1}\, dx + \int_{2l/3}^{l} -\frac{Wx}{I_2}\, dx$$

$$E[ic - i_A] = \frac{1}{I_1}\left[-\frac{Wx^2}{2} \right]_0^{2l/3} + \frac{1}{I_2}\left| -\frac{Wx^2}{2} \right|_{2l/3}^{l}$$

$$= -\frac{W}{2I_1} \times \frac{4l^2}{9} - \frac{W}{2I_2}\left(l^2 - \frac{4l^2}{9} \right) \qquad \text{But } I_2 = 16\ I_1$$

$$= -\frac{2Wl^2}{9I_1} - \frac{5}{32I_1} \times \frac{Wl^2}{9}$$

$$= -\frac{69\ Wl^2}{288\ I_1}$$

But $i_c = 0$; slope at the fixed end C is zero.

So $$-Ei_A = -\frac{69\ Wl^2}{288\ I_1}$$

or $$i_A = +\frac{23}{96}\,\frac{Wl^2}{EI_1} \qquad ...(3)$$

Deflection.

$$E\frac{d^2y}{dx^2} = -\frac{Wx}{I} \quad \text{but } I \text{ is different in two portions}$$

Multiplying both the sides by $x\ dx$ and integrating,

$$\int_0^l E\frac{d^2y}{dx^2}\cdot x\, dx = \int_0^{2l} -\frac{Wx^2dx}{I_1} + \int_{2l/3}^{l} -\frac{Wx^2dx}{I_2}$$

or $$E\left[x\frac{dy}{dx} - y \right]_0^l = \left| -\frac{Wx^3}{3I_1} \right|_0^{2l/3} + \left| -\frac{Wx^3}{3I_2} \right|_{2l/3}^{l}$$

or $$E[(l \times i_c - y_c) - (0 \times i_A - y_A)] = -\frac{8Wl^3}{81\ I_1} - \frac{3W}{3I_2}\left(l^3 - \frac{8l^3}{27} \right)$$

but $i_c = yc = 0$ at fixed end,

So $$+E\ y_A = -\frac{8Wl^3}{81\ I_1} - \frac{3Wl^3}{3I_2} \times \frac{19}{27}$$

$$= -\frac{Wl^3}{81\ I_1}\left[8 + \frac{19}{16} \right] \text{because } I_2 = 16\ I_1$$

or $$Ey_A = -\frac{147}{81\times16}\;\frac{Wl^3}{I_1}$$

Deflection at free end, $$y_A = -\frac{147}{1296}\times\frac{Wl^3}{EI_1}$$

Problem 11·17. A round tapered bar acts as a beam over a span L. The diameter at each end is D which uniformly increases to $2D$ at the centre. A load W is applied at the centre of the beam while the ends are simply supported. If E is the Young's modulus of the material, derive expression for the deflection at the centre of the beam. Compare this deflection with the deflection of a simply supported beam over span L, carrying central load W but of uniform diameter D throughout the length.

Solution. Fig. 11·43 show a beam of length L simply supported at the ends and carrying a central load. The diameter at the ends is D and at the centre it is $2D$.

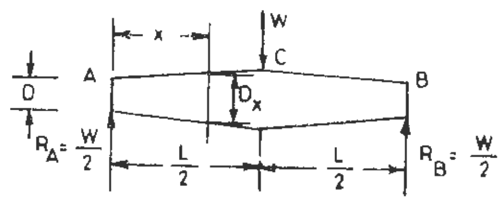


Fig. 11·43

Central load $= W$

Reactions, $R_A = R_B = \dfrac{W}{2}$

(since the beam is symmetrically loaded)

Consider a section X-X at a distance of x from the end A

B.M. at the section, $\quad M = +\dfrac{W}{2}\,x$

or $$EI_x\times\frac{d^2y}{dx^2} = \frac{W}{2}x \quad\text{where}\quad I_x = \frac{\pi}{64}(D_x)^4$$

Diameter at the section,

$$D_x = D + \frac{D}{L/2}\,.\,x = D + \frac{2D}{L}\,.\,x$$

$$= D + kx \quad\text{where}\quad k = \frac{2D}{L}$$

So $$E\,\frac{d^2y}{dx^2} = \frac{W}{2}\times\frac{x}{\pi}\times\frac{64}{D_x^4} = \frac{32}{\pi}\times\frac{Wx}{(D+kx)^4} \qquad (1)$$

Integrating $$E\,\frac{dy}{dx} = \frac{32\,W}{\pi}\int\frac{x\,dx}{(D+kx)^4}$$

$$= \frac{32\,W}{\pi}\left[\frac{x}{-3k(D+kx)^3} - \frac{1}{(-3k)(-2k)(D+kx)^2}\right] + C_1$$

where C_1 is the constant of integration.

Now $\qquad$ at $x = \dfrac{l}{2}$, $\dfrac{dy}{dx} = 0$, because of symmetry

At the centre, diameter $=2\,D$

So
$$0=\frac{32\,W}{\pi}\left[-\frac{L/2}{3k\,(2D)^3}-\frac{1}{6k^2\,(2D)^2}\right]+C_1$$

$$C_1=\frac{32\,W}{\pi}\left[\frac{L}{2\times 8D^3\times 3k}+\frac{1}{6k^2\times 4D^2}\right]$$
$$=\frac{4\,W}{\pi}\left[\frac{L}{6k\,D^3}+\frac{1}{3k^2\,D^2}\right]$$
$$=\frac{4}{3\pi}\times W\left[\frac{L}{2k\,D^3}+\frac{1}{k^2\,D^2}\right]$$

where
$$k=\frac{2D}{L}$$

Therefore
$$C_1=\frac{4\,W}{3\pi}\left[\frac{L}{2D^3}\times\frac{L}{2D}+\frac{1}{D^2}\times\frac{L^2}{4D^2}\right]$$
$$=\frac{4W}{3\pi}\times\frac{2L^2}{4D^4}=\frac{2}{3\pi}\quad\frac{WL^2}{D^4}$$

Therefore
$$E\,\frac{dy}{dx}=-\frac{32\,Wx}{3\pi k\,(D+kx)^3}\quad\frac{32\,W}{\pi 6k^2\,(D+kx)^2}+\frac{2}{3\pi}\quad\frac{WL^2}{D^4}\qquad\ldots(2)$$

Integrating equation (2)
$$Ey=-\frac{32}{3\pi k}\left[\int\frac{xdx}{(D+kx)^3}\right]-\frac{32\,W}{6\pi k^2}\int\frac{1}{(D+kx)^2}+\frac{2}{3\pi}\quad\frac{WL^2}{D^4}\;x$$
$$Ey=-\frac{32\,W}{3\pi k}\left[\frac{x}{-2k\,(D+kx)^2}-\frac{1}{(-2k)\,(-k)\,(D+kx)}\right]$$
$$-\frac{32\,W}{6\pi k^2}\times\frac{1}{(-1k)(D+kx)}+\frac{2}{3\pi}\times\frac{WL^2x}{D^4}+C_2$$

where C_2 is another constant of integration
$$Ey=+\frac{16\,Wx}{3\pi k^2\,(D+kx)^2}+\frac{16\,W}{3\pi k^3\,(D+kx)}+\frac{16}{3\pi k^3\,(D+kx)}$$
$$+\frac{2}{3\pi}\quad\frac{WL^2x}{D^4}+C_2$$
$$=\frac{16\,Wx}{3\pi k^2\,(D+kx)^2}+\frac{32\,W}{3\pi k^3\,(D+kx)}+\frac{2}{3\pi}\quad\frac{WL^2x}{D^4}+C_2$$

Moreover at $x=0$, deflection $y=0$

So
$$0=0+\frac{32\,W}{3\pi k^3\,D}+C_2$$
$$C_2=-\frac{32\,W}{3\pi k^3\,D}$$

So
$$Ey=\frac{16\,Wx}{3\pi k^2\,(D+kx)^2}+\frac{32\,W}{3\pi k^3\,(D+kx)}+\frac{2}{3\pi}\quad\frac{WL^2x}{D^4}-\frac{32\,W}{3\pi k^3\,D}$$

Deflection at the centre

$$y=y_c \quad \text{at} \quad x=\frac{L}{2}$$

$$D+k\,\frac{L}{2}=2D$$

$$Ey_c=\frac{16\,W}{3\pi}\times\frac{L}{2}\times\frac{L^2}{4D^2\,(2D)^2}+\frac{32\,W}{3\pi\,(2D/L)^3\,(2D)}$$

$$+\frac{2}{3\pi}\times\frac{WL^2}{D^4}\times\frac{L}{2}-\frac{32\,W}{3\pi k^3\,D}$$

$$=\frac{WL^3}{6\pi D^4}+\frac{32\,WL^3}{3\pi\times 16D^4}+\frac{2}{3\pi}\times\frac{WL^3}{2D^4}-\frac{32\,W}{3\pi D}\times\frac{L^3}{8D^3}$$

$$=\frac{WL^3}{3\pi D^4}\left[\frac{1}{2}+2+1-4\right]$$

$$=-\frac{WL^3}{6\pi D^4}$$

$$y_c=-\frac{WL^3}{6\pi\,ED^4}.$$

Problem 11·18. A round tapered cantilever of length 2 metres supports a load W at its free end. The diameter of the cantilever at the free end is 8 cm and at the fixed end diameter is 16 cm. What is the maximum value of W if the maximum stress developed in the section is not to exceed 800 kg/cm². Under this load what is the deflection at the free end.

$$E=2\times 10^6 \text{ kg/cm}^2$$

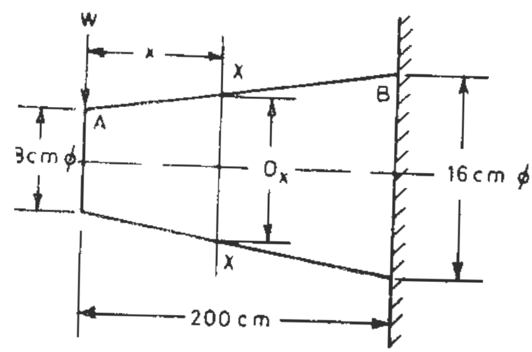


Fig. 11·44

Solution. Fig. 11·44 shows the cantilever with load W at the free end.

Maximum bending moment occurs at the fixed end and

$$M_{max}=Wl=W\times 200 \text{ kg-cm}$$

Section modulus, Z at the fixed end

$$=\frac{\pi D^3}{32}=\frac{\pi\times 16^3}{32}=402{\cdot}125 \text{ cm}^3$$

$$M_{max}=f\,Z$$

where $f=800$ kg/cm² allowable stress

$$200\,W=800\times 402{\cdot}125 \text{ cm}^3$$

$$W=4\times 402{\cdot}125=1608{\cdot}5 \text{ kg}$$

Consider a section X-X at a distance of x from the end A.

Diameter at the section,

$$D_x=8+\frac{16-8}{200}\cdot x$$

$$=(8+0.04\,x)\text{ cm}$$

Moment of inertia at the section,

$$I_x=\frac{\pi}{64}(8+0.04x)^4$$

BM at the section, $M=-Wx$

So $$EI_x\frac{d^2y}{dx^2}=-Wx \qquad \ldots(1)$$

or $$E\frac{d^2y}{dx^2}=-\frac{Wx}{\frac{\pi}{64}(8+0.04x)^4}=-\frac{64\,Wx}{\pi(8+0.04x)^4} \qquad \ldots(1)$$

Integrating equation (1),

$$E\frac{dy}{dx}=-\frac{64\,W}{\pi}\int\frac{xdx}{(8+0.04\,x)^4}+C_1 \text{ (constant of integration)}$$

Let us determine

$$\int\frac{x\,dx}{(8+0.04x)^4}=\int\frac{x\,dx}{(8+kx)^4} \quad \text{where } k=0.04$$

$$=x\int\frac{dx}{(8+kx)^4}-\int\int\frac{dx}{(8+kx)^4}=\frac{x}{-3k(8+kx)^3}+\int\frac{dx}{3k(8+kx)^3}$$

$$=-\frac{x}{3k(8+kx)^3}-\frac{1}{3k\times 2k\,(8+kx)^2}$$

$$=-\frac{x}{3k(8+kx)^3}-\frac{1}{6k^2(8+kx)^2}$$

So $$E\frac{dy}{dx}=\frac{64\,W}{\pi}\times\frac{x}{3k(8+kx)^3}+\frac{64\,W}{\pi\,(6k^2)(8+kx)^2}+C_1$$

Now at $x=200$ cm, $y=0$

$k=0.04,\quad k\times 200=8$

Therefore $$0=\frac{64\,W\times 200}{\pi\times 3k\,(16)^3}+\frac{64\,W}{\pi\times 6k^2\times 16^2}+C_1$$

$$=8.2893\,W+8.2893\,W+C_1$$

or $$C_1=-16.57860\,W$$

$\therefore$ $$E\frac{dy}{dx}=\frac{64\,W}{\pi\times 3k}\times\frac{x}{(8+kx)^3}+\frac{64\,W}{\pi\times 6k^2}\times\frac{1}{(8+kx)^2}-16.57860\,W$$

Let us put $\frac{64\ W}{3\pi k}=K_1$ a constant

$\frac{64\ W}{6\pi k^2}=K_2$ another constant

$$E\ \frac{dy}{dx}=K_1\ \frac{x}{(8+kx)^3}+K_2\ \frac{1}{(8+kx)^2}-16{\cdot}5786\ W \qquad ...(2)$$

Integrating equation (2)

$$Ey=K_1\int\frac{x\ dx}{(8+kx)^3}+K_2\int\frac{dx}{(8+kx)^2}-16{\cdot}5786\ Wx+C_2$$

(a constant of integration)

$$=K_1x\int\frac{dx}{(8+kx)^3}-K_1\int\left[\int\frac{dx}{(8+kx)^3}\right]dx+K_2\int\frac{dx}{(8+kx)^2}-16{\cdot}5786\ Wx+C_2$$

$$=-\frac{K_1\ x}{2k(8+kx)^2}-K_1\int\frac{dx}{-2k(8+kx)^2}+K_2\ \frac{1}{-k(8+kx)}-16{\cdot}5786\ Wx+C_2$$

$$=-\frac{K_1\ x}{2k(8+kx)^2}+\frac{K_1}{2k(-k)}\times\frac{1}{(8+kx)}-\frac{K_2}{k(8+kx)}-16{\cdot}5786\ Wx+C_2$$

$$=-\frac{K_1\ x}{2k(8+kx)^2}-\frac{K_1}{2k^2\ (8+kx)}-\frac{K_2}{k(8+kx)}-16{\cdot}5786\ Wx+C_2$$

Now at $x=200$ cm, $y=0$,

$8+kx=16$ cm at $x=200$ cm Putting this condition

$$0=-\frac{K_1\times200}{2k\ (16)^2}-\frac{K_1}{2k^2\ (16)}-\frac{K_2}{k\ (16)}-16{\cdot}5786\times200\ W+C_2$$

$$C_2=9{\cdot}7656\ K_1+19{\cdot}5312\ K_1+1{\cdot}5625\ K_2+3315{\cdot}72\ W$$

$$=(29{\cdot}2968)\times\frac{64\ W}{3\pi k}+\frac{1{\cdot}5625\times64\ W}{6\pi k^2}+3315{\cdot}22\ W$$

$$=4973{\cdot}5676\ W+3315{\cdot}7202\ W+3315{\cdot}22\ W$$

$$=11604{\cdot}508\ W$$

At $x=0,\quad y=y_{max}$

$$E\ y_{max}=-\frac{K_1.\times0}{2k(8)^2}-\frac{K_1}{2k^2\ (8)}-\frac{K_2}{k\ (8)}-16{\cdot}5786W\times0+11604{\cdot}508\ W$$

$$=-\frac{K_1}{16\ k^2}-\frac{K_2}{8k}+11604{\cdot}508\ W$$

$$=-\frac{64\ W}{3\pi k\times16\ k^2}-\frac{64\ W}{6\pi k^2\times8k}+11604{\cdot}508\ W$$

$$=-6631{\cdot}44\ W-6631{\cdot}44\ W+11604{\cdot}508\ W$$

$$=-1658{\cdot}37\ W$$

Deflection, $y_{max}=-\frac{1658{\cdot}37\times1608{\cdot}5}{2\times10^6}=-1{\cdot}333$ cm at the free end

Problem 11·19. A cantilever 2 metres long is of I section of depth 100 mm and $I=168$ cm⁴. A load of 20 kg is dropped at the free end of the cantilever from a height of 20 cm. What is the instantaneous deflection at the free end of the cantilever. What is the maximum stress developed in the cantilever ?

$$E=2\times 10^6 \text{ kg/cm}^2.$$

Solution. Say the instantaneous deflection at the free end of the cantilever

$$=\delta_i$$

Say P is the equivalent gradually applied load

then $$\delta_i=\frac{Pl^3}{3EI}$$

or $$P=\frac{\delta_i\times 3EI}{l^3}$$

$$P=\delta_i\times\frac{3\times 2\times 10^6\times 168}{200\times 200\times 200}$$

$$=126\ \delta_i \qquad ...(1)$$

Falling load, $W=20$ kg

Height through which load falls,

$$h=20 \text{ cm}$$

Now $$W\,(h+\delta_i)=\frac{1}{2}\ P\ \delta_i=\frac{1}{2}\times 126\ \delta_i^2$$

$$20\,(20+\delta_i)=63\ \delta_i^2$$

or $$63\ \delta_i^2-20\ \delta_i-400=0$$

$$\delta_i=\frac{20+\sqrt{400+63\times 4\times 400}}{126}=\frac{20+318{\cdot}12}{126}$$

$$=2{\cdot}6835 \text{ cm}$$

Equivalent load, $P=126\times 2{\cdot}6835=338{\cdot}12$ kg

M_{max}, Maximum bending moment

$$Pl=200\times 338{\cdot}12 \text{ kg-cm}$$

$$I=168 \text{ cm}^4$$

Depth of the section, $d=10$ cm

Maximum stress $$=\frac{M_{max}}{I}\times\frac{d}{2}=\frac{200\times 338{\cdot}12}{168}\times 5$$

$$=2012{\cdot}6 \text{ kg/cm}^2$$

Problem 11·20. A cantilever 6 m long is supported at the free end by a prop at the same level as the fixed end. A uniformly distributed load of 1 tonne/metre run is carried on the cantilever for 3 m length starting from the free end. Determine the reaction P at the prop and deflection at the centre of the cantilever. EI is the flexural rigidity of the cantilever.

Solution. Consider a section X-X at a distance of x from the end A. Extend the uniformly distributed load upto the section XX and apply load in the opposite direction also from the centre of the cantilever upto the section XX, so as to maintain equilibrium, and in reality the total load and its distribution is not changed.

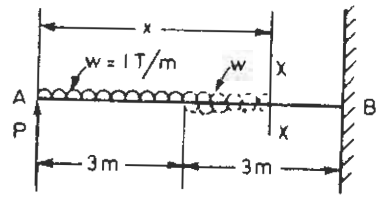


Fig. 11·45

B.M. at any section X-X,

$$M = Px - \frac{wx^2}{2} + \frac{w(x-3)^2}{2}$$

or $$EI\frac{d^2y}{dx^2} = Px - 0{\cdot}5\,x^2 + 0{\cdot}5\,(x-3)^2 - \quad (1) \text{ as } w = 1\ T/\text{m}.$$

Integrating the equation (1)

$$EI\frac{dy}{dx} = \frac{Px^2}{2} - \frac{x^3}{6} + \frac{(x-3)^3}{6} + C_1 \qquad \text{(constant of integration)}$$

at $x=6$ m, fixed end, $\frac{dy}{dx}=0$

Therefore $$0 = \frac{P\times 6^2}{2} - \frac{6^3}{6} + \frac{(6-3)^3}{6} + C_1$$

or $$C_1 = 36 - 4{\cdot}5 - 18\,P = 31{\cdot}5 - 18\,P$$

$$EI\frac{dy}{dx} = \frac{Px^2}{2} - \frac{x^3}{6} + \frac{(x-3)^3}{6} + (31{\cdot}5 - 18P) \qquad ...(2)$$

Integrating the equation (2),

$$EIy = \frac{Px^3}{6} - \frac{x^4}{24} + \frac{(x-3)^4}{24} + (31{\cdot}5 - 18\,P)x + C_2$$

(constant of integration)

at $x=6$ m, $y=0$

Therefore $$0 = \frac{P\times 6^3}{6} - \frac{6^4}{24} + \frac{(6-3)^4}{24} + (31{\cdot}5 - 18P)6 + C_2$$

$$0 = 36P - 54 + 3{\cdot}375 + 189 - 108\,P + C_2$$

or $$C_2 = 72\,P - 138{\cdot}375\,P$$

So $$EIy = \frac{Px^3}{6} - \frac{x^4}{24} + \frac{(x-3)^4}{24} + (31{\cdot}5 - 18\,P)x + (72\,P - 138{\cdot}375) \qquad ...(3)$$

Now at $x=0$, $y=0$, substituting in equation (3) we get

$$0 = 0 - 0 + 0 + 0 + 72P - 138{\cdot}375$$

or $$P = \frac{138{\cdot}375}{72} = 1{\cdot}921875 \text{ Tonnes}$$

Equation for the deflection becomes

$$EIy=\frac{Px^3}{6}-\frac{x^4}{24}+\frac{(x-3)^4}{24}+(31{\cdot}5-34{\cdot}59375)\,x$$

$$=\frac{Px^3}{6}-\frac{x^4}{24}+\frac{(x-3)^4}{24}-3{\cdot}09375\,x$$

Deflection at the centre of the cantilever *i.e.*, at $x=3$ m

$$EIy_c=1{\cdot}921875\times\frac{3^3}{6}-\frac{3^4}{24}+0-3{\cdot}09375\times3$$

$$=8{\cdot}6484375-3{\cdot}375-9{\cdot}28125$$

$$-4{\cdot}007813$$

$$y_c=\frac{4{\cdot}007813}{EI}.$$

Problem 11·21. A beam of length $3a$ simply supported over a span of a, with equal overhang on both the sides. It carries a uniformly distributed load w per unit length over the overhang portion of both the sides. Determine,

(*i*) slope and deflection at the overhang end

(*ii*) deflection at the centre of the beam

Use moment area method. EI is the flexural rigidity of the beam.

Solution. The loading diagram of the beam is shown in the Fig. 11·46.

Total load on beam$=2wa$

Beam is symmetrically loaded so the reactions, $R_B=R_D=wa$

B.M. diagram

$$M_B=M_D=-\frac{wa^2}{2}$$

Since there is no load on the portion B to D. The B.M. will remain constant between B and D. The B.M. diagram upto C can be considered into two parts A_1 and A_2 as shown.

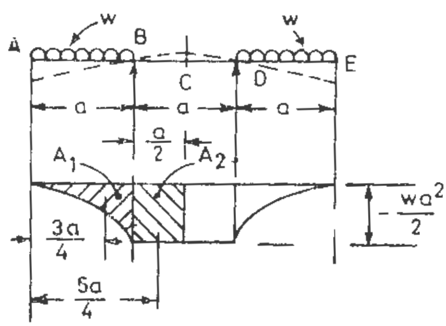


Fig. 11·46

Now area $$A_2=-\frac{wa^2}{2}\times\frac{a}{2}=-\frac{wa^3}{4}$$

Now $EI(i_C-i_B)=$area of the B.M. diagram between B and C

$$-EI\,i_B=-\frac{wa^3}{4}\text{ as slope } i_c=0 \text{ due to symmetrical loading}$$

or $$i_B=+\frac{wa^3}{4EI}$$

Similarly $\quad EI(i_B - i_A)$ = area of B.M. diagram between B and A

$$= -\frac{wa^2}{2} \times \frac{a}{3} = -\frac{wa^3}{6}$$

or
$$EI\, i_B - EI\, i_A = -\frac{wa^3}{6}$$

But
$$EI\, i_B = +\frac{wa^3}{4}$$

So
$$-EI\, i_A = -\frac{wa^3}{6} - \frac{wa^3}{4} = -\frac{5}{12}\, wa^3$$

or
$$i_A = +\frac{5}{12} \times \frac{wa^3}{EI}$$

Deflection at A

$EI[(x_B \,.\, i_B - y_B) - (x_A \,.\, i_A - y_A)$ = first moment of the area A_1 about the end A

$$= -\frac{wa^3}{6} \times \frac{3a}{4} = -\frac{wa^4}{8}$$

where
$$x_B = a, \quad y_B = 0$$

$$i_B = +\frac{wa^3}{4EI}, \quad x_A = 0$$

$$EI\left[\left(a \,.\frac{wa^3}{4EI}\right) - 0 - 0 \times i_A + y_A\right] = -\frac{wa^4}{8}$$

$$\frac{wa^4}{4} + y_A . EI = -\frac{wa^4}{8}$$

Deflection at A,
$$y_A = \frac{1}{EI}\left[-\frac{wa^4}{8} - \frac{wa^4}{4}\right] = -\frac{3wa^4}{8EI}$$

Deflection at C

$EI[(x_C \times i_C - y_C) - (x_B \times i_B - y_B)]$ = moment of the area A_2 about the end A

$$EI\left[\left(\frac{3}{2}\, a \times 0 - y_C\right) - \left(a \times \frac{wa^3}{4EI} - 0\right)\right] = -\frac{wa^3}{4} \times \left(\frac{5a}{4}\right)$$

$$-EI\, y_C - \frac{wa^4}{4} = -\frac{5}{16}\, wa^4$$

$$-y_C = \frac{1}{EI}\left[\frac{wa^4}{4} - \frac{5}{16}\, wa^4\right] = -\frac{wa^4}{16EI}$$

or $\quad$ Deflection at C,
$$y_C = \frac{wa^4}{16EI}$$

Problem 11·22. A beam 4 m long is simply supported at its ends. It carries a uniformly distributed load of 2 tonnes/metre over 2 m length starting from one end. Determine slope at ends and deflection at the centre of the beam, using conjugate beam method.

$$E=2000 \text{ tonnes/cm}^2, \ I=4000 \text{ cm}^4$$

Solution. Fig. 11·47 (*a*) shows the loading diagram of the beam of 4 m length with *udl* of $2T$/m over AC, 2 metres length. For support reactions, let us take moments of the forces about the point A.

$$2\times2\times1=4R_B$$

Reaction, $R_B=1$ Tonnes

Total vertical load $=2\times2$ Tonnes

Reaction, $R_A=4-1=3$ T

B.M. diagram

$M_A=0$

M_1, B.M at a distance of 1 m from end A

$$=3\times1-\frac{wx^2}{2}=3-\frac{2}{2}\times1^2=2 \text{ Tm}$$

$$M_C=3\times2-\frac{2}{2}\times4=6-4=2 \text{ Tm}$$

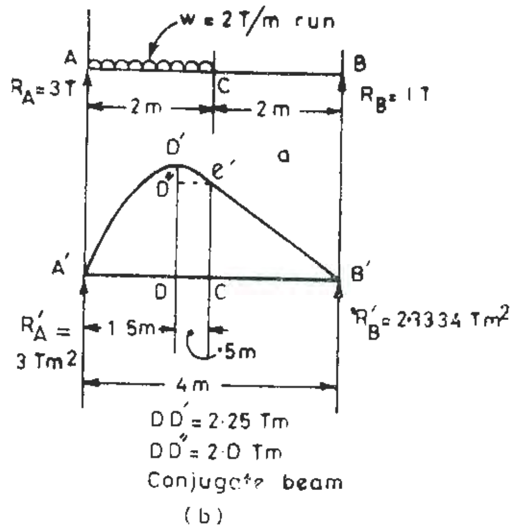


Fig. 11·47

M_{max} occurs between A and C,

$$M_x=R_A \cdot x-\frac{wx^2}{2} \quad \text{(a parabolic curve)}$$

$$=3x-x^2$$

Putting $\quad \dfrac{dM_x}{dx}=3-2x=0$ for maximum B.M.

$$x=1·5 \text{ m}$$

So $\quad M_{1·5}=3\times1·5-\dfrac{2}{2}\times1·5^2=2·25$ Tm (say occurs at point D)

Fig. 11·47(*b*) shows the conjugate beam with a loading diagram $A'D'C'B'$. Before we determine R_A' and R_B' for the conjugate beam, let us note the properties of a parabolic curve.

Fig. 11·48 shows a parabolic curve *ef*, covering an area *efg*, breadth B and Height H.

area under the parabola, $a_1=\dfrac{2}{3}BH$

area a_2, above the parabolic curve

$$=\frac{1}{3}BH$$

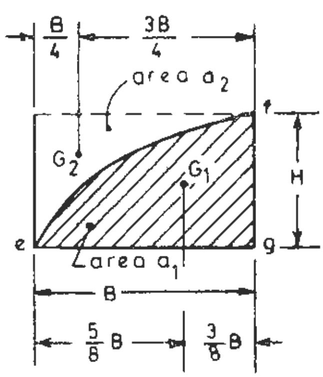


Fig. 11·48

C.G. of a_1 lies at a distance of $\frac{5}{8}$ B from C or $\frac{3}{8}B$ from g.

Similarly C.G. of area a_2 lies at a distance of $\frac{B}{4}$ from C or $\frac{3B}{4}$ from f.

Taking moments about the point A.

$$1{\cdot}5\times2{\cdot}25\times\frac{2}{3}\left(\frac{5}{8}\times1{\cdot}5\right)+(2{\cdot}25-2)(0{\cdot}5)\left(1{\cdot}5+\frac{3}{8}\times0{\cdot}5\right)\times\frac{2}{3}$$

$$+2\times0{\cdot}5\times\left(1{\cdot}5+\frac{0{\cdot}5}{2}\right)+2\times2\times\frac{1}{2}\left(2+\frac{2}{3}\right)=R_B'\times4$$

or
$$2{\cdot}109375+1{\cdot}6875\times{\cdot}0833+1{\cdot}75+5{\cdot}3333=4R_B'$$

$$2{\cdot}110+0{\cdot}1406+1{\cdot}75+5{\cdot}333=4R_B'$$

$$R_B'=\frac{9{\cdot}3336}{4}=2{\cdot}3334\ \text{Tm}^2$$

$\therefore$
$$R_A'=2{\cdot}25\times1{\cdot}5\times\frac{2}{3}+0{\cdot}25\times0{\cdot}5\times\frac{2}{3}+1+2-2{\cdot}334$$

$$=2{\cdot}250+{\cdot}0833+3-2{\cdot}334=3{\cdot}000\ \text{Tm}^2$$

Slope at the end A, $\quad i_A=-\frac{R_A'}{EI}$

$$E=2000\ \text{T/cm}^2=2000\times10^4\ \text{T/m}^2$$

$$I=4000\ \text{cm}^4=4000\times10^{-8}\ \text{m}^4$$

$$EI=800\ \text{Tm}^2$$

So $\quad i_A=-\frac{3}{800}=-0{\cdot}00375\ \text{radian}=-0{\cdot}215^\circ$

Slope at end B, $\quad i_B=+\frac{R_B'}{EI}=\frac{2{\cdot}3334}{800}$

$$=+0{\cdot}0029\ \text{radian}=+0{\cdot}167^\circ$$

Moment Mc' from conjugate beam diagram

$$=R_B'\times2-\frac{2\times2}{2}\times\frac{2}{3}$$

$$=2{\cdot}3334\times2-1{\cdot}3333=3{\cdot}3335\ \text{Tm}^3$$

Deflection at the point C,

$$y_C=\frac{Mc'}{EI}=\frac{3{\cdot}3335}{800}=0{\cdot}00417\ \text{m}=0{\cdot}417\ \text{cm}$$

SUMMARY

1. For a beam or a cantilever subjected to bending moment M,

$$EI\frac{d^2y}{dx^2}=M$$

where EI is the flexural rigidity and section is uniform

and $$E\frac{d^2y}{dx^2}=\frac{M}{I} \text{ (with variable section).}$$

2. For a beam of length l, simply supported at ends and carrying a load W at its middle

Slope at ends $=\pm\frac{Wl^2}{16\,EI}$

Maximum deflection at centre$=\frac{Wl^3}{48\,EI}$.

3. For a beam of length l, simply supported at its ends and carrying a uniformly distributed load w per unit length throughout its length

Slope at the ends $=\pm\frac{Wl^3}{24\,EI}$

Maximum deflection at centre$=\frac{5}{384}\times\frac{wl^4}{EI}$.

4. For a cantilever of length l, carrying a concentrated load W at free end,

Slope at free end$=\frac{Wl^2}{2EI}$, deflection at free end$=\frac{Wl^3}{3EI}$.

5. For a cantilever of length l, carrying a uniformly distributed load w per unit length throughout its length

Slope at free end $=\frac{wl^3}{6EI}$

Deflection at free end $=\frac{wl^4}{8EI}$.

6. For a beam of length $l=a+b$, simply supported at ends carrying a load W at a distance of a from one end,

Slope at one end $=-\frac{Wab\,(a+2b)}{6\,EI\,l}$

Slope at the other end $=+\frac{Wab\,(2a+b)}{6\,EI\,l}$

Deflection under the load$=-\frac{Wa^2b^2}{3\,EI\,l}$.

7. If a load W is allowed to fall through a height h on a beam or cantilever at a particular point and δ_i is the maximum instantaneous deflection produced then

$$W\,(h+\delta_i)=\tfrac{1}{2}\,P\,\delta_i$$

where P is the equivalent static load which when applied gradually produces deflection δ_i.

8. If a cantilever of length l, carrying uniformly distributed load w per unit length is propped at the free end such that the free end is brought to the level of the fixed end, then reaction at the prop is $3wl/8$.

9. If a beam of length l simply supported at its ends carrying uniformly distributed load w per unit length is propped at the centre, so that centre of the beam is brought to the level of the ends, then reaction at the prop is $5wl/8$.

10. If a bending moment diagram is plotted for a beam carrying transverse loads and two sections are considered at distances of x_1 and x_2 from one end

$$EI\,(i_2-i_1)=\text{area }(a)\text{ of the BM diagram between } X_2 \text{ and } X_1$$

$$EI\,[(x_2\,i_2-y_2)-(x_1\,i_1-y_1)]=A\bar{x}$$

where $\bar{x}$=distance of CG of area R, from one end.

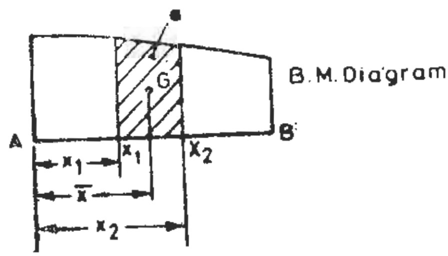


Fig. 11·49

11. If a bending moment diagram is plotted for a beam carrying transverse loads, then bending diagram shown over the length of the beam as a variable distributed load, is called a conjugate beam. The reactions at the ends obtained for the conjugate beam, divided by EI give slope at the ends. The bending moments at any section obtained for the conjugate beam divided by EI gives deflection at that section.

MULTIPLE CHOICE QUESTIONS

1. A simply supported beam of length 2 metres carries a concentrated load 3 tonnes at its centre. If $EI=5000$ tonne-metre^2 for the beam, the maximum deflection in the beam is

   (a) 1×10^{-3} m (b) 5×10^{-4} m

   (c) 2×10^{-4} m (d) 1×10^{-4} m.

2. A beam is simply supported at its ends over a span l. If the load applied at the middle of the beam is W, the minimum slope in the beam is

   (a) $\dfrac{Wl^2}{16\,EI}$ (b) $\dfrac{Wl^2}{3\,EI}$

   (c) $\dfrac{Wl^2}{2\,EI}$ (d) None of the above.

3. A beam simply supported at its ends over a span of 4 metres carries a uniformly distributed load of 1·5 tonnes/metre run throughout its length. If $EI=2500$ tonne-metre^2, the maximum deflection in the beam is

   (a) 0·2 mm (b) 0·8 mm

   (c) 1.6 mm (d) 2·00 mm.

4. A beam of length l, simply supported at its ends carries a uniformly distributed load w throughout its length. The centre of the beam is propped so that centre is brought to the level of ends. The reaction at the prop is

(a) wl (b) $0{\cdot}5\ wl$

(c) $0{\cdot}625\ wl$ (d) $0{\cdot}75\ wl$.

5. A cantilever of length L, carries a load W at the middle of its length. If EI is the flexural rigidity of the cantilever, the deflection under the load is

(a) $\dfrac{Wl^3}{24\ EI}$ (b) $\dfrac{Wl^3}{16\ EI}$

(c) $\dfrac{Wl^3}{3\ EI}$ (d) $\dfrac{Wl^3}{2\ EI}$.

6. A cantilever of length L, carries a load W at its middle. The slope at the middle of the cantilever is θ. The slope at the free end is

(a) $2\ \theta$ (b) $1{\cdot}5\ \theta$

(c) θ (d) $0{\cdot}5\ \theta$.

7. A beam of length $(a+b)$ simply supported at its ends carries a concentrated load W at a distance of a from one end. If EI is the flexural rigidity of the beam, the deflection under the load is

(a) $\dfrac{Wa^2b^2}{4\ (a+b)\ EI}$ (b) $\dfrac{Wa^2b^2}{3\ (a+b)\ EI}$

(c) $\dfrac{Wa^2}{6(a+b)\ EI}$ (d) $\dfrac{Wb^2}{6\ (a+b)\ EI}$.

8. A beam of length 6 m, carries a concentrated load W at its centre, such that BM at the centre of the beam is 6 tonne metre. If EI is the flexural rigidity of the beam, then deflection at the centre is

(a) $\dfrac{36}{EI}$ (b) $\dfrac{18}{EI}$

(c) $\dfrac{9}{EI}$ (d) None of the above.

9. A cantilever of length 4 metres carries uniformly distributed load w throughout its length. If the maximum bending moment in the cantilever is 8 tonne-metres, and EI is its flexural rigidity, the slope at the free end of the cantilever is

(a) $\dfrac{64}{3\ EI}$ (b) $\dfrac{32}{3\ EI}$

(c) $\dfrac{16}{6\ EI}$ (d) None of the above.

10. A cantilever of length 4 metres carries a uniformly distributed load of 2 kN per metre run throughout its length. The free end of the cantilever is propped such that the level of the free end is the same as that of fixed end. The reaction offered by the prop is

(a) 8 kN (b) 6 kN

(c) 3 kN (d) 2 kN

ANSWERS

1. (d)	2. (d)	3. (d)	4. (c)	5. (a)
6. (c)	7. (b)	8. (b)	9. (b)	10. (c).

EXERCISES

11·1. A beam *ABCD*, 6 metres long, simply supported at ends *A* and *D* carries concentrated loads of 2 kN and 5 kN at points *B* and *C*. Points *B* and *C* are 2 metres away from ends *A* and *D* respectively. Determine

(*i*) deflection under the loads of 4 kN and 5 kN

(*ii*) maximum deflection and its position. $E=2\times10^5$ N/mm², $I=3600$ cm⁴

[**Ans.** $-0·315$ cm, $-0·333$ cm, $y_{max}=0·373$ cm, at a distance of 3·0945 m from end *A*]

11·2. A cantilever of symmetrical cross section of length 4 metres carries a load of 30 kN at its free end. If $I=32000$ cm⁴ and depth of the section is 36 cm, determine the deflection at the free end. $E=200$ kN/mm².

What is the maximum rate of uniformly distributed load which the beam can carry (in addition to the concentrated load) over 2 m length starting from the fixed end if

(*a*) stress due to bending is not to exceed 100 N/mm²

(*b*) if the deflection at the free end is not to exceed 14 mm.

[**Ans.** 10 mm, (*a*) 28·88 kN/m (*b*) 54·857 kN/m]

11·3. A beam 6 m long, is supported at one end and at a distance of 1·5 m from the other end. It carries a concentrated load of 80 kN at over hanging end and a uniformly distributed load of 80 kN/m over a length of 4·5 m commencing from the overhanging end. Determine deflection and slope at the overhanging end of the beam.

$EI=15\times10^{12}$ N mm² [**Ans.** 16·8 mm, 0·014 radian]

11·4. A beam 6 m long, hinged at one end and is supported over a span of 4 m, with an overhang of 2 m. It carries a load 4 tonnes at the free end and a uniformly distributed load of 2 T/m run over a distance of 2 m starting from a point 2 m from the hinged end. Determine the deflection under the concentrated load.

$E=2000$ T/cm², $I=3600$ cm⁴ [**Ans.** 0·9028 cm]

11·5. A beam *AB*, 6 metres long is hinged at both the ends. A clockwise turning moment of 6 Tonne-metres is applied at a point *C* of the beam. Point *C* is at a distance of 4 metres from the end *A*. Determine the slope and deflection at the point *C*.

$E=2000$ tonnes/cm², $I=8000$ cm⁴ [**Ans.** $-0·143°$, $+3·33$ mm]

11·6. A propped cantilever of length l is fixed at one end and freely supported at the other end. The cantilever is subjected to a couple M in the vertical plane about an axis $l/2$ from one end. Determine the reaction at the prop and moment at the fixed end.

$$\left[\textbf{Ans.}\quad \frac{9}{8}\,\frac{M}{l}\,,\ \frac{M}{8}\right]$$

11·7. A vertical pole 4 m high carries a concentrated load of 80 kN inclined at an angle 45° to the axis of the pole. The pole is of uniform round section throughout. A pull *P* is applied at an angle of 45° to the axis of the pole at a distance of 2 m from the base. Determine the magnitude of *P* so that the deflection at the top of the pole is zero. Neglect the effect of axial forces in the pole. [**Ans.** 365·72 kN]

11·8. A beam 6 m long simply supported at its ends carries a uniformly increasing distributed load throughout its length. The loading rate is zero at one end and increases to 30 kN per metre at the other end. Determine (*i*) slope at the ends, (*ii*) deflection at the centre of the beam.

$E=200$ kN/mm², $I=5131{\cdot}6$ cm⁴ [**Ans.** $-0{\cdot}7°$, $+1{\cdot}2°$, 2·46 cm]

11·9. A horizontal steel beam 4 metres long carries a uniformly distributed load of 2 kN per metre run throughout its length. The beam is supported by 3 vertical steel rods, each 2 metres long, one at each end and one in the middle. The diameter of the end rods is 5 mm and that of the central rod is 8 mm. Calculate the deflection at the centre of the beam below its end points and the stress in each tie rod.

$E=200$ kN/mm², I for the beam $=750$ cm⁴

[**Ans.** 0·605 mm, 77·04 N/mm² (outer rods), 99·07 N/mm² (middle rod)]

11·10. A cantilever 4 m long, carries a uniformly distributed load of 1 tonne/metre run throughout its length. It is propped at a distance of 2·4 m from the fixed end. The reaction offered by the prop is 4 tonnes. Determine the ratio of the deflections at the free end of the propped cantilever and that of unpropped cantilever. [**Ans.** $-0{\cdot}152$]

11·11. A circular steel pipe 50 cm bore and 52 cm outside diameter is supported at each end and at the middle on a span of 10 metres. When the pipe is full of water the middle support sinks by 3 mm below the ends. Find the load on each support and draw the B.M. diagram.

$\rho_{steel}=7{\cdot}8$ g/cc, $\rho_{water}=1$ g/cc, $E=2000$ tonnes/cm²

[**Ans.** 0·253 Tonnes (central), 1·4795 Tonnes (outside supports)]

11·12. A long steel strip of uniform width and thickness 3·6 mm is lying on a level ground. Its one end is passing over a roller of 4·5 cm lying on the ground at one point. For what distance on either side of the roller will the strip be clear of the ground. What is the maximum stress induced in steel. $\rho_{steel}=7{\cdot}8$ g/cc, $E=210$ kN/mm²

[**Ans.** 176 cm, 67·22 N/mm²]

11·13. A long flat strip 50 mm wide and 3·2 mm thick is lying on a flat horizontal plane. One end of the strip is now lifted 30 mm from the plane by a vertical force applied at the end. The strip is so long that the other end remains undisturbed. Calculate (*a*) the force required to lift the end (*b*) the maximum stress in the steel.

$\rho_{steel}=7{\cdot}8$ g/cc, $E=21{,}000$ N/mm² [**Ans.** 6·97 N, 23·24 N/mm²]

11·14. A cantilever 2 m long carries a uniformly distributed load of 1 tonne/metre run throughout its length. Its free end is attached to a vertical rod 2 m long and 2 cm diameter. The bar is initially straight. Determine the load taken by the rod and the deflection of the cantilever. $E=2000$ tonnes/cm², $I=600$ cm⁴. [**Ans.** 0·739 tonne, 0·235 mm]

11·15. A cantilever of circular section of length 200 cm carries a load 1 kN at its free end. The diameter of the cantilever for half of its length starting from the free end is 4 cm and the diameter for the remaining length is 8 cm. Determine the deflection at free end. $E=200$ kN/mm². [**Ans.** 1·90 cm]

11·16. A beam of rectangular section has a uniform breadth 4 cm and depth which varies from 6 cm at each end to 18 cm at the middle of the length. *A* second beam is of the same material, of the same length and breadth but of uniform depth 18 cm throughout. Find the ratio of the maximum deflection of the first beam and that of the second beam when each is subjected to a central load W and simply supported at their ends. [**Ans.** 2·123 W]

11·17. A cantilever of length 1·6 m is of tapered square cross section throughout. The side of the square section at the free end is 8 cm and that at the fixed end is 12 cm. A load of 4 kN is applied at the free end. What are the slope and deflection at the free end. $E=200$ kN/mm². [**Ans.** +0·148°, −2·3 mm]

11·18. A cantilever of length 2·4 m is of section with depth 25 cm and $I=3717{\cdot}8$ cm⁴. How much load can be dropped onto the free end of the cantilever from a height of 15 cm, so that the maximum stress developed in the section is 80 N/mm². What is the instantaneous deflection at the free end. $E=2\times10^5$ N/mm². [**Ans.** 195·05 N, 6·144 mm]

11·19. A cantilever of length l is supported at the free end by a prop at the same level as that of the fixed end. A uniformly distributed load of w per unit length is applied on the cantilever starting from its centre and upto the fixed end. Determine the reaction of the prop and deflection at the centre of the cantilever. EI is the flexural rigidity of the cantilever.

$$\left[\textbf{Ans.}\ \frac{7}{128}\,wl,\ \frac{-13\,wl^4}{6144\,EI}\right]$$

11·20. A beam of lenth 4 metre is simply supported over a span of 2 m, with equal over hang on both the sides. It carries a uniformly distributed load of 2 tonnes/metre run on the overhang portion on both the sides. Determine (a) slope and deflection at the overhang end (b) deflection at the centre of the beam. Use moment area method. $E=2000$ T/cm², $I=2000$ cm⁴.

[**Ans.** +0·19°, −3·125 mm, +1·25 mm]

11·21. A beam $ABCD$, 6 metres long, simply supported at ends A and D carries concentrated loads of 2 kN and 5 kN at points B and C. Points B and C are 2 metres away from the ends A and D respectively. Determine (i) deflection under the loads of 2 kN and 5 kN (ii) maximum deflection and its position. $E=200$ kN/mm², $I=3600\times10^4$ mm⁴

[**Ans.** −0·315 cm, −0·333 cm, $y_{max}=-0{\cdot}373$ cm,
at a distance of 3·0945 m from end A]

12

Fixed and Continuous Beams

In the previous chapters on SF and BM diagrams and deflection, we have studied about the beams and cantilevers. Cantilever is fixed at one end and its other end is free or propped. Beams considered were either simply supported at the ends or hinged at one or both the ends. The beam at the simply supported ends or hinged ends has some slope while its deflection is zero. The cantilever at its free end has same slope and same deflection too, and at its fixed end there is fixing couple exerted by the support keeping slope and deflection zero. Now we will study about the bult in, encastre or fixed beams which are constrained at the support so that slope and deflection both remain zero at the support. The support exerts restraining couple, the direction of which is opposite to the direction of the bending moment produced by the transverse loads on the beams. There is unique value of the fixing couple required at the end, if the restraining couple exerted by the support is less than this, there will be some slope at the end and if the restraining couple is more than the required unique value, then slope at the end will be on the other side of the zero position. The fixing couples exerted by the supports can be easily worked out.

Further we will study about the continuous beams. A beam is said to be continuous when it is supported over more than two supports. Curvature of the beam at the intermediate supports will be convex upwards, therefore, support moments will be opposite in sign to the bending moments produced by transverse loads on the beam. Moreover in the case of continuous beams, the slopes at the supports are not necessarily zero.

12·1. FIXED BEAMS—B.M. DIAGRAMS

A fixed beam can be considered as equivalent to a simply supported beam plus a beam of the same length having fixing couples at the ends. Fig. 12·1 (*a*) shows a beam *ABCD* of length *l* fixed at both the ends, carrying uniformly distributed load *w* per unit length over *AB* and a point load *W* at *C*. This fixed beam is equivalent to the sum of a simply supported

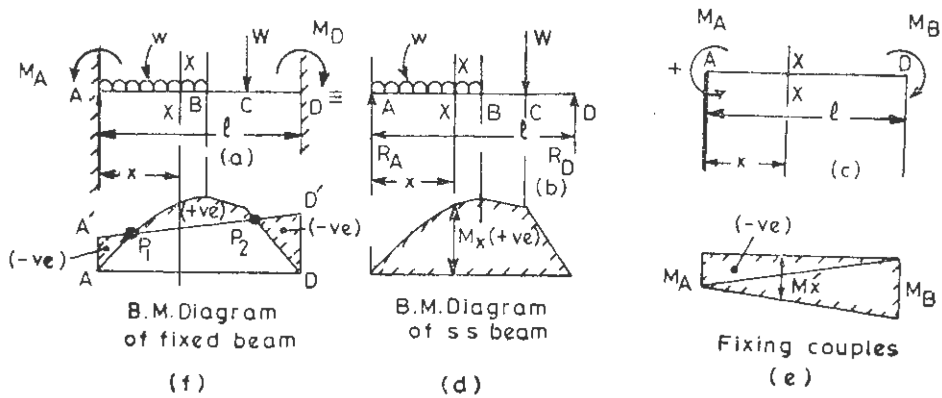


Fig. 12·1

beam $ABCD$ of same length and carrying same loads and a beam AD of length l with couples M_A and M_D applied at the ends, as shown by Fig. 12·1 (b) and (c). Fig. (d) shows the bending moment diagram of the simply supported beam. We have taken the convention that bending moments producing concavity upwards in the beam are the positive bending moments, therefore BM diagram shown in Fig. (d) is a positive bending moment diagram. Fixing couples at the ends try to bend the beam producing convexity upwards, therefore the BM diagram of fixing couples shown by (e) is negative. Fig. (f) shows the combined bending moment diagram for simply supported beam and the fixing couples or the bending moment diagram for the fixed beam. The bending moments above the line $A'D'$ are positive moments and those below the line $A'D'$ are negative moments. Points P_1 and P_2 are the points of contraflexure.

Consider a section X-X at a distance of x from the end A.

BM at the section as S.S. beam $=M_x$

BM at the section due to fixing couples$=M_x'$

$$=M_A+\frac{(M_D-M_A)}{l}\,x$$

M, Resultant bending moment at the section

$$=M_x+M_x'$$

$$=M_x+M_A+\frac{(M_D-M_A)x}{l}.$$

12·2. SUPPORT MOMENTS—FIXED BEAMS

Bending moment at any section,

$$M=M_x+M_x'$$

or $$EI\frac{d^2y}{dx^2}=M_x+M_x' \quad \text{or} \quad EI\frac{d^2y}{dx^2}\,.\,dx=M_xdx+M_x'dx \qquad ...(1)$$

Integrating over the length of the beam

$$\left| EI\frac{dy}{dx}\right|_0^l=\int_0^l M_x\,dx+\int_0^l M_x'\,dx$$

$$EI\,(i_D-i_A)=a+a' \qquad ...(2)$$

where i_D=slope at end $D=0$, as end D is fixed

i_A=slope at end $A=0$, as end A is also fixed

a=area of BM diagram considering the beam to be simply supported

a'=area of BM diagram due to fixing couples

$$=-\left(\frac{M_A+M_D}{2}\right)l \qquad ...(3)$$

or $$a+a'=0$$

$$a=\left(\frac{M_A+M_D}{2}\right)l \qquad ...(4)$$

This shows that area of the Mx diagram is numerically equal to the area of the Mx' diagram.

Consider equation (1) again and multiply both the sides by x and then integrate over the length of the beam

$$\int_0^l EI\, x\, \frac{d^2y}{dx^2}\,.\,dx=\int_0^l M_x\,.\,x\,dx+\int_0^l M_x'\,.\,x\,dx$$

or

$$EI\left|x\,\frac{dy}{dx}-y\right|_0^l=a\bar{x}+a'\bar{x}'$$

$$EI\,[(l\times i_D-y_D)-(o\times i_A-y_A)]=a\bar{x}+a\bar{x}'$$

But at the fixed ends, both the slope and deflection are zero.

Therefore $\qquad a\bar{x}+a'\bar{x}'=0 \qquad$ ...(5)

where $\qquad x$=distance of the CG of Mx diagram from end A

$\bar{x}'$=distance of the CG of the Mx' diagram from end A.

Since $\qquad a=-a'$

So $\qquad \bar{x}=\bar{x}'$

i.e., the centres of gravity of a and a' lie on the same vertical line.

Now about the origin A,

$$a'\bar{x}'=+\left[M_A\,.\,\frac{l}{2}\left(\frac{l}{3}\right)+M_D\,.\,\frac{l}{2}\left(\frac{2l}{3}\right)\right]$$

$$=+\left[M_A\,.\,\frac{l^2}{6}+M_D\,.\,\frac{l^2}{3}\right]=+\frac{l^2}{6}\,(M_A+2M_D)$$

or

$$-a\bar{x}=+a'\bar{x}'=-(M_A+2M_D)\,\frac{l^2}{6} \qquad ...(6)$$

With the help of the equations (4) and (6) fixing couples at the ends are worked out.

12·3. FIXED BEAM WITH A CONCENTRATED LOAD AT THE CENTRE

Fig. 12·2 (*a*) shows a beam ABC of length l, fixed at both the ends A and C and carrying a concentrated load W at its centre B. Fig. 12·2 (*b*) shows the SF diagram for the same as for the simply supported beam with a concentrated load W at the centre. Due to symmetrical loading about the centre of the beam fixing couples,

$$M_A=M_C$$

When the beam is simply supported at ends, maximum bending moment occurs at the centre,

$$M_B=\frac{Wl}{4}$$

ABC is the Mx diagram and $AA'C'C$ is the M_x' diagram

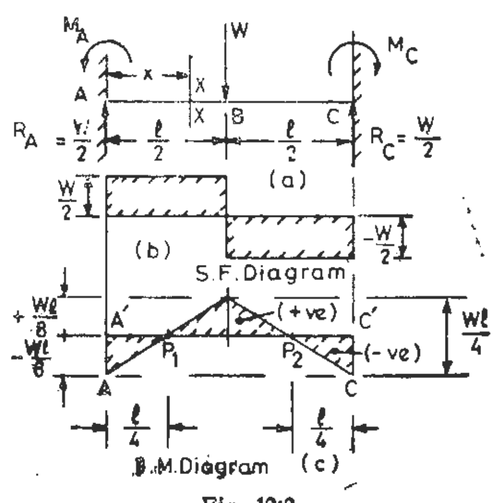


Fig. 12·2

Area $a=\frac{Wl}{4}\times\frac{l}{2}=\frac{Wl^2}{8}$

Area of M_x' diagram $a'=+M_A\,.\,l$

But $a+a'=0$

$a=-a'$

So $M_A\,.\,l=-\frac{Wl^2}{8}$

or $M_A=M_C=-\frac{Wl}{8}$

These fixing couples are equal and opposite at ends and balance each other and impose no additional reactions at the supports. Therefore the SF diagram for the fixed beam is the same as the SF diagram for a simply supported beam in this particular case. The points of contraflexure P_1 and P_2 lie at $l/4$ from each end as is obvious from the diagram (c).

Let us determine slope and deflection at any point

$$EI\frac{d^2y}{dx^2}=M=M_x+M_x'$$

$$=-\frac{Wl}{8}+\frac{Wx}{2} \qquad ...(1)$$

Integrating equation (1),

$$EI\frac{dy}{dx}=-\frac{Wl}{8}x+\frac{Wx^2}{4}+C_1 \text{ (constant of integration)}$$

at $x=0$, fixed end A, slope is zero.

Therefore $0=-0+0+C_1$ or $C_1=0$

$$EI\frac{dy}{dx}=-\frac{Wlx}{4}+\frac{Wx^2}{4} \qquad ...(2)$$

Integrating the equation (2),

$$EI\,y=-\frac{Wlx^2}{16}+\frac{Wx^3}{12}+C_2 \text{ (constant of integration)}$$

at $x=0$, fixed end A ; $y=0$

So $0=-0+0+C_2$ or $C_2=0$

$$EI\,y=-\frac{Wlx^2}{16}+\frac{Wx^3}{12} \qquad ...(3)$$

Maximum deflection will occur at the centre, $x=\frac{l}{2}$

$$EI\,y_{max}=-\frac{Wl}{16}\left(\frac{l}{2}\right)^2+\frac{W}{12}\left(\frac{l}{2}\right)^3$$

$$EI\,y_{max}=-\frac{Wl^3}{64}+\frac{Wl^3}{96}$$

$$y_{max}=-\frac{Wl^3}{192\,EI}.$$

This shows that the maximum deflection for a fixed beam of length l and a load W at the middle is only one fourth of the maximum deflection of a simply supported beam of length l, with a concentrated load W at the middle.

The obvious effect of fixing couples at the ends is to (1) to reduce the magnitude of bending moment throughout the length of the beam (2) to reduce the slopes and deflections considerably (3) to make the beam stronger and stiffer.

Example 12·3-1. A beam of length 6 m is fixed at both the ends carries a concentrated load 40 kN at its middle. Determine (*i*) fixing couples at the ends (*ii*) maximum deflection.

$$E=200 \text{ kN/mm}^2,\ I=3600 \text{ cm}^4.$$

Solution.

Length of the beam, $l=6$ m

Concentrated load at the centre,

$$W=40 \text{ kN}$$
$$E=200\times 10^6 \text{ kN/m}^2$$
$$I=3600 \text{ cm}^4=3600\times 10^{-8} \text{ m}^4$$
$$EI=200\times 10^6\times 3600\times 10^{-8}=7200 \text{ kNm}^2$$

(*i*) Fixing couple at the ends

$$=-\frac{Wl}{8}=-\frac{4\times 6}{8}=-30 \text{ kNm}$$

(*ii*) Maximum deflection $=-\dfrac{Wl^3}{192\,EI}=-\dfrac{40\times 6^3}{192\times 7200}=-0{\cdot}00625 \text{ m}=-6{\cdot}25 \text{ mm}.$

Exercise 12·3-1. A beam of length 8 m, fixed at both the ends carries a concentrated load 2·4 tonnes at its centre. Determine (*i*) fixing couples at the ends (*ii*) maximum deflection.

$$E=2000 \text{ tonnes/cm}^2,\ I=5712 \text{ cm}^4$$

[**Ans.** -2 tonne-metres, 5·6 mm]

12·4. FIXED BEAM WITH UNIFORMLY DISTRIBUTED LOAD

Fig. 12·3 shows a fixed beam of length l carrying uniformly distributed load w per unit length throughout its length.

If the beam is simply supported maximum bending moment occurs at the centre and $M_{max}=wl^2/8$. ACB is the Mx diagram with a parabolic curve. $AA'\,B'B$ is is the bending diagram due to fixing couples.

As the beam is symmetrically loaded about its centre,

Reactions $R_A=R_B=\dfrac{wl}{2}$

and fixing couples $M_A=M_B$

a, Area of Mx diagram $=\dfrac{2}{3}\times l\times \dfrac{wl^2}{8}$

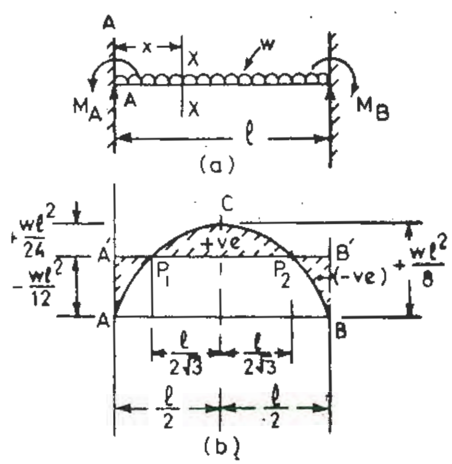


Fig. 12·3

$$=\frac{wl^3}{12}$$

a', Area of Mx' diagram$=M_A\times l=M_B\times l$

But $$a'=-a$$

$$M_A\, l=-\frac{wl^3}{12}$$

So $$M_A=-\frac{wl^2}{12}$$

Then $AA'\ P_1\ C\ P_2\ B'B$ is the resultant bending moment diagram for the fixed beam with points of contraflexure at the point P_1 and P_2. To determine the points of contraflexure consider a section X-X at a distance of x from end A.

Bending moment, $$M=-M_A+R_A\,.\,x-\frac{wx^2}{2}=0$$

$$-\frac{wl^2}{12}+\frac{wlx}{2}-\frac{wx^2}{2}=0$$

or $$6x^2-6lx+l^2=0$$

$$x=\frac{6l\pm\sqrt{36l^2-24l^2}}{12}=\frac{6l\pm2\sqrt{3}\,l}{12}=\frac{l}{2}\pm\frac{l}{2\sqrt{3}}$$

Points of contraflexure lie at a distance of $l/2\sqrt{3}$ on both the sides of the centre.

Bending moment at the centre

$$=\frac{wl^2}{8}-\frac{wl^2}{12}=+\frac{wl^2}{24}$$

BM at the ends$=$fixing moments

$$=-\frac{wl^2}{12}$$

For slope and deflection let us consider a section X-X at a distance of x from the end A.

BM at the section, $$M=-\frac{wl^2}{12}+R_A\,.\,x-\frac{wx^2}{2}$$

or $$EI\frac{d^2y}{dx^2}=-\frac{wl^2}{12}+\frac{wlx}{2}-\frac{wx^2}{2} \qquad \ldots(1)$$

Integrating equation (1), we get

$$EI\frac{dy}{dx}=-\frac{wl^2x}{12}+\frac{wlx^2}{4}-\frac{wx^3}{6}+C_1 \text{ (constant of integration)}$$

$$\text{at } x=0,\ \frac{dy}{dx}=0, \text{ (at the fixed end)}$$

$\therefore$ $$EI\times0=-0+0-0+C_1 \text{ or } C_1=0$$

So $$EI\frac{dy}{dx}=-\frac{wl^2x}{12}+\frac{wlx^2}{4}-\frac{wx^3}{6} \qquad \ldots(2)$$

Integrating the equation (2),

$$EI\,y=-\frac{wl^2x^2}{24}+\frac{wlx^3}{12}-\frac{wx^4}{24}+C_2 \text{ (constant of integration)}$$

at $x=0$, $y=0$, (fixed end)

$$EI\times 0=-0+0-0+C_2 \text{ or } C_2=0$$

$$EI\,y=-\frac{wl^2x^2}{24}+\frac{wlx^3}{12}-\frac{wx^4}{24}$$

Maximum deflection take places at the center $x=l/2$, $y=y_{max}$ (because the beam is symmetrically loaded about the centre)

$$EI\,y_{max}=-\frac{wl^4}{96}+\frac{wl^4}{96}-\frac{wl^4}{384}$$

$$y_{max}=-\frac{wl^4}{384\,EI} \text{ (indicating downward deflection).}$$

This shows that maximum deflection of a fixed beam carrying uniformly distributed load is only 1/5 the maximum deflection of a simply supported beam of same length and section and carrying uniformly distributed load throughout its length.

Example 12·4-1. A beam 6 m span has its ends built in and carries a uniformly distributed load of 500 kg per metre run. Find the maximum bending moment and the maximum deflection.

$$E=2000 \text{ tonnes/cm}^2,\ I=4800 \text{ cm}^4.$$

Solution. Span length $l=6$ m

Rate of loading, $w=500$ kg/m run$=0{\cdot}5$ tonne/metre run

$$E=2000 \text{ tonne/cm}^2=2000\times 10^4 \text{ tonne/metre}^2$$

$$I=4800 \text{ cm}^4=4800\times 10^{-8} \text{ m}^4$$

$$EI=2000\times 10^{-4}\times 4800\times 10^{-8}=960 \text{ tonne-metre}^2$$

Maximum bending moment,

$$=-\frac{wl^2}{12}=\frac{0{\cdot}5\times 6\times 6}{12}=-4{\cdot}5 \text{ tonne-metre}$$

Maximum deflection,

$$y_{max}=-\frac{wl^4}{384\,EI}=-\frac{0{\cdot}5\times 6^4}{384\times 960}=-0{\cdot}00176 \text{ m}=-1{\cdot}76 \text{ mm.}$$

Exercise 12·4-1. A fixed beam of length 8 m carries a uniformly distributed load of w kN/m run. Determine w if

(*i*) maximum bending moment is not to exceed 36 kNm

(*ii*) maximum deflection is not to exceed 1/2000 of the length.

$EI=7200$ kNm² [**Ans.** (*i*) 6·75 kN/mrun, (*ii*) 2·7 kN/m run]

12·5. FIXED BEAM CARRYING AN ECCENTRIC LOAD

Fig. 12·4 (*a*) shows a fixed beam of length l carrying a concentrated load W at a point B, at a distance a from end A. In this case, the load on the beam is not symmetrically applied about its centre, therefore M_A will not be equal to M_C.

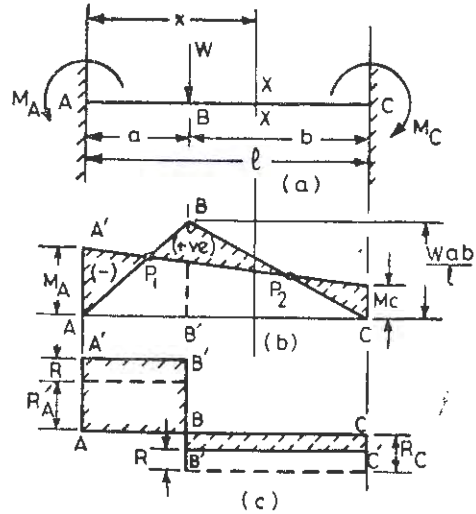


Fig. 12·4

The bending moment diagram for M_x is shown by ABC with maximum bending moment at B and equal to Wab/l.

The bending moment diagram for M_x' (due to fixing couples is shown by $AA'C'C$)

Area, $\quad a=\dfrac{Wab}{l}\times\dfrac{l}{2}=\dfrac{Wab}{2} \quad$ ...(*i*)

Area, $\quad a'=\left(\dfrac{M_A+M_C}{2}\right)l \quad$ ...(*ii*)

or $$\left(\frac{M_A+M_C}{2}\right)l=-\frac{Wab}{2} \quad \text{...(iii)}$$

To determine $a\bar{x}$, let us divide the area a into two triangles *i.e.*, ABB' and $BB'C$.

Moment of a about the end A

$$a\bar{x}=\frac{Wab}{l}\times\frac{a}{2}\left(\frac{2a}{3}\right)+\frac{Wab}{l}\times\frac{b}{2}\left(a+\frac{b}{3}\right)$$

$$=\frac{2Wa^3b}{6l}+\frac{Wab^2\,(3a+b)}{6l}$$

$$=\frac{Wab\,(2a+b)}{6l} \quad \text{since } a+b=l$$

and $$a'\bar{x}'=(M_A+2M_C)\,\frac{l^2}{6}$$

But $$a'\bar{x}'=-a\bar{x}$$

$$(M_A+2M_C)\,\frac{l^2}{6}=-\frac{Wab\,(2a+b)}{6} \quad \text{...(iv)}$$

From equations (*iii*) and (*iv*), we get

$$M_A+M_C=-\frac{Wab}{l}$$

$$M_A+2M_C=-\frac{Wab}{l^2}\,(2a+b)$$

Fixing couple, $$M_C=-\frac{Wab}{l^2}\,(2a+b)+\frac{Wab}{l}=-\frac{Wab}{l}\left(\frac{2a+b}{l}-1\right)$$

$$=-\frac{Wa^2b}{l^2}$$

Fixing couple, $$M_A=-\frac{Wab}{l^2}(2a+b)+\frac{2Wa^2b}{l^2}$$

$$=-\frac{Wab}{l^2}(2a+b-2a)=-\frac{Wab^2}{l^2}$$

In this case $b>a$, therefore $M_A > M_C$ (numerically) the unbalanced couple (M_A-M_C) will be balanced by a reaction R, upwards at A and downwards at C

$$R\times l=M_A-M_C=-\frac{Wab}{l^2}(b-a)$$

$$R=-\frac{Wab}{l^3}(b-a)$$

$$R_A'=\frac{Wb}{l}, \text{ and } R_C'=\frac{Wa}{l}$$

(for the beam simply supported at the ends). The SF diagram is shown by Fig. 12·4 (*c*) by the diagram $AA'B'BB''C'C$.

The bending moment diagram for the fixed beam is shown by Fig. (*b*), marked by $AA'\ P_1\ B\ P_2\ C'C$, with P_1 and P_2 as points of contraflexure.

Reactions at the ends

$$R_A=R_A'+R=\frac{Wb}{l}+\frac{Wab}{l^3}(b-a)$$

$$=\frac{Wbl^2+Wab\,(b-a)}{l^3}$$

$$=\frac{Wb\,[l^2+ab-a^2]}{l^3}=\frac{Wb^2\,(b+3a)}{l^3}$$

$$R_C=R_C'-R=\frac{Wa}{l}-\frac{Wab}{l^3}(b-a)$$

$$=\frac{Wa}{l}[l^2-b\,(b-a)]=\frac{Wa^2}{l^3}(a+3b)$$

For slope and deflection consider a section X-X in the portion BC at a distance of x from the end A.

BM at the section $$=R_A\,x-\frac{Wab^2}{l^2}-W\,(x-a)$$

$$EI\frac{d^2y}{dx^2}=\frac{Wb^2\,(b+3a)x}{l^3}-\frac{Wab^2}{l^2}-W\,(x-a) \quad \text{...(1)}$$

Integrating the equation (1)

$$EI\frac{dy}{dx}=\frac{Wb^2\,(b+3a)x^2}{2l^3}-\frac{Wab^2}{l^2}\,.\,x-\frac{W(x-a)^2}{2}+C_1$$

$$\frac{dy}{dx}=0 \quad \text{at} \quad x=0, \; \therefore \; C_1=0 \text{ (constant of integration)}$$

$$EI\frac{dy}{dx}=\frac{Wb^2(b+3a)x^2}{2l^3}-\frac{Wab^2x}{l^2}-\frac{W(x-a)^2}{2} \qquad ...(2)$$

Integrating the equation (2)

$$EIy=\frac{Wb^2(b+3a)x^3}{6l^3}-\frac{Wab^2x^2}{2l^2}-\frac{W(x-a)^3}{6}+C_2$$

(constant of integration)

$$y=0 \quad \text{at} \quad x=0, \quad \text{So, } C_2=0$$

Then
$$EIy=\frac{Wb^2(b+3a)x^3}{6l^3}-\frac{Wab^2x^2}{2l^2}-\frac{W(x-a)^3}{6} \qquad ...(3)$$

For the deflection to be maximum, slope has to be zero at that particular section in a beam. Let us determine the section where deflection is maximum.

Putting
$$EI\frac{dy}{dx}=0=\frac{Wb^2}{2l^3}(b+3a)x^2-\frac{Wab^2}{l^2}(x)-\frac{W(x-a)^2}{2}$$

or
$$(b^3+3ab^2)x^2-2alb^2x-l^3(x^2-2ax+a^2)=0$$

$$x^2(b^3+3ab^2-l^3)+x(2al^3-2alb^2)-a^2l^3=0$$

$$x^2(-a^3-3a^2b)+2al(l^2-b^2)x-a^2l^3=0 \qquad \text{as } l=a+b$$

or
$$-x^2(a+3b)+2l(a+2b)x-l^3=0$$

$$x^2(a+3b)-2l(a+2b)x+l^3=0$$

$$x=\frac{2l(a+2b)-\sqrt{[2l(a+2b)]^2-4l^3(a+3b)}}{2(a+3b)}$$

$$=\frac{l(a+2b)-l\sqrt{a^2+4ab+4b^2-a^2-ab-3ab-3b^2}}{(a+3b)}$$

$$=\frac{l(a+2b)-lb}{(a+3b)}=\frac{al+2lb-lb}{a+3b}$$

$$=\frac{(a+b)l}{a+3b}=\frac{l^2}{a+3b}$$

Substituting
$$x=\frac{l^2}{a+3b}, \text{ we get}$$

$$EI\,y_{max}=\frac{Wb^2(b+3a)}{6l^3}\times\frac{l^6}{(a+3b)^3}-\frac{Wab^2}{2l^2}\times\frac{l^4}{(a+3b)^2}-\frac{W}{6}\left(\frac{l^2}{a+3b}-a\right)^3$$

$$=\frac{Wb^2(b+3a)l^3}{6(a+3b)^3}-\frac{Wab^2l^2}{2(a+3b)^2}-\frac{W}{6}\left(\frac{b^2-ab}{a+3b}\right)^3$$

$$=\frac{Wb^2(b+3a)(a+b)^3-3Wab^2(a+b)^2(a+3b)-W(b^2-ab)^3}{6(a+3b)^3}$$

$$=\frac{W}{6}\left[\frac{-4b^3a^3-12a^2b^4}{(a+3b)^3}\right]$$

$$=-\frac{2}{3}W\frac{(a+3b)(b^3a^2)}{(a+3b)^3}=-\frac{2}{3}\frac{Wb^3a^2}{(a+3b)^2}$$

$$y_{max}=-\frac{2}{3}\frac{Wa^2b^3}{(a+3b)^2\,EI}$$

Deflection under the load W can be obtained by taking $x=a$ in equation (3)

$$EIy_c=\frac{Wb^2(b+3a)a^3}{6l^3}-\frac{Wab^2a^2}{2l^2}$$

$$=\frac{Wa^3b^2(b+3a)-3l(Wa^3b^2)}{6l^3}$$

$$=\frac{Wa^3b^2}{6l^3}[-2b]=-\frac{Wa^3b^3}{3l^3}$$

$$y_c=-\frac{Wa^3b^3}{3l^3\,EI}\text{ (indicating downward deflection)}$$

Points of Inflexion

portion AB. B.M. at any section in portion AB

$$M=\frac{Wb^2(b+3a)x}{l^3}-\frac{Wab^2}{l^2}=0$$

$$x=\frac{al}{b+3a},$$ distance of the point of contraflexure from end A.

Portion BC. B.M. at any section,

$$M=\frac{Wb^2(b+3a)x}{l^3}-\frac{Wab^2}{l^2}-W(x-a)=0$$

$$Wb^2(b+3a)x-Wlab^2-W(x-a)l^3=0$$

$$(b^3+3ab^2)x-(a+b)\,ab^2-(x-a)l^3=0$$

or $$(b^3+3ab^2-l^3)\,x=ab^2(a+b)-al^3$$

$$(-a^3-3a^2b)x=a[b^2l-l^3]\qquad\text{since } l=a+b$$

$$x=\frac{l[b^2-l^2]}{-a^2-3ab}=\frac{l[l^2-b^2]}{a(a+3b)}=\frac{l(a+2b)}{a+3b}$$

distance of point of contraflexure from end **A**.

Example 12·5-1. A built in beam of 6 m span carries a concentrated load of 60 kN at a distance of 2 metres from the left hand end Find the position and amount of maximum deflection. $E=200$ kN/mm², $I=13600$ cm⁴.

Solution.

Length of the beam, $l=6$ m

Distances $a=2$ m

$b=4$ m

Central load $W=60$ kN

$$E=200\times10^6 \text{ kN/m}^2 \qquad I=13600\times10^{-8} \text{ m}^4$$

$$EI=200\times13600\times10^{-2} \text{ kNm}^2=27200 \text{ kNm}^2$$

For y_{max}, $x=\dfrac{l^2}{a+3b}$ from left hand end

$$=\frac{6^2}{2+3\times4}=\frac{36}{14}=2{\cdot}57 \text{ m}$$

$$y_{max}=-\frac{2}{3}\,\frac{Wa^2b^3}{(a+3b)^2EI}=-\frac{2}{3}\times\frac{60\times2^2\times4^3}{(2+3\times4)^2\times27200}$$

$$=-\frac{2}{3}\times\frac{60\times4\times64}{14\times14\times27200}=-{\cdot}00192 \text{ m}$$

$$=-1{\cdot}92 \text{ mm}$$

Exercise 12·5-1 A fixed beam of length 7 metres carries a concentrated load of 3 tonnes at a distance of 3 metres from left hand end. Determine (*i*) support moments (*ii*) position of the points of contraflexure from left hand end (*iii*) deflection under the load. Given $EI=1600$ T-m^2.

[**Ans.** (*i*) $-2{\cdot}938$ Tm, $-2{\cdot}122$ Tm (*ii*) 1·615 m, 5·133 m from left hand end (*iii*) $-3{\cdot}15$ mm]

12·6. ALTERNATE METHOD FOR DETERMINING SUPPORT—MOMENTS, SLOPE AND DEFLECTIONS FOR FIXED BEAMS

To determine support moments and deflections etc., for fixed beams carrying any type of loading it is not necessary to draw M_x and M_x' diagrams as shown in articles 12·1. One can assume the support moments and support reactions and determine their values using the end conditions. Further the slope and deflection at any section of the beam can be determined. Consider the case of a fixed beam of length l carrying an eccentric load W, at a distance of a from the left hand end A as shown in the Fig. 12·5. Say the support moments at A and C are M_C and M_A and reaction are R_A and R_C respectively.

Consider a section X-X at a distance of x from the end A.

B.M. at the section, $M=M_A+R_A.x-W(x-a)$

or
$$EI\,\frac{d^2y}{dx^2}=M_A+R_Ax-W(x-a) \qquad \text{...(1)}$$

Integrating the equation (1),

$$EI\,\frac{dy}{dx}=M_Ax+R_A\,\frac{x^2}{2}-\frac{W(x-a)^2}{2}+C_1$$

where C_1 is the constant of integration.

Now at $x=0$, fixed end A, $\dfrac{dy}{dx}=0$

$\therefore$ $0=0+0-(\text{omitted term})+C_1$

$C_1=0$

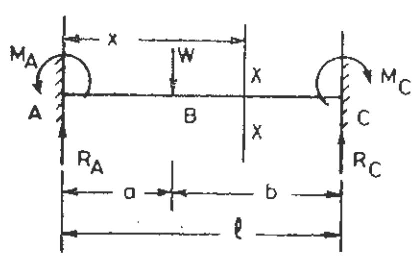


Fig. 12·5

So $$EI\frac{dy}{dx}=M_A.x+R_A\ \frac{x^2}{2}-\frac{W}{2}\ (x-a)^2 \qquad ...(2)$$

Integrating the equation (2),

$$EIy=M_A\ .\ \frac{x^2}{2}+R_A.\ \frac{x^3}{6}-\frac{W}{6}\ (x-a)^3+C_2$$

where C_2 is the constant of integration

at $x=0$, fixed end A, $y=0$

$\therefore$ $$0=0+0-(\text{omitted term})+C_2 \text{ or } C_2=0$$

So $$EIy=M_A\ \frac{x^2}{2}+R_A\ \frac{x^3}{6}-\frac{W}{6}\ (x-a)^3 \qquad ...(3)$$

Now at $x=l$, fixed end B, $\frac{dy}{dx}=0$ and $y=0$

Substituting in equations (2) and (3), we get

$$0=M_A.l+R_A\ .\ \frac{l^2}{2}-\frac{W}{2}\ b^2 \qquad ...(4)$$

as $(l-a)=b$

$$0=M_A\ .\ \frac{l^2}{2}+R_A\ \frac{l^3}{6}-\frac{Wb^3}{6} \qquad ...(5)$$

Solving the equation (4) and (5), we get

$$M_A=-\frac{Wab^2}{l^2},\ R_A=\frac{Wb^2(b+3a)}{l^3}$$

This is what we have obtained in article 12·5. Equation (1) now becomes

$$EI\ \frac{d^2y}{dx^2}=-\frac{Wab^2}{l^2}+\frac{Wb^2(b+3a)}{l^3}\ x-W(x-a)$$

Successive integration of this equation will yield the equation for slope and then for deflection and we can derive expressions for y_{max} and y_B etc.

Example 12·6-1. A fixed beam 8 m long carries a uniformly distributed load of 2 tonnes/metre run over 4 m length starting from left hand end and a concentrated load of 4 tonnes at a distance of 6 m from the left hand end. Determine (*i*) support moments, (*ii*) deflection at the centre of the beam. $EI=1500$ Tm².

Solution. Fig. 12·6 shows a fixed beam *ABCD* carrying uniformly distributed load of 2T/m from *A* to *B*, 4 m length and 4 T load at *C*, 6 m from end *A*. Let us assume that R_A and R_D are the support reactions and M_A and M_D are the support moments. Taking the origin at *D* and *x* positive towards left, consider a section *X-X* at a distance of *x* from the end *D*, in the portion *BA*.

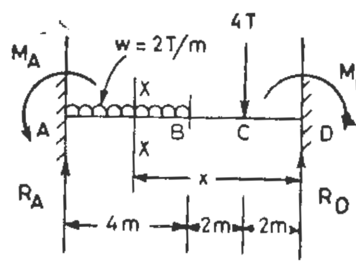


Fig. 12·6

B.M. at the section,

$$M=M_D+R_D\times x-4(x-2)$$
$$-\frac{w}{2}\ (x-4)^2 \quad \text{where } w=2\text{T/m}$$

or $$EI\frac{d^2y}{dx^2}=M_D+R_D.x-4(x-2)-(x-4)^2 \quad ...(1)$$

Integrating the equation (1), we get

$$EI\frac{dy}{dx}=M_D.x+R_D.\frac{x^2}{2}-2(x-2)^2-\frac{(x-4)^3}{3}+C_1$$

$$x=0,\ \frac{dy}{dx}=0 \text{ at fixed end } D$$

$\therefore$ $$0=0+0+\text{omitted terms}+C_1$$

or $$C_1=0$$

So $$EI\frac{dy}{dx}=M_D\,.\,x+R_D\,.\,\frac{x^2}{2}-2(x-2)^2-\frac{(x-4)^3}{3} \quad ...(2)$$

Integrating the equation (2), we get

$$EIy=M_D\frac{x^2}{2}+R_D\frac{x^3}{6}-\frac{2}{3}(x-2)^3-\frac{(x-4)^4}{12}+C_2$$

But at $x=0$, $y=0$, at fixed end D

$$0=0+0-\text{omitted terms}+C_2 \text{ (constant of inegration)}$$

$$C_2=0$$

$\therefore$ $$EIy=M_D\,.\,\frac{x^2}{2}+R_D\frac{x^3}{6}-\frac{2}{3}(x-2)^3-\frac{(x-4)^4}{12} \quad ...(3)$$

Now at $x=8$ m, at end A, $\frac{dy}{dx}=0$, $y=0$

Substituting in equations (2) and (3), we get

$$0=8\ M_D+32\ R_D-72-\frac{64}{3} \quad ...(4)$$

$$0=32\ M_D+\frac{512}{6}R_D-144-\frac{64}{3} \quad ...(5)$$

From equations (4) and (5),

$$R_D=4{\cdot}875 \text{ Tonnes}$$
$$M_D=-7{\cdot}833 \text{ Tonne-metres}$$
$$R_A=4\times2+4-4{\cdot}875=7{\cdot}125 \text{ Tonnes}$$

Equation for B.M. $$M=M_D+R_D.x-4(x-2)-(x-4)^2$$
$$=-7{\cdot}833+4{\cdot}875\ x-4(x-2)-(x-4)^2$$

at $x=8$ m, $M=M_A$

So $$M_A=-7{\cdot}833+4{\cdot}875\times8-4(6)-(8-4)^2$$
$$=-7{\cdot}833+39-24-16$$
$$=-8{\cdot}833 \text{ T-m}$$

So the support moments are $M_A=-8{\cdot}833$ T-m and $M_D=-7{\cdot}833$ Tm

To determine the deflection at the centre, $x=4$ m from equation (3)

$$EIy_C=M_D.\frac{4^2}{2}+R_D.\frac{4^3}{6}-\frac{2}{3}(2)^3-0$$

$$EIy_C=-7{\cdot}833\times 8+4{\cdot}875\times\frac{64}{6}-\frac{16}{3}$$

$$=-62{\cdot}664+52-5{\cdot}333=-15{\cdot}997$$

$$y_c=-\frac{15{\cdot}997}{1500}=-0{\cdot}0106 \text{ m}=-10{\cdot}6 \text{ mm}$$

Exercise 12·6-1. A fixed beom 6 m long carries point loads of 40 kN each at a distance of 2 m from each end. Determine support reactions, support moments and deflection at the centre of the beam $E=205$ kN/mm², $I=3200$ cm⁴.

[**Ans.** 40 kN each ; $-53{\cdot}333$ kNm at both the end ; 11·18 mm]

12·7. EFFECT OF SINKING OF SUPPORT IN A FIXED BEAM

If one of the supports of a fixed beam sinks, its effect on support reactions and support moments can be calculated.

Let us consider a fixed beam of length l and flexural rigidity EI as shown in Fig. 12·7 (*a*). Say the support B sinks by δ below the level of support A.

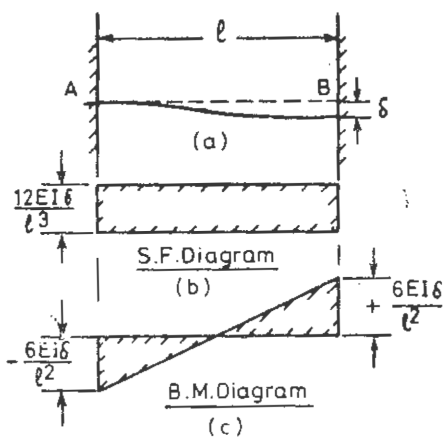


Fig. 12·7

We know that

$$EI\frac{d^4y}{dx^4}=\text{rate of loading}$$

$$=0 \quad \text{(in this case)}$$

$$EI\frac{d^3y}{dx^3}=C_1 \text{ (constant of integration)}$$

at $x=0$, say reaction is R_A

$$EI\frac{d^3y}{dx^3}=R_A \qquad \ldots(1)$$

Integrating further

$$EI\frac{d^2y}{dx^2}=R_A.x+C_2 \qquad \ldots(2)$$

Say at $x=0$, end A, support moment is M_A

$$C_2=M_A$$

$$EI\frac{d^2y}{dx^2}=R_A.x+M_A \qquad \ldots(3)$$

Integrating again

$$EI\frac{dy}{dx}=R_A\frac{x^2}{2}+M_A.x+C_3$$

at $x=0, \frac{dy}{dx}=0$, slope is zero at fixed end

$\therefore$ $0=0+0+C_3$ or $C_3=0$

So $$EI\frac{dy}{dx}=R_A.\frac{x^2}{2}+M_A.x \quad ...(4)$$

Integrating again

$$EIy=R_A\frac{x^3}{6}+M_A\frac{x^2}{2}+C_4$$

at $x=0$ $y=0$,

$\therefore$ $0=0+0+C_4$

Therefore $$EIy=R_A\frac{x^3}{6}+M_A\frac{x^2}{2}$$

But at the end B, $y=-\delta$ (sinking of the support downwards)

$$\therefore \quad -EI\,\delta=R_A.\frac{l^3}{6}+M_A.\frac{l^2}{2} \quad ...(5)$$

Moreover at $x=l$, $\frac{dy}{dx}=0$, at the fixed end B

$$0=R_A.\frac{l^2}{2}+M_A.l \quad ...(6)$$

or $$R_A=-\frac{2M_A}{l}$$

From equation (5)

$$-EI\,\delta=-\frac{2M_A}{l}\times\frac{l^3}{6}+M_A.\frac{l^2}{2}=+\frac{M_A.l^2}{6}$$

$$M_A=-\frac{6\,EI\,\delta}{l^2}$$

$$R_A=-\frac{2M_A}{l}=+\frac{12EI\,\delta}{l^3}$$

For equilibrium, $$R_B=-R_A=-\frac{12EI\,\delta}{l^3}$$

Using equation (3) we can find the support moment at end B *i.e.*, at $x=l$

$$M_B=\frac{12EI\,\delta}{l^3}\times l-\frac{6EI\,\delta}{l^2}=+\frac{6EI\,\delta}{l^2}$$

Fig. 12·7 (*b*) shows the S F. diagram and 12·7(*c*) shows the B.M. diagram for the beam with one support sinked.

If a fixed beam carrying a central load W has one support lower than the other by δ, then the support moment at the higher end will be $-\frac{Wl}{8}-\frac{6EI\delta}{l^2}$ and at the lower end it will be $-\frac{Wl}{8}+\frac{6EI\delta}{l^2}$.

Example 12·7-1. A beam of 7 m span is built in at both the ends. A uniformly distributed load of 20 kN per metre run is placed on the beam, the level of right hand support sinks by 1 cm below that of the left hand end. Find the support reactions, support moments and deflection at the centre. $E=208$ kN/mm², $I=4520$ cm⁴.

Solution. Fig. 12·8 shows a fixed beam of length 7 m, with uniformly distributed load of 20 kN/m throughout its length.

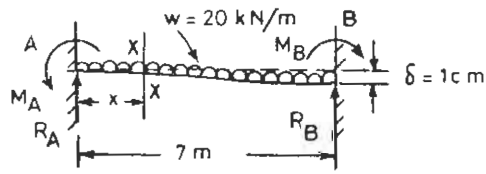


Fig. 12·8

The support B sinks by δ,

$$\delta=1 \text{ cm}=\cdot 01 \text{ m}$$

$$E=208 \text{ kN/mm}^2=208\times 10^6 \text{ kN/m}^2$$

$$I=4520 \text{ cm}^4=4520\times 10^{-8} \text{ m}^4$$

$$EI=208\times 4520\times 10^{-2}=208\times 45\cdot 2 \text{ kNm}^2$$

If the beam support does not sink, then

$$M_A'=M_B'=-\frac{wl^2}{12}=-\frac{20\times 7^2}{12}=-81\cdot 666 \text{ kNm}$$

$$R_A'=R_B'=\frac{wl}{2}=\frac{20\times 7}{2}=70 \text{ kN}$$

After the sinking of the support B

Support moment, $$M_A=-\frac{wl^2}{12}-\frac{6EI\,\delta}{l^2}$$

$$=-81\cdot 666-\frac{6\times 208\times 45\cdot 2\times\cdot 01}{49}$$

$$=-81\cdot 666-11\cdot 512=-93\cdot 178 \text{ kNm}$$

$$M_B=-\frac{wl^2}{12}+\frac{6EI\,\delta}{l^2}=-81\cdot 666+11\cdot 512$$

$$=-70\cdot 154 \text{ kNm}$$

Reactions, $$R_A=\frac{wl}{2}+\frac{12\,EI\,\delta}{l^3}$$

$$=70+\frac{12\times 208\times 45\cdot 2\times\cdot 01}{7\times 7\times 7}$$

$$=70+3\cdot 29 \text{ kN}=73\cdot 29 \text{ kN}$$

$$R_B=70-3\cdot 29=66\cdot 71 \text{ kN}$$

Consider a section X-X at a distance of x from the end A

B.M. at the section, $M = M_A + R_A . x - \frac{wx^2}{2}$

or
$$EI \frac{d^2y}{dx^2} = -93.178 + 73.29\ x - \frac{20x^2}{2} \quad ...(1)$$

Integrating equation (1)

$$EI \frac{dy}{dx} = -93.178\ x + 73.29\ \frac{x^2}{2} - \frac{10x^3}{3} + 0$$

$$\left(\text{constant of integration is zero because } \frac{dy}{dx} = 0 \text{ at } x = 0 \right)$$

Integrating further $\quad EIy = -93.178\ \frac{x^2}{2} + 73.29\ \frac{x^3}{6} - \frac{10x^4}{12} + 0$

(constant of integration is zero because $y=0$ at $x=0$)

At $y = 3.5$ m

$$EIy_c = -93.178 \times \frac{3.5^2}{2} + \frac{73.29 \times 3.5^3}{6} - \frac{10 \times 3.5^4}{12}$$
$$= -570.715 + 523.718 - 125.052$$
$$= -172.049$$

Deflection at the centre,

$$y_c = -\frac{172.049}{208 \times 45.2} = -0.0183 \text{ m}$$
$$= -1.83 \text{ cm}$$

Exercise 12 7-1. A fixed beam 4 m span is built in at both the ends. A concentrated load 6 Tonnes is placed on the middle of the beam, the level of the right hand support is 6 mm below the level of left hand support. Find support reactions, support moments and deflection at the centre of the beam. Given $E = 2 \times 10^7$ Tonnes/m², $I = .6 \times 10^{-4}$ m⁴

[**Ans.** 4·35 Tonnes, 1·65 Tonnes ; −5·7 T-m, −0·3 Tm ; −4·67 mm]

12·8. CONTINUOUS BEAMS

The analysis of continuous beams is similar to that of fixed beams. Each span of a continuous beam is considered separately and M_x diagram is plotted *i.e.*, B.M. diagram considering the ends of each span as simply supported. Upon these M_x diagrams of all the spans (though considered separately), M_x' diagram for support moments is superimposed. M_x' diagram for each span is of opposite sign to M_x diagram. As soon as the support moments are evaluated, the resultant bending moment diagram $M_x + M_x'$ is completed for whole of the beam. Considering the bending moment at any section *i.e.*, $M_x + M_x'$, is determined. Then successive integrations of bending moment equation give slope and deflection at any section.

In the case of fixed beams we considered slope at the fixed ends to be zero but in the case of continuous beams, slopes at the supports are not necessarily zero.

To obtain support moment we will use the Clapeyron's theorem of 3 moments, which gives a relationship between the support moments at 3 consecutive supports of a continuous beam.

Let us consider first the continuous beams carrying only the uniformly distributed loads. Consider two consecutive spans AB and BC of a continuous beam, of lengths l_1 and l_2 and carrying uniformly distributed load w_1 and w_2 per unit length respectively as shown in Fig. 12·9.

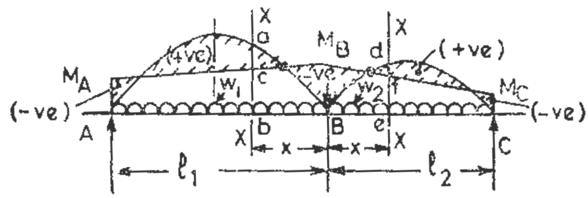


Fig. 12·9

Say the support moments at A, B and C are M_A, M_B and M_C respectively, these support moments are of negative sign.

Span BA. When this span is considered independently, there will be reaction at the supports A and B, equal to $\frac{w_1 l_1}{2}$ and maximum bending moment will occur at the centre, equal to $\frac{w_1 l_1^2}{8}$. The B.M. diagram M_x will be a parabola as shown. Taking the origin at B and x positive towards left

B.M. at the section, $M_x = \frac{w_1 l x}{2} - \frac{w_1 x^2}{2}$ (positive bending moment)

M_x is shown by the line ab on the diagram 12·9

B.M. at the section due to fixing couples,

$$M_x' = M_B + \frac{(M_A - M_B)}{l_1} x$$

(negative bending moment shown by cb on the Fig.)

Resultant bending moment at the section $= M_x - M_x'$

or
$$EI\frac{d^2y}{dx^2} = \frac{w_1 l_1 x}{2} - \frac{w_1 x^2}{2} - M_B - \left(\frac{M_A - M_B}{l_1}\right) x \quad \ldots(1)$$

Integrating the equation (1)

$$EI\frac{dy}{dx} = \frac{w_1 l_1 x^2}{4} - \frac{w_1 x^3}{6} - M_B.x - \left(\frac{M_A - M_B}{l_1}\right)\frac{x^2}{2} + C_1 \quad \ldots(2)$$

(C_1 is constant of integration)

At $x=0$ *i.e.*, at the end B, $\frac{dy}{dx} = +i_B$

So $\qquad C_1 = EI\, i_B$

$$EI\frac{dy}{dx} = \frac{w_1 l_1 x^2}{4} - \frac{w_1 x^3}{6} + M_B.x + \left(\frac{M_A - M_B}{l_1}\right)\frac{x^2}{2} + EI\, i_B \quad \ldots(2)$$

Integrating equation (2)

$$EIy=\frac{w_1 l_1 x^3}{12}-\frac{w_1 x^4}{24}+M_B\ \frac{x^2}{2}$$

$$+\frac{(M_A-M_B)}{l_1}\times\frac{x^3}{6}+EIi_B\ .\ x+C_2 \text{ (constant of integration)}$$

at $x=0$, $y=0$ at the end B

$$0=0-0-0+0+C_2$$

$$C_2=0$$

So
$$EIy=\frac{w_1 l_1 x^3}{12}-\frac{w_1 x^4}{24}+\frac{M_B x^2}{2}+\frac{(M_A-M_B)}{l_1}\ .\ \frac{x^3}{6}+EI\,i_B\ .\ x$$

at $x=l_1$ $y=0$ *i.e.*, at the end A

$$0=\frac{w_1 l_1^4}{12}-\frac{w_1 l_1^4}{24}+M_B\ \frac{l_1^2}{2}+\frac{(M_A-M_B)}{l_1}\times\frac{l_1^3}{6}+EI\,i_B\ .\ l_1 \quad ...(3)$$

$$0=+\frac{w_1 l_1^4}{24}+M_B\ .\ \frac{l_1^2}{2}+M_A\ .\ \frac{l_1^2}{6}-\frac{M_B . l_1^2}{6}+EI\,i_B\ .\ l_1$$

$$0=\frac{w_1 l_1^3}{24}+\frac{M_B l_1}{3}+\frac{M_A l_1}{6}+EI\ .\ i_B$$

or
$$+6EI\,i_B+(2M_B+M_A)\ l_1=-\frac{w_1 l_1^3}{4} \quad ...(4)$$

Span BC. When this span is also considered independently, there will be reactions at B and C equal to $\frac{w_2 l_2}{2}$ and B.M. diagram will be parabolic with maximum bending moment at the centre and equal to $\frac{w_2 l_2^2}{8}$. Taking the origin at B and x positive towards right, consider a section X-X at a distance of x from the end B.

B.M. M_x at the section $=+\frac{w_2 l_2}{2}\ x-\frac{w_2 x_2^2}{2}$ (positive B.M.)

M_x' at the section, *i.e.*, bending moment due to support moments M_B and M_C,

$$M_x'=M_B+\frac{(M_c-M_B)}{l_2}\ .\ x \qquad \text{(negative B.M.)}$$

Bending moment M_x is shown by *de* and M_x' is shown by *fe* in the diagram
Resultant bending moment $=M_x-M_x'$

or
$$EI\ \frac{d^2y}{dx^2}=\frac{w_2 l_2 x}{2}-\frac{w_2 x_2^2}{2}+M_B+\frac{(M_C-M_B)}{l_2}\ x \quad ...(5)$$

Integrating equation (5)

$$EI\ \frac{dy}{dx}=\frac{w_2 l_2 x^2}{2\times 2}-\frac{w_2 x_2^3}{2\times 3}+M_B . x+\frac{(M_C-M_B)}{l_2}\ \frac{x^2}{2}+C_3$$

where C_3 is the constant of integration at the end B, $x=0$, $\dfrac{dy}{dx}=i_B'$ (say)

So $$EI\,.\,i_B'=C_3$$

$$EI\,\frac{dy}{dx}=\frac{w_2l_2x^2}{4}-\frac{w_2x_2^3}{6}+M_B.x$$

$$+\frac{(M_C-M_B)}{l_2}\times\frac{x^2}{2}+EI.i_B' \qquad ...(6)$$

Integrating the equation (6) also

$$EIy=\frac{w_2l_2x^3}{12}-\frac{w_2x^4}{24}+M_B\,.\,\frac{x^2}{2}$$

$$+\frac{(M_C-M_B)}{l_2}\times\frac{x^3}{6}+EI\,i_B'\,x+C_4 \qquad ...(7)$$

$$C_4=0 \text{ as at } x=0 \quad y=0$$

Moreover at $x=l_2$, *i.e.*, end C, $y=0$

$$0=\frac{w_2l_2^4}{12}-\frac{w_2l_2^4}{24}+M_B.\frac{l_2^2}{2}+\frac{(M_C-M_B)}{l_2}\times\frac{l_2^3}{6}+EI\,.\,i_B'\,l_2$$

$$0=\frac{w_2l_2^3}{24}+\frac{M_B\,l_2}{2}+\frac{M_C\,l_2}{6}-\frac{M_B\,.\,l_2}{6}+EI\,.\,i'_B$$

$$0=\frac{w_2l_2^3}{24}+\frac{M_B\,l_2}{3}+\frac{M_C\,l_2}{6}+EI\,i_B'$$

or $$+6EI\,i_B'+(2M_B+M_C)\,l_2=-\frac{w_2\,l_2^3}{4} \qquad ...(8)$$

Adding the equations (4) and (8)

$$+6EI\,(i_B+i_B')-2M_B\,(l_1+l_2)+M_A\,l_1+M_C\,l_2$$

$$=-\frac{w_1\,l_1^3}{4}-\frac{w_2\,l_2^3}{4} \qquad ...(9)$$

But $i_B'=-i_B$ as the direction of x has been reversed. In the portion BA we took x positive towards left and in the position BC we took x positive towards right.

Therefore $$M_A\,.\,l_1+2M_B\,(l_1+l_2)+M_C\,l_2$$

$$=-\left(\frac{w_1\,l_1^3}{4}+\frac{w_2\,l_2^3}{4}\right) \qquad ...(10)$$

This is the well known Clapeyron's theorem for three moments for support moments for 3 consecutive supports for a continuous beam carrying uniformly distributed loads. If there are n supports for a continuous beam, the two ends being simply supported. Then three will be $(n-2)$ intermediate supports and $(n-2)$ equations will be formed so as to determine the support moments at $(n-2)$ supports.

Example 12·8-1. A continuous beam of length $2l$ is supported over 3 supports with span l each. A uniformly distributed load w per unit length acts throughout the length of the continuous beam. Determine support moments, support reactions and maximum deflection in the beam. EI is the flexural rigidity of the beam.

Solution. Fig. 12·10 shows a continuous beam ABC of length $2l$, span lengths AB and BC of l each, carrying a uniformly distributed load w per unit length.

The beam is supported only on two spans AB and BC, the support moments at A and B will be zero.

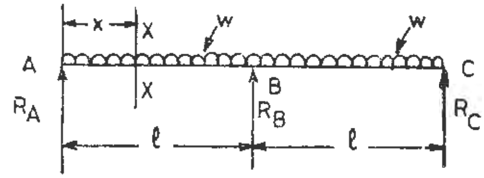


Fig. 12·10

$$M_A = M_C = 0$$

Moreover $\quad l_1 = l_2 = l,\ w_1 = w_2 = w$ as given.

Using the equation No. (10)

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = -\frac{w_1 l_1^3}{4} - \frac{w_2 l_2^3}{4}$$

In this case $\quad 2M_B\,(l+l) = -\dfrac{wl^3}{2}$

$$M_B = -\frac{wl^3}{8l} = -\frac{wl^2}{8}$$

while plotting the Mx diagram for each span, maximum bending moment $+wl^2/8$ occurs at the centre of the spans AB and BC and BM diagram is parabolic as shown. $AB'C$ is the Mx' diagram for bending moment due to support moments. The resultant bending moment diagram is shown with two points of contraflexure.

Support reactions. We know the magnitude of support moment M_B, therefore taking moments of the forces about the point B

$$+R_A\,.\,l - wl\,.\,\frac{l}{2} = -\frac{wl^2}{2} \text{ i.e., } M_B$$

$$R_A{}^2 = \frac{1}{l}\left(\frac{wl^2}{2} - \frac{wl^2}{8}\right) = \frac{3wl}{8}$$

Due to symmetry about the central support B

$$R_C = R_A = \frac{3wl}{8}$$

Total load $\quad = wl + wl = 2wl$

$\therefore$ Reaction, $\quad R_B = 2wl - \dfrac{3}{8}\,wl - \dfrac{3}{8}\,wl = \dfrac{5}{4}\,wl$

Fig. 12·11 (*a*) shows the BM diagram for the continuous beam having support moment

$$M_B = -\frac{wl^2}{8}.$$

Consider a section X-X at a distance of x from the end A in the span AB.

SF at the section $F = +R_A - wx$ (note that upward forces on the left side of the section are positive)

$$F = +\frac{3wl}{8} - wx$$

$$F = +\frac{3wl}{8} \quad \text{at} \quad x = 0$$

$$F = -\frac{5wl}{8} \quad \text{at} \quad x = l$$

$$F' = -\frac{5}{8}wl + \frac{5}{4}wl = +\frac{5}{8}wl$$

Similarly SF can be calculated in the portion BC.

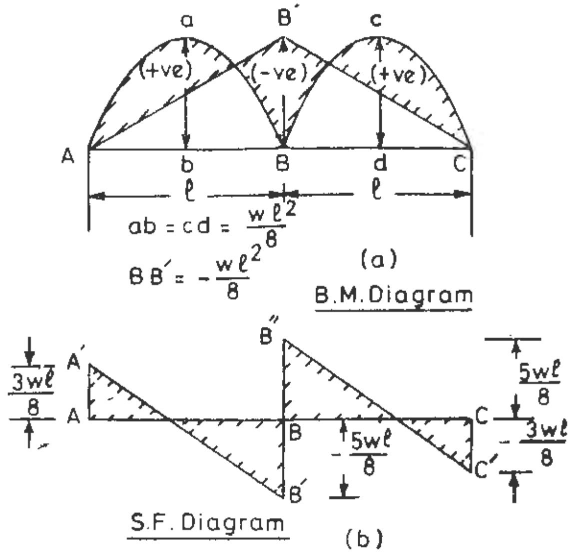


Fig. 12·11

Fig. 12·11 (*b*) shows the SF diagram for the continuous beam. To determine slope and deflection let us consider only the portion AB, since the beam is symmetrically loaded about its centre B and deflection curve for BC will be the same as for AB.

BM at the section $= R_A . x - \frac{wx^2}{2}$

or
$$EI\frac{d^2y}{dx^2} = \frac{3wl}{8}x - \frac{wx^2}{2} \quad \text{.. (1)}$$

Integrating equation (1) two times

$$EI\frac{dy}{dx} = \frac{3wlx^2}{16} - \frac{wx^3}{6} + C_1 \quad \text{...(2)}$$

and
$$EI\,y = \frac{3wlx^3}{48} - \frac{wx^4}{24} + C_1x + C_2 \quad \text{...(3)}$$

at $x=0$, $y=0$, so $C_2=0$

at $x=l$, $y=0$, end B

$$\therefore \quad 0 = \frac{wl^4}{16} - \frac{wl^4}{24} + C_1 l$$

$$C_1 = -\frac{wl^3}{48}$$

$$EI\frac{dy}{dx} = \frac{3wlx^2}{16} - \frac{wx^3}{6} - \frac{wl^3}{48}$$

For the deflection to be maximum, $dy/dx = 0$ at the particular section.

So $$\frac{3wlx^2}{16} - \frac{wx^3}{6} - \frac{wl^3}{48} = 0$$

or $$9lx^2 - 8x^3 - l^3 = 0$$

$$(x-l)(l^2 + lx - 8x^2) = 0$$

So $$x = l$$

or $$8x^2 - lx - l^2 = 0$$

$$x = \frac{l + \sqrt{l^2 + 32l^2}}{16} = \frac{l + l \times 5{\cdot}744}{16}$$

$$= 0{\cdot}4215\ l$$

Substituting this value in equation (3)

$$EI\ y_{max} = \frac{wl}{16}(0{\cdot}4215\ l)^3 - \frac{w}{24}(0{\cdot}4215\ l)^4 - \frac{wl^3}{48}(0{\cdot}4215\ l)$$

$$= wl^4\,[0{\cdot}00468 - 0{\cdot}0013 - 0{\cdot}00878]$$

$$= -0{\cdot}0054\ wl^4$$

$$y_{max} = -\frac{0{\cdot}0054\ wl^4}{EI}$$ (indicating downward deflection).

Example 12·8-2. A continuous beam $ABCD$, 13 metres long simply supported with spans $AB = 4$ m, $BC = 5$ m, $CD = 4$ m carries uniformly distributed load of 1·2 tonnes/metre run over AB and CD and 1·6 tonnes/metre run over BC. Determine (1) support reactions, (2) support moments.

Determine the deflection at the centre of the portion BC.

Given $E = 2100$ tonnes/cm², $I = 4000$ cm⁴.

Solution. In this problem

$l_1 = 4$ m, $l_2 = 5$ m, $l_3 = 4$ m

Rate of loading

$w_1 = 1{\cdot}2$ T/m, $w_2 = 1{\cdot}6$ T/m

$w_3 = 1{\cdot}2$ T/m.

Since the continuous beam is symmetrically loaded about its mid point, the support moments $M_B = M_C$ and at ends of the beam, moments, $M_A = M_D = 0$.

Moreover reactions

$R_A = R_D$ and $R_B = R_C$

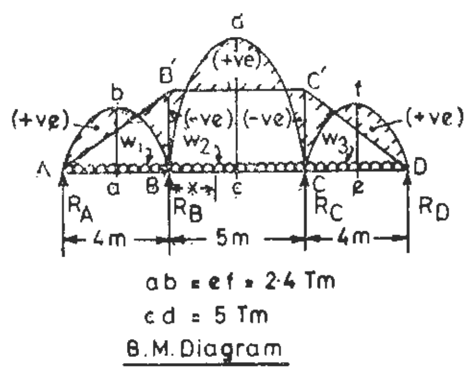


Fig. 12·12

Using the theorem of 3 moments for the spans AB and BC

$$M_A = 0$$

$$M_A \cdot l_1+2M_B(l_1+l_2)+M_C \cdot l_2=-\frac{w_1l_1^3}{4}-\frac{w_2l_2^3}{4}$$

$$0\times 4+2M_B(4+5)+5\,M_C=-\frac{1{\cdot}2\times 4^3}{4}-\frac{1{\cdot}6\times 5^3}{4}$$

$$18\,M_B+5\,M_C=-19{\cdot}2-50=-69{\cdot}2$$

But $M_B=M_C$ (due to symmetrical loading about the centre of the continuous beam)

$$\therefore \qquad 23\,M_B=-69{\cdot}2$$

$$M_B=3{\cdot}009 \text{ T-m}=M_C$$

To draw the BM diagram, let us find out.

Maximum BM, M_x for span AC

$$=\frac{w_1l_1^2}{8}=\frac{1{\cdot}2\times 4^2}{8}=+2{\cdot}4 \text{ T-m}$$

Maximum BM, M_x for span BC

$$=\frac{w_2l_2^2}{8}=\frac{1{\cdot}6\times 5^2}{8}=+5 \text{ T-m}$$

Maximum BM, M_x for span CD

$$=\frac{w_3l_3^2}{8}=\frac{1{\cdot}2\times 5^2}{8}=+2{\cdot}4 \text{ T-m}$$

Fig. 12·12 shows the BM diagram where $AB'C'D$ is the M_x' diagram due to support moments and $A\,b\,B$, $B\,d\,C$, $C\,f\,D$ are the M_x diagrams for spans AB, BC and CD respectively. The resultant bending moment diagram is shown by positive and negative areas.

Support reactions. Taking moments of the forces about the point B

$$R_A\times 4-\frac{w_1\times l_1^2}{2}=M_B$$

$$4\,R_A-\frac{1{\cdot}2\times 4^2}{2}=-3{\cdot}009$$

$$4\,R_A-9{\cdot}6=3{\cdot}009$$

$$R_A=1{\cdot}65 \text{ Tonnes}=R_D$$

Total vertical load on beam

$$=4\times 1{\cdot}2+5\times 1{\cdot}6+4\times 1{\cdot}2=17{\cdot}6 \text{ Tonnes}$$

Reaction, $\quad R_B=\text{Reaction, } R_C=\dfrac{17{\cdot}6-1{\cdot}65-1{\cdot}65}{2}=7{\cdot}15 \text{ Tonnes}$

To determine the deflection at the centre of the span BC, let us consider a section at a distance of x from the end B, in span BC.

B.M. at the section $\quad =M_B+R_B.x-\dfrac{w_2x^2}{2}=-3{\cdot}009+7{\cdot}15\,x-\dfrac{1{\cdot}6x^2}{2}$

or
$$EI\,\frac{d^2y}{dx^2}=-3{\cdot}009+7{\cdot}15x-0{\cdot}8\,x^2 \qquad \ldots(1)$$

Integrating equation (1) we get

$$EI\,\frac{dy}{dx}=-3{\cdot}009\,x+7{\cdot}15\,\frac{x^2}{2}-0{\cdot}8\,\frac{x^3}{3}+C_1$$

(constant of integration)

At the centre of the span BC, $\frac{dy}{dx}=0$, because the continuous beam is symmetrically loaded about this point. So taking $x=2\cdot5$ m

$$0=-3\cdot009\times2\cdot5+7\cdot15\times\frac{2\cdot5^2}{2}-\frac{0\cdot8\times2\cdot5^3}{3}+C_1$$

$$C_1=4\cdot167+7\cdot5225-22\cdot343=-10\cdot653$$

Therefore, $$EI\frac{dy}{dx}=-3\cdot009\,x+7\cdot15\,\frac{x^2}{2}-\frac{0\cdot8\,x^3}{3}-10\cdot653 \quad ...(2)$$

Integrating equation (2),

$$EIy=-3\cdot009\,\frac{x^2}{2}+\frac{7\cdot15\,x^3}{6}-\frac{0\cdot8\times x^4}{12}-10\cdot653x+C_2$$

$C_2=0$ because at $x=0$, $y=0$

So $$EIy=-3\cdot009\,\frac{x^2}{2}+\frac{7\cdot15\,x^3}{6}-\frac{0\cdot8\times x^4}{12}-10\cdot653\,x$$

Maximum deflection occurs at the centre *i.e.*, at $x=2\cdot5$ m

$$\therefore \quad EIy_{max}=-3\cdot009\times\frac{2\cdot5^2}{2}+\frac{7\cdot15\times2\cdot5^3}{6}-\frac{0\cdot8\times2\cdot5^4}{12}-10\cdot653\times2\,5$$

$$=-9\cdot403+19\cdot401-2\cdot604-26\cdot632=-19\cdot238$$

$$EI=2100\times10^4\times40{,}000\times10^{-8}\ \text{Tm}^2=8400\ \text{Tm}^2$$

$$y_{max}=-\frac{19\cdot238}{8400}=-0\cdot0023=-2\cdot3\ \text{mm}$$

Exercise 12·8-1. A continuous beam ABC, 10 m long simply supported over A, B and C with $AB=4$ m, $BC=6$ m carries uniformly distributed load of 12 kN/m over span AB and 10 kN/m over span BC. Determine the support reactions and support moments.

[**Ans.** $R_A=14\cdot85$ kN, $R_B=69\cdot25$ kN, $R_C=23\cdot9$ kN, $M_B=-36\cdot6$ kNm]

Exercise 12·8-2. A continuous beam $ABCD$ of length $3l$ supported over 3 equal spans, carries uniformly distributed load w per unit length. Determine support reactions and support moments.

$\left[\textbf{Ans.}\ 0\cdot4\,wl,\ 1\cdot1\,wl,\ 1\cdot1\,wl,\ 0\cdot4\,wl\ ;\ 0,\ -\frac{wl^2}{10},\ -\frac{wl^2}{10},\ 0\right]$

12·9. THEOREM OF THREE MOMENTS—ANY LOADING

Let us consider two spans of a continuous beam. Spans AB and BC of lengths l_1 and l_2 respectively carrying uniformly distributed load and concentrated loads as shown. To determine support moments, first of all let us construct the M_x diagrams of both the spans *i.e.* the B.M. diagrams considering the spans AB and BC independently and ends are simply supported. Diagram $A\,a\,B$ is the M_x diagram for span AB and diagram $B\,cd\,C$ is the M_x diagram for span BC. Say M_A, M_B and M_C are the moments at the supports A, B and C respectively. Diagram $AA'B'C'C$ is the M_x' diagram due to the support moments.

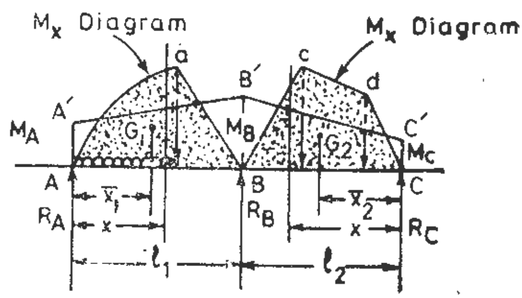


Fig. 12·13

Consider the span AB, and a section at a distance of x from end A (*i.e.*, origin at A and x positive towards right)

B.M. at the section $=M_x+M_x'$

or $EI\,\dfrac{d^2y}{dx^2}=M_x+M_x'$, multiplying both the sides by x and integrating

or
$$EI\int_0^{l_1}\frac{d^2y}{dx^2}\,x\,dx=\int_0^{l_1}M_x\,x\,dx+\int_0^{l_1}M_x'.xdx \quad ...(1)$$

$$EI\left|x\frac{dy}{dx}-y\right|_0^{l_1}=\int_0^{l_1}M_x.x\,dx+\int_0^{l_1}M_x'.xdx$$

$$EI[(l_1\,i_B-0)-(0\times i_A-0)]=a_1\bar{x}_1+a_1'\bar{x}' \quad ...(2)$$

or $EI\,l_1\,i_B=a_1\bar{x}_1+a_1'\bar{x}'$

where
a_1 = area of the M_x diagram for span AB ($A\ a\ B$)
$\bar{x}_1$ = distance of C.G. of a_1 from the end A
a_1' = area of M_x' diagram ($AA'B'B$)
$\bar{x}_1'$ = distance of C.G. of the diagram a_1' from the end A

or $a_1'\bar{x}_1'=(M_A+2M_B)\,\dfrac{l_1^2}{6}$

So $EI\,l_1\,i_B=a_1\bar{x}_1+(M_A+2M_B)\,\dfrac{l_1^2}{6}$

or
$$EI\,i_B=\frac{a_1\bar{x}_1}{l_1}+(M_A+2M_B)\,\frac{l_1}{6} \quad ...(3)$$

Now consider the span CB, origin at C and x positive towards left, a section at a distance of x from end C.

B.M. at the section, $EI\,\dfrac{d^2y}{dx^2}=M_x+M_x'$

Multiplying both the sides by xdx and integrating over the length of the span BC

$$EI\int_0^{l_2}\frac{d^2y}{dx^2}\,x\,dx=\int_0^{l_2}M_x\,xdx+\int_0^{l_2}M_x'\,x\,dx \quad ...(4)$$

or
$$EI\left|x\frac{dy}{dx}-y\right|_0^{l_2}=a_2\bar{x}_2+a_2'\bar{x}_2'$$

$$EI[(l_2\,.\,i_B'-0)-(0\times i_B-0)]=a_2\bar{x}_2+a'\bar{x}^{2\prime}$$

or $EI\,l_2\,i_B'=a_2\bar{x}_2+a_2'\bar{x}_2'$

where
a_2 = area of the M_x diagram for the span CB ($C\ dc\ B$)
$\bar{x}_2$ = distance of C.G. of a_2 from end C
a_2' = area of M_x' diagram ($CC'B'B$)
$\bar{x}_2'$ = distance of C.G. of a_2' from the end C

or $$EIi_B' = \frac{a_2\bar{x}_2}{l_2} + \frac{a_2'\bar{x}_2'}{l_2} = \frac{a_2\bar{x}_2}{l_2} + (M_A+2M_B)\,\frac{l_2}{6} \qquad ...(5)$$

Adding the equations (4) and (5),

$$EI\,i_B + EI\,i_B' = \frac{a_1\bar{x}_1}{l_1} + \frac{a_2\bar{x}_2}{l_2} + \frac{M_A l_1}{6} + \frac{2M_B(l_1+l_2)}{6} + \frac{M_B l_2}{6} \qquad ...(6)$$

But slope $i_B = -i_B'$ because we have taken x positive towards right in span AB and x positive towards left in span CB.

So $$M_A l_1 + 2M_B(l_1+l_2) + M_B l_2 + \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2} = 0 \qquad ...(7)$$

The areas a_1 and a_2 are positive as per the conventions we have taken, so the support moments M_A, M_B and M_C will be negative.

If the span AB carries only the uniformly distributed load w_1 per unit length and span CB carries only the uniformly distributed load w_2 per unit length then

$$\frac{6a_1\bar{x}_1}{l_1} = 6\times\frac{2}{3}\times l_1 \times \frac{w_1 l_1^2}{8} \times \frac{l_1}{2} \times \frac{1}{l_1} = \frac{w_1 l_1^3}{4} \text{ as } \bar{x}_1 = \frac{l_1}{2}$$

$$\frac{6a_2\bar{x}_2}{l_2} = 6\times\frac{2}{3}\times l_2 \times \frac{w_2 l_2^2}{8} \times \frac{l_2}{8} \times \frac{1}{l_2} = \frac{w_2 l_2^3}{4} \text{ as } \bar{x}_2 = \frac{l_2}{2}$$

and we obtain

$$M_A l_1 + 2M_B(l_1+l_2) + M_C l_2 = -\frac{w_1 l_1^3}{4} - \frac{w_2 l_2^3}{4}$$

as already derived in article 12·8.

Example 12·9-1. A continuous beam $ABCD$, 12 m long supported over spans $AB = BC = CD = 4$ m, carries a uniformly distributed load of 3 tonnes/metre run over span AB, a concentrated load of 4 tonnes at a distance of 1 m from point B on support BC and a load of 3 tonnes at the centre of the span CD. Determine support moments and draw the B.M. diagram for the continuous beam.

Solution. Fig. 12·14 shows the continuous beam $ABCD$, with equal spans, span AB with a uniformly distributed load of 3T/m, span BC with a concentrated load 4T at a distance of 1 m from B and span CD with a central load of 3 tonnes.

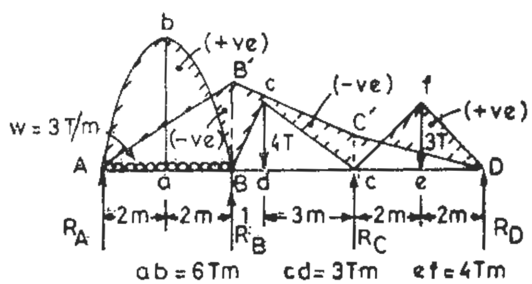


Fig. 12·14

Let us construct M_x diagram for each span.

Span AB. ab, Maximum bending moment $= \frac{wl^2}{8} = \frac{3\times4^2}{8} = 6$ Tm

area $a_1 = 6\times4\times\frac{2}{3} = 16$ Tm²

$a_1\bar{x}_1 = 16\times2 = 32$ Tm³

(origin at A and x positive towards right)

Span CB. M_{max}, $cd = \frac{4\times1\times3}{4} = 3$ Tm

Taking again at C and x positive towards left.

$$a_2\bar{x}_2 = \frac{3\times3}{2}(2) + \frac{3\times1}{2}\left(3+\frac{1}{3}\right) = 9 + \frac{3}{2}\times\frac{10}{3} = 14 \text{ Tm}^3$$

Problem 13·3. A hollow circular shaft is required to transmit 300 metric horse power at 200 r.p.m. The maximum torque developed is 1·5 times the mean torque. Determine the external diameter of the shaft if it is double the internal diameter if the maximum shear stress is not to exceed 800 kg/cm². Given $G=820$ tonnes/cm².

Solution. Mean torque, $T_{mean}=\dfrac{300\times4500}{2\pi\times200}=1074{\cdot}3$ kg-metre

Maxmimum torque, $T_{max}=1{\cdot}5\ T_{mean}=1611{\cdot}45$ kg-metres$=161145$ kg cm

Say external diameter $=D$ cm

Thus internal diameter $=0{\cdot}5\ D$

Polar moment of inertia. $J=\dfrac{\pi}{32}\left[D^4-(0{\cdot}5\ D)^4\right]=\dfrac{\pi\times15D^4}{16\times32}$

Now $\dfrac{T}{J}=\dfrac{q}{D/2}$ (q is maximum shear stress)

$$\frac{100\times1611{\cdot}45\times16\times32}{15\pi D^4}=\frac{2q}{D}=\frac{2\times800}{D}$$

or $$D^3=\frac{100\times1611{\cdot}45\times512}{1600\times15\times\pi}=1094{\cdot}27$$

Shaft diameter, $D=10{\cdot}3$ cm.

Problem 13·4. A solid bar of a metal of diameter 50 mm and length 200 mm is tested under tension. A load of 10,000 N produces an extension of 0·0051 mm. When the same bar is tested as a shaft, a torque of 4000 Nm produces an angular twist of 1 degree. Determine the Young's modulus, shear modulus and Poisson's ratio of the shaft.

Solution. Diameter of tbe shaft$=50$ mm

Direct stress, $f=\dfrac{4\times10000}{\pi\times50\times50}=5{\cdot}093$ N/mm²

Change in length, $\delta l=0{\cdot}0051$ mm

Strain, $\epsilon=\dfrac{0{\cdot}0051}{200}=2{\cdot}55\times10^{-5}$ N/mm²

Young's modulus, $E=\dfrac{5{\cdot}093}{2{\cdot}55\times10^{-5}}=1{\cdot}997\times10^5$ N/mm²

Torque, $T=4000$ Nm$=4\times10^6$ Nmm

Polar moment of inertia,

$$J=\frac{\pi d^4}{32}=\frac{\pi\times50^4}{32}=61{\cdot}359\times10^4\ \text{mm}^4$$

Shear modulus, $G=\dfrac{Tl}{J\theta}$ where $\theta=\dfrac{\pi}{180}$ radian

$$=\frac{4\times10^6\times200\times180}{61{\cdot}359\times10^4\times\pi}=0{\cdot}747\times10^5\ \text{N/mm}^2$$

Poisson's ratio $=\dfrac{E}{2G}-1=\dfrac{1{\cdot}997}{2\times0{\cdot}747}-1=1{\cdot}337-1=0{\cdot}337$

Efficiency $=60\%$

Input supplied by shaft$=\dfrac{5000\times 300}{0\cdot 6}=25\times 10^5$ kg-m

R.P.M.$=200$

Torque on the shaft, $T=\dfrac{25\times 10^5}{2\times\pi\times 200}=1989\cdot 43$ kg-m$=198943$ kg-cm

Say the diameter $=d$ cm

Maximum shear stress, $q=500$ kg/cm^2

$$T=\frac{\pi}{16}d^3\times q$$

or

$$d^3=\frac{16\times 198943}{\pi\times 500}=2026\cdot 4141 \text{ cm}^3$$

Shaft diameter, $d=12\cdot 65$ cm.

Problem 13·2. A solid circular steel shaft is transmitting 200 Horse power at 300 rpm. Determine the diameter of the shaft if the maximum shear stress is not to exceed 80 N/mm^2 and angular twist per metre length of the shaft is not to exceed 1°.

$$G=80{,}000 \text{ N/mm}^2$$

Solution. H.P.$=200$

Power transmitted $=200\times 746$ Watts per second

$=200\times 746$ Nm per second

Angular speed, $\omega=\dfrac{2\times\pi\times 300}{60}=31\cdot 416$ rad/second

Torque, $T=\dfrac{200\times 746}{31\cdot 416}=4749\cdot 2$ Nm

$=474\cdot 92\times 10^4$ Nmm

(*a*) Considering the maximum shearing stress as the criterion for design

$$d^3=\frac{16\times 474\cdot 92\times 10^4}{\pi\times 80}=302\cdot 34\times 10^3$$

$$d=67\cdot 15 \text{ mm}$$

(*b*) Considering the maximum angular twist as the criterion for design

$$\frac{T}{\pi d^4}\times 32=\frac{G\theta}{l}$$

$$\theta=1^\circ=\frac{\pi}{180}\text{ radian}$$

$$l=1\text{ m}=1000\text{ mm}$$

$$d^4=\frac{32\,Tl}{\pi G\theta}$$

$$=\frac{32\times 474\cdot 92\times 10^4\times 1000\times 180}{\pi\times 80000\times\pi}=3464\cdot 58\times 10^4$$

$$d=7\cdot 672\times 10=76\cdot 72\text{ mm}$$

The diameter of the shaft should not be less than 76·72 mm so as to keep the maximum stress and angular twist within the limits.

Now $$T=2\tau_1\times A_1+2\tau_2\times A_2$$

$$4\times10^6 \text{ Nmm}=2\times15\times10^3\times\tau_1+2\times7{\cdot}5\times10^3\times\tau_2$$

$$2000=15\ \tau_1+7{\cdot}5\ \tau_2 \qquad \ldots(1)$$

But θ in cell 1 $=\theta$ in cell 2

So $$\frac{1}{2A_1G}\ (a_1\tau_1-a_{12}\tau_2)=\frac{1}{2A_2G}\ (a_2\tau_2-a_{12}\tau_1)$$

$$130\ \tau_1-60\ \tau_2=\frac{A_1}{A_2}\ (160\ \tau_2-60\ \tau_1)$$

But $$\frac{A_1}{A_2}=2$$

So $$130\ \tau_1-60\ \tau_2=320\ \tau_2-120\ \tau_1$$

$$250\ \tau_1=380\ \tau_2 \quad \text{or} \quad \tau_1=1{\cdot}52\ \tau_2 \qquad \ldots(2)$$

Substituting in equation (1),

$$15\times1{\cdot}52\ \tau_2+7{\cdot}5\ \tau_2=2000$$

Shear flow, $$\tau_2=\frac{2000}{30{\cdot}3}=66{\cdot}00 \text{ N/mm}$$

Shear flow, $$\tau_1=1{\cdot}52\times66=100{\cdot}33 \text{ N/mm}$$

Shear stress in rectangular part $$=\frac{100{\cdot}33}{5}=20{\cdot}06 \text{ N/mm}^2$$

Shear stress in triangular part $$=\frac{66}{2{\cdot}5}=26{\cdot}4 \text{ N/mm}^2$$

Shear stress in web $$=\frac{\tau_1-\tau_2}{2{\cdot}5}=\frac{100{\cdot}33-66}{2{\cdot}5}=13{\cdot}73 \text{ N/mm}^2$$

Exercise 13·15-1. A steel girder is of the section shown in Fig. 13·36. It has uniform thickness of 1·2 cm throughout. What is the allowable torque if the maximum shear stress is not to exceed 300 kg/cm². What will be the angular twist per metre length of the girder. What is the stress in the middle web of the section $G=820$ tonnes/cm²

[**Ans.** 2073·6 kg-metre, 0·26°, stress in middle web is zero]

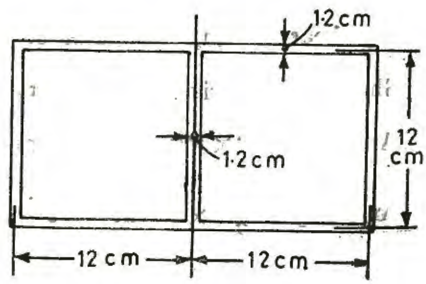


Fig. 13·36

Problem 13·1. A circular shaft running at 200 rpm transmits power to a crane lifting a load of 5 tonnes at a speed of 5 metres/second. If the efficiency of the gearing system of the crane is 60%, determine the diameter of the shaft if the maximum shearing stress is not to exceed 500 kg/cm².

Solution.

Load lifted, $W=5$ tonnes $=5000$ kg

Speed of lifting $=5$ metres/second

Output per minute $=5000\times5\times60$ kg-metre where $2\tau_2 A_1'$ is the moment due to the shear flow τ_2 in the middle web

Total twisting moment, $T = T_1 + T_2$

$$= 2\tau_1 A_1 + 2\tau_2 A_2 \qquad \ldots(4)$$

For continuity the angular twist per unit length in each cell will be the same. For closed thin sections.

$$\theta = \frac{\tau}{2AG} \oint \frac{ds}{t}$$

but in this case shear flow is changing, so

$$\theta = \frac{1}{2AG} \int \frac{\tau ds}{t}$$

Say $a_1 = \oint \frac{ds}{t}$ for cell 1 including the web

$a_2 = \oint \frac{ds}{t}$ for cell 2 including the web

$a_{12} = \oint \frac{ds}{t}$ for the web

For cell 1, $\theta = \frac{1}{2A_1G}(a_1\tau_1 - a_{12}\tau_2) \qquad \ldots(5)$

For cell 2, $\theta = \frac{1}{2A_2G}(a_2\tau_2 - a_{12}\tau_1) \qquad \ldots(6)$

From equations (4), (5) and (6) shear flow τ_1, τ_2 and angular twist θ can be worked out.

Example 13·15-1. The Fig. 13·35 shows the dimensions of a double-celled cross section in the form of a rectangle and a triangle. A torque of 4 kNm is applied ; calculate the shear stress in each part and the angle of twist per metre length. $G = 82000$ N/mm².

Say shear flow in rectangular cell $= \tau_1$

Shear flow in triangular cell $= \tau_2$

Area $A_1 = 150 \times 100 = 15 \times 10^3$ mm²

Area $A_2 = \frac{150}{2} \times \sqrt{125^2 - 75^2}$

$$= \frac{150 \times 100}{2} = 7.5 \times 10^3 \text{ mm}^2$$

Line integrals

$$a_1 = \frac{150}{5} + \frac{100}{5} + \frac{100}{5} + \frac{150}{2.5} = 130$$

$$a_2 = \frac{150}{2.5} + \frac{125}{2.5} + \frac{125}{2.5} = 160$$

$$a_{12} = \frac{150}{2.5} = 60$$

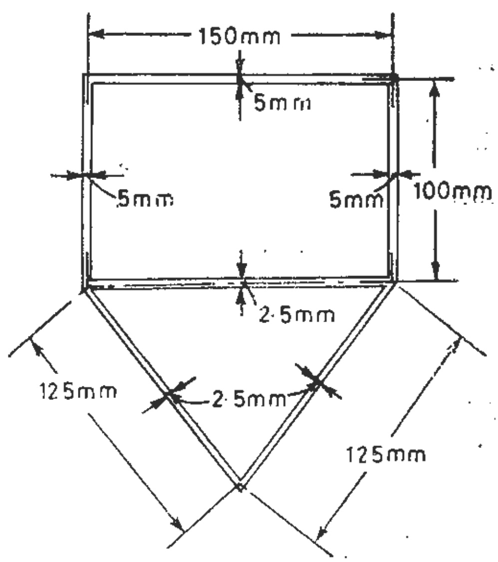


Fig. 13·35

$$=\frac{150\times10^{-2}}{8\times188}\text{ radian/cm length}$$

$$=0{\cdot}1\times10^{-2}\text{ radian/cm length}$$

$$=0{\cdot}1\text{ radian/metre length}=5{\cdot}7^\circ\text{ per metre length.}$$

Exercise 13·14-1. A *T* section with flange 10 cm×1 cm and web 19 cm×0·8 cm is subjected to a torque 2000 kg-cm. Find the maximum shear stress and angle of twist per metre length. G=82,0000 kg/cm².

[**Ans.** 270·78 kg/cm², 1·275° per metre length]

13·15. TORSION OF THIN WALLED MULTI-CELL SECTIONS

The analysis of thin walled closed sections can be extended to multi-cell sections. Consider a two cell section as shown in the Fig. 13·34. Say the shear flow in cell 1 is τ_1, in cell 2 is τ_2 and in the web shear flow is τ_3. Now consider the equilibrium of shear forces at

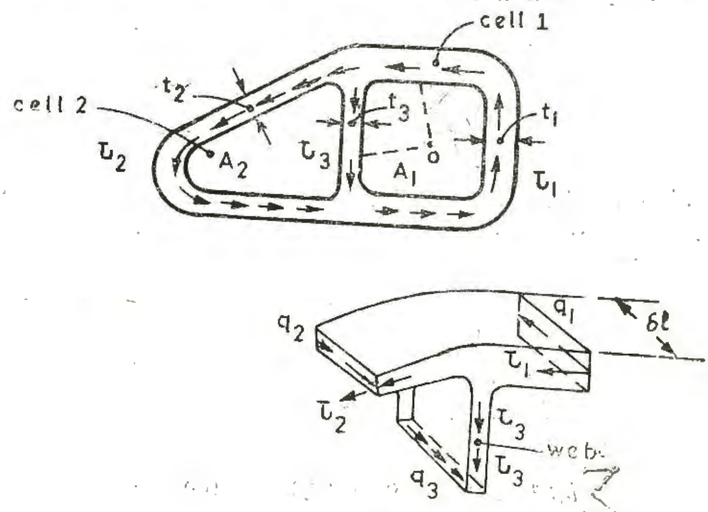


Fig. 13·34

the junction of the two cells, taking a small length δl. The complementary shear stresses q_1, q_2 and q_3 are shown on the longitudinal sections of length δl each but thicknesses t_1, t_2 and t_3 respectively. For the equilibrium in the direction of the axis of the tube.

$$q_1t_1\delta l-q_2t_2\delta l-q_3t_3\,\delta l=0$$

or $$q_1t_1=q_2t_2+q_3t_3 \quad\text{or}\quad \tau_1=\tau_2+\tau_3 \qquad ...(1)$$

Shear flow τ_1=shear flow τ_2+shear flow τ_3. This is equivalent to fluid flow dividing itself into two streams.

Moreover shear flow in web, $\tau_3=\tau_1-\tau_2$

Now twisting moment T_1 about O due to τ_1 flowing in cell 1

$$T_1=2\,\tau_1\,A_1 \qquad ...(2)$$

where A_1=area of the cell 1

Twisting moment T_2 about O, due to τ_2 in cell 2

$$T_2=2\tau_2(A_2+A_1')-2\tau_2A_1' \qquad ...(3)$$

In the case of channel section and I section,

Torque, $T=\frac{1}{3}\,G\theta\,(b_1t_1^3+b_2t_2^3+b_3t_3^3)$

Angle of twist per unit length,

$$\theta=\frac{3T}{G\sum_{i=1}^{3} b_i\,t_i^3}$$

q=maximum shear stress

$$=\frac{3T}{(b_1t_1^2+b_2t_2^2+b_3t_3^2)}=\frac{3T}{G\sum_{i=1}^{3} b_i\,t_i^2}$$

In the case of Angle Section and T Section,

$$\theta=\frac{3T}{G\sum_{i=1}^{2} b_i\,t_i^3}\ ,\ \text{and}\ \ q=\frac{3T}{\sum_{i=1}^{3} b_i\,t_i^2}$$

Example 13·14-1. An I section with flanges 10 cm$\times$2 cm and web 28 cm$\times$1 cm is subjected to a torque T=5 kNm. Find the maximum shear stress and angle of twist per unit length. G=80,000 N/mm².

Solution. $G=8\times10^4$ N/mm²$=8\times10^6$ N/cm²

Torque $T=5$ kNm$=5\times10^5$ N cm

2 Flanges, $b=10$ cm, $t=2$ cm

1 Web $b'=28$ cm, $t'=1$ cm

$$\sum_{i=1}^{3} b_it_i^2=10\times4+10\times4+28\times1=108\ \text{cm}^3$$

$$\sum_{i=1}^{4} b_it_i^3=10\times8+10\times8+28\times1=188\ \text{cm}^4$$

Maximum shear stress, $q=\dfrac{3T}{\sum_{i=1}^{3} b_i\,t_i^2}=\dfrac{3\times5\times10^5}{108}$

$$=1{\cdot}39\times10^4\ \text{N/cm}^2=139{\cdot}0\ \text{N/mm}^2$$

Angular twist per unit length, $\theta=\dfrac{3T}{G\sum_{i=1}^{3} b_i\,t_i^3}=\dfrac{3\times5\times10^5}{8\times10^6\times188}$

Exercise 13·13-1. A thin rectangular steel section is shown in the Fig. 13·31. Determine the torque if maximum shear stress is 35000 kPa. If the length of the shaft is 2 metres, find the angular twist.

$G=82000$ N/mm².

Ans. [2.94 kNm, 1·2°]

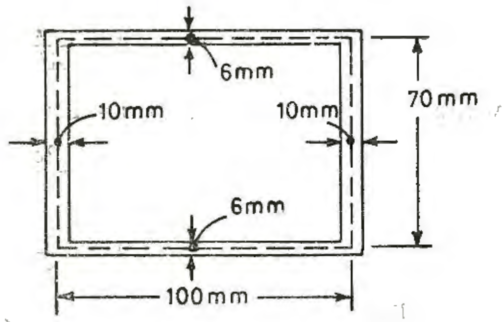


Fig. 13·31

Exercise 13·13-2. A tubular circular section of mean radius 50 mm and wall thickness 5 mm is subjected to twisting moment such that maximum shear stress developed is 80 N/mm². Another tubular square section of same wall thickness 5 mm and same circumference as that of circular section is also subjected to the same twisting moment. Find the maximum shear stress developed in square section. [**Ans.** 101·86 N/mm²]

13·14. TORSION OF THIN RECTANGULAR SECTIONS

Figure 13·32 shows a thin rectangular section subjected to the torque T. Thickness t of the section is small in comparison to its width b. This section consists only one boundary. In this case maximum shear stress occurs at $y=\pm\frac{t}{2}$.

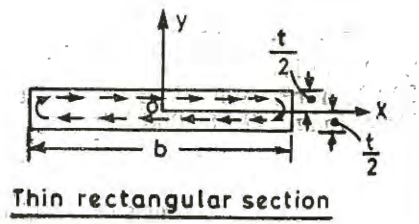


Fig. 13·32

If θ=angular twist per unit length

T=Torque on the section

$$T=\tfrac{1}{3}\,bt^3\,G\theta \qquad \text{...(1)}$$

Angle of twist per unit length,

$$\theta=\frac{1}{G}\cdot\frac{3T}{bt^3} \qquad \text{...(2)}$$

$$q=\text{maximum shear stress}=\pm\frac{3T}{bt^2}$$

These results can be applied to sections built up of rectangular strips and having only one boundary such as Angle Section, I section, T and channel sections as shown in Fig 13·33.

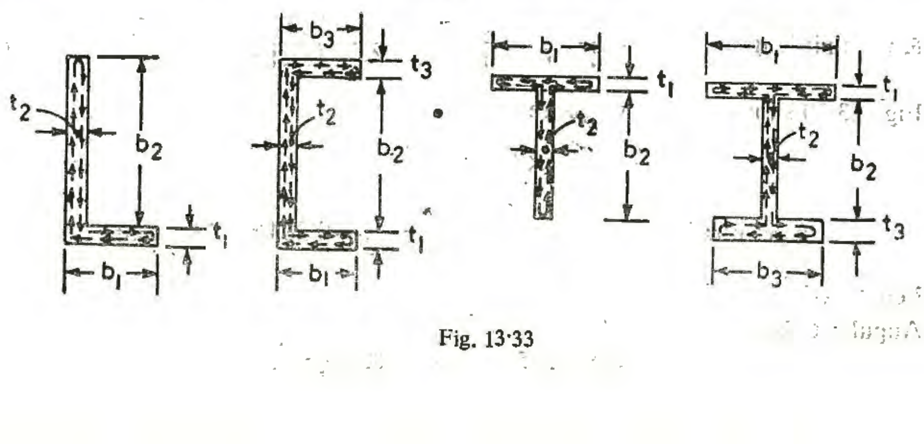


Fig. 13·33

Angular twist for the thin box section,

$$\theta = \frac{T}{4A^2 G} \oint \frac{ds}{t}$$

where area $A = 3a \times 2a = 6a^2$

$$\oint \frac{ds}{t} = \frac{3a}{t} + \frac{2a}{t} + \frac{2a}{t} + \frac{3a}{t} = \frac{10a}{t}$$

$$\theta = \frac{T}{4 \times (6a^2)^2 \cdot G} \times \frac{10a}{t} = \frac{10\ Ta}{144a^4\ Gt} = \frac{10\ T}{144\ Ga^3 t} \quad ...(2)$$

But $\theta' = \theta$

So
$$\frac{32}{G\pi \times 81\ a^4} = \frac{10T}{144\ Ga^3 t}$$

$$32 \times 144a^3\ t = 10\pi \times 81\ a^4$$

$$t = \frac{10\pi \times 81\ a}{32 \times 144} = 0{\cdot}55\ a$$

Example 13·13-2. A shaft of hollow square section of outer side 48 mm and inner side 40 mm is subjected to a twisting moment such that the maximum shear stress developed is 200 N/mm². What is the torque acting on the shaft and what is the angular twist if the shaft is 1·6 m long. $G = 80{,}000$ N/mm².

Solution. Outer side = 48 mm

Inner side = 40 mm

Mean side = 44 mm

Thickness, $t = 4$ mm

Area, $A = 44 \times 44$ mm²

Maximum shear stress,

$q = 200$ N/mm²

Shear flow, $\tau = q \times t$

$= 200 \times 4 = 800$ N/mm length

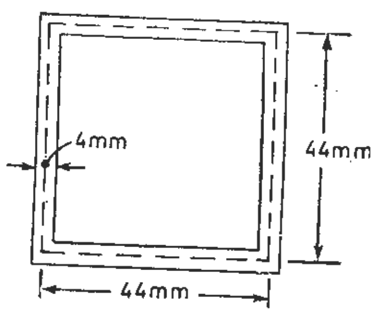


Fig. 13·30

Torque on the shaft $T = 2A\tau$

$= 2 \times 44 \times 44 \times 800$ N mm

$= 1548{\cdot}8$ Nm $= 1{\cdot}5488$ kNm

θ, per unit length
$$= \frac{q}{2AG} \oint \frac{ds}{t} = \frac{200}{2 \times 44 \times 44 \times 8 \times 10^4} \oint \frac{ds}{t}$$

Fig. 13·30 shows a hollow square section.

$$\frac{ds}{t} = \frac{44}{4} \times 4 = 44$$

$$\theta \text{ per mm length} = \frac{200 \times 44}{2 \times 44 \times 44 \times 8 \times 10^4} = 0{\cdot}284 \times 10^{-4} \text{ radian}$$

Length of the shaft $= 1{\cdot}6$ m $= 1600$ mm

Angular twist in shaft $= 0{\cdot}284 \times 10^{-4} \times 1600$

$= 0{\cdot}04545$ radian $= 2{\cdot}60$ degree.

Example 13·13-1. A thin walled box section $3a\times2a\times t$ is subjected to a twisting moment T. A solid circular section of diameter d is also subjected to the same twisting moment. Determine the thicknees of the box section (*a*) if the maximum shear stress developed in box section is the same as that in solid circular section, and $d=3a$, and (*b*) if the stiffness for both is the same under the same torque.

Solution. Fig. 13·29 shows the thin walled section $3a\times2a\times t$ and a solid circular section of diameter d.

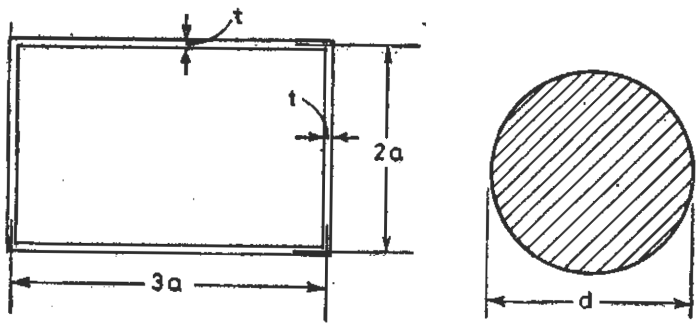


Fig. 13·29

Torque$=T$

Maximum shear stress in circular section,

$$q=\frac{16T}{\pi d^3}$$

Shear flow in box section, $\tau=\dfrac{T}{2A}$

$$2A=2\times3a\times2a=12a^2$$

taking $a>>t$.

Maximum shear stress in box section,

$$q'=\frac{\tau}{t}=\frac{T}{2At}=\frac{T}{t\times12a^2}=\frac{T}{12a^2t}$$

But $$q'-q$$

$$\frac{16T}{\pi d^3}=\frac{T}{12a^2t}$$

or $$192\,a^2t=\pi d^3=\pi(3a)^3$$

$$t=\frac{27\,\pi a}{192}=0{\cdot}44\,a$$

Angular twist for solid circular shaft $\left(\dfrac{\theta}{l}\right)$

or $$\theta'=\frac{T}{GJ}\quad\text{where } J=\frac{\pi d^4}{32}$$

but $$d=3a$$

So $$\theta'=\frac{T}{G}\times\frac{32}{\pi\times81\,a^4} \qquad ...(1)$$

$$\delta T = q \,.\, t \,.\, h\, \delta s$$

$= \tau \,.\, (h\, \delta s)$ where δs is the base and h is the altitude of the triangle shown shaded in the Fig. 13·28.

$\delta T = \tau \,.\, 2\delta_A$ where δ_A is the area of the triangle.

Total torque, $T = \Sigma \tau \,.\, 2\delta_A = 2\tau \,.\, \Sigma\delta_A = 2 \,.\, \tau \,.\, A.$

A=area enclosed by the centre line of the tube. This is an equation generally known as Bredt-Batho formula.

To determine the angular twist, consider the twist in the small element of peripheral length δs and axial length δl and thickness t as shown in Fig. 13·28.

Shear force on the small element,

$$\delta Q = \tau \,.\, \delta s = q \,.\, t \,.\, \delta s$$

displacement at the edge bc or $ad = \delta$ (say)

shear strain, $\gamma = \dfrac{\delta}{\delta l}$

δu, strain energy for the small element

$$= \tfrac{1}{2}\delta Q \,.\, \delta = (\tfrac{1}{2}\, q \,.\, t \,.\, \delta s \,.\, \delta) = \tfrac{1}{2}\, q \,.\, t \,.\, \delta s\, r\, \delta l = \tfrac{1}{2}\, q \,.\, \gamma \,.\, t\, \delta s \,.\, \delta l$$

$$= \frac{q^2}{2G} \times (t \,.\, \delta s\, \delta l)$$

because shear strain, $\gamma = \dfrac{q}{G}$

Now $\quad q = \dfrac{\tau}{t} = \dfrac{\text{shear flow}}{\text{thickness}}$ or $\tau = q \,.\, t$

Substituting for q,

Shear strain energy, $\quad \delta u = \dfrac{\tau^2}{2G} \times \delta l \,.\, \dfrac{\delta s}{t}$

Moreover $\quad \tau = \dfrac{T}{2A} = \dfrac{\text{Torque}}{2 \times \text{area enclosed by the centre line of the tube}}$

$$\delta u = \frac{T^3}{8A^2G} \times \delta l \,.\, \frac{\delta s}{t}$$

Let us take $\quad \delta l = 1,$

δu for the small element of unit axial length $= \dfrac{T^2}{8A^2G} \times \dfrac{\delta s}{t}$

Total strain energy per unit length of tube, $u = \dfrac{T^2}{8A^2G} \oint \dfrac{ds}{t}$

Using the Castiglianos' theorem

angular twist, $\theta = \dfrac{\partial u}{\partial T}$

$$\theta = \text{Angular twist per unit length} = \frac{T}{4A^2G} \oint \frac{ds}{t}$$

Again $\quad \dfrac{T}{2A} = \tau$ (shear flow)

$$\theta = \frac{\tau}{2AG} \oint \frac{ds}{t}.$$

13·13. TORSION OF THIN WALLED SECTIONS

Consider a thin walled tube subjected to twisting moment. The thickness of the tube need not be uniform along its periphery. Let q is the shear stress and t is the thickness at any point in the boundry. Consider a small element $abcd$ of length δs along the periphery. Say the thickness at cd is t_1 and shear stress at cd is q_1, the thickness at ab is t_2 and the shear stress at ab is q_2. There will be complementary shear stresses in the direction parallel to the axis OO'. Say the thickness along the axis remains the same. Consider a small length δl along the axis. (See Fig. 13·27)

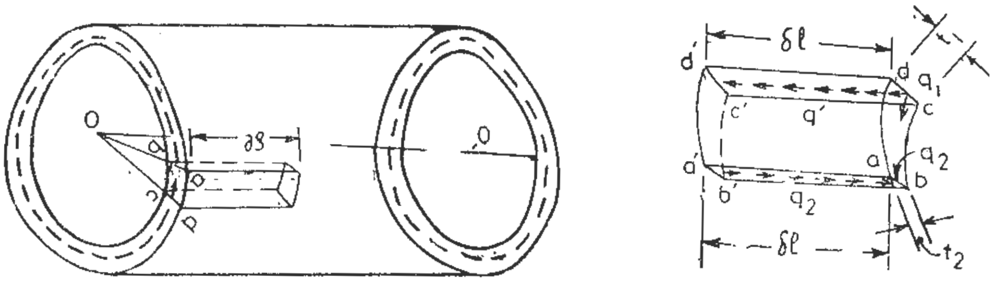


Fig. 13·27

Complementary shear stress on surface, $a'b'ba=q_2$

Complementary shear stress as surface, $c'cdd'=q_1$

For equilibrium, $$q_2.t_2.\delta l-q_1t_1\delta l=0$$

or $$q_2t_2=q_1t_1=\tau, \text{ a constant=shear force per unit lengtl}$$

The quantity $q.t$, a constant is called shear flow τ. Now consider the torque due tc the shear force on a small element of length, δs. Say the shear stress is q and thickness is t.

Shear force acting on the small element,

$$\delta Q=\tau.\delta s=q.t.\delta s \quad \text{(as shown in the Fig. 13·28)}$$

Moment of the force δQ at the centre $O=\delta T=h\delta Q$
where h is the moment arm of force, δQ about O.

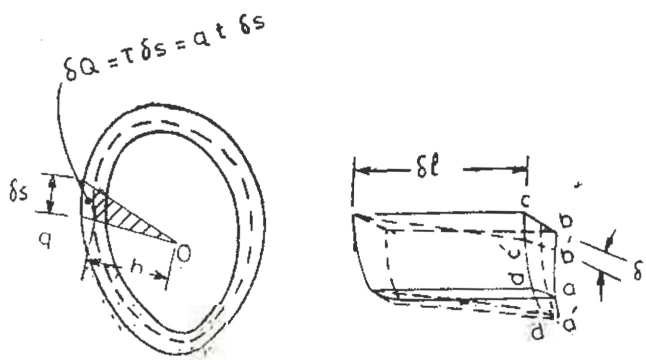


Fig. 13·28

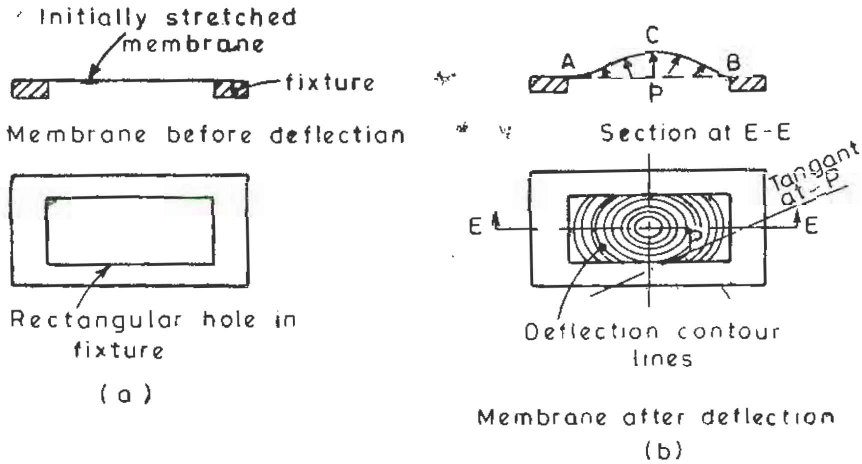


Fig. 13·25

Fig. 13·26 (a) shows a membrane of any shape stretched initially with tension T, subjected to internal pressure p and the membrane is blown up as shown. Initial tension T on the element $abcd$ is shown in Figs. (a) and (b). Say slope at the end a is α and at b its slope is $\alpha+\Delta\alpha$. Shear stress q is shown perpendicular to the slope at different points.

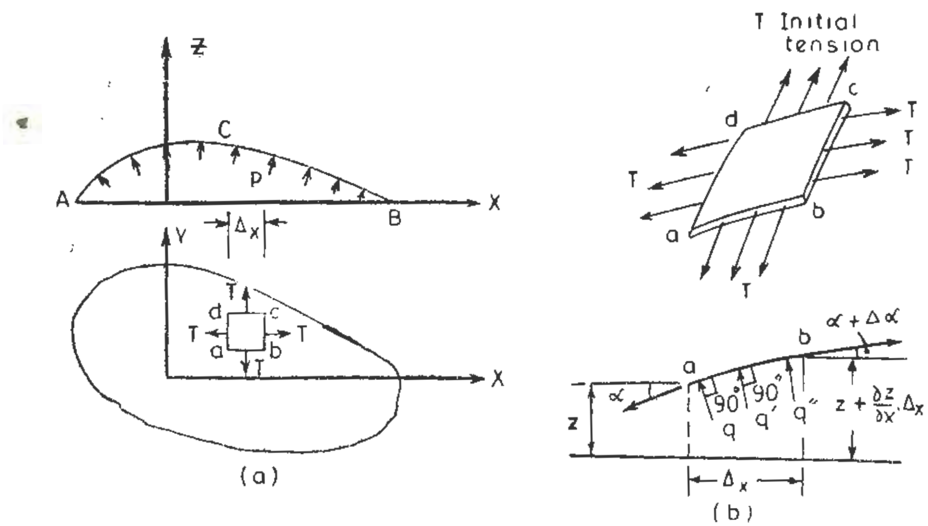


Fig. 13·26

The thickness of the rubber membrane, the pressure used and the pretension affect the deflections. With the help of travelling microscope, deflection at grid points on the membrane are noted down and deflection contours are plotted. From deflection contours, slope at any point can be determined. Knowing the results of a circular cross section, the membrane is calibrated for a circular cross section under given pretension T, internal pressure p and membrane thickness t.

Example 13·11-1. A shaft of equilateral triangular section of side 40 mm, is subjected to an axial twisting moment T. Determine the magnitude of T if the maximum shear stress is not to exceed 100 N/mm². What will be angular twist in 2 metres length of the shaft ? G=80,000 N/mm².

Solution. $q=100$ N/mm²

Side, $a=40$ mm

$$q=\frac{15\sqrt{3}T}{2a^3} \quad \text{or} \quad T=\frac{2a^3\times q}{15\sqrt{3}}$$

$$=\frac{2\times 40^3\times 100}{15\sqrt{3}}=\frac{128\times 10^5}{15\sqrt{3}}=4\text{·}927\times 10^5 \text{ Nmm}$$

Angular twist per metre length,

$$\theta=\frac{15\sqrt{3}\ T}{Ga^4}=\frac{15\times\sqrt{3}\times 4\text{·}927\times 10^5}{80{,}000\times(40)^4}$$

$$=0\text{·}0625\times 10^{-3} \text{ radian/mm length}$$

$$=0\text{·}0625 \text{ radian/m length}$$

Angular twist in 2 metres length

$$=0\text{·}0625\times 2=0\text{·}125 \text{ radian}=7\text{·}162 \text{ degrees.}$$

Exercise 13·11-1. A shaft of equilateral triangular section of the side 6 cm is subjected to the torque of 4×10^6 Nmm. Determine (*i*) maximum shear stress developed (*ii*) angular twist over 1 metre length of the shaft. G=84,000 N/mm²

[**Ans.** (*i*) 240·55 N/mm² (*ii*) 5·73°]

13·12. MEMBRANE ANALOGY

The analysis for the determination of shear stress and angular twist in non-circular shafts is quite complicated and involved. In such cases, simple experimental techniques can be used for the analysis. Prandtl has introduced membrane analogy for non-circular sections, in which a thin rubber sheet initially stretched to a uniform tension is fixed at its edges. The area bound by edges is of the shape of the non-circular section. This stretched rubber sheet or membrane is subjected to an internal pressure and the membrane is deflected. If the slopes deflections of the membrane are taken at different points, then following observations are made.

(1) Slope of the membrane at any point is proportional to the magnitude of shear stress at that point.

(2) The direction of shear stress at any point is obtained by drawing a tangent to the deflection contour lines of the membrane at that point or in other words the direction of the shear stress is perpendicular to that of the slope.

(3) The twisting moment is numerically equivalent to twice the volume under the memberane *i.e.*, volume under the bulge ACB shown in the Fig. 13·25 (*b*). Figures above show a thin rubber stretched under initial tension in a fixture with a rectangular cut out Initial tension T on the membrane is large enough to ignore its change when the membrane is blown up by a small internal pressure p.

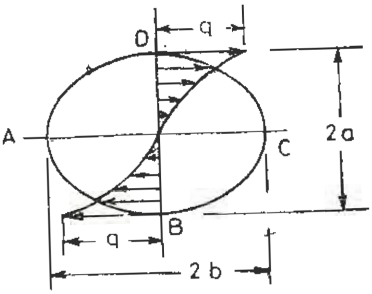


Fig. 13·22

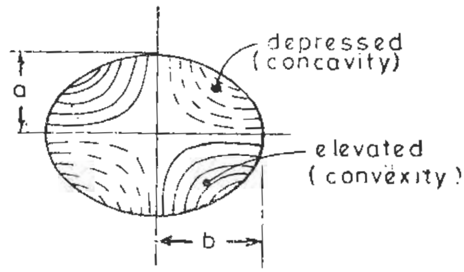


Fig. 13·23

Example 13·10-1. A shaft of elliptical section with minor axis $2a$ and major axis $2b$ is subjected to a torque of 2 kNm. If the maximum shear stress in the shaft is not to exceed 80 N/mm², determine the major and minor axis, if $b=1·5\ a$. What will be the angular twist in a metre length in this shaft under the given torque ? $G=80{,}000$ N/mm².

Solution. $b=1·5\ a$

Maximum shear stress, $q=\dfrac{2T}{\pi a^2 b}=\dfrac{2\times 2\times 10^6}{\pi\times a^2\times 1·5\ a}=80$

or $a^3=\dfrac{4\times 10^6}{\pi\times 120}=10·61\times 10^3,\quad a=22$ mm

Minor axis $=44$ mm

Major axis $=66$ mm

Angular twist per mm length

$$=\frac{T}{G}\times\frac{a^2+b^2}{\pi a^3 b^3}=\frac{2\times 10^6(22^2+33^2)}{80{,}000\times 22^3\times 33^3}=0·102\times 10^{-3}\text{ radian}$$

θ per metre length $=0·102$ radian $=5·89$ degrees

Exercise 13·10-1. A shaft of elliptical section ; major axis 6 cm, minor axis 4 cm, is subjected to an axial twisting moment of 200 kg-metre. What is the maximum stress developed in the section and what is the angular twist per metre length $G=400$ tonnes/cm².

[**Ans.** 1061 kg/cm², 17·24°]

13·11. TORSION OF A SHAFT WITH EQUILATERAL TRIANGULAR SECTION

Fig. 13·24 shows an equilateral triangle section of a shaft subjected to the twisting moment T. Say a is the side of the equilateral triangle. Maximum shear stress occurs at the centre of the sides *i.e.*, at point D, E and F as shown in the Fig.

Angular twist per unit length

$$=\frac{15\sqrt{3}T}{Ga^4}$$

Maximum shear stress,

$$q=\frac{15\sqrt{3}T}{2a^3}$$

At the corners of the triangle *i.e.*, at A, B and C shear stress is zero.

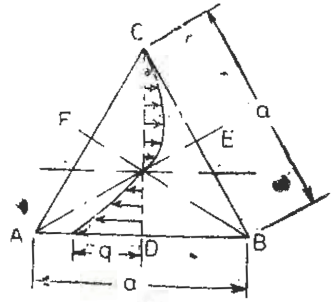


Fig. 13·24

Example 13·9-2. A rectangular shaft 6 cm × 4 cm made of steel is subjected to a torque of 300 kg-metre. What is the maximum shear stress developed in the shaft and what is the angular twist per metre length. $G=0·8\times10^6$ kg/cm². Use approximate relationship.

Solution.

Torque, $T=300 \text{ kg-m}=3\times10^4$ kg-cm

Longer side, $2b=6$ cm, $b=3$ cm

Shorter side, $2a=4$ cm, $a=2$ cm

$G=0·8\times10^6$ kg/cm²

Maximum shear stress, $q=\dfrac{T(3b+1·8a)}{8a^2b^2}=\dfrac{3\times10^4(3\times3+1·8\times2)}{8\times9\times4}$

$=1312·5$ kg/cm²

Angular twist per cm length

$$=k.\frac{a^2+b^2}{16a^3b^3}\times\frac{T}{G}$$

where $k=3·645-0·06\times\dfrac{3}{2}=3·555$

θ per cm length $=3·555\times\dfrac{(4+9)\times3\times10^4}{16\times27\times8\times·8\times10^6}=0·050\times10^{-2}$ radian

θ per metre length $=0·050\times10^{-2}\times100$ radian

$=0·050$ radian $=2·865$ degrees.

Exercise 13·9-1. A rectangular shaft of sides 60 mm × 24 mm made of steel is subjected to a torque such that the maximum stress developed is 800 kg/cm². What is the magnitude of the torque and what will be the angular twist in a length of 1 metre of the shaft. $G=800$ tonnes/cm². Take values of constants from the table.

[**Ans.** 71 40 kg-metre, 3·3 degree]

Exercise 13·9-2. A rectangular shaft of steel 9·6 cm × 6 cm is subjected to a torque such that the angular twist in a length of 1 metre is 1·5 degree. What is the magnitude of the torque and what will be the maximum shear stress developed in the shaft. $G=80{,}000$ N/mm² use approximate analysis. [**Ans.** 8·8 kNm, 291·76 N/mm²]

13·10. TORSION OF ELLIPTICAL SECTION SHAFT

For the elliptical section shaft, the expressions for maximum shear stress and angular twist per unit length are

$$q=\frac{2T}{\pi a^2b} \quad\text{and}\quad \theta=\frac{T}{GJ}=\frac{T}{G}\times\frac{a^2+b^2}{\pi a^3b^3}$$

Maximum shear stress occurs at the ends of the minor axis as shown in Fig. 13·22. *i.e.*, at the points B and D. Fig. 13·23 shows the contour lines of constant displacement. The convex portions of the cross section where displacements in the direction of axis of the shaft are positive. Where the surface is depressed, depressions are shown by dotted lines.

$$=\frac{T}{G}\cdot\frac{a^2+(1\cdot5)^2a^2}{16a^3\,(1\cdot5)^3\,a^3}\,k=\frac{T}{G\times16\cdot615\,a^4}\,k$$

$$k=3\cdot645-0\cdot06\times1\cdot5=3\cdot645-0\cdot09=3\cdot555$$

$$\theta=\frac{T}{G\times16\cdot615}\times\frac{3\cdot555}{a^4}=\frac{1}{4\cdot674}\times\frac{T}{Ga^4}$$

If we compare the results of maximum shear stress and angular twist, from two analysis, we can find some negligible difference between the two cases.

The maximum intensity of shear stress q, occurs at the centre of the longer side as shown in the Fig. 13·20. Fig. 13·21 shows the distortion of the ends of a shaft of square section.

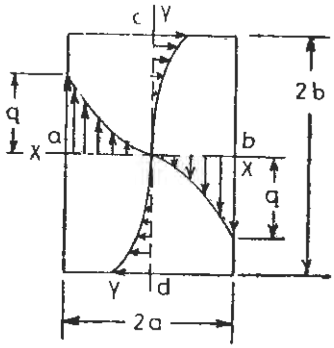


Fig. 13·20

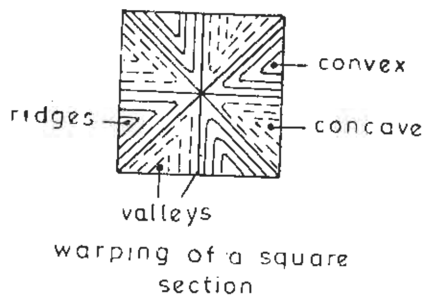


Fig. 13·21

Example 13·9-1. A 50 mm×25 mm rectangular steel shaft is subjected to a torque of 1·8 kNm. What is the maximum shear stress developed in the shaft and what is the angular twist per unit length. $G=80$ GN/m².

Solution. $G=80$ GN/m² $=80\times10^9$ N/m² $=80\times10^3$ N/mm²

Longer side, $2b=50$ mm, $b=25$ mm

Shorter side, $2a=25$ mm, $a=12\cdot5$ mm

Torque, $T=1\cdot8$ kNm $=1\cdot8\times10^6$ Nmm

Maximum shear stress, $q=K_2\,\dfrac{T}{a^2b}$

From tables for $\dfrac{b}{a}=2$, $K_2=0\cdot508$, $K=3\cdot664$

$$q=\frac{0\cdot508\times1\cdot8\times10^6}{12\cdot5^2\times25}=234\cdot08\text{ N/mm}^2$$

Angular twist per unit length,

$$\theta=\frac{T}{GKa^3b}=\frac{1\cdot8\times10^6}{80\times10^3\times3\cdot664\times12\cdot5^3\times25}$$

$=0\cdot1257\times10^{-3}$ radian per mm length

$=0\cdot1258$ radian/metre length

$=7\cdot2$ degrees per metre length

TABLE 13·1

b/a	K	K_1	K_2
1	2·250	1·350	0·600
1·2	2·656	1·518	0·571
1·5	3·136	1·696	0·541
2·0	3·664	1·860	0·508
2·5	3·984	1·936	0·484
3·0	4·208	1·970	0·468
4·0	4·496	1·994	0·443
5·0	4·656	1·998	0·430
10·0	4·992	2·000	0·401
∞	5·328	2·000	0·375

Expressions for θ and q can be approximately given as follows and one does not need to refer to table of constants

$$q=\frac{T(3b+1\cdot 8a)}{8a^2b^2}$$

$$\theta=k\,.\,\frac{a^2+b^2}{16a^3b^3}\times\frac{T}{G}$$

where $$k=\left(3\cdot 645-0\cdot 06\,\frac{b}{a}\right) \text{ approximately.}$$

For the sake of comparison let us take $\frac{b}{a}=1\cdot 5$

From tables $K=3\cdot 136$, $K_1=1\cdot 696$, $K_2=0\cdot 541$.

Maximum shear stress, $q=K_2\,\frac{T}{a^2b}=0\cdot 541\times\frac{T}{a^2\times 1\cdot 5\,a}=0\cdot 360\times\frac{T}{a^3}$

Angular twist per unit length

$$\theta=\frac{T}{GKa^3b}=\frac{T}{G\times 3\cdot 136\times a^3\times 1\cdot 59}=\frac{T}{4\cdot 704\,\mathbf{G}a^4}$$

From approximate analysis

Maximum shear stress, $q=\frac{T(3b+1\cdot 8a)}{8a^2b^2}=\frac{T(3\times 1\cdot 5a+1\cdot 8a)}{8\times a^2\times(1\cdot 5a)^2}=0\cdot 35\times\frac{T}{a^3}$

Angular twist per unit length

$$\theta=k\,\frac{a^2+b^2}{16a^3\,b^3}\times\frac{I}{G}$$

All non circular sections are distorted under torsion to a greater or lesser degree. For sections nearer to circle, these effects of distortion are less marked as in the case of elliptical section.

The detailed analysis of the torsion of non-circular shafts which includes the warping of the sections is beyond the scope of this text book. However the results of the theory developed by St. Venant and Prandtl for the calculation of maximum shear stress and angular twist are summarised in this chapter.

Rectangular Sections

Torque $T=GJ\theta$

where GJ=Torsional rigidity of the shaft

θ=angular twist per unit length

$J=Ka^3 b$

Angular twist,

$$\theta=\frac{T}{GKa^3 b}$$

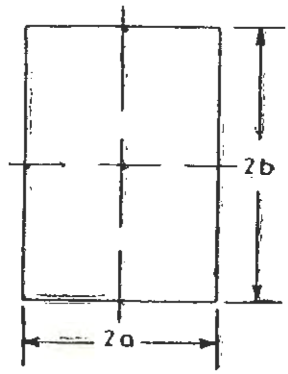


Fig. 13·19

The value of constant K depends upon the ratio of $\frac{b}{a}$, where $2b$ is the longer side of the rectangular section and $2a$ is the shorter side of the section as shown in Fig. 13·19. The values of K for various ratios of $\frac{b}{a}$ are shown in the Table 13·1.

Maximum shear stress, $q=K_1\frac{Ta}{J}$

where K_1 is another constant again depending upon the ratio of $\frac{b}{a}$.

or $$q=K_1\frac{Ta}{Ka^3b}=\frac{K_1}{K}\cdot\frac{T}{a^2b}=K_2\frac{T}{a^2b}$$

where $$K_2=\frac{K_1}{K}$$

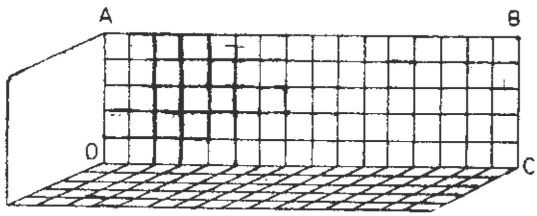
The constants K, K_1, and K_2 are shown in Table 13·1.

Bearing stress in key, $f=\dfrac{Q}{t/2\times l}=\dfrac{2148{\cdot}5}{0{\cdot}5\times 10}=429{\cdot}7$ kg/cm²

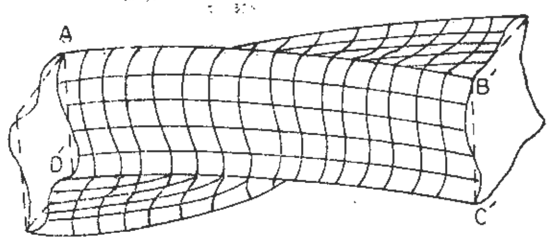
Exercise 13·7-1. A shaft of diameter 6 cm is transmitting 20 horse power at 300 r.p.m. The shaft is connected to a pulley of axial width 12 cm with the help of a key 14 mm × 12 mm (deep). Determine the shear and compressive stresses developed in the key. Take 1 H.P.=746 watts per second. [**Ans.** 9·423 N/mm², 21·98 N/mm²]

13·9. TORSION OF NON-CIRCULAR SHAFTS

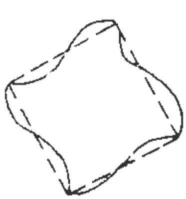
In the case of circular shafts, we assumed that sections transverse to the axis of the shaft are plane before the application of a twisting moment and remain plane after the shaft is twisted by a twisting moment. Moreover we have proved that volumetric strain in a shaft is zero and a circular section remains a circular section of the same original diameter when the shaft is subjected to torsion. But when shafts of non-circular sections are subjected to torsion, plane sections before torsion do not remain plane after a twisting moment is applied on them. The plane sides also do not remain plane and high and low areas exist giving a series of valleys and ridges. Fig. 13·18 (*a*) shows a rectangular shaft with grid lines drawn it. Fig. 13·18 (*b*) shows the same rectangular shaft after torsion and warping. The distortion of a plane surface by the formation of high and low regions is termed as warping.

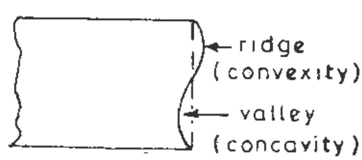


13·18 (a) Rectangular shaft before torsion and warping



13·18 (b) Rectangular shaft after torsion and warping.


Change of shape of cross section (c)



warping (d)

Fig. 13·18.

13·8. STRESSES DEVELOPED IN KEY

Shafts transmit power through gears, flywheels, pulleys etc. which are keyed to the shaft with the help of various types of keys. Here we will discuss only rectangular type key as shown in the Fig. 13·17. The breadth and depth of the key are b and t respectively and half the key is embedded in a keyway provided in the shaft and half the key is embedded in the key way provided in the hub of the pulley, flywheel, gear etc.

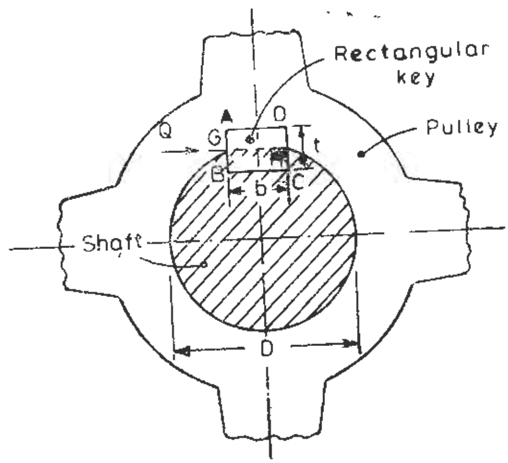


Fig. 13·17

Say the shaft is transmitting torque T.

Shaft diameter$=D$

Then tangential force or the shear force acting on the periphery of the shaft,

$$Q=\frac{T}{D/2}=\frac{2T}{D} \quad ...(1)$$

Say the length of the key$=l$

Section of the key subjected to shear force$=b\times l$ (shown by section GH)

$$\text{Shear stress developed in the key}=\frac{Q}{bl}=\frac{2T}{Dbl} \quad ...(2)$$

Section of the key subjected to compressive force$=\frac{t}{2}\times l$ (shown by section GB)

Bearing stress or compressive stress in key

$$=\frac{Q}{tl/2}=\frac{2Q}{tl}=\frac{4T}{Dtl} \quad ...(3)$$

Same shear force Q acts as a compressive force on faces BG and DH.

Example 13·7-1. A shaft 5 cm diameter, 1·5 metres long is transmitting 15 horse power at 200 r.p.m. The shaft is connected to the pulley of axial width 10 cm by a key of breadth 1·2 cm and depth 1·0 cm. Determine the shear and compressive stresses developed in the section of the key.

Solution. H.P.$=15$

R.P.M.$=200$

Torque on the shaft, $T=\frac{15\times 4500}{2\pi\times 200}$ kgm

$=53{\cdot}714$ kg-m$=5371{\cdot}4$ kg-cm

Shaft diameter, $D=5$ cm

Tangential force on shaft,

$$Q=\frac{5371{\cdot}4}{2{\cdot}5}=2148{\cdot}5 \text{ kg}$$

Width of the key, $b=1{\cdot}2$ cm

Length of the key, $l=10$ cm

Thickness of the key, $t=1$ cm

Shear stress in key, $q=\frac{Q}{l\times b}=\frac{2148{\cdot}5}{10\times 1{\cdot}2}=179{\cdot}05$ kg/cm²

Then q_r at any radius $r=\frac{q}{R_2}\times r$

where q=maximum shear stress at R_2

Strain energy for the elementary cylinder

$$=\frac{4\pi q^2 l r^3 dr}{GD_2^2} \quad \text{where} \quad D_2=2R_2, \text{ outer diameter}$$

Strain energy for the whole shaft,

$$U=\frac{4\pi q^2 l}{GD_2^2}\int_{R_1}^{R_2} r^3\,dr$$

$$=\frac{\pi q^2 l}{GR_2^2}\int_{R_1}^{R_2} r^3\,dr=\frac{\pi q^2 l}{GR_2^2}\left[\frac{R_2^4}{4}-\frac{R_1^4}{4}\right]$$

$$=\frac{\pi q^2 l}{4GR_2^2}(R_2^2-R_1^2)(R_2^2+R_1^2)$$

$$=\frac{q^2}{4G}\left(\frac{R_2^2+R_1^2}{R_2^2}\right)[\pi l(R_2^2-R_1^2)]$$

$$=\frac{q^2}{4G}\times\frac{R_2^2+R_1^2}{R_2^2}\times\text{Volume of the shaft}$$

$$=\frac{q^2}{4G}\left(\frac{D_2^2+D_1^2}{D_2^2}\right)\times\text{Volume of the shaft}$$

Example 13·7-1. A solid circular steel shaft of diameter 50 mm and length 1 metre is subjected to a twisting moment of 5000 Nm. Determine the strain energy absorbed by the shaft.

G for steel=78400 N/mm²

Solution. Maximum shear stress developed in the shaft

$$q=\frac{16\,T}{\pi d^3}=\frac{16\times5000\times1000}{\pi\times(50)^3}=203{\cdot}718 \text{ N/mm}^2$$

Volume of the shaft $=\frac{\pi}{4}(50)^2\times1000=1963500$ mm³

$=1963{\cdot}5\times10^3$ mm³

Strain energy absorbed, $U=\frac{q^2}{2G}\times$Volume of the shaft

$$=\frac{(203{\cdot}718)^2}{2\times78400}\times1963{\cdot}5\times10^3 \text{ Nmm}=267{\cdot}554 \text{ Nm}$$

Exercise 13·7-1. A hollow circular steel shaft of outside diameter 40 mm and inside diameter 20 mm, length 1200 mm is subjected to a twisting moment so that the maximum stress developed in the shaft is 150 N/mm². Determine the shear strain energy developed in the shaft. G=78400 N/mm²

[**Ans.** 101·43 Nm]

Equivalent twisting moment,

$$T_e = \sqrt{M^2+T^2} = \sqrt{10^2+33{\cdot}42^2} = 34{\cdot}88 \text{ kNm}$$

(*ii*) Maximum principal stress,

$$p_1 = \frac{32\ M_e}{\pi d^3} = \frac{32 \times 22{\cdot}44 \times 1000 \times 1000}{\pi \times 180 \times 180 \times 180} = 39{\cdot}19 \text{ N/mm}^2$$

(*iii*) Maximum most shear stress,

$$q_{max} = \frac{16\ T_e}{\pi d^3} = \frac{16 \times 34{\cdot}88 \times 1000 \times 1000}{\pi \times 180 \times 180 \times 180} = 30{\cdot}46 \text{ N/mm}^2$$

Exercise 13·6-1. A solid shaft of diameter d is subjected to a bending moment M=15 kNm and a twisting moment T=25 kNm. What is the minimum diameter of the shaft if the shear stress is not to exceed 160 N/mm² and direct stress is not to exceed 200 N/mm². [**Ans.** 97·6 mm]

13·7. TORSIONAL RESILIENCE OF SHAFTS

Consider a circular shaft of diameter D and length l subjected to twisting moment.

Strain energy per unit volume $= \dfrac{q'^2}{2G}$

where q'=shear stress

G=modulus of rigidity

Now consider an elementary cylinder at radius r and radial thickness dr and length l, as shown in Fig. 13·16.

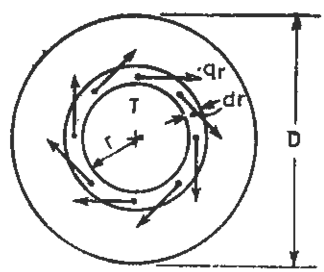


Fig. 13·16

Shear stress at any radius, $q_r = \dfrac{q}{D/2} \times r = \dfrac{2qr}{D}$

where q is the maximum shear stress.

Strain energy per unit volume $= \dfrac{q_r^2}{2G} = \dfrac{2q^2r^2}{GD^2}$

Volume of the elementary cylinder $= 2\pi rl\ dr$

Strain energy for the elementary cylinder

$$= \frac{2q^2r^2}{GD^2} \times 2\pi r\ l\ dr = \frac{4\pi q^2 l}{GD^2} r^3\ dr$$

Strain energy for the solid shaft,

$$U = \frac{4\pi q^2 l}{GD^2} \int_0^R r^3\ dr = \frac{4\pi q^2 l}{GD^2} \times \frac{R^4}{4}$$

$$= \frac{q^2}{4G} \times \left(\frac{\pi}{4}\ D^2 l \right) = \frac{q^2}{4G} \times \text{Volume of the shaft}$$

For a hollow shaft, maximum shear stress occurs at the outer radius. Say R_1 is the inner radius and R_2 is the outer radius.

Maximum principal stress at the section

$$p_1=\frac{f}{2}+\sqrt{\left(\frac{f}{2}\right)^2+q^2}=\frac{16M}{\pi d^3}+\sqrt{\left(\frac{16M}{\pi d^3}\right)^2+\left(\frac{16T}{\pi d^3}\right)^2}$$

$$=\frac{16}{\pi d^3}\left[M+\sqrt{M^2+T^2}\right]$$

or $$p_1\times\frac{\pi d^3}{16}=M+\sqrt{M^2+T^2}$$

or $$p_1\times\frac{\pi d^3}{32}=\frac{M+\sqrt{M^2+T^2}}{2}$$

The bending moment corresponding to the maximum principal stress is termed as Equivalent Bending Moment, M_e.

or $$M_e=\frac{M+\sqrt{M^2+T^2}}{2}$$

Similarly maximum most shear stress developed at the section

$$q_{max}=\sqrt{\left(\frac{f}{2}\right)^2+q^2}=\sqrt{\left(\frac{16M}{\pi d^3}\right)^2+\left(\frac{16T}{\pi d^3}\right)^2}=\frac{16}{\pi d^3}\left[\sqrt{M^2+T^2}\right]$$

or $$\frac{\pi d^3}{16}\times q_{max}=\sqrt{M^2+T^2}.$$

The twisting moment corresponding to the maximum most shear stress on the surface of the shaft is termed as Equivalent Twisting moment, T_e

So $$T_e=\sqrt{M^2+T^2}$$

Example 13·6-1. A solid shaft of diameter 180 mm is transmitting 700 kW at 200 rpm. It is subjected to a bending moment of 10 kNm. Determine

(*i*) Equivalent bending moment and twisting moment

(*ii*) Maximum principal stress on the surface of the shaft

(*iii*) Maximum most shear stress.

Solution.

Power transmitted $=700$ kW$=700$ kNm per second

Speed $=200$ RPM

$$=\frac{200\times 2\pi}{60}\text{ rad/second}=20\cdot944\text{ rad/sec}$$

Torque on the shaft, $T=\dfrac{700}{20\cdot944}=33\cdot42$ kNm

Bending moment, $M=10$ kNm

(*i*) Equivalent bending moment,

$$M_e=\frac{M+\sqrt{M^2+T^2}}{2}$$

$$=\frac{10+\sqrt{10^2+33\cdot42^2}}{2}=22\cdot44\text{ kNm}$$

F_n, Normal force on inclined plane

$$=Q_1 \sin\theta + Q_2 \cos\theta \qquad \text{(compressive for equilibrium)}$$

$$F_n = -q\times BC\times \sin\theta - q\times AC\cos\theta.$$

Normal stress on the plane AB

$$f_n = -q\,\frac{BC}{AB}\times\sin\theta - q\,\frac{AC}{AB}\cos\theta$$

$$= -q\cos\theta\sin\theta - q\sin\theta\cos\theta = -q\sin 2\theta$$

Normal stress is maximum when $\theta=45°$, $135°$

$$f_{n_{max}} = -q,\ +q$$

Tangential force on the inclined plane, $F_t = Q_1\cos\theta - Q_2\sin\theta$.

If f_t is the tangential stress, then

$$f_t = q\times\frac{BC}{AB}\times\cos\theta - q\,\frac{AC}{AB}\sin\theta = q\cos^2\theta - q\sin^2\theta = q\cos 2\theta$$

Shear stress is maximum when $\theta=0$, $90°$ $f_{t_{max}} = +q - q$

Moreover shear stress is zero when $\theta=45°$, $135°$.

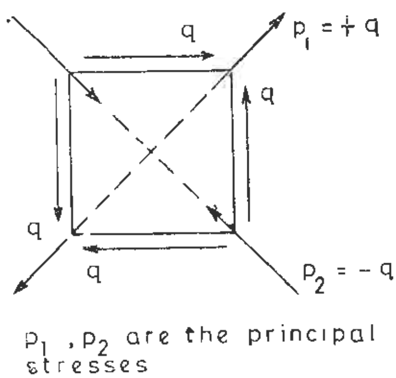
This means that $f_{n_{max}} = +q$, $-q$ are the principal stresses acting on the surface of the shaft as shown in Fig. 13·14.

Principal stresses are $+q$, $-q$, 0 at a point on the surface of the shaft

Principal strains are $\epsilon_1 = \frac{q}{E} + \frac{q}{mE}$ $\epsilon_2 = \frac{-q}{E} - \frac{q}{mE}$ $\epsilon_3 = \frac{-q}{mE} + \frac{q}{mE} = 0$

Volumetric strain, $\epsilon_v = \epsilon_1 + \epsilon_2 + \epsilon_3 = 0$.

This shows that in a shaft subjected to pure torque, there is no change in volnme.



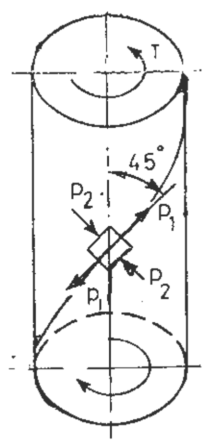


Fig. 13·14

(*ii*) **Shafts subjected to twisting moment and bending moment simultaneously.** Shafts transmitting power are subjected to bending moments due to belt tensions on pulleys, normal force on the gears etc. in addition to the twisting moment. Fig. 13·15 shows a portion of a shaft transmitting power, subjected to twisting moment T and a bending moment M. Say the diameter of the shaft is d. Maxm. direct stress, f is developed on the surface of the shaft and a maximum shear stress q is also developed on the surface of the shaft due to T.

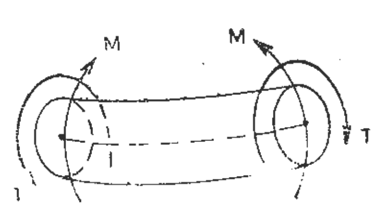


Fig. 13.15

$$f = \frac{32M}{\pi d^3} \text{ and } q = \frac{16T}{\pi d^3}$$

(*a*) The magnitude of T if the maximum stress in steel is not to exceed 90 N/mm².

(*b*) The maximum shear stress in the aluminium shaft.

Show that the shear strain at a point is linearly proportional to its distance from the centre of the shaft.

Given $G_{steel}=3{\cdot}1\ G_{aluminium}$

[**Ans.** (*a*) 12·087 kNm (*b*) 48·3 N/mm²]

13·6. STRESSES IN SHAFTS SUBJECTED TO TORQUE

(*i*) **Shaft Subjected to Twisting Moment only.** When a shaft is subjected to pure torsion *i.e.*, only to the twisting moment, maximum shear stress is developed on the surface on

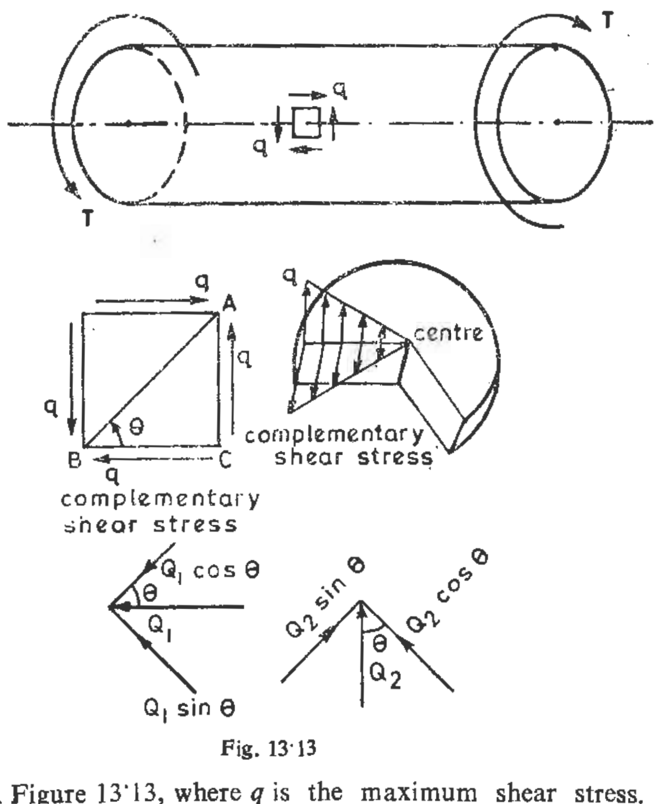


Fig. 13·13

the shaft as shown in Figure 13·13, where q is the maximum shear stress. The longitudinal shear stress q shown is the complementary shear stress.

Consider a small element of unit thickness on the surface of the shaft. The stresses on the planes AC and BC are the shear stresses q each.

Shear force on plane BC,

$$Q_1=q\times BC\times 1$$

Shear force on plane AC,

$$Q_2=q\times AC\times 1,$$

Resolving these forces parallel and perpendicular to the inclined plane AB.

(*b*) Angular twist per unit length $= \frac{\theta s}{ls} = \frac{Ts}{JsGs}$

$$= \frac{3.5 \times 10^5}{128\ \pi \times 80 \times 10^5} \quad \text{where} \quad Gs = 80 \times 10^5\ \text{N/cm}^2$$

$= 1.09 \times 10^{-4}$ radian $= 0.00623$ degree.

(*c*) When the torque is applied at a centre of the shaft, the torque will be equally divided but the nature of the torque in one half portion of the shaft will be opposite to the nature of the torque in the other half because the net angular twist between the fixed ends is $\theta_A + \theta_B = 0$, or $\theta_A = -\theta_C$, as shown in figure 13·11.

So, now the torque on steel shaft

$Ts' = 1.75 \times 10^5$ N cm

Torque on copper shaft

$Tc' = 2.25 \times 10^5$ N cm

Maxm. stress in steel shaft,

$$qs' = \frac{34.815}{2}$$

$= 17.407$ N/mm²

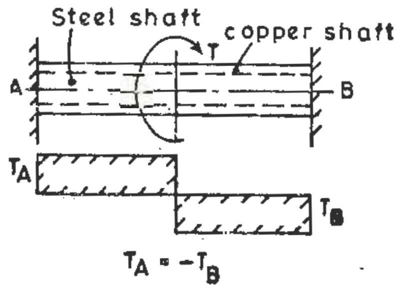


Fig. 13·11

Maximum stress in copper shaft,

$$qc' = \frac{23.90}{2} = 11.95\ \text{N/mm}^2$$

Exercise 13·5-1. A horizontal shaft 200 cm long, rigidly fixed at both the ends is subjected to an axial twisting moment $T_1 = 30$ tonne-cm and $T_2 = 30$ tonne-cm at distance of 80 and 150 cm from one end. Both the twisting couples are acting in the same direction. Determine the end fixing couples in magnitude and direction and find the diameter of the shaft if the maximum shearing stress is not to exceed 1000 kg/cm².

Determine also the section where the shaft suffers no angular twist.

[**Ans.** 10·5 tonne-cm, 10·5 tonne-cm; 4·63 cm-shaft diameter, 123·07 cm from the end from which the distances for moments are given]

Exercise 13·5-2. A shaft of 2m length is of different sections as shown in the figure 13·12. Determine the ratio of the torques in the portions I and II, if the diameter of the portion II is 1·5 times the diameter of the portion I. Determine the value of the torque T, if the maximum shear stress in the shaft is not to exceed 80 N/mm² and diameter of the portion I is 40 mm.

[**Ans.** $\frac{T_I}{T_{II}} = \frac{16}{81}$, 4·063 k Nm

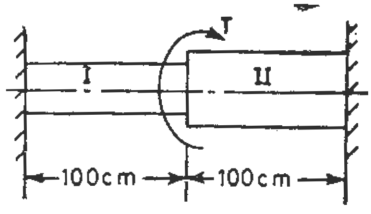


Fig. 13·12

Exercise 13·5-3. A solid circular steel shaft is enclosed in an aluminium hollow shaft so as to make a compound shaft.

The diameter of the steel shaft is 60 mm and the outside diameter of the aluminium shaft is 100 mm. This compound shaft is subjected to an axial torque T. Determine

qs, Maximum shear stress in steel shaft

$$= \frac{16\ Ts}{\pi \times 4^3} = \frac{19{\cdot}364 \times 16}{\pi \times 64} = 1{\cdot}5409 \text{ tonnes/cm}^2$$

Maximum shear stress in copper

$$= \frac{Tc}{Jc} \times \frac{5}{2} = \frac{20{\cdot}636}{17\ \pi} \times \frac{5}{2} = 0{\cdot}966 \text{ tonne/cm}^2$$

Example 13·5-3. A solid circular steel shaft is encased in a copper hollow shaft so as to make a compound shaft. The diameter of the steel shaft is 8 cm and the outside diameter of the copper shaft is 11 cm. The compound shaft of length 200 cm is subjected to an axial torque of 8 kNm. Determine

(*a*) Maximum shear stress in steel and copper

(*b*) Angular twist per unit length

(*c*) What will be the maximum stresses developed in the steel and copper if the torque acts in the centre of the shaft and both the ends are securely fixed.

Given G steel$=2$ G copper, G steel$=80$ kN/mm²

Solution. Torque on the composite shaft.

$$=80 \text{ kNm} = 8 \times 10^5 \text{ N cm}^3$$

Say Ts = Torque shared by steel shaft

Tc = Torque shared by copper shaft

$$\frac{Ts\ ls}{Gs\ Js} = \frac{Tc\ lc}{Gc\ Jc} \quad \text{but} \quad ls = lc \text{ in this case and } Gs = 2Gc$$

$$\frac{Ts}{Tc} = \frac{Gs}{Gc} \times \frac{Js}{Jc} = 2\frac{Js}{Jc}$$

Polar moment of inertia, $Js = \frac{\pi}{32}(8^4) = 128\ \pi \text{ cm}^4$

Moment of inertia, $Jc = \frac{\pi}{32}(11^4 - 8^4) = 329{\cdot}53\ \pi \text{ cm}^4$

So $$\frac{Ts}{Tc} = 2 \times \frac{128\ \pi}{329{\cdot}53\ \pi} = 0{\cdot}777$$

$$Ts = 0{\cdot}777\ Tc$$

$$T = Ts + Tc$$

$\therefore$ $$0{\cdot}777\ Tc + Tc = 8 \times 10^5 \text{ N cm}$$

or $$Tc = 4{\cdot}50 \times 10^5 \text{ N cm} \quad \text{and} \quad Ts = 3{\cdot}50 \times 10^5 \text{ N cm}$$

(*a*) **Maximum Shear Stress**

In steel shaft, $$q_s = \frac{Ts}{Js} \times 4 = \frac{3{\cdot}5 \times 10^5}{128\ \pi} \times$$

$$= 3481{\cdot}5 \text{ N/cm}^2 \text{ or } 34{\cdot}815 \text{ N/mm}^2$$

In copper shaft, $$q_c = \frac{Jc}{Jc} \times 5{\cdot}5 = \frac{4{\cdot}5 \times 10^5 \times 5{\cdot}5}{329{\cdot}53\pi}$$

$$= 2390{\cdot}72 \text{ N/cm}^2 \quad \text{or} \quad 23{\cdot}90 \text{ N/mm}^2$$

Angular twist at D, $\theta_D=\dfrac{T_B\times l_3}{GJ}=\dfrac{1{\cdot}5625\times 0{\cdot}6}{GJ}$

Say CC_1 represents $\theta_C \propto {\cdot}0625$

and DD_1 represents $\theta_D \propto 1{\cdot}5625$

The position of the section where angular twist is zero can be determined from end A

$$x=0\ 6+0{\cdot}4\left(\frac{{\cdot}0625}{1{\cdot}5625+{\cdot}0625}\right)=0{\cdot}6+0{\cdot}0154$$

$$=0{\cdot}6154 \text{ m}=61{\cdot}54 \text{ cm from end } A.$$

Fig. 13·9 shows the torque distribution diagram and angular twist variation diagram.

Example 13·5-2. A composite shaft is made by joining an 80 cm long solid steel shaft with 80 cm long hollow copper shaft as shown in the Fig. 13·10. The diameter of the solid shaft is 4 cm, while the external and internal diameters of hollow shaft are 5 cm and 3 cm respectively. Determine the maximum shear stress developed in steel and copper shaft if the torque T applied at the junction is 40 tonne-cms.

Given G steel $=2\,G$ copper

Solution. Polar moment of inertia of solid shaft,

$$J_S=\frac{\pi\times 4^4}{32}=8\pi \text{ cm}^4$$

length, $l_S=80$ cm

Polar moment of inertia of copper shaft,

$$J_C=\frac{\pi}{32}(5^4-3^4)$$

$$=17\,\pi \text{ cm}^4$$

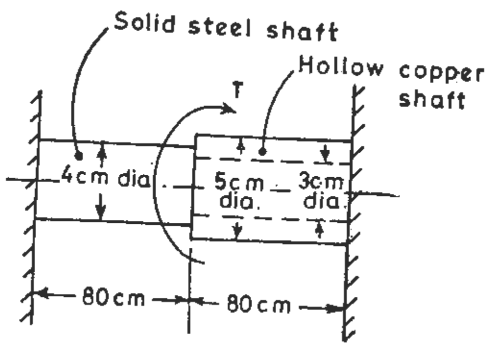


Fig. 13·10

Angular twist in steel shaft

$\theta_S=\theta_C$, angular twist in copper shaft

Say T_S = Torque shared by steel shaft

T_C = Torque shared by copper shaft

Now $$\frac{T_S}{J_S}\times\frac{l_S}{G_S}=\frac{T_C}{J_C}\times\frac{l_C}{G_C}$$

but $l_S=l_C$ and $G_S=2\,G_C$

So $$\frac{T_S}{T_C}=\frac{J_S}{J_C}\times\frac{G_S}{G_C}=\frac{8\pi}{17\,\pi}\times 2=\frac{16}{17}$$

$$T_S=\frac{16}{17}\,T_C$$

B $T_S+T_C=40$ tonne-em'

$$\frac{16}{17}T_C+T_C=40 \quad\text{or}\quad T_C=20{\cdot}636 \text{ tonne-cms}$$

$$T_S=40-20{\cdot}636=19{\cdot}364 \text{ tonne-cms}$$

and 100 cm from the end A and are in the anticlockwise direction looking from the end A. Determine the end fixing couples in magnitude and direction and calculate the diameter of the shaft if the maximum shearing stress is not to exceed 80 N/mm².

Determine also the section where the shaft suffers no angular twist.

Solution.

Say the end fixing couple at $A = T_A$

end fixing couple at $B = T_B$

Torque on the portion CD

$= T_A - 2{\cdot}5$ kNm

Torque on the portion DB

$= T_A - 2{\cdot}5 + 4{\cdot}0$

$= 1{\cdot}5 + T_A$ kNm

Total angular twist between A and B

$= \theta_1 + \theta_2 + \theta_3$

$= 0$ (as both the ends are fixed)

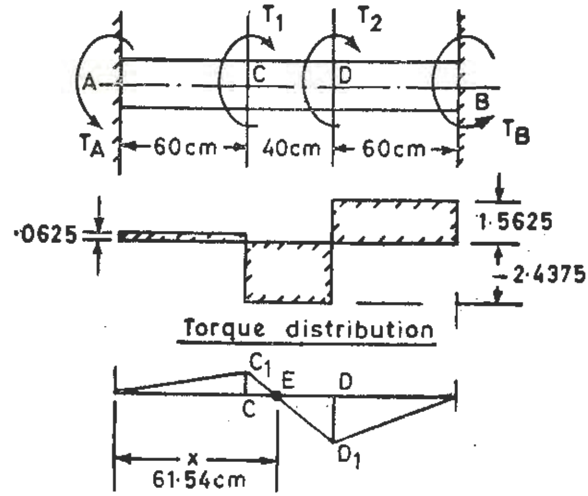


Fig. 13·9

So
$$0 = \frac{T_A \cdot l_1}{GJ} + \frac{(T_A - 2{\cdot}5)l_2}{GJ} + \frac{(T_A + 1{\cdot}5)\,l_3}{GJ}$$

where $l_1 = 0{\cdot}6$ m, $l_2 = 0{\cdot}4$ m, $l_3 = 0{\cdot}6$ m

$$T_A \times 0{\cdot}6 + (T_A - 2{\cdot}5) \times 0{\cdot}4 + (T_A + 1{\cdot}5) \times 0{\cdot}6 = 0$$

$$1{\cdot}6\, T_A - 1 + 0{\cdot}9 = 0$$

$$T_A = \frac{0{\cdot}1}{1{\cdot}6} = +{\cdot}0625 \text{ kNm}$$

$$T_B = 1{\cdot}5 + T_A = 1{\cdot}5625 \text{ kNm}$$

Twisting moment in the middle portion CD

$$= 0{\cdot}0625 - 2{\cdot}5 = -2{\cdot}4375 \text{ kNm}$$

Therefore maximum twisting moment

$$= 2{\cdot}4375 \text{ kNm} = 2{\cdot}4375 \times 10^6 \text{ Nmm}$$

Say the shaft diameter $= D$ mm

Maximum shear stress, $q = 80$ N/mm²

$$D^3 = \frac{16\, T}{\pi \times q} = \frac{16 \times 2{\cdot}4375 \times 10^6}{\pi \times 80}$$

$$= 155{\cdot}1757 \times 10^3 \text{ mm}^3$$

$$D = 5{\cdot}372 \times 10 \text{ mm}$$

Shaft diameter, $D = 53{\cdot}72$ mm

Angular twist at C, $\theta_C = \dfrac{T_A \cdot l_1}{GJ} = \dfrac{0{\cdot}0625 \times 0{\cdot}6}{GJ}$

Angular twist in both the shafts at the junction is the same *i.e.*,

$$\theta_A=\theta_B \qquad \text{...(1)}$$

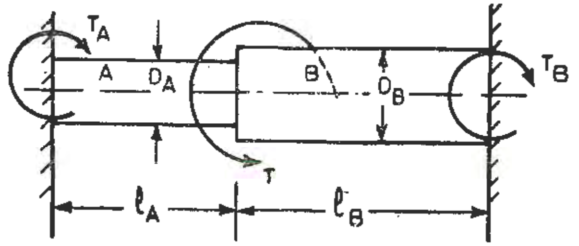


Fig. 13·7

Say G_A and G_B are the modulii of rigidity of the materials of the shafts A and B ; J_A and J_B are their polar moments of inertia. Then,

$$\theta_A=\frac{T_A\, l_A}{G_A\, J_A}, \qquad \theta_B=\frac{T_B l_B}{G_B J_B}$$

or
$$\frac{T_A l_A}{G_A\, J_A}=\frac{T_B l_B}{G_B J_B} \quad \text{or} \quad \frac{T_A}{T_B}=\frac{J_A}{J_B}\cdot\frac{G_A}{G_B}\cdot\frac{l_B}{l_A}$$

If the two portions are of solid circular sections as shown then,

$$J_A=\frac{\pi D_A{}^4}{32} \quad \text{and} \quad J_B=\frac{\pi D_B{}^4}{32}$$

Now consider a shaft consisting of a solid round bar encased in a tube or hollow sbaft which may be of different material as shown in Fig. 13·8, subjected to the torque T.

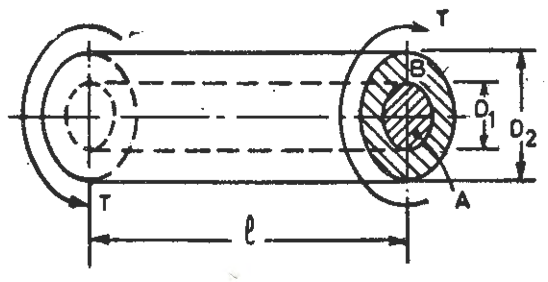


Fig. 13·8

External torque T

$=T_A+T_B$

=resisting torque of shaft A
+resisting torque of hollow shaft B

Again the angular twist in both the solid and hollow shafts is the same because both are rigidly fixed together and there is no relative displacement between them when the torque is applied,

So
$$\theta_A=\theta_B$$

$$\frac{T_A\, l_A}{J_A G_A}=\frac{T_B l_B}{J_B G_B} \quad \text{but } l_A=l_B \text{ in this case}$$

$$\frac{T_A}{T_B}=\frac{G_A}{G_B}\times\frac{J_A}{J_B}$$

Where
$$J_A=\frac{\pi D_1{}^4}{32} \quad \text{and} \quad J_B=\frac{\pi}{32}\,(D_2{}^4-D_1{}^4)$$

Example 13·5-1. A horizontal shaft 160 cm long, rigidly fixed at both the ends is subjected to axial twisting moments of $T_1=2·5$ kNm and $T_2=4·0$ kNm at distances of 60 cm

Determine the total angular twist in the shaft if $G=8\times10^5$ kg/cm²

Solution. The dimensioned sketch of the shaft is shown in the Fig. 13·6.

Taking lengths $l_1=50$ cm, $l_2=40$ cm, $l_3=60$ cm

Diameters $D_1=3$ cm, $D_2=5$ cm, $D_3=4$ cm, $D_4=6$ cm

Polar moment of Inertia

$$J_1=\frac{\pi\times3^4}{32}=\frac{81\,\pi}{32}\text{ cm}^4$$

$$J_2=\frac{\pi\times5^4}{32}=\frac{625\,\pi}{32}\text{ cm}^4$$

$$J_3=\frac{\pi}{32}\,(6^4-4^4)=\frac{1040\,\pi}{32}\text{ cm}^4$$

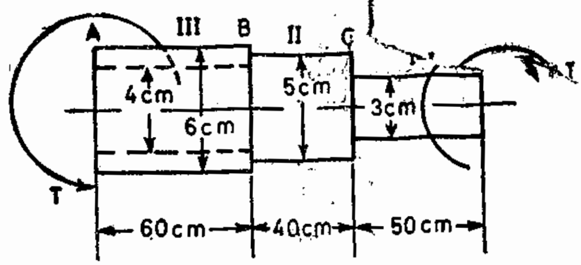


Fig. 13·6

T, torque, $=100$ kg-metre$=10{,}000$ kg-cm

Angular twist, $$\theta=\frac{T}{G}\left[\frac{l_1}{J_1}+\frac{l_2}{J_2}+\frac{l_3}{J_3}\right]$$

$$=\frac{10000}{8\times10^5}\left[\frac{50\times32}{\pi\times81}+\frac{40\times32}{\pi\times625}+\frac{60\times32}{\pi\times1040}\right]$$

$$=\frac{1}{80}\,[6{\cdot}288+0{\cdot}652+0{\cdot}588]$$

$$=\frac{7{\cdot}528}{80}=0{\cdot}0941\text{ radian}=5{\cdot}39^\circ$$

Exercise 13·4-1. A tapered circular shaft 2 m long, having 80 mm diameter at one end and 50 mm diameter at the other end is subjected to a torque 5 kNm. Determine the angular twist in the shaft if $G=84$ kN/mm². [**Ans.** 4·66 degrees]

Exercise 13·4-2. A circular shaft ABC subjected to a twisting moment T has the following dimensions.

Length $AB=50$ cm, a hollow portion with external diameter 8 cm and internal diameter 4 cm

Length $BC=50$ cm, a tapered portion, 8 cm diameter at one end B and 4 cm diameter at the other end C.

Determine the torque T to produce an angular twist of 3° in the shaft if $G=84$ kN/m[illegible]

[**Ans.** 6·85 k

13·5. COMPOUND SHAFTS

Consider a compound shaft consisting of two portions A and B, which may different materials and different dimensions. The external torque T acts at the junct[illegible] the two portions.

External torque, $T=T_A+T_B$

$=$Resisting torque at fixed end $A+$Resisting torque a[illegible]
end B

Angular twist $$\theta=\frac{32\ Tl}{3\pi\ G}\times\frac{(D_2^2+D_2D_1+D_1^2)}{D_1^3\ D_2^3}$$

Now let us consider another case of a shaft of varying diameters in steps as shown in Fig. 13·5 subjected to a twisting moment T. There are three portions of the shafts with diameters D_1, D_2 and D_3 and axial lengths l_1, l_2 and l_3 respectively.

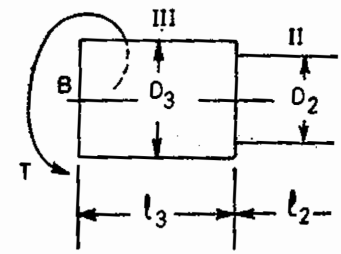


Fig. 13·5

Polar moment of inertia of the sections are

$$J_1=\frac{\pi D_1^4}{32}\qquad J_2=\frac{\pi D_2^4}{32}$$

$$J_3=\frac{\pi D_3^4}{32}$$

The total angular twist between the ends A and B is

$$\theta=\theta_1+\theta_2+\theta_3=\frac{Tl_1}{GJ_1}+\frac{Tl_2}{GJ_2}+\frac{Tl_3}{GJ_3}$$
$$=\frac{T}{G}\left[\frac{l_1}{J_1}+\frac{l_2}{J_2}+\frac{l_3}{J_3}\right]$$

Example 13·4-1. A tapered circular shaft, 100 cm long having 6 c end and 4 cm diameter at the other end is subjected to a torque so as to p twist of 1·5°. Determine the required torque if G=80,000 N/mm²

Solution.

Length of the shaft $=100$ cm

Diameter, $D_1=4$ cm

$D_2=6$ cm

Modulus of rigidity, $G=8\times10^6$ N/mm²

Angular twist, $\theta=1{\cdot}5^\circ=1{\cdot}5\ \times\frac{\pi}{180}=\frac{\pi}{120}$ radian

$$=\frac{32\ Tl}{3\pi\ G}\left(\frac{D_1^2+D_1D_2+D_2^2}{D_1^3D_2^3}\right)$$

Substituting the values we get

$$\frac{\pi}{120}=\frac{32\ T\times100}{3\pi\times8\times10^6}\left(\frac{36+24+16}{216\times64}\right)$$

Torque, $$T=\frac{3\pi^2\times8\times10^6}{120\times3200}\times\frac{216\times64}{76}=112202{\cdot}32\ \text{Ncn}$$

$$=1122{\cdot}02\ \text{Nm}$$

Example 13·4-2. A circular shaft $ABCD$, subjected to a twisting metr has the following dimensions.

Length AB=60 cm, a hollow portion with exter meters 6 cm and 4 cm

BC=40 cm, a solid portion with diameter

CD=50 cm, a solid portion with diameter 3

$$12^4 - D_1^4 = \frac{143000 \times 192}{\pi \times 600} = 14566$$

$$D_1^4 = 20736 - 14566 = 6170$$

Internal diameter $D_1 = 8{\cdot}86$ cm.

Exercise 13·3-1. A solid circular shaft of diameter 6 cm is transmitting 80 horse power at 200 revolutions per minute. Determine the maximum shear stress developed in the shaft. [**Ans.** 1350·95 kg/cm²)

Exercise 13·3-2. A hollow shaft of internal diameter 50 mm and external diameter 100 mm is transmitting horse power at 250 revolutions per minute. What maximum horse power can be transmitted if the maximum shear stress in the shaft is not to exceed 70 N/mm². [**Ans.** 45·22]

13·4. SHAFTS OF VARYING DIAMETERS

Let us consider a circular shaft of length l, tapered from a diameter D_2 to D_1 over the length l, as shown in the Fig. 13·4. The shaft is subjected to an axial torque T.

Consider a small strip of length dx at a distance of x from the end of diameter D_2.

Diameter of the shaft at the section,

$$D_x = D_2 - \left(\frac{D_2 - D_1}{l}\right) x$$

$$= D_2 - kx$$

where $k = \dfrac{D_2 - D_1}{l}$

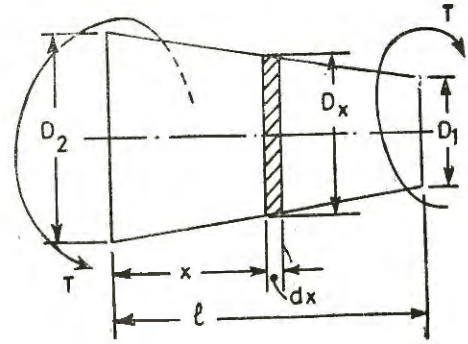


Fig. 13·4

Polar moment of inertia of strip section,

$$J_x = \frac{\pi D_x^4}{32} = \frac{\pi (D_2 - kx)^4}{32}$$

Angular twist over the length dx,

$$d\theta = \frac{T\,dx}{G\,J_x} = \frac{32\,T\,dx}{G \times \pi (D_2 - kx)^4}$$

Total angular twist, $$\theta = \int_0^l \frac{32\,T\,dx}{G\pi (D_2 - kx)^4}$$

$$= \frac{32\,T}{G\pi} \left| \frac{(D_2 - kx)^{-3}}{3k} \right|_0^l = \frac{32\,T}{3\,G\pi k} \left[\frac{1}{(D_2 - kl)^3} - \frac{1}{D_2^3} \right]$$

$$= \frac{32\,T}{3\,G\pi k} \left[\frac{1}{D_1^3} - \frac{1}{D_2^3} \right] = \frac{32\,T\,(D_2^3 - D_1^3)}{3\pi\,Gk\,D_1^3\,D_2^3}$$

Putting the value of k

$$\theta = \frac{32\,Tl}{3\pi\,G(D_2 - D_1)} \left[\frac{(D_2 - D_1)(D_2^2 + D_2 D_1 + D_1^2)}{D_1^3\,D_2^3} \right]$$

Exercise 13·2-2. A torsion test specimen of gauge length 20 cm and diameter 2 cm, when tested under torsion failed at a torque of 4250 kg-metre. Determine the modulus of rupture of the material. [**Ans.** 2705·6 kg/cm²]

13·3. HORSE POWER TRANSMITTED BY SHAFT

If a shaft subjected to torque T is rotating at N revolutions per minute, then power transmitted by the shaft in one minute $=2\pi NT$

HP, Horse power transmitted by the shaft $=\dfrac{2\pi NT}{746\times 60}$ if T is in Nm as 1 watt $=1$ Nm

metric HP $=\dfrac{2\pi NT}{4500}$ if T is in kg-metre.

There is slight difference between metric horse power and horse power.

The maximum shear stress developed on the surface of the shaft will be

$$q=\frac{16T}{\pi D^3} \text{ in case of solid shaft of diameter } D$$

$$q=\frac{16D_2T}{\pi(D_2^4-D_1^4)} \text{ in case of hollow shaft of diameters } D_1 \text{ and } D_2.$$

Example 13·3-1. The maximum shearing stress developed in 80 mm steel shaft is 60 N/mm². If the shaft rotates at 300 revolutions per minute, find the horse power transmitted by the shaft.

Solution. D, shaft diameter $=80$ mm

Maximum shearing stress, $q=60$ N/mm²

Torque on the shaft, $T=\dfrac{\pi D^3 q}{16}=\dfrac{\pi\times 80^3}{16}\times 60$

$=6031{\cdot}8\times 10^3$ Nmm $=6031{\cdot}8$ Nm

Number of revolutions/ per minute, $N=300$

HP transmitted $=\dfrac{2\pi NT}{746\times 60}=\dfrac{2\pi\times 300\times 6031{\cdot}8}{746\times 60}=254.$

Example 13·3-2. A hollow shaft of external diameter 12 cm is transmitting 400 metric horse power at 200 revolutions per minute. Determine the internal diameter if the maximum stress in the shaft is not to exceed 600 kg/cm².

Solution. Metric horse power $=400$

Revolutions per minute $=200$

Torque, $T=\dfrac{m\text{HP}\times 4500}{2\pi N}=\dfrac{400\times 4500}{2\pi\times 200}$

$=1430$ kg-metres $=143{,}000$ kg-cm.

Now $T=\dfrac{\pi(D_2^4-D_1^4)}{16\,D_2}\times q$

$$143{,}000=\frac{\pi(12^4-D_1^4)}{16\times 12}\times 600$$

13·2. MODULUS OF RUPTURE

The torsion formula $\frac{T}{J}=\frac{G\theta}{l}=\frac{q}{R}$ has been derived taking the assumption that shear stress is proportional to shear strain, *i.e.*, the proportional limit of the material is not exceeded. Many a times it is desired to, determine the maximum torque at which the shaft fails by fracture.

It has been observed experimentally that even after crossing the proportional limit, a circular section remains a circular section of the same diameter upto the angular twist at the ultimate stage. This shows that the strain at a point is still proportional to its distance from the centre while the stress at a point is no longer proportional to its distance from the centre of the shaft.

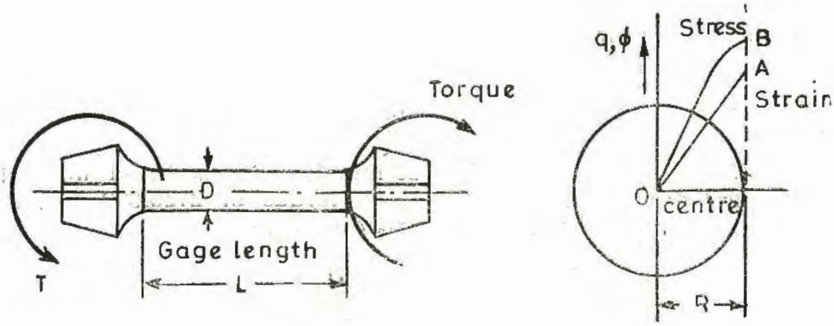


Fig. 13·3

Fig. 13·3 shows a torsion test specimen of diameter D and gage length L and shear stress and shear strain distribution along the radius of the shaft at the stage of fracture.

Say T_{max}=maximum torque at which the test piece is broken

Max. shear stress, $q'=\frac{16\,T_{max}}{\pi D^3}$ (for a solid shaft)

$=\frac{16T_{max}\,D_2}{\pi(D_2^4-D_1^4)}$ (for a hollow shaft)

This maximum shear stress calculated by using the original Torsion formula is termed as modulus of rupture. It can be observed that it is not the actual shear stress at the surface but it is a hyphothetical stress which would exist if the shear stress-shear strain curve is a straight line or the shear stress distribution is linear along the radius of the shaft. However if the modulus of rupture is known for a material, the torque required to produce fracture in the shaft can be determined.

Example 13·2-1. A torsion test specimen of gage length 250 mm and diameter 25 mm, when tested under torsion failed at a torque of 828 Nm. Determine the modulus of rupture of the material.

Solution. Torque at failure, $T=828$ Nm$=828000$ Nmm

Shaft diameter, $D=25$ mm

Modulus of rupture, $q=\frac{16T}{\pi D^3}=\frac{16\times 828000}{\pi\times(25)^3}=269{\cdot}9$ N/mm²

$$J=\frac{\pi}{2}\ (6^4-4^4)=1633{\cdot}63\ \text{cm}^4$$

Angular twist $\theta=1{\cdot}5^\circ=\frac{1{\cdot}5\times\pi}{180}=\frac{\pi}{120}$ radian

Modulus of rigidity, $G=8\times10^5$ kg/cm²

Torque, $T=\frac{G\theta J}{l}=\frac{8\times10^5\times\pi}{200\times120}\times1633{\cdot}63$

$=1{\cdot}71\times10^5$ kg-cm

Shear stress at outer surface,

$$q=\frac{T}{J}\times R_2=\frac{1{\cdot}71\times10^5}{1633{\cdot}63}\times6=628{\cdot}05\ \text{kg/cm}^2$$

Shear stress at inner surface,

$$q'=\frac{T}{J}\times R_1=\frac{1{\cdot}71\times10^5}{1633{\cdot}63}\times4=418{\cdot}7\ \text{kg/cm}^2$$

Example 13·1-3. A circular shaft of 25 mm diameter is tested under torsion. The gauge length of the test specimen is 250 mm. The torque of 2120,000 Nmm produces an angular twist of 1°. Determine the modulus of rigidity of the shaft.

Solution. Diameter of the shaft$=25$ mm$=D$

Polar moment of inertia, $J=\frac{\pi D^4}{32}=\frac{\pi\times25^4}{32}=38349{\cdot}6$ mm⁴

Angular twist, $\theta=1^\circ=\frac{\pi}{180}$ radian.

Torque, $T=2120{,}000$ Nmm

Length $=250$ mm

Modulus of rigidity, $G=\frac{Tl}{\theta J}=\frac{2120000\times250\times180}{\pi\times38349{\cdot}6}=79{\cdot}18\times10^6$ N/mm².

Exercise 13·1-1 A steel shaft for which the modulus of rigidity is $8{\cdot}2\times10^5$ kg/cm² is twisted by 2° in a length of 250 cm. The diameter of the shaft is 8 cm. Determine the torque required and the maximum shear stress developed. [**Ans.** 449 kg-metre, 132 kg/cm²]

Exercise 13·1-2. A hollow circular steel shaft having 100 mm external diameter and 60 mm internal diameter is subjected to a twisting moment of 6 kNm. Determine

(*a*) Shear stress at the inner and outer surfaces of the shaft.

(*b*) Angular twist over 2 metres length of the shaft.

G for steel$=80$ kN/mm². [**Ans.** 21·06 N/mm², 35·1 N/mm², 1·01 degree]

Exercise 13·1-3. A hollow shaft having 4 cm external diameter and 2 cm internal diameter is tested under torsion. The gage length of the test specimen is 40 cm. A torque of 6200 kg-cm produces an angular twist of 1·5°. Determine the modulus of rigidity of the shaft.

[**Ans.** $4{\cdot}02\times10^5$ kg/cm²]

=Polar moment of inertia of a hollow circular section

$$\frac{T}{J}=\frac{G\theta}{l}=\frac{q}{R_2}=\frac{q_r}{r} \qquad \ldots(3)$$

Fig. 13·2 shows the shear stress distribution along the radius of the hollow shaft. Maximum shear stress occurs at the outer radius R_2 and minimum shear stress occurs at the inner radius R_1.

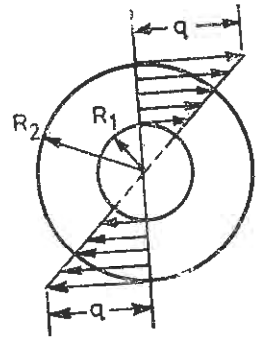


Fig. 13·2

Example 13·1-1. A circular steel shaft of 30 mm diameter is subjected to a torque of 0·56 kNm. Determine,

(*a*) The maximum shear stress developed in the shaft

(*b*) Angular twist over 1 metre length of the shaft

(*c*) The shear stress at a point which is at a distance of 1 cm from the centre of the shaft. G for steel$=82\times10^3$ N/mm².

Solotion.

Diameter, $D=30$ mm

Radius, $R=15$ mm

Torque, $T=0·56$ kNm$=0·56\times10^6$ Nmm

Polar moment of incrtia$=\dfrac{\pi\times15^4}{2}=79521·75$ mm⁴

(*a*) Maximum shear stress,

$$q=\frac{T}{J}\times R=\frac{0·56\times10^6}{79521·75}\times15=105·63 \text{ N/mm}^2$$

(*b*) Length of the shaft $=1$ m$=1000$ mm

Angular twist, $$\theta=\frac{Tl}{GJ}=\frac{0·56\times10^6\times1000}{82\times10^3\times79521·75}=0·0858 \text{ radian}$$
$$=4·92°$$

(*c*) Shear stress at $r=1$ cm$=10$ mm

$$q_r=\frac{q}{R}\times r=\frac{105·63}{15}\times10=70·42 \text{ N/mm}^2$$

Example 13·1-2. A steel shaft for which the modulus of rigidity is 8×10^5 kg/cm² is twisted by 1°30′ in a length of 2 metres. The shaft is hollow with inside diameter 8 cm and outside diameter 12 cm. Determine the torque required.

Calculate the stresses at the inner and outer surfaces of the shaft.

Solution. Polar moment of inertia,

$$J=\frac{\pi}{2}\,(R_2^4-R_1^4)$$

$R_2=6$ cm, $R_1=4$ cm, length $l=2$ m$=200$ cm

Shear stress developed at radius r

$$=q_r=q\frac{r}{D/2}=\frac{q\,.\,r}{R}$$

Shear force on the elementary ring,

$$\delta F=2\pi r\,dr\,\frac{q.r}{R}=\frac{q2\pi r^2}{R}\,dr$$

Moment of the shear force on ring about the centre of the shaft,

$$\delta T=\frac{2\pi r^2}{R}\,q.dr.r=\frac{2\pi r^3}{R}q\,dr$$

Total twisting moment of resistance,

$$T=\int_0^R\frac{2\pi r^3}{R}\,q\,.\,dr=\frac{2\pi q}{R}\int_0^R r^3.dr$$

$$=\frac{2\,q}{R}\times\frac{R^4}{4}=\frac{q}{R}\left(\frac{\pi R^4}{2}\right)$$

But $\frac{\pi R^4}{2}=J$, polar moment of inertia of the solid circular section

Therefore $$\frac{T}{J}=\frac{q}{R}=\frac{G\theta}{l}=\frac{q_r}{r} \quad ...(2)$$

where q=maximum shear stress at radius R

q_r=shear stress at radius r

R=Radius of the shaft

$$J=\frac{\pi R^4}{2}=\frac{\pi D^4}{32}$$

Hollow Shaft

Say R_1=inner radius or D_1=inner diameter

R_2=outer radius D_2=outer diameter

Twisting moment of resistance,

$$T=\int_{R_1}^{R_2}\frac{2\pi r^3}{R_2}\,q\,dr$$

because maximum shear q occurs at the maximum radius which is R_2 *i.e.*, the outer radius in a hollow shaft

$$T=\frac{2\pi q}{R_2}\left|\frac{r^4}{4}\right|_{R_1}^{R_2}=\frac{q}{R_2}\left[\frac{\pi}{2}\,(R_2^4-R_1^4)\right]$$

$$=\frac{q}{R_2}\times J$$

Where $$J=\frac{\pi}{2}\,(R_2^4-R_1^4)=\frac{\pi}{32}\,(D_2^4-D_1^4)$$

(*b*) The shaft is not distorted initially.

(*c*) The displacement at a point in the shaft is proportional to its distance from the centre of the shaft or consequently the shear strain at a point is proportional to its distance from the centre of the shaft.

(*d*) Cross sections perpendicular to the axis of the shaft which are plane before the shaft is twisted remain plane after the shaft is subjected to the twisting moment.

(*e*) The twist is uniform along the length of the shaft.

13·1 SHEAR STRESS AND ANGULAR TWIST IN SHAFT

If a shaft, is acted upon by a pure torque T (meaning thereby that there is no bending moment M on the shaft) about its polar axis OO_1, shear stress will be set up on all transverse sections as shown, in Fig. 13·1.

To investigate the shear strain, shear stress and angular twist produced in the shaft under torque T, consider a circular shaft of diameter D, length l fixed at one end and subjected to the torque T on the other end. At the fixed end there will be equal and opposite reaction to torque T. A line CA initially drawn on the shaft parallel to the shaft axis has taken the new position CA' after the shaft is twisted. The angle $\angle ACA'$ is called the shear angle ϕ or shear strain and the angle $\angle AOA'$ is called the angular twist, θ.

At any radius r at a distance of l from the fixed end, the peripheral displacement

$$BB'=r\theta$$

Shear angle, or shear strain

$$\phi_r=\frac{BB'}{l}=r\,\frac{\theta}{l} \quad \text{or} \quad \phi_r \propto r$$

Say the shear stress at radius r is q_r

Then $$\frac{q_r}{G}=r\,\frac{\theta}{l} \quad \text{where } G=\text{Modulus of rigidity of the material}$$

$$q_r=r\,\frac{\theta}{l}\,.\,G \quad \text{or} \quad q_r \propto r$$

i.e., shear strain ϕ_r and shear stress q_r at any radius are proportional to r. The maximum value of shear stress will correspond to the maximum value of r *i.e.*, at the surface of the shaft.

Say shear stress at the surface $=q$ at $r=\dfrac{D}{2}$

So $$q=\frac{D}{2}\times\frac{\theta}{l}\,.\,G$$

or $$\frac{q}{D/2}=\frac{\theta}{l}\times G \quad \text{or} \quad \frac{q}{R}=\frac{G\theta}{l} \quad \text{where} \quad R=\frac{D}{2} \qquad \ldots(1)$$

Moreover it has been shown in Fig. 13·1 (*c*) that the angular twist at any section is proportional to its distance from the fixed end and θ/l is a unique value for a particular shaft subjected to a certain amount of torque. Therefore, the shear stress at a particular radius on all transverse sections of the shaft is the same.

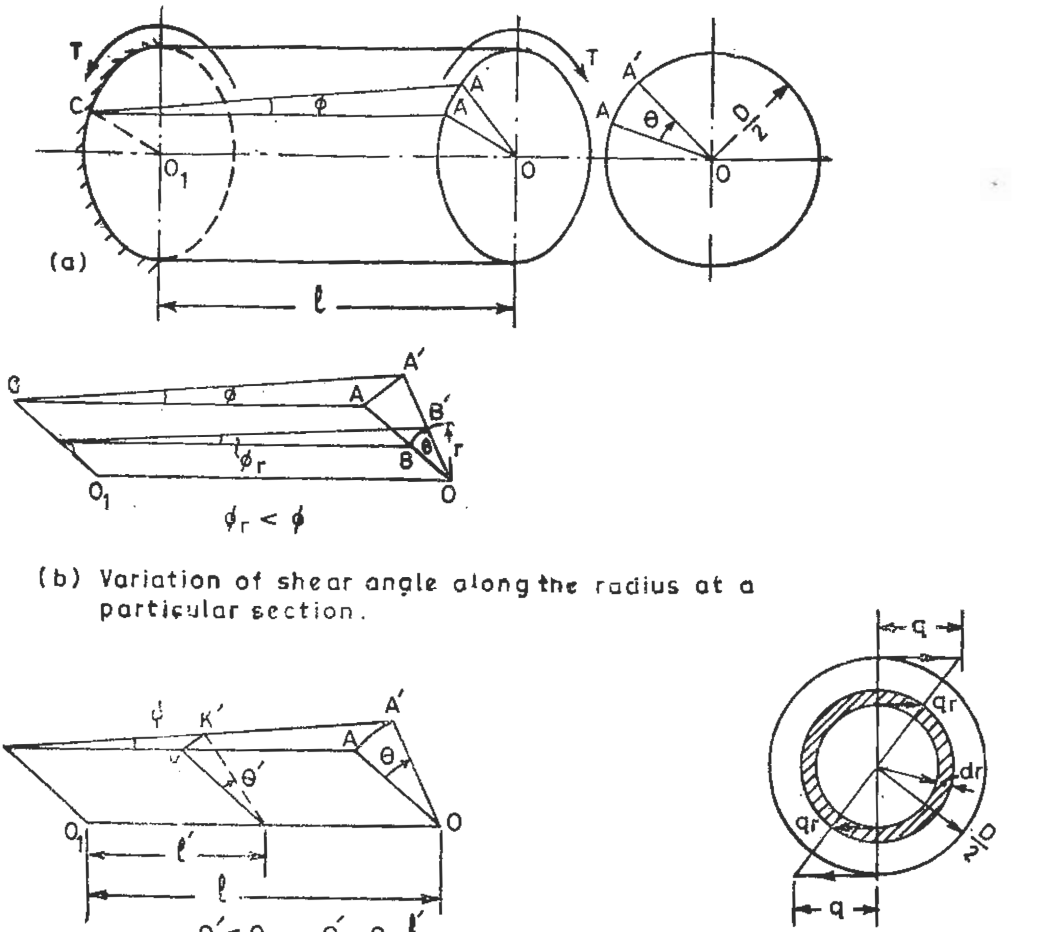
Let us determine the torque which is transmitted from section to section. Consider an elementary ring of thickness δ_r at the radius r [Fig. 13·1 (*d*)].

13

Torsion

The shafts carrying the pulleys, gears etc., and transmitting power are subjected to the twisting moments. The shaft is distorted when it is transmitting power. To determine the angular twist and the shear strain developed in the shaft under the twisting moment or the torque, following assumptions are taken :

(*a*) The material is homogeneous and isotropic *i.e.*, its elastic properties are the same at all the points of the body and in all the directions.



(b) Variation of shear angle along the radius at a particular section.

(c) Variation of angular twist along the length.

Variation of shear stress along the radius

(d)

Fig. 13·1

mid span. On CD, there is another concentrated load of 8 tonnes at a distance of 1 m from end D. Determine support moments and support reactions. Draw a dimensioned B.M. diagram for the beam. [**Ans.** -4 Tm, $-4{\cdot}625$ Tm, 0 ; 8·843 T, 8·314 T, 4·843 T]

12·11. A continuous beam ABC, having equal spans $AB=BC=l$ carries a uniformly distributed load of w per unit length on whole of its length. The beam is simply supported at the ends. If the support B sinks by δ below the level of the supports A and C, show that reaction at B is

$$R_B=\frac{5wl}{4}-\frac{6\,EI\,\delta}{l^3}$$

12·12. A continuous beam ABC of length 8 m is supported over two spans AB and BC of equal lengths. A concentrated load 20 kN is applied at the mid point of AB and a concentrated load of 60 kN is applied at the mid point of BC. Determine the slope at the supports A, B and C. $E=200$ kN/mm^2, $I=20{,}000$ cm^4

[**Ans.** 0, $-0{\cdot}5\times10^{-3}$ radian, $+1{\cdot}0\times10^{-3}$ radian]

12·13. A continuous beam $ABCD$, 10 m long fixed at end A, supported over spans AB and BC, both 4 m long has an overhang $CD=2$ m. There is a concentrated load of 5 tonnes at the centre of AB and a uniformly distributed load of 1·5 tonnes/metre run from B to D. While the supports A and C remain at one level, support B sinks by 2 mm. The moment of inertia of the beam from A to B is 2000 cm^4 and from B to D it is 1500 cm^4. If $E=2000$ tonnes/cm^2, determine the support moments and support reactions.

[**Ans.** $-3{\cdot}38$ Tm, $-1{\cdot}04$ Tm, -3 Tm ; 3·085 T, 4·425 T, 6·490 T]

12·2. A steel girder 20 cm deep has a span of 4 metres and is rigidly built in at both the ends. The loads on the girder consists of a uniformly distributed load of 30 kN/metre run on the whole span and a point loan W at the centre of the span. Find the required point load if the maximum stress due to bending is not to exceed 80 N/mm².

The section of the girder is symmetrical about XX and YY axis. I_{xx}=8000 cm⁴. Determine also the maximum deflection. E=200 kN/mm².

[**Ans.** 48 kN ; −2·25 mm]

12·3. A beam of span l is fixed at both the ends. A couple M is applied to the beam at its centre, about a rule horizontal axis at right angles to the beam. Determine the fixing couples at each support and slope at the centre of the beam. EI is the flexural rigidity of the beam.

$$\left[\textbf{Ans.}\quad \frac{M}{4} \text{ (in the same direction at both the ends)}, \quad \frac{Ml}{16\,EI}\right]$$

12·4. A rung of a vertical ladder is in the horizontal plane. Rung is perpendicular to the vertical sides of the ladder. Length of the rung is $4B$ and distance between the rungs is B. Ladder is made of a circular steel section. If a vertical load W is carried in the middle of a particular rung, find the twisting moment at the ends of the rung. E_{steel}=208 kN/mm², G_{steel}=80 kN/mm²

[**Ans.** 0·303 WB]

12·5. A fixed beam of length l carries a linearly increasing distributed load of intensity zero at the left hand end to w per unit length at the right hand end. Determine (*i*) support reactions and (*ii*) support moments. Given EI is the flexural rigidity of the beam.

$$\left[\textbf{Ans.}\quad 0{\cdot}15\, wl,\ 0{\cdot}35\, wl\ ;\ -\frac{wl^2}{30},\ -\frac{wl^2}{20}\right]$$

12·6. A continuous beam ABC, 12 m long is supported on two spans $AB=BC$=6 m. Span AB carries a uniformly distributed load of 1·5 T/m run and span BC carries a uniformly distributed load of 2·4 T/m run. The moment of inertia of the beam for the span AB is I_1 and that for the span BC is I_2. If $I_1=\frac{1}{2}I_2$, determine support reactions and support moments.

[**Ans.** 3·15 T, 14·4 T, 5·85 T ; M_B=8·1 Tm]

12·7. A continuous beam ABC of length 12 m, with span AB and BC each 6 m long, carries a uniformly distributed load of 1·5 T/m run from the end A upto the centre of AB and from the end C upto the centre of CB. Determine the reactions and moments at the supports.

[**Ans.** 2·885 T, 3·230 T, 2·885 T ; M_B=−2·954 Tm]

12·8. A continuous beam $ABCD$, 18 metres long, rests on supports A, B and C at the same level ; AB=6 m, BC=10 m. The loading is 2 tonnes/metre run throughout and in addition a concentrated load of 4 tonnes acts in the centre of the span BC and a load of 2 tonnes acts at D. Determine the reactions and moments at the supports. Draw the B.M. diagram. [**Ans.** 2·75, 22·4, 16·85 T ; 0, −19·5 Tm, −8 Tm]

12·9. A continuous beam $ABCD$, 15 m long supported over 3 equal spans AB, BC and CD. Span AB carries a point load of 8 Tonnes at its centre. Span BC carries a uniformly distributed load of 2 tonnes/metre run throughout its length and span CD carries a point load of 6 Tonnes at its centre. The level of the support C is 5 mm below the levels of A, B and D. Determine support moments and support reactions. E=2000 tonnes/cm², I=12000 cm⁴

[**Ans.** M_B=−5·366 Tm, M_C=−3·156 Tm ; 2·927, 10·515, 8·189 T and 2·369 T]

12·10. A continuous beam $ABCD$, 10 m long supported over two spans BC and CD, each 4 m long, has an overhang AB of 2 m. The overhang portion is loaded with a uniformly distributed load of 2 tonnes/metre run. On BC, there is a concentrated load of 10 Tonnes at the

5. A beam fixed at both the ends carries a uniformly distributed load of 10 k N/m over its entire span of 6 m. The bending moment at the centre of the beam, is

(*a*) 15 kNm (*b*) 30 kNm
(*c*) 45 kNm (*d*) 60 kNm

6. A beam of length 4 m, fixed at both the ends carries a concentrated load at its centre. If $W=6$ tonnes and EI for the beam is 2000 tonne-metre², deflection at the centre of the beam is

(*a*) 0·1 mm (*b*) 1·0 mm
(*c*) 10 mm (*d*) None of the above

7. A continuous beam 8 m long, supported over two spans 4 m long each, carries a uniformly distributed load of 1 tonne/metre run over its entire length. The support moment at the central support is

(*a*) 2 Tm (*b*) 3 Tm
(*c*) 4 Tm (*d*) 4·5 Tm (Tonne-metres)

8. A continuous beam 12 m long, supported over two spans 6 m each, carries a concentrated load 40 kN each at the centre of each span. The bending moment at the central support is

(*a*) 90 kNm (*b*) 60 kNm
(*c*) 45 kNm (*d*) 30 kNm

9. A fixed beam of length l, sinks at one end by an amount δ, If EI is the flexural rigidity of the beam, the fixing couple at the ends is

(*a*) $EI\,\delta/l^2$ (*b*) $3\,EI\,\delta/l^2$
(*c*) $6\,EI\,\delta/l^2$ (*d*) $6\,EI\,\delta/l^3$

10. A continuous beam 8 m long, supported over two spans 4 m each, carries a uniformly distributed load of 1 tonne/metre run, over its entire length. If the reaction at one end supports is 2·5 Tonne, the reaction at the central support will be

(*a*) 1·5 Tonnes (*b*) 3 Tonnes
(*c*) 4 Tonnes (*d*) 5 Tonnes

ANSWERS

1. (*c*)	2. (*c*)	3. (*a*)	4. (*b*)	5. (*a*)
6. (*b*)	7. (*a*)	8. (*c*)	9. (*c*)	10. (*b*)

EXERCISE

12·1. A girder of 9 m span is fixed horizontally at the ends. A downward vertical load of 4 Tonnes acts on the girder at a distance of 3 metres from the left hand end and an upward vertical force of 4 Tonnes acts on the girder at a distance of 3 metres from the right hand end. Determine the reactions and fixing couples. Draw the SF and BM diagrams for the beam. [**Ans.** $\pm$1·926 Tonnes ; $\mp$2·667 Tonne-meters]

8. For the two consecutive spans $AB=l_1$ and $BC=l_2$ of a continuous beam carrying any type of loading, theorem of 3 moments is as follows (when A, B and C supports are at the same level)

$$(i)\quad M_A\, l_1+2M_B\,(l_1+l_2)+M_C l_2+\frac{6a_1\bar{x}_1}{l_1}+\frac{6a_2\bar{x}_2}{l_2}=0$$

where $a_1\,\bar{x}_1$=first moment of M_x bending moment diagram for span AB, considering the origin at A. (In M_x diagram span AB is considered independently as simply supported)

$a_2\bar{x}_2$=first moment of M_x bending moment diagram for span BC, considering C to be the origin.

(*ii*) if support B is below the level of A by δ_1 and below the level of C by δ_2, theorem of 3 moments will be

$$M_A\,.\,l_1+2M_B(l_1+l_2)+M_C l_2+\frac{6a_1\bar{x}_1}{l_1}+\frac{6a_2\bar{x}_2}{l_2}-\frac{6EI\delta_1}{l_1}-\frac{6EI\delta_2}{l_2}=0.$$

9. For the two consecutive spans $AB=l_1$ and $BC=l_2$ of a continuous beam, if the end A is fixed and spans carry any type of loading, an imaginary span AA' of length zero can be considered by the side of BA and theorem of 3 moments is modified as follows (for span $A'A$ (zero length) and AB) :—

$$2M_A l_1+M_B l_1+\frac{6a_1(l_1-\bar{x}_1)}{l_1}=0$$

where a_1=area of M_x bending moment diagram for span AB

$\bar{x}_1$=distance of CG of area a_1 from end A.

MULTIPLE CHOICE QUESTIONS

1. A beam of length 6 metres carries a concentrated load 60 kN at its centre. The beam is fixed at the both the ends. The fixing couple at the ends is

   (*a*) 90 kNm (*b*) 60 kNm
   (*c*) 45 kNm (*d*) 30 kNm

2. A beam of length l fixed at both the ends carries a uniformly distributed load w per unit length, throughout the span. The bending moment at the ends is

   (*a*) $wl^2/4$ (*b*) $wl^2/8$
   (*c*) $wl^2/12$ (*d*) $wl^2/16$

3. A beam of length l, fixed at both the ends carries a concentrated load W at its centre. If EI is the flexural rigidity of the beam, the maximum deflection in the beam is

   (*a*) $Wl^3/192\ EI$ (*b*) $Wl^3/96\ EI$
   (*c*) $Wl^3/48\ EI$ (*d*) None of the above

4. A beam of length l, fixed at both the ends carries a uniformly distributed load of w per unit length. If EI is the flexural rigidity of the beam, then maximum deflection in the beam is

   (*a*) $5\ wl^4/384\ EI$ (*b*) $wl^4/384\ EI$
   (*c*) $wl^4/48\ EI$ (*d*) $wl^4/192\ EI$

SUMMARY

1. If a beam is fixed at both the ends, then slope and deflection at both the ends are zero.

2. For a fixed beam AB, of length l $\quad a\bar{x}+a'\bar{x}'=0$

where $\quad a\bar{x}$=first moment of the area a of the B.M. diagram about the point A (considering the beam to be simply supported at the ends)

$a'\bar{x}'$=first moment of area a' of the B.M. diagram due to support moments, about the point A

$$=(M_A+2M_B)\,\frac{l^2}{6} \text{ and } a'=(M_A+M_B)\,\frac{l}{2}.$$

3. For a beam AB of length l, fixed at both the ends carrying a concentrated load W at its centre,

Fixing couples, $M_A=M_B=\frac{-Wl}{8}$

B.M. at the centre of the beam$=+\frac{Wl}{8}$

Deflection, $y_{max}=\frac{-Wl^3}{192\,EI}$ (at the centre).

4. For a beam AB of length l, fixed at both the ends carrying a uniformly distributed load w throughout its length

Fixing couples, $M_A=M_B=-\frac{wl^2}{12}$

B.M. at the centre of the beam$=+\frac{wl^2}{24}$

Deflection, $y_{max}=-\frac{wl^4}{384\,EI}$ (at the centre).

5. For a beam AB of length l, fixed at both the ends, carrying a load W at a distance of a from end A, $\left(a<\frac{l}{2}\right)$

Fixing couples, $M_A=-\frac{Wab^2}{l^2}$, $\quad M_B=-\frac{Wa^2b}{l^2}$

Reactions, $R_A=\frac{Wb^2(b+3a}{l^3}$ and $R_B=\frac{Wa^2(a+3b)}{l^3}$

Deflection, $y_{max}=-\frac{2}{3}\;\frac{Wa^2b^3}{(a+3b)^2\,EI}$.

6. In a fixed beam, if one support sinks by δ, then fixing couple at the ends due to sinking of support is $\frac{6EI\delta}{l^2}$ where l is the length between the supports.

7. For the two consecutive spans AB and BC of a continuous beam, carrying uniformly distributed load w_1 over AB of length l_1 and w_2 over BC of length l_2, Clapeyron's theorem of 3 moments gives the expression for support moments M_A, M_B, M_C as

$$M_Al_1+2M_B(l_1+l_2)+M_Cl_2+\frac{w_1l_1^3}{4}+\frac{w_2l_2^3}{4}=0.$$

So $$10\,M_A+5M_B=-\frac{260{\cdot}416\times6}{5}-\frac{6\times210\times10^6\times18000\times10^{-8}\times{\cdot}001}{5}$$

$$10\,M_A+5M_B=-312{\cdot}5-45{\cdot}36=-357{\cdot}86 \qquad ...(2)$$

Again using the theorem of 3 moments for the spans AB and BC and noting that B is below A and C by 1 mm and moment of inertia for AB is 18000 cm^4 and for BC it is 12000 cm^4.

$$\frac{5M_A}{I_1}+\frac{2M_B\times5}{I_1}+\frac{2M_B\times4}{I_2}+\frac{4M_C}{I_2}$$

$$=-\frac{6a_1\bar{x}_1}{I_1l_1}-\frac{6a_2\bar{x}_2}{I_2l_2}+\frac{6EI_1\delta_1}{I_1l_1}+\frac{6EI_2\delta_2}{I_2l_2} \text{ where } \delta_1=\delta_2={\cdot}001 \text{ m}$$

Multiplying throughout by I_1 we get

$$5M_A+10M_B+8M_B\times\frac{I_1}{I_2}+4M_C\times\frac{I_1}{I_2}=-\frac{6a_1\bar{x}_1}{l_1}-\frac{6a_2\bar{x}_2}{l_2}\times\frac{I_1}{I_2}+\frac{6EI_1\,\delta}{l_1}+\frac{6EI_1\,\delta}{l_2} \qquad ...(3)$$

Substituting the values in equation (3)

$$5M_A+10M_B+8M_B\times\left(\frac{18000}{12000}\right)+4M_C\times\left(\frac{18000}{12000}\right)$$

$$=-\frac{6\times260{\cdot}416}{5}-\frac{6\times160}{4}\times\left(\frac{18000}{12000}\right)$$

$$+6\times210\times10^6\times18000\times10^{-8}\times{\cdot}001\left(\frac{1}{5}+\frac{1}{4}\right)$$

or $$5M_A+22M_B+6M_C=-312{\cdot}5-360+102{\cdot}06$$

$$5M_A+22M_B+6(-20)=-570{\cdot}44$$

$$5M_A+22M_B=-570{\cdot}44+120=-450{\cdot}44 \qquad ...(4)$$

From equation (2)

$$5M_A+2{\cdot}5M_B=-178{\cdot}93 \qquad ...(5)$$

or From equations (4) and (5)

$$19{\cdot}5\,M_B=-271{\cdot}51 \quad \text{or} \quad M_B=-13{\cdot}92 \text{ kNm}$$

$$M_A=-28{\cdot}83 \text{ kNm} \quad \text{and} \quad M_C=-20 \text{ kNm.}$$

Support reactions. Taking moments about the point B,

$$-20\times5+4R_C-40\times2=M_B, \quad \text{or} \quad 4R_C=-13{\cdot}92+180$$

Reaction, $R_C=41{\cdot}52$ kN

Taking moments about the point A

$$-20\times10+9R_C+5R_B-40\times7-5\times10\times2{\cdot}5=M_A$$

$$-200+9\times41{\cdot}52+5R_B-280-125=-28{\cdot}83$$

$$5R_B=-28{\cdot}83+605-373{\cdot}68 \quad \text{or} \quad 5R_B=202{\cdot}49$$

Reaction, $R_B=40{\cdot}5$ kN

So Reaction, $R_A=5\times10+40+20-41{\cdot}52-40{\cdot}50=110-82{\cdot}02=27{\cdot}98$ kN

$$EI\, i_B = 0 - 0 - \frac{Wl^2}{8}, \text{ or } i_B = -\frac{Wl^2}{8EI}$$

But the slope will be positive at B, *i.e.*, $i_B = +\frac{Wl^2}{8EI}$ because we have reversed the direction of x.

Problem 12·15. A continuous beam $ABCD$, 10 m long, fixed at end A, supported over spans AB and BC ; $AB=5$ m and $BC=4$ m with overhang $CD=1$ m. There is a uniformly distributed load of 10 kN/m over AB, a concentrated load 40 kN at the centre of BC and a concentrated load 20 kN at the free end D. While the supports A and C remain at are level and the support B sinks by 1 mm. The moment of inertia of the beam from A to B is 180,00 cm^4 and from B to D is 12000 cm^4. If $E=210$ GN/m^2, determine the support moments and support reactions.

Solution. Fig. 12·34. shows a continuous beam $ABCD$, fixed at A supported over AB and BC with overhang CD. The span $AB=5$ m, $BC=4$ m and overhang $CD=1$ m. On span AB, a *udl* of 10 kN/m and on span BC, a point load 40 kN acts. At the free end there is a concentrated load of 20 kN.

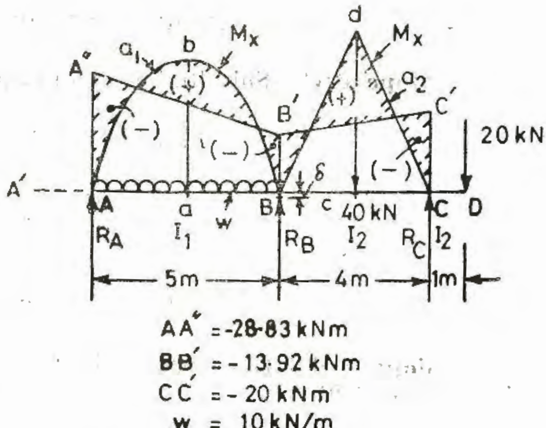


Fig. 12·34

First of all let us construct the M_x diagrams for AB and BC.

B.M. at the centre of AB

$$=\frac{wl^2}{8}=\frac{10\times 5^2}{8}=31·25 \text{ kNm}=a\,b$$

as shown in the Figure. The curve AbB is a parabola.

B.M. under the central load, span BC

$$=\frac{Wl}{4}=\frac{40\times 4}{8}=40 \text{ kNm}=cd$$

as shown in the Fig. 12·34.

Origin at B, $a_1\bar{x}_1=31·25\times 5\times \frac{2}{3}\times 2·5=260·416$ kNm^3

Origin at A, $a_1\bar{x}_1=260·416$ kNm^3

Origin at B, $a_2\bar{x}_2=40\times \frac{4}{2}\times 2=160$ kNm^3

Origin at C, $a_2\bar{x}_2=40\times \frac{4}{2}\times 2=160$ kNm^3

Support Moments. Support moment at C, $M_C=-20\times 1=-20$ kNm

Consider an imaginary span $A'A$ of length 0 using the theorem of 3 moments (with sinking suport) for spans $A'AB$.

$$\frac{M_{A'}\times 0}{I_1}+2M_A\frac{(0+5)}{I_1}+\frac{5M_B}{I_1}=-\frac{6a_1\bar{x}_1}{I_1 l_1}+\frac{6EI_1\delta}{I_1 l_1} \qquad \ldots(1)$$

because moment of inertia is different in two portions, the theorem of 3 moments has been modified as above. Moreover level of B is 1 mm below the level of A or in other words leve of A is 1 mm higher them the level of B so $\delta=-1$ mm. $E=210\times 10^9$ N/m^2$=210\times 10^6$ kN/m^2

$$-\frac{3}{32}W(1+n)=-\frac{3}{8}W$$

$$1+n=4 \quad \text{or} \quad n=3$$

So $$M_C=-\frac{3}{32}Wl(1+n)=-\frac{3Wl}{8}$$

Reaction, $$R_A=\frac{W}{8}$$

To determine reaction R_B let us take moments about the point C

$$R_C\times l-nW\times\frac{l}{2}=M_C$$

$$R_C\times l-1\cdot5\,Wl=-\cdot375\,Wl, \quad \text{or} \quad R_C=1\cdot125\,W$$

Slope at C. Substituting $x=l$ in equation (4)

$$EI\,i_C=R_A\,.\,\frac{l^2}{2}-\frac{W}{2}\times\frac{l^2}{4}+\frac{Wl^2}{48}-\frac{Wl^2}{48}$$

$$=\frac{W}{8}\times\frac{l^2}{2}-\frac{Wl^2}{8}=-\frac{Wl^2}{16}$$

$$i_C=-\frac{Wl^2}{16\,EI}$$

Slope at B. To determine slope at B, let us consider a section in the portion CB, taking origin at B

$$EI\frac{d^2y}{dx^2}=R_C\,.\,x-(3W)\left(x-\frac{l}{2}\right) \text{ as } n=3$$

$$=1\cdot125\,Wx-3W\left(x-\frac{l}{2}\right) \qquad \ldots(6)$$

Integrating two times the equation (6)

$$EI\frac{dy}{dx}=R_C.\,\frac{x^2}{2}-\frac{3W}{2}\left(x-\frac{l}{2}\right)^2+C_3 \qquad \ldots(7)$$

$$EIy=1\cdot125\,W\frac{x^3}{6}-\frac{3W}{6}\left(x-\frac{l}{2}\right)^3+C_3x+C_4 \qquad \ldots(8)$$

$C_4=0$ because $y=0$ at $x=0$ at end B.

Moreover $y=0$ at $x=l$ at point C

So $$0=1\cdot125\frac{Wl^3}{6}-\frac{W}{2}\times\left(\frac{l^3}{8}\right)+C_3l$$

$$C_3=\frac{Wl^2}{16}-\frac{1\cdot125\,Wl^2}{6}=-\frac{12Wl^2}{96}=-\frac{Wl^2}{8}$$

$$EI\frac{dy}{dx}=R_C\,.\,\frac{x^2}{2}-\frac{3W}{2}\left(x-\frac{l}{2}\right)^2-\frac{Wl^2}{8}$$

But at $x=0$, $\frac{dy}{dx}=i_B$, omitting $\left(x-\frac{l}{2}\right)$ term.

$$M_C = -\frac{3}{32}\, Wl\,(1+n) \quad \ldots(1)$$

where $M_A = M_B = 0$

as the beam is simply supported at the ends

Taking moments about C

$$R_A \,.\, l - \frac{Wl}{2} = M_C = -\frac{3}{32}\, Wl\,(1+n)$$

$$R_A l = \frac{Wl}{2} - \frac{3}{32}\, Wl\,(1+n)$$

$$R_A = \frac{W}{2} - \frac{3W}{32}\,(1+n) \quad \ldots(2)$$

Now consider a section X-X at a distance of x from end A

B.M. on the section as shown $= R_A \,.\, x - \frac{W}{2}\left(x - \frac{l}{2}\right)$

or
$$EI\frac{d^2y}{dx^2} = R_A x - \frac{W}{2}\left(x - \frac{l}{2}\right) \quad \ldots(3)$$

Integrating equation (3)

$$EI\frac{dy}{dx} = RA\,\frac{x^2}{2} - \frac{W}{2}\left(x - \frac{l}{2}\right)^2 + C_1 \quad \ldots(4)$$

Integrating equation (4) also

$$EIy = RA\,.\,\frac{x^3}{6} - \frac{W}{6}\left(x - \frac{l}{2}\right)^3 + C_1x + C_2 \quad \ldots(5)$$

Constant of integration, $C_2 = 0$ because at $x=0$, $y=0$

Moreover at $x = l$, $y = 0$

So
$$0 = R_A\,.\,\frac{l^3}{6} - \frac{W}{6}\left(\frac{l}{2}\right)^3 + C_1 l$$

or Constant of integration, $C_1 = \frac{Wl^2}{48} - R_A.\,\frac{l^2}{6}$

$$EIy = R_A\,\frac{x^3}{6} - \frac{W}{6}\left(x - \frac{l}{2}\right)^3 + \left(\frac{Wl^2}{48} - R_A.\,\frac{l^2}{6}\right)x \quad \ldots(5)$$

Moreover
$$EI\,\frac{dy}{dx} = R_A.\,\frac{x^2}{2} - \frac{W}{2}\left(x - \frac{l}{2}\right)^2 + \left(\frac{Wl^2}{48} - R_A.\,\frac{l^2}{6}\right) \quad \ldots(4)$$

But at $x=0$, $\frac{dy}{dx} = 0$ as given in the problem.

Omitting the term $\left(x - \frac{l}{2}\right)$

$$EI \times 0 = 0 - 0 + \frac{Wl^2}{48} - R_A\,\frac{l^2}{6} \quad \text{or} \quad R_A = \frac{W}{8}$$

But
$$R_A = \frac{W}{2} - \frac{3W}{32}\,(1+n) = \frac{W}{8}$$

Reaction, $R_C = \frac{M_B}{l_2} + \frac{w_2 l_2}{2}$

Reaction, $R_B = w_1 l_1 + w_2 l_2 - R_A - R_C = \frac{w_1 l_1}{2} + \frac{w_2 l_2}{2} - \frac{M_B}{l_1} - \frac{M_B}{l_2}$

$$= \frac{w_1 l_1}{2} + \frac{w_2 l_2}{2} + \frac{w_1 l_1^2}{8(l_1+l_2)} + \frac{w_2 l_2^3}{8l_1(l_1+l_2)} - \frac{6EI\delta}{2l_1^2 l_2}$$

$$+ \frac{w_1 l_1^3}{8(l_1+l_2)l_2} + \frac{w_2 l_1^2}{8(l_1+l_2)} - \frac{6EI\delta}{2l_1 l_2^2}$$

but $w_1 = w_2 = w$

$$R_B = \frac{w(l_1+l_2)}{2} + \frac{wl_1^2 + wl_2^2}{8(l_1+)l_2} + \frac{wl_2^3}{8(l_1+l_2)l_1} + \frac{wl_1^3}{8(l_1+l_2)l_2} - \frac{3E\delta I}{l_1 l_2}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)$$

$$R_B = \frac{w(l_1+l_2)}{2} + \frac{w(l_1^3+l_2^3)}{8l_1 l_2} - \frac{3E\delta\,(l_1+l_2)}{l_1^2 l_2^2}$$

Problem 12·14. A beam AB of length $2l$, simply supported at its ends is propped at the mid point C to the same level as the ends. A concentrated load W is applied at the mid point of AC and a concentrated load nW at the mid point of CB. For what value of n, slope at the end A will be zero. Determine the slopes at B and C. Given EI is the flexural rigidity of the beam.

Solution. Figure 12·33 shows a continuous beam of length $2l$, with span $AC=l$ and $CB=l$.

Load at the mid point of AC is W and load at the mid point of CB is nW.

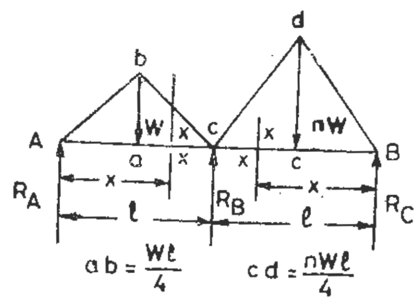


Fig. 12·33

Let us construct M_x diagram

M_{max} on span AC, $ab = \frac{Wl}{4}$

$$a_1\bar{x}_1 = \frac{Wl}{4} \times \frac{l}{2} \times \frac{l}{2} = \frac{Wl^3}{16}$$

(about the origin A)

Span CB

$$M_{max} = cd = \frac{nWl}{4}$$

$$a_2\bar{x}_2 = \frac{nWl}{4} \times \frac{l}{2} \times \frac{l}{2} = \frac{nWl^3}{16}$$

(about the origin C)

Using the theorem of 3 moments

$$M_A l + 2M_C(2l) + M_B l = -\frac{6a_1\bar{x}_1}{l_1} - \frac{6a_2\bar{x}_2}{l_2} = -\frac{6Wl^3}{16l} - \frac{6nWl^3}{16l}$$

$$4M_C l = -\frac{6Wl^2}{16}(1+n)$$

Problem 12·13. A continuous ABC having two spans $AB=l_1$ and $BC=l_2$ carries a uniformly distributed load of w per unit length on its whole length. The beam simply rests on the end supports. If the support B sinks by an amount δ below the level of the supports A and C, show that the reaction at B is

$$R_B=\frac{w(l_1+l_2)}{2}+\frac{w(l_1^3+l_2^3)}{8l_2l_2}-\frac{3EI\delta(l_1+l_2)}{l_1^2l_2^2}.$$

Solution. Let us first of all construct the M_x diagram for both the spans AB and BC, *i.e.*, drawing B.M. diagrams considering the spans AB and BC independently. Maximum

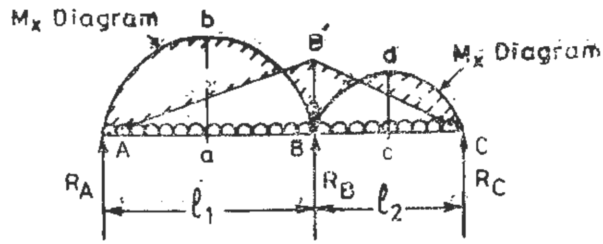


Fig. 12·32

bending moment $\frac{w_1l_1^2}{8}$ occurs at the centre of AB and AbB is the B.M. curve which is parabolic with $ab=\frac{w_1l_1^2}{8}$.

Similarly maximum B.M. $\frac{w_2l_2^2}{8}$ occurs at the centre of BC and BdC is the parabolic B.M. curve with $cd=\frac{w_2l_2^2}{8}$.

Now
$$a_1\bar{x}_1=\frac{2}{3}\times\frac{w_1l_1^2}{8}\times l_1\times\frac{l_1}{2}=\frac{w_1l_1^4}{24}$$
$$a_2\bar{x}_2=\frac{2}{3}\times\frac{w_2l_2^2}{8}\times l_2\times\frac{l_2}{2}=\frac{w_2l_2^4}{24}.$$

Support moments $M_A=M_C=0$ since the beam is simply supported at the ends. Using the equation of 3 moments with a sinked support

$$2M_B(l_1+l_2)+\frac{6a_1\bar{x}_1}{l_1}+\frac{6a_2\bar{x}_2}{l_2}-\frac{6EI\delta}{l_1}-\frac{EI\delta}{l_2}=0$$

$$2M_B(l_1+l_2)+\frac{6\times w_1l_1^4}{24l_1}+\frac{6w_2l_2^4}{24l_2}-6EI\delta\left(\frac{l_1+l_2}{l_1l_2}\right)=0$$

$$M_B=-\frac{w_1l_1^3}{8(l_1+l_2)}-\frac{w_2l_2^3}{8(l_1+l_2)}+\frac{6EI\delta}{2l_1l_2}$$

Taking moments of the forces about the point B

$$R_Al_1-\frac{w_1l_1^2}{2}=M_B,\quad R_Cl_2-\frac{w_2l_2^2}{2}=M_B$$

or Reaction,
$$R_A=\frac{M_B}{l_1}+\frac{w_1l_1}{2}$$

Span DE, B.M. under the load$=\dfrac{20\times4}{4}=+20$ kNm

Taking origin at B, $a_2\bar{x}_2$ for span $BC=\dfrac{30\times4}{2}\times2=120$ kNm³

origin at D, $a_3\bar{x}_3$ for span $CD=\dfrac{2}{3}\times30\times4\times2=160$ kNm³

origin at C, $a_3\bar{x}_3$ for span $CD=160$ kNm³

origin at D, $a_4\bar{x}_4$ for span $DE=20\times\dfrac{4}{2}\times2=80$ kNm³

Using the Clapeyron's theorem of 3 moments for spans BC and DC,

$$M_B.4+2M_C(4+4)+M_D\times4+\frac{6\times a_2\bar{x}_2}{l_2}+\frac{6a_3\bar{x}_3}{l_3}=0$$

where $l_2=4$ m, $l_3=4$ m

$$4\times(-30)+16\,M_C+4M_D+\frac{6}{4}\times120+\frac{6}{4}\times160=0$$

$$-120+16\,M_C+4\,M_D+180+240=0$$

or $$16\,M_C+4\,M_D=-300 \qquad ...(1)$$

Again using the theorem of 3 moments for spans CD and ED,

$$4M_C+2M_D(4+4)+\frac{6a_3\bar{x}_3}{l_3}+\frac{6a_4\bar{x}_4}{l_4}=0$$

Since $M_E=0$, as the beam is simply supported at the end E.

$$4M_C+16\,M_D+\frac{6\times160}{4}+\frac{6\times80}{4}=0$$

$$4M_C+16\,M_D+240+120=0$$

or $$4M_C+16\,M_D=-360 \qquad ...(2)$$

or $$M_C+4\,M_D=-90 \qquad ...(3)$$

From equations (1) and (2),

$$15\,M_C=-210,\quad M_C=-14\text{ kNm},\quad M_D=-19\text{ kNm}$$

Diagram $AB'C'D'E$ is the M_x' diagram for support moments. Resultant bending moment diagram is shown by positive and negative area.

Support reactions. Taking moments about the point C

$$-10\times7-30\times2+R_B\times4=M_C$$

$$-70-60+4R_B=-14$$

Reaction, $R_B=29$ kN

Also $-10\times11-30\times6-15\times4\times2+8\,R_B+4\,R_C=M_D$

$$-110-180-120+8\times29+4R_C=-19$$

$$4\,R_C=+178-19=-159$$

Reaction, $R_C=39{\cdot}75$ kN

$$4\,R_E-20\times2=M_D$$

$$4R_E=40-19=21$$

Reaction, $R_E=5{\cdot}25$ kN

Reaction, $R_D=10+30+15\times4+20-29-39{\cdot}75-5{\cdot}25=46$ kN

Since $M_D=0$ as D is the simply supported end

$$14M_C+4M_B+48+\frac{64}{3}+1{\cdot}5\times3000\times{\cdot}006=0$$

$$14M_C+4M_B+69{\cdot}333+27=0$$

$$14M_C+4M_B=-96{\cdot}333 \qquad \ldots(2)$$

From equations (1) and (2),

$$M_B=+1{\cdot}233 \text{ Tm}$$

$$M_C=-7{\cdot}2333 \text{ Tm}$$

In the Fig. $AB'C'D$ is the M_x' diagram or the diagram of support moments. The resultant bending moment diagram is shown with positive and negative areas.

Support reactions. Taking moments of the forces about the point B on both the sides.

$$R_A\times3-10\times2=1{\cdot}233$$

$$R_A=\frac{21{\cdot}233}{3}=7{\cdot}077 \text{ T}$$

$$R_D\times7+R_C\times4-6\times8-3\times4\times2=1{\cdot}233$$

$$7R_D+4R_C=73{\cdot}233$$

Moments about the point C,

$$3R_D-8\times2=M_c=-7{\cdot}2333$$

$$R_D=\frac{16-7{\cdot}2333}{3}=2{\cdot}922 \text{ T}$$

$$4R_C=73{\cdot}233-7R_D=73{\cdot}233-7\times2{\cdot}922$$

$$=73{\cdot}233-20{\cdot}454=13{\cdot}194 \text{ T}$$

$$R_B=10+8+12-7{\cdot}077-2{\cdot}922-13{\cdot}194=6{\cdot}807 \text{ T}$$

Problem 12·12. A continuous beam $ABCDE$ is 15 metres long. It has an overhang of 3 m length AB and supported on 3 spans 4 m each. A concentrated load of 30 kN acts at the middle of span BC ; a uniformly distributed load of 15 kN/m run acts over the span CD and a load of 20 kN acts at the middle of span DE. A concentrated load of 10 kN acts the free end of the beam. Determine the support moments and support reactions.

Solution. Fig. 12·31 shows the continuous beam $ABCDE$, 15 m long carrying loads as given in the problem.

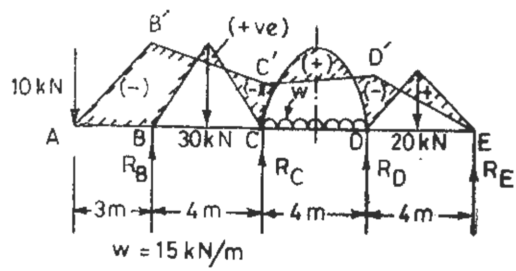


Fig. 12·31

B.M. at $A=0$

B.M. at $B=-10\times3$

$$M_B=-30 \text{ kNm}$$

Let us construct M_x *diagrams* for spans BC, CD and DE.

Span BC, B.M. under the load$=\dfrac{30\times4}{4}=+30$ kNm

Span CD, B.M. at the centre$=\dfrac{w\times4^2}{8}=\dfrac{15\times4^2}{8}=+30$ kNm

(a parabolic curve)

M_x Diagrams. Span AB

B.M. under the load $= \frac{10\times 1\times 2}{3} = \frac{20}{3}$ Tm (shown by *ab*)

$a_1\bar{x}_1$ about the point A $= \frac{20}{3}\times\frac{1}{2}\times\left(\frac{2}{3}\right)+\frac{20}{3}\times\frac{2}{2}\times\left(1+\frac{2}{3}\right)$

$= \frac{20}{9}+\frac{100}{9}=\frac{40}{3}\text{Tm}^3$

Span BC

Max. B.M. at the centre$=\frac{3\times 4^2}{8}=6$ Tm (shown by *cd*)

$a_2\bar{x}_2$ about the point C $=6\times 4\times\frac{2}{3}\times 2=32\text{ Tm}^3$

Moreover $a_2\bar{x}_2$ about the point $B=32\text{ Tm}^3$

Span CD

B.M. under the load $=\frac{8\times 1\times 2}{3}=\frac{16}{3}$ Tm (shown by *ef*)

$a_3\bar{x}_3$ about the point $D=\frac{16}{3}\times\frac{1}{2}\times\frac{2}{3}+\frac{16}{3}\times\frac{2}{2}\times\left(1+\frac{2}{3}\right)$

$=\frac{16}{9}+\frac{80}{9}=\frac{96}{9}=\frac{32}{3}\text{ Tm}^3$

Now $EI=2000\times 10^4\times 15000\times 10^{-8}\text{ Tm}^2=3000\text{ Tm}^2$

Using the equation of 3 moments with a sinking support, for the spans AB and CB.

$$2M_B(3+4)+M_C\times 4+\frac{6}{3}\times\frac{40}{3}+\frac{6\times 32}{4}-\frac{6\,EI\,\delta_1}{3}-\frac{6\,EI\,\delta_2}{4}=0$$

Since $M_A=0$ as A is simply supported end

$$14M_B+4M_C+\frac{80}{3}+48-2\times 3000\,\delta_1-1{\cdot}5\times 3000\,\delta_2=0$$

where $\delta_1=\delta_2=6\text{ mm}=0{\cdot}006\text{ m}$

B is below the level of A by $6\text{ mm}=\delta_1=0{\cdot}006\text{ m}$

B is below the level of C by $6\text{ mm}=\delta_2=0{\cdot}006\text{ m}$

So $\quad 14M_B+4M_C+\frac{80}{3}+48-2\times 3000\times 0{\cdot}006-1{\cdot}5\times 3000\times{\cdot}006=0$

or $\quad 14M_B+4M_C+74{\cdot}667-36-27=0$

$$14M_B+4M_C=-11{\cdot}667 \qquad ...(1)$$

Again using the equation of 3 moments with a sinking support for the spans BC and DC. (Support C is 6 mm higher than the support B or $\delta_2=-6$ mm, support C and D are at the same level so $\delta_3=0$)

$$2M_C(4+3)+M_B\times 4+\frac{6\times 32}{4}+\frac{6\times 32}{3\times 3}-\frac{6EI(-{\cdot}006)}{4}-0=0$$

Support reactions. Taking moments about the point B

$$12R_A - 4{\cdot}5\times16\times8 = M_B = -123$$

$$12R_A = 576 - 123 = 453$$

Reaction $\quad R_A = 37{\cdot}75$ T

Again, $16R_C - 6\times20\times10 = M_B = -123$

$$16R_C = 1200 - 123 = 1077$$

Reaction, $\quad R_C = 67{\cdot}31$ T

Reaction, $\quad R_B = 16\times4{\cdot}5 + 20\times6 - 37{\cdot}75 - 67{\cdot}31 = 86{\cdot}94$ T.

SF diagram

Portion EA $\quad F = -w_1x$

$= 0$ at $x = 0$

$= -4{\cdot}5\times4 = -18$ T $\quad$ at $x = 4$ m

Portion AB $\quad F = -w_1x + R_A = -4{\cdot}5x + R_A$

$= -18 + 37{\cdot}75 = +19{\cdot}75$ T $\quad$ at $x = 4$

$= -4{\cdot}5\times8 + 37{\cdot}75 = +1{\cdot}75$ T $x = 8$ m

$= -4{\cdot}5\times12 + 37{\cdot}75 = -16{\cdot}25$ T $\quad$ at $x = 12$ m

$= -4{\cdot}5\times16 + 37{\cdot}75 = -34{\cdot}25$ T $\quad$ at $x = 16$ m.

Portion BC $\quad F = -4{\cdot}5\times16 - w_2(x-16) + R_A + R_B$

$= -72 - w_2(x-16) + 37{\cdot}75 + 86{\cdot}94 = 52{\cdot}69 - w_2(x-16).$

$= 52{\cdot}59$ T $\quad$ at $x = 16$ m

$= 52{\cdot}69 - 6(20-16)$

$= +28{\cdot}69$ T $\quad$ at $x = 20$ m

$= 52{\cdot}69 - 6(24-16) = +4{\cdot}69$ T $\quad$ at $x = 24$ m

$= 52{\cdot}69 - 6(28-16) = -19{\cdot}31$ T $\quad$ at $x = 28$ m

$= 52{\cdot}69 - 6(32-16) = -43{\cdot}31$ T $\quad$ at $x = 32$ m.

Portion CE $\quad F = -16\times4{\cdot}5 + R_A + R_B + R_C - w_2(x-16) = +120 - 6(x-16)$

$= 120 - 6(32-16) = -24$ T $\quad$ at $x = 32$ m

$= 0$ $\quad$ at $x = 36$ m

The shear force diagram is shown in the Fig. 12·29 (*b*).

Problem 12·11. A beam $ABCD$, 10 m long is supported over 3 spans, $AB = 3$ m, $BC = 4$ m and $CD = 3$ m. On AB there is a point load of 10 tonnes at a distance of 1 m from A. On BC, there is a uniformly distributed load of 3 tonnes/metre run throughout and on CD there is a point load of 8 tonnes at a distance of 1 m from D. The level of support B is 6 mm below the levels of A, C and D supports. Determine support moments and support reactions. $E = 2000$ tonnes/cm², $I = 15000$ cm⁴.

Solution. Fig. 12·30 shows a continuous beam $ABCD$, over three spans AB, BC and CD as given in the problem. Load on AB is 10 T at a distance of 1 m from A, on BC, there is a uniformly distributed load of 3T/m and on CD, a point load of 8 tonnes at a distance of 1 m from end D. Let us first draw M_x diagrams for each span independently.

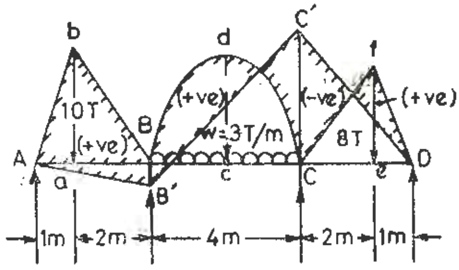


Fig. 12·30

Again, $R_C \times 7 - 4 \times 5 = -2{\cdot}045$, $7R_C = 20 - 2{\cdot}045 = 17{\cdot}955$

Reaction, $R_C = 2{\cdot}565$ T

Reaction, $R_B = 2 \times 1 + 2 \times 2 + 4 - 1{\cdot}988 - 2{\cdot}565 = 5{\cdot}447$ T.

Problem 12·10. A continuous beam of length 36 metres is supported at A, B and C. The span AB is 12 m long and the span BC is 16 metres long. The overhang is equal on both the sides. The beam carries a uniformly distributed load of 4·5 tonnes/metre run from one end (near the point A) upto the point B and a uniformly distributed load of 6 tonnes/metre run from the other end upto the point B. Determine the reactions and moments at the supports. Draw the B.M. and SF diagrams.

Solution. Figure 12·29 (a) shows a continuous beam 36 m long, supported at A, B and C as given in the problem. Say the end points of the beam are E and F.

Uniformly distributed load from E to $B = 4{\cdot}5$ T/m. Uniformly distributed load from F to $B = 6$ T/m.

Bending moment at A,

$$M_A = -\frac{w_1 \times 4^2}{2} = -4{\cdot}5 \times \frac{4^2}{2}$$

$= -36$ Tm

Bending moment at C,

$$M_C = -\frac{w_2 \times 4^2}{2} = -6 \times \frac{4^2}{2}$$

$= -48$ Tm.

To determine support moment at B, let us first draw the M_x *diagram* for AB and BC.

Bending moment at the centre of AB

$$= \frac{4{\cdot}5 \times 12^2}{8} = +81 \text{ Tm}$$

Bending moment at the centre of BC

$$= \frac{6 \times 16^2}{8} = +192 \text{ Tm}$$

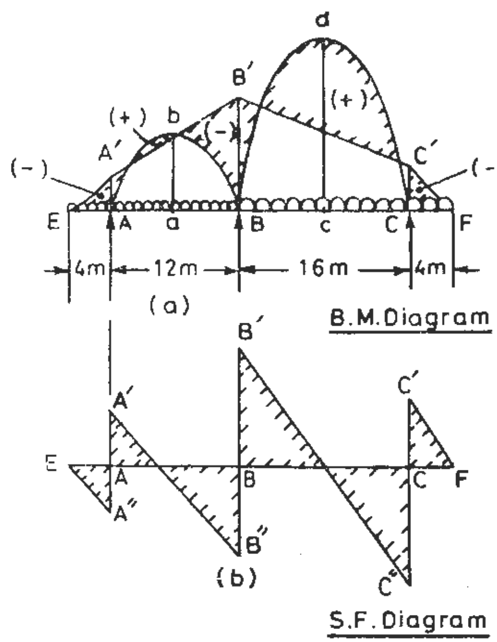


Fig. 12·29

Taking origin at A, $a_1\bar{x}_1 = 81 \times 12 \times \frac{2}{3} \times 6 = 3888$ Tm³

Taking origin at C_1, $a_2\bar{x}_2 = 192 \times 16 \times \frac{2}{3} \times 8 = 16384$ Tm³

Using the Clapeyron's theorem of 3 moments

$$12M_A + 2M_B(12+16) + 16M_C + \frac{6 \times 3888}{12} + \frac{6 \times 16384}{16} = 0$$

$$-36 \times 12 + 56M_B - 48 \times 16 + 1944 + 6144 = 0$$

$$56M_B = -8088 + 432 + 768 = -6888$$

$$M_B = -123 \text{ Tm}$$

$EA'B'C'F$ shows the M_x' diagram due to support moments. Diagram with positive and negative areas shown is the resultant bending moment diagram.

$$+\int_2^4 \left\{ R_A' \, x - w_1 \times 2(x-1) - \frac{w_2}{2} (x-2)^2 \right\} x \, dx$$

$$= \int_0^2 \left(2{\cdot}5 \, x^2 - \frac{x^3}{2} \right) dx + \int_2^4 \left\{ 2{\cdot}5x^2 - 2x(x-1) - x(x-2)^2 \right\} dx$$

as $w_1 = 1$ T/m and $w_2 = 2$ T/m

$$a_1 \bar{x}_1 = \int_0^2 \left(2{\cdot}5x^2 - \frac{x^3}{2} \right) dx + \int_2^4 \left(4{\cdot}5x^2 - x^3 - 2x \right) dx$$

$$= \left| 2{\cdot}5 \frac{x^3}{3} - \frac{x^4}{8} \right|_0^2 + \left| 4{\cdot}5 \frac{x^3}{3} - \frac{x^4}{4} - x^2 \right|_2^4$$

$$= +\frac{20}{3} - 2 + \left(4{\cdot}5 \times \frac{64}{3} - \frac{256}{4} - 16 \right) - \left(\frac{4{\cdot}5 \times 8}{3} - \frac{16}{4} - 4 \right)$$

$$= \frac{14}{3} + 96 - 64 - 16 - 12 + 4 + 4 = +\frac{50}{3} \text{ Tm}^3$$

Span BC, B.M. under the load $= \frac{4 \times 2 \times 5}{7} = \frac{40}{7}$ Tm

Taking origin at C, $a_2 \bar{x}_2 = \frac{40}{7} \times \frac{2}{2} \times \left(\frac{4}{3} \right) + \frac{40}{7} \times \frac{5}{2} \times \left(2 + \frac{5}{3} \right)$

$$= \frac{160}{21} + \frac{100}{7} \times \frac{11}{3} = \frac{490}{21} = \frac{70}{3} \text{ Tm}^3.$$

Using the Clapeyron's theorem of 3 moment

$$2M_B(4+7) = -\frac{6 \times a_1 \bar{x}_1}{l_1} - \frac{6 a_2 \bar{x}_2}{l_2}$$

$$22 M_B = -6 \times \frac{50}{3} \times \frac{1}{4} - 6 \times \frac{70}{3} \times \frac{}{7}$$

$$= -25 - 20 = -45$$

$$M_B = -2{\cdot}045 \text{ Tm.}$$

Now if the M_x diagrams for spans AB and BC are plotted then resultant B.M. diagram is shown by the positive and negative areas.

Support reactions. Taking moments about the point B

$$R_A \times 4 - 2 \times 1 \times 3 - 2 \times 2 \times 1 = -2{\cdot}045$$

$$4R_A = 10 - 2{\cdot}045 = 7{\cdot}955$$

Reaction, $R_A = 1{\cdot}988$ T

Taking origin at C, $a_2\bar{x}_2=20\times\frac{4}{2}\times 2=80$ kNm³ (for span BC).

The moment of inertia of the beam is different in two spans, the theorem of 3 moments can be modified as follows :

$$\frac{M_A\,.\,l}{I_1}+\frac{2M_Bl_1}{I_1}+\frac{2M_Bl_2}{I_2}+\frac{M_C\,.\,l_2}{I_2}+\frac{6a_1\bar{x}_1}{I_1l_1}+\frac{6a_2\bar{x}_2}{I_2l_2}=0 \quad ...(1)$$

but $M_A=M_C=0$, as the beam is simply supported at the ends A and C.

and $I_1=2I_2$

Equation (1), can be written as

$$\frac{2M_B\times 6}{2I_2}+\frac{2M_B\times 4}{I_2}+\frac{6\times 648}{2I_2\times 6}\times\frac{6\times 80}{I_2\times 4}=0$$

or $6M_B+8M_B+324+120=0$

$14M_B=-444$, or $M_B=-31{\cdot}71$ kNm

$AB'C$ shows the M_x' diagram for support moments. The bending moment diagram with positive and negative areas shows the resultant bending moments.

Support reactions. Taking moments at the point B

$R_A\times 6-12\times 6\times 3=-31{\cdot}71,\quad 6R_A=216-31{\cdot}71$

Reaction, $R_A=30{\cdot}71$ kN.

Again, $R_C\times 4-20\times 2=-31{\cdot}71$, or $R_C\times 4=40-31{\cdot}71$

Reaction, $R_C=2{\cdot}07$ kN

Reaction, $R_B=12\times 6+20-30{\cdot}71-2{\cdot}07=59{\cdot}22$ kN.

Problem 12·9. A continuous beam ABC of length 11 m is loaded as shown in the Fig. 12·28. Determine the reactions at the supports A, B and C and the support moment at B.

Solution. Figure 12·28 (*a*) shows a continuous beam 11 m long supported at A, B and C and carrying loads. 1 T/m from A for 2 m length and 2 T/m from B for 2 m length on span AB. On span BC, there is a concentrated load 4T at a distance of 2 m from end C.

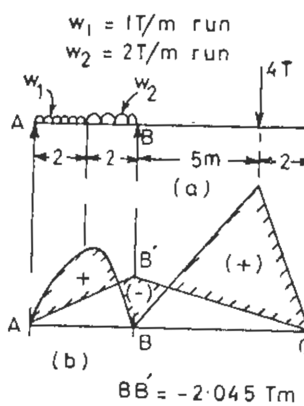


Fig. 12·28

Support moments $M_A=M_C=0$ as the beam is simply supported at the ends.

M_x diagrams. Considering the beam AB independently, taking moments about the point B,

Reaction

$$R_A'=\frac{1\times 2\times 3+2\times 2\times 1}{4}=2{\cdot}5\text{ T}$$

$$R_B'=2+4-2{\cdot}5=3{\cdot}5\text{ T}$$

Taking origin at A, $a_1\bar{x}_1=\int_0^2\left(R_A'x-\frac{w_1x^2}{2}\right)x\,dx$

$$=\frac{-w}{al^2}\int_0^a (lx^3-x^4)\ dx.$$

$$=-\frac{w}{al^2}\left[\frac{la^4}{4}-\frac{a^5}{5}\right]=-\frac{w}{20al^2}\left[5la^4-4a^5\right]=-\frac{wa^3}{20l^2}(5l-4a).$$

Support reactions. Total load on beam $=\frac{wa}{2}$

Taking moments about the point A,

$$M_A=+M_B+R_Bl-\frac{wa}{2}\left(l-a+\frac{a}{3}\right)$$

or
$$\frac{-wa^3}{20l^2}(5l-4a)+R_B\times l-\frac{wa}{2}\left(l-\frac{2a}{3}\right)=-\frac{wa^2}{30l^2}(10l^2+6a^2-5al)$$

$$R_Bl=+\frac{wal}{2}-\frac{wa^2}{3}+\frac{wa^3}{4l}-\frac{wa^4}{5l^2}-\frac{wa^2}{3}-\frac{wa^4}{5l^2}+\frac{wa^3}{6l}$$

$$=\frac{wal}{2}-\frac{2wa^2}{3}-\frac{2wa^4}{5l^2}+\frac{5wa^3}{12l}$$

Reaction,
$$R_B=\frac{wa}{2}-\frac{2wa^2}{3l}-\frac{2wa^4}{5l^3}+\frac{5}{12}\ \frac{wa^3}{l^2}$$

$$=\frac{wa}{60l^3}(30l^3-40al^2-24a+25a^2l)$$

Similarly taking moments of the forces about the point B

$$M_B=M_A+R_A\ .\ l-\frac{wa}{2}\times\frac{a}{3}$$

or
$$-\frac{wa^3}{20l^2}(5l-4a)=\frac{-wa^2}{30l^2}(10l^2+6a^2-5al)+R_Al-\frac{wa^2}{6}$$

$$-\frac{wa^3}{4l}+\frac{wa^4}{5l^2}=\frac{-wa^2}{3}-\frac{wa^4}{5l^2}+\frac{wa^3}{6}+R_Al-\quad\frac{wa^2}{6}-\frac{5wa^3}{12l}+\frac{2wa^4}{5l^2}+\frac{wa^2}{2}=R_A\ .\ l$$

Reaction,
$$R_A=\frac{-5wa^3}{12l^2}+\frac{2wa^4}{5l^3}+\frac{wa^2}{l}=\frac{wa^2}{60l^3}(-25al+24a^2+30l^2).$$

Problem 12·8. A continuous beams ABC, 10 m long is supported on two spans AB=6 m and BC=4 m. Span AB carries a uniformly distributed load of 12 kN per metre run and span BC carries a concentrated load of 20 kN as its centre. The moment of inertia of the section of the beam for the span AB is I_1 and that for the span BC is I_2. If $I_1=2I_2$, determine support moments and support reactions.

Solution. Let us construct the M_x diagram for both the spans AB and BC as shown in the Fig. 12·27.

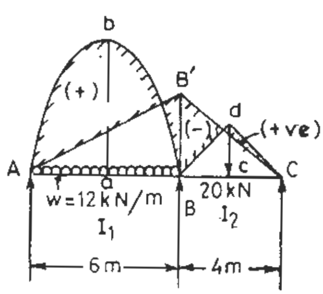


Fig 12·27

Span AB, maxm. bending moment occurs at the centre $=\frac{wl^2}{8}=\frac{12\times6^2}{8}=54$ kNm

Span BC, B.M. under the load

$$=\frac{Wl}{4}=\frac{20\times4}{4}=20\text{ kNm}$$

Taking origin at A, $a_1\bar{x}_1=\frac{2}{3}\times54\times6\times3$

$=648$ kNm³ (for span AB)

From equations (1) and (2),

$$\frac{Tb}{GJ}=\frac{9Wb^2}{16EI}-\frac{3Tb}{2EI}$$

$$T=\frac{9}{16}\times W\ \frac{G}{E}\times\frac{J}{I}\ .\ \frac{b^2}{b}-\frac{3}{2}\times T\ .\ \frac{b}{b}\ .\ \frac{G}{E}\times\frac{J}{I}$$

$$=\frac{9}{16}\ Wb\ .\ \frac{840}{2100}\times 2-\ \frac{3}{2}\times T\times\frac{840}{2100}\times 2$$

$$=0{\cdot}45\ Wb-1{\cdot}2\ T$$

or $$2{\cdot}2T=0{\cdot}45\ Wb,\ T=0{\cdot}2045\ Wb.$$

Problem 12·7. A fixed beam of length l carries a linearly increasing distributed load of intensity zero at the left hand end to w upto a distance a from the same end. Determine (1) support reactions, (2) support moments. Given EI is the flexural rigidity of the beam.

Solution. Figure 12·26 shows a fixed beam of length l carrying a linearly increasing distributed load of intensity zero at end A and w at a distance of a form end A.

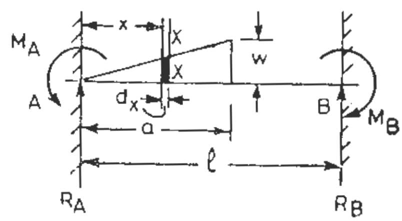


Fig. 12·26

Consider a section X-X at a distance of x from the end A.

Rate of loading at the section $=\frac{x}{a}\ .\ w$

Elementary load for small length $dx=\frac{w}{a}\ .\ x\,dx$

Refer to article 12·5, and considering $\frac{wx\,dx}{a}$ as eccentric load

Support Moments, $$dM_A=\frac{-\left(\frac{w}{a}x\,dx\right)x\,(l-x)^2}{l^2}$$

$$dM_B=\frac{\left(-\frac{w}{a}\ .\ x\,dx\right)(x^2)(l-x)}{l^2}$$

Support moment, $$M_A=-\int_0^a\frac{w}{al^2}\ x^2(l-x)^2\ dx=-\frac{w}{al^2}\int_0^a(x^2l^2+x^4-2lx^3)\ dx$$

$$=\frac{-w}{al^2}\left[\frac{a^3}{3}\,l^2+\frac{a^5}{5}-\frac{2l\ .\ a^4}{4}\right]$$

$$=-w\left[\frac{a^2}{3}+\frac{a^4}{5l^2}-\frac{a^3}{2l}\right]$$

$$=\frac{-wa^2}{30l^2}\,[10l^2+6a^2-15al]$$

Similarly the support moment, $$M_B=-\int_0^a\frac{\left(\frac{w}{a}\ .\ xd\,x\right)(x^2)(l-x)}{l^2}$$

Problem 12·6. A rung of a vertical ladder is in the horizontal plane. Rung is perpendicular to the vertical sides of the ladder. Length of the rung is $3b$ and distance between the rungs is b. Ladder is made of steel of circular section. If a vertical load W is carried in the middle of a particular rung find the twisting moment at the ends of the rung.

$E=2100$ tonnes/cm² (for steel), $G=840$ tonnes/cm² (for steel).

Solution. Figure 12·25 (*a*) shows a ladder in which length of rung is $3b$ and distance between rungs is b. When a load W is applied at the middle of the rung, it will bend and the

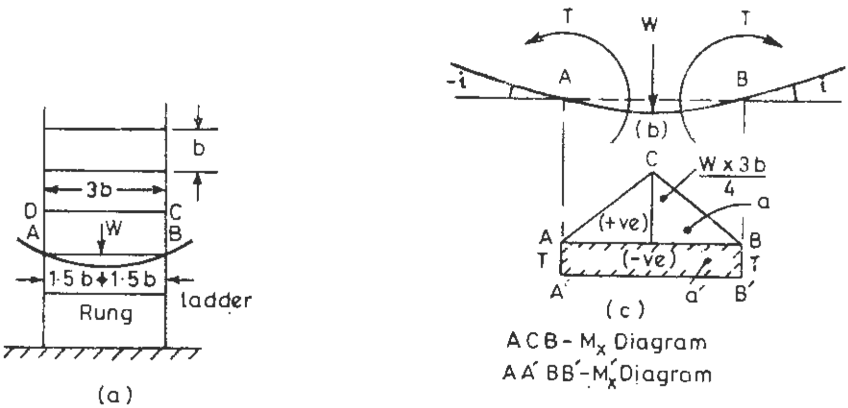


Fig. 12·25

slope at the ends will not necessarily be zero, but say the slope at A is $-i$ and at B slope is $+i$, as shown in Fig. 12·25 (*b*). There are fixing couples T each at the ends, which do not completely fix the beam AB but a small amount of slope remains at both the ends. Fig. 12·25 (*c*) shows the BM diagram M_x and M_x'. *i.e.*, ABC is M_x diagram, *i.e.*, considering the beam to be simply supported and diagram $A'ABB'$ is the M_x' diagram, *i.e.*, BM diagram due to fixing couples.

Now $\quad EI\,(i_B-i_A)=$area of M_x diagram$-$area of M_x' diagram

$$EI(+i+i)=\frac{3Wb}{4}\times\frac{3b}{2}-3b\times T$$

$$2EIi=\frac{9Wb^2}{8}-3Tb \qquad ...(1)$$

Considering the twisting of shorter side of ladder *i.e.*, AD or BC.

$$\frac{T}{J}=\frac{Gi}{b}$$ where $G=$Modulus of rigidity

$i=$angular twist

$b=$length of the circular bar AD under twisting moment T

Now $\quad J=2I$

for a circular section. $J=$Polar moment of inertia

(Detailed study of twisting of members will be made in Chapter 13 on Torsion)

So $$i=\frac{Tb}{GJ} \qquad ...(2)$$

Integrating equation (1), we get

$$EI\frac{dy}{dx}=-Mi\,x+\frac{Wx^2}{4}-\frac{W}{2}\left(x-\frac{l}{2}\right)^2+C_1$$

At $x=0$, slope $\frac{dy}{dx}=-i.$

Therefore $-EIi=0+0-\text{omitted term}+C_1$

or $C_1=-EIi$

So
$$EI\frac{dy}{dx}=-Mix+\frac{Wx^2}{4}-\frac{W}{2}\left(x-\frac{l}{2}\right)^2-EIi \quad ...(2)$$

Moreover at $x=\frac{l}{2}$, $\frac{dy}{dx}=0$ because the beam is symmetrically loaded about the centre.

Putting $x=\frac{l}{2}$ in equation (2)

$$0=-Mi\,\frac{l}{2}+\frac{Wl^2}{16}-0-EIi$$

or
$$i\left(\frac{Ml}{2}+EI\right)=\frac{Wl^2}{16},\ i\left(\frac{Ml+2EI}{2}\right)=\frac{Wl^2}{16},\ i=\frac{Wl^2}{8(Ml+2EI)}$$

or the restraining couple at the end $=Mi=\frac{MWl^2}{8(Ml+2EI)}$

Rewriting the equation (2),

$$EI\frac{dy}{dx}=-M\left(\frac{Wl^2}{8(Ml+2EI)}\right)x+\frac{Wx^2}{4}-\frac{W}{2}\left(x-\frac{l}{2}\right)^2-EIi \quad ...(2)$$

Integrating the equation (2) again

$$EIy=-M\left(\frac{Wl^2}{8(Ml+2EI)}\right)\frac{x^2}{2}+\frac{Wx^3}{12}-\frac{W}{6}\left(x-\frac{l}{2}\right)^3-EIix+0 \quad ...(3)$$

(Constant of integration is zero because at $x=0$, $y=0$ and the term $\left(x-\frac{l}{2}\right)$ is to be omitted).

Deflection at the centre $y=y_B$ at $x=\frac{l}{2}$. Substituting in equation (3), we get

$$EIy_B=-M\left(\frac{Wl^2}{8(Ml+2EI)}\right)\frac{l^2}{8}+\frac{Wl^3}{96}-0-EI\,\frac{l}{2}\left(\frac{Wl^2}{8(Ml+2EI)}\right)$$

$$=-\frac{MWl^4}{64(Ml+2EI)}+\frac{Wl^3}{96}-EI\left(\frac{Wl^3}{16(Ml+2EI)}\right)$$

$$=-\frac{Wl^3}{64(Ml+2EI)}(Ml+4EI)+\frac{Wl^3}{96}$$

$$=\frac{Wl^3}{192}\left[2-\frac{3(Ml+4EI)}{Ml+2EI}\right]$$

$$=\frac{Wl^3}{192}\left[\frac{2Ml+4EI-3Ml-12EI}{Ml+2EI}\right]=\frac{Wl^3}{192}\left[\frac{-Ml-8EI}{Ml+2EI}\right]$$

$$y_B=-\frac{Wl^3}{192EI}\left[\frac{Ml+8EI}{Ml+2EI}\right]$$

negative sign indicates downward deflection.

$$0=M_A \cdot l+R_A \cdot \frac{l^2}{2}-M\left(\frac{2l}{3}\right) \quad ...(4)$$

$$0=M_A \cdot \frac{l^2}{2}+R_A \cdot \frac{l^3}{6}-\frac{2Ml^2}{9} \quad ...(5)$$

From equations (4) and (5),

$$R_A=+\frac{4}{3}\frac{M}{l}, \quad M_A=0$$

For equilibrium $\qquad R_B=-\frac{4}{3}\frac{M}{l}$

Take equation (1) and put $x=l$, B.M. at $C=M_C$

$$M_C=0+l\left(\frac{4}{3}\frac{M}{l}\right)-M=\frac{4}{3}M-M=\frac{M}{3}$$

Fixing couples at support A, $\qquad M_A=0$

at support C, $\qquad M_C=+\frac{M}{3}$

To determine slope at the point B, let us use equation (2) and put $x=l/3$

$$EIi_B=0\times\frac{l}{3}+\frac{4}{3}\frac{M}{l}\times\frac{l^2}{18}-0=\frac{2Ml}{27}$$

Slope at B, $\qquad i_B=\frac{2Ml}{27EI}$.

Problem 12·5. A beam of span l carries a load W at its middle. It is so constrained at the ends that when the end slope is i, the restraining couple couple is Mi. Prove that magnitude of restraining couple at each end is $\frac{MWl^2}{8}\Big/(Ml+2EI)$ and that the magnitude of central deflection is $\frac{Wl^3}{192EI}\left(\frac{Ml+8EI}{Ml+2EI}\right)$.

Solution. Figure 12·24 shows a beam of length l with a central concentrated load W. The beam has end slopes $-i$ at A and $+i$ at C. As per the condition given

Restraining couple at $A=M_A=-iM$.

Since the beam is loaded symmetrically about the centre, reactions

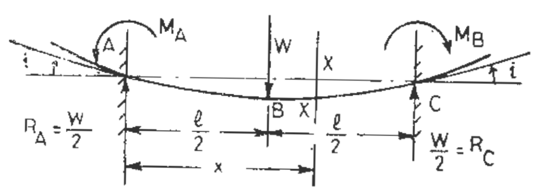


Fig. 12·24

$$R_A=R_C=\frac{W}{2}$$

Consider a section X-X at a distance of x from the end A in the portion BC,

B.M. at the section $\qquad =-Mi+\frac{W}{2}(x)-W\left(x-\frac{l}{2}\right)$

or $\qquad EI\frac{d^2y}{dx^2}=-Mi+\frac{W}{2}(x)-W\left(x-\frac{l}{2}\right) \quad ...(1)$

Maximum deflection will occur at the centre of beam.

Substituting $x=4$ m in equation (5), we get

$$E\,.\,I\,.\,y_{max}=M_A\,.\,\frac{4^2}{2}+R_A\,.\,\frac{4^3}{6}-\frac{w\times 4^4}{24}-\frac{3{\cdot}59}{6}(2)^3$$

$$=8\left(-\frac{32}{3}-2{\cdot}5W\right)+\frac{64}{6}\left(8+1{\cdot}5\times 3{\cdot}59\right)-\frac{2\times 256}{24}-\frac{8\times 3{\cdot}59}{6}$$

$$=8(-10{\cdot}667-2{\cdot}5\times 3{\cdot}59)+\frac{64}{6}(13{\cdot}385)-21{\cdot}333-4{\cdot}787$$

$$=-157{\cdot}136+142{\cdot}773-21{\cdot}333-4{\cdot}787=-40{\cdot}485 \text{ T-m}^3$$

$$EI=2000\times 10^{+4}\times 60{,}000\times 10^{-8}=12000 \text{ Tm}^2$$

$$y_{max}=-\frac{40{\cdot}485}{12000}=-0{\cdot}00337 \text{ m}$$

$$=-3{\cdot}37 \text{ mm (showing downward deflection).}$$

Problem 12·4. A beam of span l is fixed at both ends. A couple M is applied to the beam at a distance of $l/3$ from left hand end, about a horizontal axis at right angles to the beam. Determine the fixing couples at each support and slope at the point where couple is applied.

Solution. Figure 12·23 shows a fixed beam of length l. At the point B an anti-clockwise moment M is applied. Point B is at a distance of $l/3$ from end A. Let us say R_A and R_C are the reactions at supports A and C and M_A and M_C are the support moments at A and C respectively.

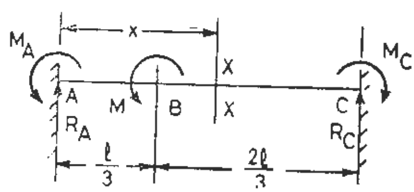


Fig. 12·23

Consider a section X-X at a distance of x from the end A

B.M. at the section $\quad = M_A+R_A\,.\,x-M\left(x-\frac{l}{3}\right)^{\circ}$

$$EI\,\frac{d^2y}{dx^2}=M_A+R_Ax-M\left(x-\frac{l}{3}\right)^{\circ} \qquad ...(1)$$

Integrating equation (1),

$$EI\,\frac{dy}{dx}=M_Ax+R_A\,\frac{x^2}{2}-M\left(x-\frac{l}{3}\right)+0 \qquad ...(2)$$

(Constant of integration is zero because $\frac{dy}{dx}=0$ at $x=0$, at end A)

Integrating further equation (2),

$$EIy=M_A\,.\,\frac{x^2}{2}+R_A\,\frac{x^3}{6}-\frac{M}{2}\left(x-\frac{l}{3}\right)^2+0 \qquad ...(3)$$

(Constant of integration is zero, because $y=0$ at $x=0$, fixed end A).

At the end C, $\quad x=l,\ \frac{dy}{dx}=0,\ y=0$

Substituting in equations (2) and (3), we get or $$0=8M_A+32R_A-\frac{512}{3}-18W-8W-2W \quad ...(4)$$

$$0=32M_A+\frac{256}{3}R_A-\frac{1024}{3}-36W-\frac{32}{3}W-\frac{4W}{3} \quad ...(5)$$

or $$0=8M_A+\frac{64}{3}R_A-\frac{256}{3}-9W-\frac{8}{3}W-\frac{W}{3} \quad ...(5)$$

[Dividing equation (5) by 4 throughout]

Equating the equations (4) and (5), we get

$$32R_A-\frac{512}{3}-28W=\frac{64}{3}R_A-\frac{256}{3}-\frac{36W}{3}$$

or $$96R_A-512-84W=64R_A-256-36W$$

$$32R_A=+256+48W$$

$$R_A=8+1{\cdot}5W \quad ...(6)$$

This is what we have considered.

Again taking equation (4) and dividing throughout by 8, we get

$$M_A+4R_A-\frac{64}{3}-3{\cdot}5W=0$$

$$M_A=\frac{64}{3}+3{\cdot}5W-4(8+1{\cdot}5W)$$

$$=\frac{64}{3}+3{\cdot}5W-32-6W$$

$$=-\frac{32}{3}-2{\cdot}5W=-\left(\frac{32}{3}+2{\cdot}5W\right) \quad ...(7)$$

To determine the maximum let us determine the B.M. at the centre of the beam

$$M_C=M_A+R_A\times4-\frac{w\cdot4^2}{2}-W(4{\cdot}2)$$

$$=-\frac{32}{3}-2{\cdot}5W+4(8+1{\cdot}5W)-16-2W \quad \text{since } w=2 \text{ tonnes/m}$$

$$=-\frac{32}{3}-2{\cdot}5W+32+6W-16-2W=\frac{16}{3}+1{\cdot}5W.$$

This shows that support moment is maximum.

So $$M_{max}=\frac{32+7{\cdot}5W}{3} \text{ tonne-metres}$$

$$=\frac{(32+7{\cdot}5\,W)100}{3} \text{ T-cm}=f\times Z$$

$$Z=\text{section modulus}=\frac{60{,}000}{15}=\frac{I}{d/2}=4000 \text{ cm}^3$$

$$f=0{\cdot}5 \text{ tonne/cm}^2$$

$$\therefore \quad \frac{(32+7\,5W)}{3}\times100=0{\cdot}5\times4000=2000$$

or $$32+7{\cdot}5W=60$$

$$W=\frac{60-32}{7{\cdot}5}=\frac{28}{7{\cdot}5}=3{\cdot}59 \text{ tonnes, each.}$$

Problem 12·3. A steel girder 30 cm deep has a span of 8 metres and is rigidly built in at both the ends. The loading on the girder consists of a uniformly distributed load of 2 tonnes/metre run on the whole span and 3 equal point loads at the centre and the quarter span points. Find the magnitudes of the point loads if the maximum stress due to bending is 0·5 tonnes/cm².

The section of the girder is symmetrical about X-X and YY axis and $I_{xx}=60{,}000$ cm⁴.

Determine also the maximum deflection $E=2000$ tonnes/cm².

Solution. Figure 12·22 shows a fixed beam 8 m long carrying uniformly distributed load of 2 tonnes/metre run throughout its length. Then it carries 3 point loads say W each, at distances of 2 m, 4m and 6m from the end A.

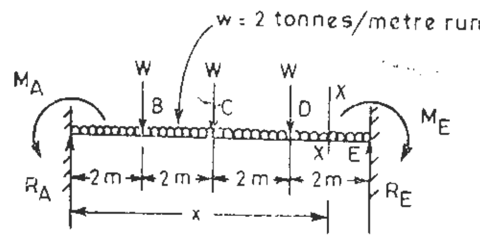


Fig. 12·22

Since the beam is symmetrically loaded.

Reactions $R_A=R_E$, $R_A=\dfrac{2\times 8+3W}{2}$

$=8+1{\cdot}5W$

Fixing couples, $M_A=M_E$

Consider a section at a distance of x from the end A.

BM at the section is $M=M_A+R_Ax-\dfrac{wx^2}{2}-W(x-2)-W(x-4)-W(x-6)$

$$EI\frac{d^2y}{dx^2}=M_A+R_A\,.\,x-\frac{wx^2}{2}-W(x-2)-W(x-4)-W(x-6) \qquad ...(1)$$

Integrating equation (1),

$$EI\frac{dy}{dx}=M_A\,.\,x+\frac{R_A\,x^2}{2}-\frac{wx^3}{6}-\frac{W(x-2)^2}{2}-\frac{W(x-4)^2}{2}-\frac{W(x-6)^2}{2}+0 \quad ...(2)$$

(Constant of integration is zero because at $x=0$, $\dfrac{dy}{dx}=0$ at fixed end and terms $(x-2)$, $(x-4)$, $(x-6)$ are omitted).

Again integrating equation (2),

$$EIy=M_A\frac{x^2}{2}+R_A\frac{x^3}{6}-\frac{wx^4}{24}-\frac{W}{6}(x-2)^3-\frac{W}{6}(x-4)^3-\frac{W}{6}(x-6)^3+0 \quad ...(3)$$

[Constant of integration is again zero because at $x=0$, $y=0$ at the fixed end and terms $(x-2)$, $(x-4)$, $(x-6)$ are omitted].

At $x=8$ m, $\dfrac{dy}{dx}=0$, $y=0$ at fixed end. Substituting these values in equations (2) and (3) we get

$$0=8M_A+32R_A-\frac{2\times 8^3}{6}-\frac{W}{2}(6)^2-\frac{W}{2}(4)^2-\frac{W}{2}(2)^2 \qquad ...(4)$$

$$0=32M_A+\frac{8^3R_A}{6}-\frac{2\times 8^4}{24}-\frac{W}{6}\times 6^3-\frac{W}{6}\times 4^3-\frac{W}{6}\times 2^3 \qquad ...(5)$$

(Constant of integration is zero, because $y=0$ at $x=0$ and terms $(x-4)$ and $(x-6)$ are to be ommitted at $x=0$)

At the end D, $x=12$ m, $\frac{dy}{dx}=0$ and $y=0$.

Substituting these conditions in equations (2) and (3) we get

$$0=12M_A+72R_A-256+72 \quad \ldots(4)$$

$$0=72M_A+288R_A-\frac{2048}{3}+144 \quad \ldots(5)$$

Dividing equation (4) throughout by 12 and equation (5) throughout by 72 we get

$$M_A+6R_A-15{\cdot}3333=0 \quad \ldots(6)$$

$$M_A+4R_A-7{\cdot}4815=0 \quad \ldots(7)$$

From these equations

or $2R_A=7{\cdot}8518$ or $R_A=3{\cdot}9259$ Tonnes

$M_A=15{\cdot}3333-6R_A=+15{\cdot}3333-6\times 3{\cdot}9259$

$=15{\cdot}3333-23{\cdot}5554=-8{\cdot}2221$ Tm

Reactions $R_A+R_D=8-4=4$T

$R_D=4-3{\cdot}9259=0{\cdot}0741$ Tonnes.

Shear Force Diagram

$F_{AB}=+3{\cdot}9259$ Tonnes

$F_{BC}=3{\cdot}9259-8=-4{\cdot}0741$ Tonnes

$F_{CD}=-4{\cdot}0741+4=-0{\cdot}0741$ Tonnes.

Bending Moment Diagram

$M_A=-8{\cdot}2221$ Tonne-metres

$M_B=-8{\cdot}2221+R_A\times 4=-8{\cdot}221+4\times 3{\cdot}9259$

$=-8{\cdot}2221+15{\cdot}7036=7{\cdot}4815$ Tonne-metres

$M_C=-8{\cdot}2221+R_A\times 6-8(6-4)$

$=-8{\cdot}2221+3{\cdot}9259\times 6-16=-24{\cdot}2221+23{\cdot}5554$

$=-0{\cdot}6667$ tonne-metres

$M_D=-8{\cdot}2221+R_A\times 12-8(12-4)+4(12-6)$

$=-8{\cdot}2221+47{\cdot}1108-64+24=-1{\cdot}1113$ tonne-metres.

Fig. 12·21 (*a*) shows the SF diagram, 12·21 (*b*) shows the B.M. diagram where P_1 and P_2 are the points of contraflexure.

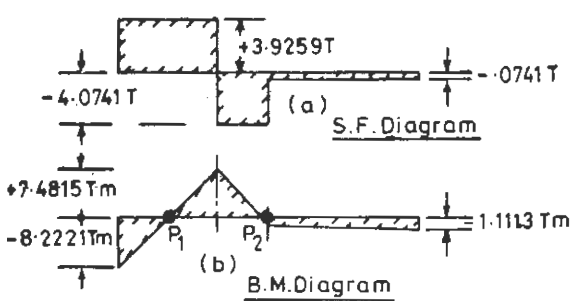


Fig. 12·21

Further integrating the equation (2)

$$EIy=-\frac{Wa(l-a)x^2}{2l}+\frac{Wx^3}{6}-\frac{W(x-a)^3}{6}+0$$

(Constant of integration is zero because $y=0$ at $x=0$)

At $x=a$, deflection $y=y_B$ under is the load W

So
$$EIy_B=-\frac{Wa(l-a)a^2}{2l}+\frac{Wa^3}{6}=-\frac{Wa^3l}{2l}+\frac{Wa^4}{2l}+\frac{Wa^3}{6}$$

$$=-\frac{Wa^3}{3}+\frac{Wa^4}{2l}=-\frac{Wa^3}{6l}(2l-3a)$$

$$y_B=-\frac{Wa^3(2l-3a)}{6EI\ l}\quad\text{(indicating downward deflection)}$$

At $x=\frac{l}{2}$, deflection at the centre is y_{max}

$$EIy_{max}=-\frac{Wa(l-a)}{2l}\times\frac{l^2}{4}+\frac{W}{6}\times\frac{l^3}{8}-\frac{W}{6}\left(\frac{l}{2}-a\right)^3$$

$$=-\frac{Wa(l-a)l}{8}+\frac{Wl^3}{48}-\frac{W}{6}\left(\frac{l^3}{8}-a^3-\frac{3}{4}al^2+\frac{3}{2}a^2l\right)$$

$$y_{max}=-\frac{W(3l-4a)a^2}{24\ EI}\quad\text{(indicating downward deflection)}$$

Problem 12·2. A girder 12 m span is fixed horizontally at the ends. A downward vertical load of 8 tonnes acts on the girder at a distance of 4 m from the left hand end and an upward vertical force of 4 tonnes acts at a distance of 6 m from the right hand end. Determine end reactions and fixing couples and draw the bending moment and shearing force diagrams for the girder.

Solution. Figure 12·20 shows a beam $ABCD$, 12 m long, carrying a downward load 4T at a distance of 4 m from end A and an upward load 4T at a distance of 6 m from end D, as given in the problem. Let us assume R_A and R_D are the support reactions and M_A, M_D are the support moments.

Consider a section X-X at a distance of x from end A.

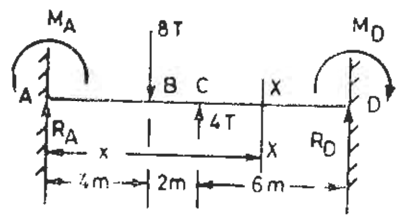


Fig. 12·20

B.M. at the section $=M_A+R_Ax-8(x-4)+4(x-6)$

or
$$EI\frac{d^2y}{dx^2}=M_A+R_A\,.\,x-8(x-4)+4(x-6)\qquad ...(1)$$

Integrating equation (1)

$$EI\frac{dy}{dx}=M_A\,.\,x+R_A\frac{x^2}{2}-\frac{8}{2}(x-4)^2+\frac{4}{2}(x-6)^2+0\qquad ...(2)$$

(Constant of integration is zero because $\frac{dy}{dx}=0$ at $x=0$ and the terms $(x-4)$ and $(x-6)$ arc to be omitted)

Integrating equation (2) also

$$EIy=M_A\,.\,\frac{x^2}{2}+R_A\,.\,\frac{x^3}{6}-\frac{8}{6}(x-4)^3+\frac{4}{6}(x-6)^3+0\qquad ...(3)$$

Problem 12·1. A beam of span l is fixed at both the ends. Two loads W each are placed at distance a from both the ends. Show that (*i*) the bending moment at the centre is $\frac{Wa^2}{l}$, (*ii*) deflection under either load is $\frac{Wa^3(2l-3a)}{6EI\,l}$, (*iii*) deflection at the middle of the beam is $\frac{Wa^2(3l-4a)}{24EI}$.

Solution. Fig. 12·19 (*a*) shows a fixed beam of length l carrying concentrated loads W each at a distance of a from each end. Since the beam is symmetrically loaded about its centre

Reactions, $R_A = R_D = W$ each

Fixing couples $M_A = M_D$.

Figure $ABCD$ shows the M_x diagram *i.e.*, when the beam is considered to be simply supported, with BM at B and C equal to Wa.

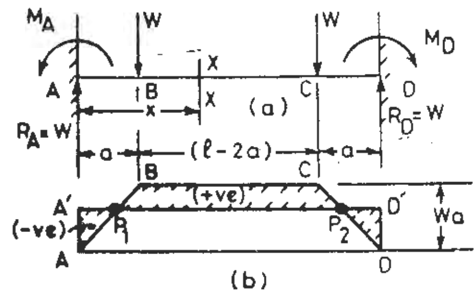


Fig. 12·19

Area
$$A = Wa \times \frac{a}{2} + Wa \times \frac{a}{2} + Wa(l-2a)$$
$$= Wa^2 + Wal - 2Wa^2 = Wa(l-a) \qquad \ldots(1)$$

Then diagram $AA'D'D$ is the M_x' diagram *i.e.*, BM diagram due to fixing couples A and D.

Area
$$A' = M_A \times l = M_D \times l$$
$= -A$ (area of the B.M. diagram considering the beam to be simply supported)
$$= -Wa(l-a)$$

or
$$M_A = -\frac{Wa(l-a)}{l} = M_D.$$

Therefore, B.M. at the centre $= Wa - \frac{Wa(l-a)}{l} = \frac{Wal - Wal + Wa^2}{l} = \frac{Wa^2}{l}$.

Since the beam is symmetrically loaded about its centre, the slope at the centre of the beam will be zero. Let us consider a section X-X at a distance of x from the end A in the portion BC of the beam. (We need not consider a section in the portion CD).

B.M. at the section, $M = M_A + R_A\, x - W(x-a)$

$$EI\,\frac{d^2y}{dx^2} = -\frac{Wa(l-a)}{l} + Wx - W(x-a) \qquad \ldots(1)$$

Integrating the equation (1), we get

$$EI\,\frac{dy}{dx} = -\frac{Wa(l-a)x}{l} + \frac{Wx^2}{2} - \frac{W(x-a)^2}{2} + 0 \qquad \ldots(2)$$

(Constant of integration is zero because $\frac{dy}{dx} = 0$ at $x=0$ and the term $(x-a)$ is omitted at $x=0$).

Substituting in equation (2)

$$-40-3M_B+20M_B+4M_C=-88$$

$$17M_B+4M_C=-48 \quad \ldots(3)$$

Again applying the theorem of 3 moments for spans BC and CD

$$4M_B+2M_C(4-14)+4M_D=-\frac{6\times16}{4}-\frac{6\times24}{4}=-60 \text{ Tm}^2$$

But $M_D=0$ since the end D is simply supported

So $4M_B+16M_C=-60$

or $M_B+4M_C=-15 \quad \ldots(4)$

From equations (3) and (4), $16M_B=-48+15=-33$

$$M_B=-2{\cdot}0625 \text{ Tm} \quad \ldots(5)$$

$$4M_C=-15-M_P=-15+2{\cdot}0625=-12{\cdot}9375$$

$$M_C=-3{\cdot}234 \text{ Tm}$$

From equation (1), $12M_A=-80-6M_B=-80+2{\cdot}0625\times6=-67{\cdot}625$

$$M_A=-5{\cdot}635 \text{ Tm}$$

Figure $AA'B'C'D$ is the M_x' diagram due to support moments. Figure with positive and negative areas shows the resultant bending moment diagram.

Support reactions. Taking moments about the point C

$$4R_D-6\times2=M_C=-3{\cdot}234$$

$$4R_D=12-3{\cdot}234=8{\cdot}766$$

Reaction, $R_D=2{\cdot}19$ T

Now taking moments about the point B

$$8R_D+4R_C-6\times6-6\times2=M_B=-2{\cdot}0625$$

$$8\times2{\cdot}19+4R_C-48=-2{\cdot}0625$$

$$4R_C=48-2{\cdot}0625-17{\cdot}52=28{\cdot}4175$$

Reaction, $R_C=7{\cdot}10$ Tonnes

Taking moments about tbe point A.

$$14R_D+10R_C+6R_B-12\times6-6\times8-6\times2=M_A=-5{\cdot}635$$

$$14\times2{\cdot}19+10\times7{\cdot}10+6R_B-72-48-12=-5{\cdot}635$$

$$30{\cdot}66+71{\cdot}0+6R_B-132=-5{\cdot}635$$

$$6R_B=132-5{\cdot}635-71-30{\cdot}66$$

$$6R_B=24{\cdot}705$$

Reaction, $R_B=4{\cdot}11$ T

Reaction, $R_A=6+6+6-2{\cdot}19-7{\cdot}10-4{\cdot}11=4{\cdot}6$ T.

Exercise 12·11-1. A continuous beam $ABCD$, fixed at end A is supported over points B, C and end D. The lengths of the spans are $AB=BC=CD=6$ m each. The beam carries a uniformly distributed load of 20 kN/m run throughout its length. Determine support moments and reactions.

[**Ans.** $-62{\cdot}31$, $-55{\cdot}38$, $-76{\cdot}154$, 0 kNm ; 61·16, 115·37, 136·17, 47·3 kN]

$$M_A'\times 0+2M_A(0+l_1)+M_B\,.\,l_1=0-\frac{6(a_1)(l_1-\bar{x}_1)}{l_1}$$

or $$2M_A\,.\,l_1+M_B\,.\,l_1+\frac{6a_1(l_1-\bar{x}_1)}{l_1}=0 \qquad ...(1)$$

If the other end of the continuous beam is also fixed, a similar equation can be obtained by imagining a zero span to the right of the fixed end and then applying the theorem of three moments.

Example 12·11-1. A continuous beam *ABCD*, 14 m long rests on supports *A*, *B*, *C* and *D* all at the same level. $AB=6$ m, $BC=4$ m and $CD=4$ m. It carries two concentrated loads of 6 Tonnes each at a distance of 2 m from end *A* and end *D*. There is a uniformly distributed load of 1.5 Tonnes/metre over the span *BC*. Support *A* is fixed but support *D* is free. Find the moments and reactions at the supports.

Solution. Fig. 12·18 shows a continuous beam *ABCD* fixed at the end *A* and simply supported at the end *D*, with three spans *AB*, *BC*, *CD* of 6 m, 4 m and 4 m respectively. Let us first draw the M_x diagrams for the 3 spans.

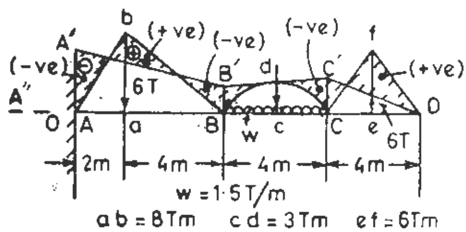


Fig. 12·18

Span AB

B.M. under the load $=\frac{6\times2\times4}{6}=8$ Tm

Taking origin at *A*, $a_1\bar{x}_1=\frac{8\times2}{2}\left(\frac{4}{3}\right)+\frac{8\times4}{2}\left(2+\frac{4}{3}\right)=\frac{32}{3}+\frac{160}{3}=64$ Tm³

Taking origin at *B*, $a_1\bar{x}_1=\frac{8\times4}{2}\left(\frac{8}{3}\right)+\frac{8\times2}{2}\left(4+\frac{2}{3}\right)=\frac{128}{3}+\frac{112}{3}=\frac{240}{3}=80$ Tm³.

Span BC

Taking origin at *B*, $a_2\bar{x}_2=\frac{3\times2}{3}\times4\times2=16$ Tm³

Taking origin at *C*, $a_2\bar{x}_2=16$ Tm³.

Span CD

Taking origin at *D*, $a_2\bar{x}_3=\frac{6\times4}{2}\times2=24$ Tm³.

Consider the imaginary span $A''A=0$ length. Applying the theorem of 3 moments

$$M_A''\times0+2M_A(0+6)+M_B\times6=0-\frac{6a_1\bar{x}_1}{l_1} \text{ (about the origin } B)$$

$$12M_A+6M_B=-\frac{6\times80}{6}=-80 \text{ Tm}^2$$

$$12M_A+6M_B=-80 \text{ Tm}^2 \qquad ...(1)$$

Applying the theorem of 3 moments for the spans *AB* and *BC*

$$6M_A+2M_B(6+4)+4M_C=-\frac{6\times64}{6}-\frac{6\times16}{4}$$

$$6M_A+20M_B+4M_C=-88 \text{ Tm}^2 \qquad ...(2)$$

From equation (1), $6M_A=-40-3M_B$

12·11. CONTINUOUS BEAM WITH FIXED ENDS

If a continuous beam has fixed ends, then equations for support moments can be derived considering that slope and deflection at the fixed end are zero. Fig. 12·17 shows a continuous beam supported at A (a fixed end) and other supports B, C, D etc. In the analysis let us consider only two spans AB and BC. M_x diagrams for AB and BC are constructed, *i.e.*, BM diagrams are plotted considering the spans AB and BC independently. Say for the span AB and BC carrying any type of transverse loads, M_x diagrams are $A\,e\,B$ and $B\,f\,C$ respectively.

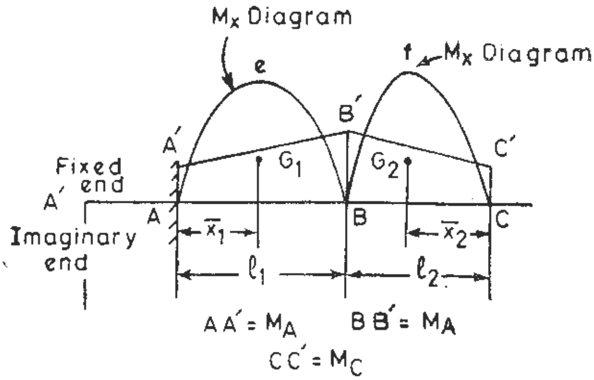


Fig. 12·17

Considering the origin at B, x positive towards left, BM at any section at a distance of x from B

$$=M_x+M_x'$$

or $$EI\frac{d^2y}{dx^2}=M_x+M_x' \qquad \ldots(1)$$

Multiplying both the sides by xdx and integrating over the length l_1

$$EI\int_0^{l_1}\frac{d^2y}{dx^2}\cdot xdx=\int_0^{l_1}M_x\,.\,xdx+\int_0^{l_1}M_x'\,.\,xdx$$

$$EI\left|x\frac{dy}{dx}-y\right|_0^{l_1}=a_1\,(l_1-\bar{x}_1)+\frac{l_1^2}{6}\,(M_B+2M_A)$$

But at the fixed end, at $x=l_1$,

$$\frac{dy}{dx}=0,\ y=0$$

at B, $x=0$, $y=0$,

thus making the left hand side of the equation equal to zero.

Therefore, $$2M_A\,l_1+M_B\,l_1+\frac{6\,a_1\,(l_1-\bar{x}_1)}{l_1}=0 \qquad \ldots(1)$$

This relationship can also be obtained by considering Clapeyron's an imaginary span $A'A$ of length zero and bending moment at A', $M_A'=0$ and using the Clapeyron's theorem of 3 moments.

Curve $A\ b\ B$ is parabolic.

Taking origin at A and x positive towards right

$a_1\bar{x}_1$ = first moment of M_x diagram for span AB (considering the beam AB independently)

$$=\frac{2}{3}\times 12\text{·}5\times 10\times 5=416\text{·}67\ \text{Tm}^3.$$

Span BC. Maximum bending moment occurs at the centre

$$M_{max}=\frac{Wl}{4}=\frac{12\times 5}{4}=15\ \text{Tm}$$

M_x diagram is a triangle $A\ d\ C$.

Taking origin at C and x positive towards left

$$a_2\bar{x}_2=\frac{15\times 5}{2}\times 2\text{·}5=93\text{·}75\ \text{Tm}^3$$

Using the equation (5) of article 12·10,

$$\frac{6\times 416\text{·}67}{10}+\frac{6\times 93\text{·}75}{5}+2M_B\ (10+5)-\frac{6\ EI\ \delta_1}{l_1}-\frac{6\ EI\ \delta_2}{l_2}=0$$

where
$$\delta_1=\delta_2=0\text{·}01\ \text{m}$$
$$l_1=10\ \text{m},\ l_2=5\ \text{m}$$
$$EI=2000\times 10^4\times 30{,}000\times 10^{-8}\ \text{Tm}^2=6000\ \text{Tm}^2$$
$$EI\ \delta=6000\times 0\text{·}01=60\ \text{Tm}^3$$

So
$$250+112\text{·}5+30\ M_B-\frac{6\times 60}{10}-\frac{6\times 60}{5}=0$$
$$30\ M_B=-362\text{·}5+108$$

or
$$M_B=-8\text{·}483\ \text{Tm}$$

$AB'C$ shows the M_x diagram for the continuous beam. The resultant BM diagram is shown by the Fig. 12·16 with positive and negative areas.

Support moments. Taking moments of the forces about the support B

$$R_A\times 10-10\times 5=-8\text{·}483\ (M_B)$$

Reaction, $R_A=4\text{·}151$ T

Also $R_C\times 5-12\times 2\text{·}5=-8\text{·}483$

$$R_C\times 5=-8\text{·}483+30$$

Reaction, $R_C=4\text{·}303$ T

Reaction, $R_B=10+12-4\text{·}151-4\text{·}303=13\text{·}546$ T.

Exercise 12·10-1. A girder 10 m long is supported at the ends and has an intermediate support at 6 m from one end. It carries a concentrated load of 12 tonnes at the middle of each span. The intermediate support is 1·2 cm lower than the end supports, calculate the reactions at the supports.

$$I=25000\ \text{cm}^4,\ E=2100\ \text{tonnes/cm}^2$$

[**Ans.** 5·3625, 13·5938, 5·0437 T]

or $$EI\,l_1\,i_B+EI\,\delta_1=a_1\,\bar{x}_1+a_1'\,\bar{x}_1'$$

or $$EI\,i_B=\frac{a_1\bar{x}_1}{l_1}+\frac{a_1'\bar{x}_1'}{l_1}-\frac{EI\,\delta_1}{l_1} \qquad \ldots(2)$$

where $a_1\bar{x}_1$ = first moment of the area of M_x diagram

$a_1'\bar{x}_1$ = first moment of the area of M_x' diagram

$$=(M_A+2M_B)\,\frac{l_1^2}{6}$$

or $$EI\,i_B=\frac{a_1\bar{x}_1}{l_1}+(M_A+2M_B)\,\frac{l_1}{6}-\frac{EI\,\delta_1}{l_1}$$

or $$6\,EI\,i_B=\frac{6\,a_1\bar{x}_1}{l_1}+(M_A+2M_B)\,l_1-\frac{6EI\,\delta_1}{l_1} \qquad \ldots(3)$$

Similarly considering the span CB, with origin at C and x positive towards left we can write

$$6\,EI\,i_B'=\frac{6\,a_2\bar{x}_2}{l_2}+(M_A+2M_B)\,l_2-\frac{6\,EI\,\delta_2}{l_2} \qquad \ldots(4)$$

where i_B' is the slope at B, taking x positive towards left, therefore, $i_B'=-i_B$, *i.e.*, slope at B taking x positive towards right.

Adding equations (3) and (4)

$$0=\frac{6\,a_1\bar{x}_1}{l_1}+\frac{6\,a_2\bar{x}_2}{l_2}+M_A\,.\,l_1+2M_B\,(l_1+l_2)+M_C\,.\,l_2$$
$$-\frac{6\,EI\,\delta_1}{l_1}-\frac{6\,EI\,\delta_2}{l_2} \qquad \ldots(5)$$

From this equation, support moments are determined.

Example 12·10-1. A girder 15 m long is supported at the ends and has an intermediate support at 10 m from one end. It carries a concentrated load of 12 tonnes at the middle of 5 m span and a uniformly distributed load of 1 tonne/metre run over a span of 10 m. The central support is 1 cm lower than the end supports. Calculate support moments and support reactions.

$$E=2000\ \text{T/cm}^2,\ I=30{,}000\ \text{cm}^4.$$

Solution. Fig. 12·16 shows a continuous beam ABC, 15 m long, span $AB=10$ m carrying uniformly distributed load of 1 tonne/metre and span $BC=5$ m carries a concentrated load of 12 tonnes at its centre. The level of support B is 1 cm below the levels of A and C.

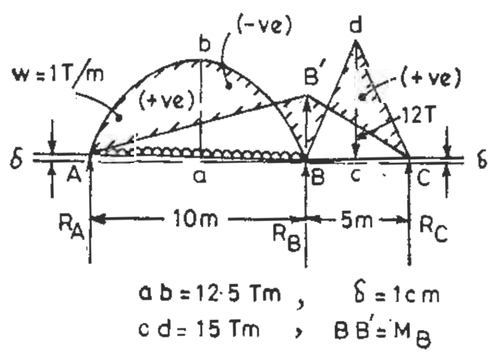


Fig. 12·16

Since the beam is simply supported at the ends, moments

$$M_A=M_C=0$$

Let us construct the M_x diagrams

Span AB. Maximum bending moment occurs at the centre

$$M_{max}=\frac{wl^2}{8}=\frac{1\times 10^2}{8}=12{\cdot}5\ \text{T-m}$$

Moreover $M_B = R_D \times 8 - 3 \times (8-2) + R_C \times 4 - 4(8-7)$

$-4{\cdot}05 = 1{\cdot}2375 \times 8 - 18 + 4R_C - 4$

$R_C = 2{\cdot}0125$ T

Total vertical load on continuous beam $= 3 \times 4 + 4 + 3 = 19$ Tonnes

Reaction $R_B = 19 - R_A - R_D - R_D = 19 - 4{\cdot}9875 - 1{\cdot}2375 - 2{\cdot}0125$

$= 10{\cdot}7625$ Tonnes

Exercise 12·9-1. A continuous beam *ABCD*, 16 m long, supported over spans $AB = 6$ m, $BC = 4$ m and $CD = 6$ m, carries a concentrated load of 6 tonnes at a distance of 2 m from end *A*, a uniformly distributed load of 1·5 tonne/metre run over the span *BC* and a concentrated load of 6 tonnes at a distance of 4 m from the point *C*. Determine support reactions and support moments. Draw the resultant BM diagram.

[**Ans.** 2·278, 6·722, 6·722, 2·278 T ; 0, −10·33, −10·33, 0 Tm]

12·10. SUPPORTS NOT AT THE SAME LEVEL IN A CONTINUOUS BEAM

Consider two spans *AB* and *BC* of a continuous beam of lengths l_1 and l_2 respectively. Say the support *B* is below the support *A* by δ_1 and below the support *C* by δ_2. These

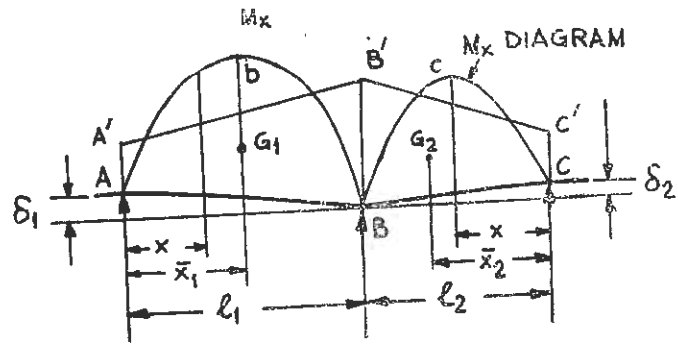


Fig. 12·15

level differences are very small as compared to span lengths and are not so large as shown in the diagram. For span *AB*, M_x diagram *A b B* and for span *BC*, M_x diagram *B c C* are constructed. Say M_A, M_B and M_C are the support moments and *AA′ B′B* is M_x' diagram for span *AB* and *BB′ C′C* is the M_x' diagram for the span *BC*.

Span AB. Consider a section at a distance of x from the end *A* (*i.e.*, taking origin at *A* and x positive towards right),

BM at the section $= M_x + M_x'$

or
$$EI \frac{d^2y}{dx^2} = M_x + M_x' \qquad \ldots(1)$$

Multiplying both the sides by xdx and integrating,

$$\int_0^{l_1} EI \frac{d^2y}{dx^2} \cdot xdx = \int_0^{l_1} M_x\, x \,.\, dx + \int_0^{l_1} M_x'\, xdx$$

$$EI \left| x \frac{dy}{dx} - y \right|_0^{l_1} = a_1\, \bar{x}_1 + a_1'\, \bar{x}_1'$$

or
$$EI\,(l_1 \times i_B + \delta_1) - EI\,(0 \times i_A - 0) = a_1\, \bar{x}_1 + a_1' \bar{x}_1'$$

(because downward deflection is taken as negative)

Using the equation for 3 moments

$$M_A\times4+2M_B(4+4)+M_C\times4=-\frac{6\times32}{4}-\frac{6\times14}{4}$$

But $M_A=0$ because the beam is simply supported at end A

So $\quad 16\,M_B+4\,M_C=-69\text{ Tm}^3 \quad$ ...(1)

Span BC. Taking origin at B and x positive towards right

$$a_2\bar{x}_2=3\times\frac{1}{2}\times\frac{2}{3}+\frac{3\times3}{2}\times(1+1)=1+9=10\text{ Tm}^3$$

Span DC. Maximum bending moment occurs at the centre,

$$M_{max}=\frac{Wl}{4}=\frac{3\times4}{4}=3\text{ Tm}$$

$$a_3\bar{x}_3=\frac{3\times4}{2}\times2=12\text{ Tm}^3$$

Using the equation of 3 moments

$$M_B\times4+2M_C(4+4)+M_D\times4=-\frac{6a_2\bar{x}_2}{4}-\frac{6a_3\bar{x}_3}{4}=-6\left(\frac{10}{4}+\frac{12}{4}\right)=-33$$

But $M_D=0$ because the beam is simply supported at end D

So $\quad 4M_B+16\,M_C=-33 \quad$ ...(2)

From equations (1) and (2)

$$16\,M_B+4M_C=-69 \quad \text{...(1)}$$

$$16\,M_B+64\,M_C=-132 \quad \text{...(2)}$$

Substracting equation (1) from equation (2)

$$60\,M_C=-63$$

$$M_C=-1{\cdot}05\text{ Tm}$$

Substituting the value of M_C in equation (1)

$$16\,M_B-4\times1{\cdot}05=-69$$

$$16\,M_B=-69+4{\cdot}2=-64{\cdot}8$$

$$M_B=-4{\cdot}05\text{ Tm}$$

M_x' diagram $AB'C'D$ is drawn with $BB'=-4{\cdot}05$ Tm, $CC'=-1{\cdot}05$ Tm.
Resultant bending moment diagram is shown in the Figure with +ve and −ve areas.

Support reactions. Taking moments of the forces about the point B

$$M_B=R_A\times4-\frac{w\times4\times4}{2}$$

$$-4{\cdot}05=R_A\times4-3\times8$$

$$R_A=\frac{24-4{\cdot}05}{4}=4{\cdot}9875\text{ Tonnes}$$

Similarly taking moments of the forces about the point C

$$M_C=R_D\times4-3\times2$$

$$-1{\cdot}05=4R_D-6$$

$$R_D=1{\cdot}2375\text{ Tonnes}$$

Problem 13·5. A solid marine propeller shaft is transmitting power at 1000 r.p.m. The vessel is being propelled at a speed of 20 kilometers per hour for the expenditure of 5000 horse power. If the efficiency of the propeller is 70% and the greatest thrust is not to exceed 600 kg/cm², calculate the shaft diameter and the maximum shearing stress developed in the shaft.

Solution. Say the direct thrust $=P$ kg

Useful work done per second as the output

$$=\frac{P\times 20\times 1000}{3600}=5{\cdot}5555\ P \text{ kg-metre}=555{\cdot}55\ P \text{ kg-cm}$$

Efficiency $=70\ \%$

Input work $=\dfrac{555{\cdot}55\ P}{0{\cdot}7}=793{\cdot}643\ P$ kg-cm

H.P. $=5000$

Work done per second $=\dfrac{5000\times 4500}{60}=375000$ kg-m

$=375\times 10^3$ kg-m $=375\times 10^5$ kg-cm

So $793{\cdot}643P=375\times 10^5$

$P=0{\cdot}472\times 10^5$ kg

Allowable direct stress in shaft $=600$ kg/cm²

Shaft diameter, $d=\sqrt{\dfrac{4P}{\pi\times 600}}=\sqrt{\dfrac{4\times 0{\cdot}472\times 10^5}{\pi\times 600}}=10{\cdot}0$ cm

Torque on the shaft, $T=\dfrac{5000\times 4500}{2\pi\times 1000}=3{\cdot}58\times 10^3$ kg-m $=3{\cdot}58\times 10^5$ kg-cm

Maximum shearing stress,

$$q=\frac{16\ T}{\pi d^3}=\frac{16\times 3{\cdot}58\times 10^5}{\pi\times 10^3}=1823{\cdot}27 \text{ kg/cm}^2.$$

Problem 13·6. A circular copper shaft is required to transmit 60 horse power at 200 rpm. Determine the diameter of the shaft if the maximum shear stress is not to exceed 60 N/mm².

The solid shaft is now replaced by a hollow copper shaft with the internal diameter equal to 75 % of the external diameter. Determine the external diameter of the shaft if it is required to transmit the same horse power at the same rpm and the maximum shear stress produced is also the same. Find the weight of the material saved per metre length of the shaft, if copper weighs 8·9 gm/c.c.

Solution. H.P. $=60$

Work done $=746\times 60\times 60$ Nm per minute

RPM $=200$

Torque, $T=\dfrac{746\times 60\times 60}{2\pi\times 200}=2207{\cdot}4$ Nm

$=2207{\cdot}4\times 10^3$ Nmm

Maximum shear stress, $q=\dfrac{16\ T}{\pi d^3}$ where d = shaft diameter

$$d^3=\frac{16\times 2207{\cdot}4\times 10^3}{\pi\times 60}=187{\cdot}369\times 10^3$$

Shaft diameter, $d=57{\cdot}2$ mm

Hollow Shaft

Say external diameter $=D$

Internal diameter $={\cdot}75\ D=\frac{3}{4}D$

Polar moment of inertia,

$$J=\frac{\pi}{32}\left[D^4-\frac{81}{256}\times D^4\right]$$

$$=\frac{\pi\times 175}{32\times 256}.\ D^4=0{\cdot}067\ D^4$$

Torque, $T=2207{\cdot}4\times 10^3$ Nmm

Maximum shear stress, $q=60$ N/mm²

$$\frac{T}{J}=\frac{q}{D/2}$$

$$\frac{2207{\cdot}4\times 10^3}{0{\cdot}067\ D^4}=\frac{60}{0{\cdot}5D}$$

$$D^3=\frac{2207{\cdot}4\times 10^3}{0{\cdot}067\times 120}=274{\cdot}55\times 10^3$$

External diameter, $D=6{\cdot}5\times 10=65$ mm

Internal diameter, $d=48{\cdot}75$ mm

Area of cross section of solid shaft

$$=\frac{\pi}{4}\ (5{\cdot}72)^2=25{\cdot}697\ \text{cm}^2$$

Area of cross section of hollow shaft

$$=\frac{\pi}{4}\ (6{\cdot}5^2-4{\cdot}875^2)=14{\cdot}517\ \text{cm}^2$$

Weight per metre length of solid shaft $=\rho\times 100\times 25{\cdot}697$ gm

Weight per metre length of hollow shaft $=\rho\times 100\times 14{\cdot}517$ gm

Saving in weight per metre length

$$=\rho\times 100\ (25{\cdot}697-14{\cdot}517)=\rho\times 100\times 11{\cdot}18=9{\cdot}5\ \text{kg}$$

(taking $\rho=0{\cdot}0089$ kg/cm³)

A hollow shaft is better than a solid shaft because the material near the axis of the solid shaft is not stressed upto the economical limit (*i.e.* the maximum allowable stress).

Problem 13·7. A solid alloy shaft of diameter 6 cm is coupled to a hollow steel shaft of same external diameter. If the angular twist per unit length of hollow shaft is 80 per cent of the angular twist of alloy shaft, determine the internal diameter of the steel shaft. At what speed the shafts will transmit power of 200 kW. The maximum shearing stress in steel shaft is not to exceed 1000 kg/cm², while in alloy shaft it is not to exceed 600 kg/cm²

G for steel$=2{\cdot}0\ G$ for alloy

Solution. The torque transmitted by the hollow steel shaft and the solid alloy shaft is the same, since they are coupled together.

$$\frac{T_S}{J_S}=\frac{G_S\theta_S}{l_S} \quad \text{and} \quad \frac{T_A}{J_A}=\frac{G_A\theta_A}{l_A}$$

But $\qquad T_S=T_A$

So $\qquad \dfrac{J_S G_S \theta_S}{l_S}=\dfrac{J_A G_A \theta_A}{l_A}$

But again $\qquad \dfrac{\theta_S}{l_S}=\dfrac{\theta_A}{l_A}\times 0{\cdot}8 \quad$ (as given)

So $\qquad 0{\cdot}8\ J_S\ G_S=J_A\ G_A$

$$J_S=\frac{J_A}{0{\cdot}8}\times\frac{G_A}{G_S}=\frac{J_A}{1{\cdot}6} \quad \text{as } G_S=2\ G_A$$

$$=\frac{\pi\times 6^4}{32\times 1{\cdot}6}=\frac{\pi}{32}\ (6^4-d^4)$$

where $\qquad d$=internal diameter of hollow shaft

or $\qquad 6^4=6^4\times 1{\cdot}6-1{\cdot}6\ d^4$

$$d^4=\frac{777{\cdot}6}{1{\cdot}6}=486 \quad \text{or} \quad d=4{\cdot}695 \text{ cm}$$

Maximum allowable stress in steel=1000 kg/cm²

Maximum torque which can be transmitted by steel shaft

$$=J_S\times\frac{1000}{3}=\frac{\pi}{32}\ (6^4-4{\cdot}695^4)\times\frac{1000}{3}=26507{\cdot}25 \text{ kg-cm}$$

Similarly the maximum torque which can be transmitted by the alloy shaft

$$=\frac{\pi}{32}\ (6^4)=\frac{600}{3}=25446{\cdot}96 \text{ kg-cm}=254{\cdot}47 \text{ kg-meter}$$

So the maximum allowable torque=25446·96 kg-cm

Power transmitted $\qquad$ =200 kW

$$1 \text{ kW}=102 \text{ kg-m}$$

$$200\times 102=254{\cdot}47\times\frac{2\pi\times N}{60}$$

Revolutions per minute,

$$N=\frac{200\times 102\times 60}{2\pi\times 254{\cdot}47}=765{\cdot}48 \text{ RPM.}$$

Problem 13·8. A solid circular steel shaft is rigidly connected to a copper tube to make a torsional spring as shown in the Fig. 13·37. The useful length of the shaft and the tube is 40 cm. The diameter of the shaft is 2 cm and the internal diameter of the tube is 2·4 cm and external diameter is 2·8 cm. Determine the maximum stresses in steel and copper if torque T=1000 kg-cm is applied at the end A and find the total angular twist.

Given G_{steel}=800 tonnes/cm², G_{copper}=400 tonnes/cm²

Solution.

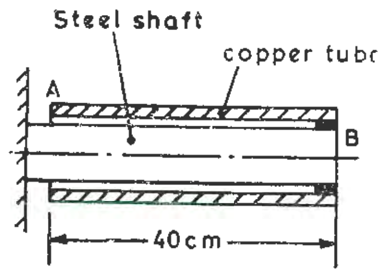


Fig. 13·37

Diameter of the steel shaft = 2 cm

External diameter of copper tube = 2·8 cm

Internal diameter of the copper tube = 2·4 cm

Polar moment of inertia of copper tube,

$$J_c = \frac{\pi}{32}(2.8^4 - 2.4^4)$$
$$= 2.7744 \text{ cm}^4$$

Polar moment of inertia of steel shaft,

$$J_s = \frac{\pi}{32} \times 2^4 = 1.5708 \text{ cm}^4$$

Torque, $T = 1000$ kg-cm

Maximnm stress in steel shaft

$$= \frac{T}{J_s} \times \frac{2}{2} = \frac{4000}{1.5708}$$
$$q_s = 636.62 \text{ kg/cm}^2$$

Maximum stress in copper tube,

$$q_c = \frac{T}{J_c} \times \frac{2.8}{2}$$
$$= \frac{1000}{2.7744} \times 1.4 = 504.61 \text{ kg/cm}^2$$

Total angular twist, $\theta = \theta_s + \theta_c = \dfrac{Tl}{G_s J_s} + \dfrac{Tl}{G_c J_c}$

$$= 1000 \times 40 \left[\frac{1}{800 \times 1000 \times 1.5708} + \frac{1}{400 \times 1000 \times 2.7744}\right]$$
$$= [0.0318 + 0.0360] \text{ radian} = 0.0678 \text{ radian} = 3.885 \text{ degree.}$$

Problem 13·9. A solid circular steel shaft is surrounded by a thick copper tube so as to form a compound shaft. If the copper tube shares double the torque shared by the steel shaft, determine the ratio of the external diameter to the internal diameter of the compound shaft. $G_{steel} = 2\, G_{copper}$.

Solution. The angular twist per unit length in the compound shaft is the same in steel shaft and in copper shaft.

Say diameter of steel shaft $= d$

External diameter of copper shaft $= D$

Polar moment of inertia of steel shaft,

$$J_s = \frac{\pi d^4}{32}$$

Polar moment of inertia of copper shaft,

$$J_c = \frac{\pi}{32}(D^4 - d^4)$$

Torque on copper shaf, $T_c = 2 \times$ Torque on steel shaft, T_s

or $$J_c \,.\, G_c \,.\, \frac{\theta_c}{l_c} = 2\, J_s \,.\, G_s\, \frac{\theta_s}{l_s}$$

but $\frac{\theta}{l}$ is the same for both

So $$J_c G_c = 2 J_s\, G_s \quad \text{but} \quad G_s = 2\, G_c$$

So $$J_c = 4\, J_s$$

$$\frac{\pi}{32}\,(D^4 - d^4) = 4 \times \frac{\pi}{32}\,(d^4)$$

or $$D^4 = 5\, d^4$$

Ratio of diameters, $\frac{D}{d} = 1{\cdot}495.$

Problem 13·10. A steel shaft of diameter 200 mm runs at 300 r.p.m. This steel shaft has a 30 mm thick bronze bushing shrunk over its entire length of 8 metres. If the maximum shearing stress in the steel shaft is not to exceed 12 N/mm², find (*a*) power of the engine, (*b*) torsional rigidity of the shaft.

$$G_{steel} = 84{,}000 \text{ N/mm}^2$$
$$G_{bronze} = 42{,}000 \text{ N/mm}^2$$

Solution. Steel shaft is encased in bronze shaft, therefore, angular twist due to the twisting moment will be same in both the shafts.

θ_s, angular twist in steel shaft

$$= \frac{T_s}{J_s} \times \frac{l_s}{G_s} \qquad ...(1)$$

where T_s = torque shared by the steel shaft

J_s = polar moment of inertia of steel shaft

G_s = shear modulus of steel

l_s = length of steel shaft = 8000 mm

Similarly, θ_b angular twist in bronze shaft

$$= \frac{T_b}{J_b} \times \frac{l_b}{G_b} \qquad ...(2)$$

But $$\theta_s = \theta_b$$

So $$\frac{T_s}{J_s} \times \frac{l_s}{G_s} = \frac{T_b}{J_b} \times \frac{l_b}{G_b}$$

$$\frac{T_s}{T_b} = \frac{J_s}{J_b} \times \frac{G_s}{G_b} \quad \text{because } l_s = l_b$$

$$J_s = \frac{\pi}{32}(200) = \frac{\pi}{32} \times 16 \times 10^8$$

$$J_b = \frac{\pi}{32}\,(260^4 - 200^4)$$

where 260 mm-Outer dia. of bronze shaft

$$=\frac{\pi}{32}\times 29{\cdot}698\times 10^8$$

Therefore, $\dfrac{T_s}{T_b}=\dfrac{\pi}{32}\times\dfrac{16\times 10^8}{\pi}\times\dfrac{32}{29{\cdot}698\times 10^8}\times\dfrac{84{,}000}{42{,}000}=1{\cdot}0775$

q, Maximum shear stress in steel shaft$=12$ N/mm²

$$q=\frac{16\ T_s}{\pi d^3}\quad \text{where } d=\text{diameter of the steel shaft}$$

$$12=\frac{16\times T_s}{\pi\times(200)^3}$$

$$T_s=\frac{\pi\times 12\times(200)^3}{16}=18{\cdot}85\times 10^6\ \text{Nmm}$$

$$T_b=\frac{T_s}{1{\cdot}0775}=17{\cdot}494\times 10^6\ \text{Nmm}$$

Total Torque $=T_s+T_b=(18{\cdot}850+17{\cdot}494)\times 10^6$ Nmm

$T=36{\cdot}344\times 10^6$ Nmm

Angular twist, $\theta=\dfrac{T_s\ l_s}{J_s\ G_s}=\dfrac{18{\cdot}85\times 10^6\times 8000}{\frac{\pi}{2}\times 10^8\times 84000}$

$=0{\cdot}01143$ radian$=$ 6508 degree

Torsional Rigidity $=\dfrac{T}{\theta}=\dfrac{36{\cdot}344\times 10^6}{0{\cdot}01143}=3179{\cdot}7\times 10^6$ Nmm/radian

Power of the Engine $=2\pi$ NT Nmm

$=2\times\pi\times 300\times 36{\cdot}344\times 10^6$ Nmm where N=300 rpm

$=600\ \pi\times 36{\cdot}344\times 10^3$ Nm/minute

$=10\pi\times 36{\cdot}344\times 10^3$ Nm/second

$=1141{\cdot}78$ kNm/second$=1141{\cdot}78$ k Watt.

Problem 13·11. A solid circular uniformly tapered shaft of length l, with a small angle of taper is subjected to a torque T. The diameter at the small end is D and that at the big end is $1{\cdot}1\ D$. Determine the error introduced if the angular twist for a given length is determined on the basis of the mean diameter of the shaft.

Solution.

Small end diameter $=D$

Big end diameter $=1{\cdot}1\ D$

Torque $=T$

Length $=l$

Say modulus of rigidity $=G$

Angular twist in tapered shaft,

$$\theta=\frac{32\ Tl}{3\pi G}\left(\frac{D_2^2+D_2D_1+D_1^2}{D_1^3D_2^3}\right)$$

$$=\frac{32\,Tl}{3\pi G}\left[\frac{(1\cdot1\ D)^2+1\cdot1\ D^2+D^2}{(1\cdot1)^3\ D^6}\right]=\frac{32\,Tl}{3\pi G}\times\frac{3\cdot31}{1\cdot331\ D^4}$$

$$=\left(\frac{32\,Tl}{\pi G}\right)\frac{3\cdot31}{3\cdot993\ D^4}=\left(\frac{32\,Tl}{\pi GD^4}\right)(0\cdot8290)$$

Mean diameter, $D_m=1\cdot05\ D$

Angular twist in uniform shaft of diameter D_m

$$\theta'=\frac{32\,Tl}{\pi G}\times\frac{1}{(1\cdot05\ D)^4}$$

$$=\frac{32\,Tl}{\pi G}\times\frac{1}{1\cdot2155\ D^4}=\frac{32\,Tl}{\pi GD^4}\ (0\cdot8227)$$

Percentage error $=\frac{0\cdot8290-0\cdot8227}{0\cdot8290}\times100=0\cdot76\ \%$

Problem 13·12. A vessel having a single propeller shaft 250 mm in diameter running at 200 rpm is re-engined to two propeller shafts of equal cross sections and producing 50 % more horse power at 500 r.p.m. If the working shearing stress in these shafts is 20 per cent more than the single shaft, determine the diameter of the shafts.

Solution. Say the metric horse power developed by the single shaft

$$=H_1=\frac{2\pi\times200\times T_1}{4500}$$

Torque, $T_1=\frac{4500\times H_1}{400\ \pi}=3\cdot58\ H_1$

Metric horse power developed by two shafts $=1\cdot5\ H_1$

m HP developed by each shaft $=0\cdot75\ H_1$

Torque, $T_2=\frac{0\cdot75\ H_1\times4500}{2\pi\times500}=1\cdot0743\ H_1$

$$\frac{T_1}{T_2}=\frac{3\cdot58}{1\cdot0743}=3\cdot33$$

Now $T_1=\frac{\pi}{16}\ d_1^3\times q$

$T_2=\frac{\pi}{16}\ d_2^3\times q'$ where $q'=1\cdot2\ q$ (as given)

$=\frac{\pi}{16}\ d_2^3\times1\cdot2\ q$ or $\frac{T_1}{T_2}=\frac{d_1^3}{d_2^3}\times\frac{1}{1\cdot2}$

So $3\cdot33=\frac{(250)^3\times1}{d_2^3\times1\cdot2}$

$$d_2^3=\frac{(250)^3}{1\cdot2\times3\cdot33}=3910\cdot16\times10^3$$

Diameter of shafts, $d_2=15\cdot75\times10=157\cdot5$ mm.

Problem 13·13. A steel shaft 10 cm diameter is solid for a certain part of its length and hollow for the remainder part of its length, with an inside diameter of 4 cm. If a pure torque is applied of such a magnitude that yielding just occurs on the surface of the solid part of the shaft, determine the depth of yielding in the hollow part of the shaft. Determine also the angles of twist per unit length for the two parts of the shaft.

Solution. Solid shaft is of radius 5 cm, but hollow portion is of outer radius 5 cm and inner radius 2 cm. Therefore, polar moment of inertia of hollow portion is less than the polar, moment of inertia of the solid portion. So when yield stress has just reached the surface of the solid shaft yielding has occured at radius r in hollow shaft or between r and 5 cm, material of the shaft has yielded and say the stress in this region is uniform q_y.

Torque on the solid part,

$$T=\frac{\pi}{2}\times 5^3\times q_y=62{\cdot}5\pi\ q_y \quad ...(1)$$

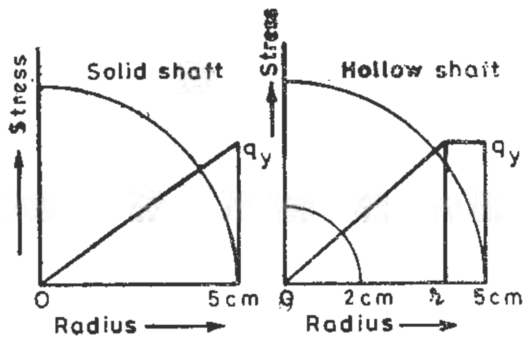


Fig. 13·38

Torque on the hollow part, $T=T_1+T_2$

T_1=torque on the section upto radius r (elastic region)

T_2=torque on the section between r and 5 cm (yielded region)

$$T_1=\frac{\pi}{2}\ \frac{(r^4-2^4)}{r}\ .\ q_y=\frac{\pi}{2r}\ (r^4-16)q_y$$

$$T_2=\int_r^5 2\pi r^2 q_y\ .\ dr\ .=q_y\int_r^5 2\pi r^2\,dr$$

$$=2\pi q_y\left[\frac{r^3}{3}\right]_r^5=2\pi q_y\left[\frac{125}{3}-\frac{r^3}{3}\right]=\frac{2\pi}{3}\ q_y[125-r^3]$$

So

$$\frac{\pi}{2r}\ (r^4-16)q_y+\frac{2\pi}{3}\ q_y\,[125-r^3]=62{\cdot}5\pi\ q_y \quad ...(2)$$

$$\frac{r^4-16}{2r}+\frac{2}{3}\ (125-r^3)=62{\cdot}5$$

$$3r^4-48+500r-4r^4=375r$$

$$r^4-125r+48=0$$

$r=4{\cdot}865$ cm by trial and error

Depth of yielding $=5-4{\cdot}865=0{\cdot}135$ cm$=1{\cdot}35$ mm.

Ratio of angular twist (say based on the elastic part of both the solid and hollow parts)

$$\frac{G\theta_1}{l_1}=\frac{q_y}{r} \text{ in hollow shaft}$$

$$\frac{G\theta_2}{l_2}=\frac{q_y}{5} \text{ in solid shaft}$$

or

$$\frac{\theta_1/l_1}{\theta_2/l_2}=\frac{5}{r}=\frac{5}{4{\cdot}865}=1{\cdot}028$$

Ratio of angular twist per unit length $=1{\cdot}028$.

Problem 13·14. A flanged coupling has n bolts of 25 mm diameter arranged symmetrically along a bolt circle of diameter 300 mm. If the diameter of the shaft is 100 mm and it is stressed upto 100 N/mm², determine the value of n if the shear stress in the bolts is not to exceed 50 N/mm².

Solution. Diameter of the shaft

$=100$ mm

Maximum stress in shaft

$q=100$ N/mm²

Torque on shaft

$=\frac{\pi}{16}\times(100)^3\times 100$

$=19{\cdot}635\times 10^6$ N mm

Radius of the bolts circle

$=150$ mm

Shear force on all the bolts

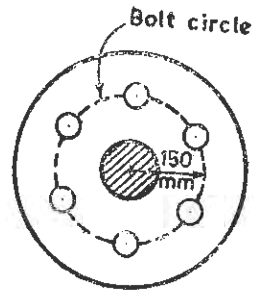


Fig. 13·39

$$Q=\frac{19{\cdot}635\times 10^6}{150}=13{\cdot}09\times 10^4 \text{ N}$$

Say the number of bolts $=n$

Area of each bolt $=\frac{\pi}{4}\ (25)^2=490{\cdot}88$ mm²

Maximum stress in bolt $=50$ N/mm²

Therefore $\frac{Q}{n\times 490{\cdot}88}=50$

Number of bolts $=\frac{13{\cdot}09\times 10^4}{50\times 490{\cdot}88}=5{\cdot}33$ say 6 bolts.

Problem 13·15. Fig. 13·40 shows a vertical shaft with pulleys keyed on it. The shaft is rotating with a uniform velocity at 2000 r.p.m. The belt pulls are indicated and the 3 pulleys are rigidly keyed to the shaft. If the maximum shear stress in the shaft is not to exceed 50 N/mm², determine the necessary diameter of a solid circular shaft. The shaft is supported in bearings near the pulleys and the bending of the shaft may be neglected.

Solution. Torque on pulley A

$=(3000-900)\times 10$ N cm

$=21000$ N cm

Torque on pulley B

$=(1800-1000)\times 12{\cdot}5$ N cm

$=10000$ N cm

Torque on pulley C

$=(2000-1000)\times 11$ N cm

$=11000$ N cm

This shows that shaft is receiving power at pulley A and is transmitting power to machines through pulleys B and C.

So the maximum torque on the shaft

$=21000$ Ncm

$=21\times 10^3$ N cm $=21\times 10^4$ Nmm

Shear stress permissible,

$q=50$ N/mm²

So the shaft diameter,

$$d^3=\frac{16\ T_{max}}{\pi q}=\frac{16\times 21\times 10^4}{\pi\times 50}$$

$=2{\cdot}139\times 10^4$

Shaft diameter,

$d=2{\cdot}776\times 10=27{\cdot}76$ mm.

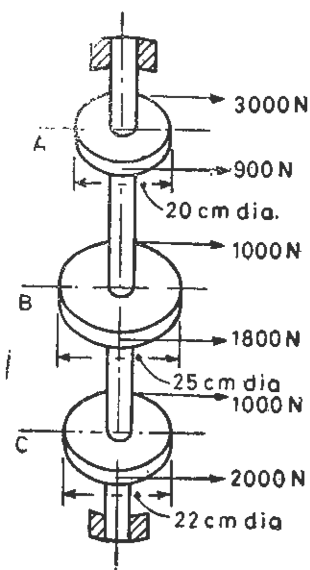


Fig. 13·40

Problem 13·16. A solid shaft of diameter 11 cm is transmitting 700 kW at 200 r.p.m. It is also subjected to a bending moment 15 k Nm and an end thrust. If the maximum principal stress developed in the shaft is 200 N/mm², determine the magnitude of end thrust.

Solution. Power transmitted = 700 kW = 700 k Nm/second.

Speed $= 200$ r.p.m.

$$= \frac{200 \times 2\pi}{60} = 20·944 \text{ rad/sec}$$

Torque, $$T = \frac{700 \times 1000}{20·94} \text{ Nm} = 33422·459 \text{ Nm} = 33·4 \times 10^6 \text{ Nmm}$$

Shaft diameter, $d = 11$ cm $= 110$ mm

Shear stress developed in the shaft,

$$q = \frac{16T}{\pi d^3} = \frac{16 \times 33·4 \times 10^6}{\pi \times (110)^3} = 127·8 \text{ N/mm}^2$$

Bending moment on shaft, $M = 15$ kNm $= 15 \times 10^6$ Nmm

Say f_1 = direct stress due to bending developed in the shaft

Then $$M = \frac{\pi d^3}{32} \times f_1 \text{ or } 15 \times 10^6 = \frac{\pi \times (110)^3}{32} \times f_1$$

or $$f_1 = \frac{15 \times 32 \times 10^6}{\pi \times (110)^3} \text{ N/mm}^2$$

$$f_1 = 114·792 \text{ N/mm}^2.$$

Say the direct stress due to end thrust is f_2 then resultant direct stress

$$f = f_1 - f_2 \quad (\text{when } f_1 \text{ is tensile})$$
$$= f_1 + f_2 \quad (\text{when } f_2 \text{ is compressive})$$

Now maximum principal stress,

$$p_{max} = \frac{f}{2} + \sqrt{\left(\frac{f}{2}\right)^2 + q^2}\,; \text{ or } 200 = \frac{f}{2} + \sqrt{\left(\frac{f}{2}\right)^2 + (127·8)^2}$$

or $$\left(200 - \frac{f}{2}\right)^2 = \left(\frac{f}{2}\right)^2 + (127·8)^2$$

$$4 \times 10^4 + \left(\frac{f}{2}\right)^2 - 200f = \left(\frac{f}{2}\right)^2 + (127·8)^2$$

or $$200f = 4 \times 10^4 - 1·278^2 \times 10^4 = 2·367 \times 10^4$$

$$f = 118·39 \text{ N/mm}^2.$$

Taking f_1 and f_2 both compressive

$$f_2 = 118·390 - 114·792 = 3·598 \text{ N/mm}^2$$

Area of cross section of shaft $= \frac{\pi}{4}(110)^2 = 0·950 \times 10^4$ mm²

End thrust, $$P = 0·950 \times 10^4 \times 3·598$$
$$= 3·418 \times 10^4 \text{ N} = 34·18 \text{ kN}.$$

Now consideriug $f_2 = f + f_1$, the maximum principal stress developed in the shaft would be more than 200 N/mm² so not permissible.

Problem 13·17. A shaft is subjected to bending and twisting moments. The greater principal stress developed in the shaft is numerically 6 times the minor principal stress. Determine the ratio of bending moment and twisting moment and the angle which the plane of greater principal stress makes with the plane of bending stress.

Solution. Say M=Bending moment

T=Twisting moment on the section of the shaft

d=diameter of the shaft

f=stress due to bending moment

q=shear stress due to twisting moment

$$p_1, \text{ greater principal stress} = \frac{f}{2} + \sqrt{\left(\frac{f}{2}\right)^2 + q^2} \quad \ldots(1)$$

$$p_2, \text{ minor principal stress} = \frac{f}{2} - \sqrt{\left(\frac{f}{2}\right)^2 + q^2}$$

In Fig. 13·41, AC is the plane of bending stress of the shaft. BC is the plane parallel to the axis of the shaft.

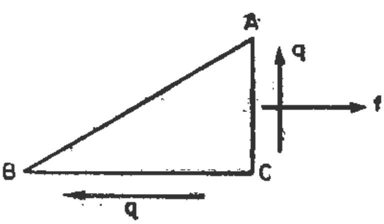


Fig. 13·41

From (1), $\frac{p_1}{p_2} = -6$

since p_2 will be negative if p_1 is positive, as is obvious from the expressions (1)

or $$\frac{M+\sqrt{M^2+T^2}}{M-\sqrt{M^2+T^2}} = -6 \quad \ldots(2)$$

or $$M+\sqrt{M^2+T^2} = -6M+6\sqrt{M^2+T^2}$$

or $$-5\sqrt{M^2+T^2} = -7\,M \quad \text{or} \quad 25(M^2+T^2) = 49\,M^2$$

or $$25\,T^2 = 24\,M^2$$

or $$\frac{M}{T} = \sqrt{\frac{25}{24}} = 1{\cdot}020$$

$$M = 1{\cdot}020\,T \quad \ldots(3)$$

Now bending stress, $f = \dfrac{32\,M}{\pi d^2}$

Shear stress, $q = \dfrac{16\,T}{\pi d^3}$

Say θ=angle which the plane of greater principal stress makes with the plane of bending stress

$$\tan 2\theta = \frac{2q}{f} = \frac{32\,T}{\pi d^3} \times \frac{\pi d^3}{32\,M}$$

$$= \frac{T}{M} = \frac{1}{1{\cdot}020} = 0{\cdot}980$$

$$2\theta = 44°\,24' \quad \text{or} \quad \theta = 22°\,12'.$$

Problem 13·18. A solid shaft transmits 2000 kW at 200 revolutions per minute. The maximum torque developed in the shaft is 1·8 times the mean torque. The distance between the bearings is 1·8 metres with a flywheel weighing 5000 kg midway between the bearings, Fig. 13·42. Determine the shaft diameter if (*a*) the maximum permissible tensile stress is 60 N/mm² (*b*) the maximum permissible shearing stress is 40 N/mm².

Solution.

Power developed $=2000$ kW

$=2000$ kNm/second

Speed $=200$ r.p.m.

$$=\frac{200}{60}\times 2\pi \text{ rad/sec}$$

$=20·944$ rad/second

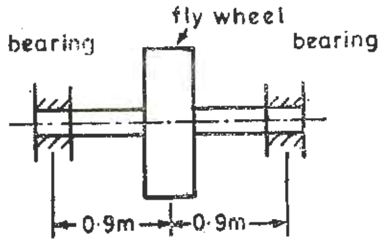


Fig. 13·42

Therefore mean torque,

$$T_m=\frac{2000\times 1000}{20·944}$$

$=95492·7$ Nm

The maximum bending moment due to flywheel occurs at the centre of the shaft

$$M_{max}=\frac{W\times l}{4}$$

where W=weight of the flywheel

l=distance between the bearings

$$M_{max}=\frac{5000\times 1·8}{4}=2250 \text{ kg-m}$$

$=2250\times 9·8$ Nm $=22050$ Nm

$M_{max}=22·050\times 10^3$ Nm

Maximum torque, $T_{max}=1·8$ Tm $=1·8\times 95492·7$ Nm

$=171886·86$ Nm $=171·866\times 10^3$ Nm

Now equivalent bending moment

$$M_e=\frac{M_{max}+\sqrt{M_{max}^2+T_{max}^2}}{2}$$

$$=\frac{22·050\times 10^3+\sqrt{(22·050\times 10^3)^2+(171·886\times 10^3)^2}}{2}$$

$$=\frac{22·050\times 10^3+10^3\sqrt{486·20+29544·79}}{2}$$

$$=\frac{22·050\times 10^3+10^3\times 173·294}{2}$$

$=97·672\times 10^3$ Nm $=97·672\times 10^6$ Nmm

Equivalent twisting moment,

$$T_e=\sqrt{M_{max}^2+T_{max}^2}$$

$=\sqrt{(22·050\times 10^3)^2+(171·886\times 10^3)^2}$

$=173·294\times 10^3$ Nm $=173·294\times 10^6$ Nmm

(a) Say d=diameter of the solid shaft

f=allowable tensile stress

Then $$M_e=\frac{\pi d^3}{32}\times f, \quad \text{or} \quad 97{\cdot}672\times10^6=\frac{\pi d^3}{32}\times60$$

$$d^3=\frac{97{\cdot}672\times32}{60\pi}\times10^6=16{\cdot}5812\times10^6$$

$$d=249{\cdot}7 \text{ mm}$$

(b) Say q=allowable shearing stress

$$T_e=\frac{\pi d^3}{16}\times q$$

$$10^6\times173{\cdot}294=\frac{\pi d^3}{16}\times40$$

$$d^3=\frac{173{\cdot}294\times16\times10^6}{40\pi}=22{\cdot}064\times10^6$$

$$d=280.5 \text{ mm}$$

Therefore the shaft diameter to withstand the allowable stresses is 280·5 mm.

Problem 13·19. A shaft of rectangular section is transmitting power at 200 r.p.m. lifting a load of 6 tonnes at a speed of 10 metres per minute. The efficiency of the crane gearing is 60 % and the maximum permissible shear stress in shaft is 40 N/mm². If the ratio of breadth to depth is 1·5, determine the size of the shaft and the angle of twist in a length of 4 metres.

If $G=78400$ N/mm².

Solution. Work done per minute$=6000\times9{\cdot}8\times10\times1000=58{\cdot}8\times10^7$ Nmm

Efficiency of gearing $=60$ %

Input work per minute $=\dfrac{58{\cdot}8\times10^7}{0{\cdot}6}=98\times10^7$ Nmm

$$N=200 \text{ r.p.m.}$$

Therefore torque $$T=\frac{98\times10^7}{2\pi\times200}=779857 \text{ Nmm}$$

For the rectangular section, longer side

$=1{\cdot}5\times$shorter side, or $a=1{\cdot}5\,b$

Moment of resistance $$=\frac{a^2b^2}{3a+1{\cdot}8b}\times q$$

$$=\frac{(1{\cdot}5\,b)^2\times b^2}{3\times1{\cdot}5\,b+1{\cdot}8\,b}\times q=\frac{2{\cdot}25\,b^4}{6{\cdot}3\,b}\times40$$

or $$779857=0{\cdot}357\,b^3\times40$$

$$b^3=\frac{779857}{40\times0{\cdot}357}=54590 \text{ mm}^3$$

Shorter side, $b=37{\cdot}93$ mm

Longer side, $a=37{\cdot}93\times1{\cdot}5=56{\cdot}895$ mm

Angle of twist, $\theta = k \cdot \dfrac{a^2+b^2}{a^3b^3} \times \dfrac{Tl}{G}$

where $k = 3{\cdot}645 - 0{\cdot}06 \times \dfrac{a}{b} = 3{\cdot}645 - 0{\cdot}06 \times \dfrac{1{\cdot}5\,b}{b} = 3{\cdot}555$

$$\theta = 3{\cdot}555 \times \frac{56{\cdot}895^2 + 37{\cdot}93^2}{56{\cdot}895^3 \times 37{\cdot}93^3} \times \frac{779857 \times 4000}{78400}$$

$$= 0{\cdot}0658 \text{ radian} = 3{\cdot}77 \text{ degree.}$$

Problem 13·20. A shaft of elliptical section with major axis 40 mm and minor axis 25 mm is subjected to a twisting moment of 250 Nm. Determine the maximum shear stress developed in the shaft and the angle of twist in a length of 1 metre. $G=78400$ N/mm²

Solution. Torque, $T = 250$ Nm $= 250 \times 1000$ Nmm

Minor axis, $a = 25$ mm

Major axis, $b = 40$ mm

$$T = \frac{\pi}{16}\, ba^2\, q$$

where q = maximum shear stress at the edges of minor axis

$$q = \frac{16 \times 250{,}000}{\pi \times 40 \times 25^2} = 50{\cdot}93 \text{ N/mm}^2$$

Angular twist, $\theta = \dfrac{16}{\pi} \dfrac{(a^2+b^2)}{a^3b^3} \times \dfrac{Tl}{G}$

$$= \frac{16}{\pi} \times \frac{(25^2+40^2)}{25^3 \times 40^3} \times \frac{250{,}000 \times 1000}{78400}$$

$$= 0{\cdot}036 \text{ radian} = 2{\cdot}07 \text{ degree.}$$

Problem 13·21. A closed tubular section of mean radius R and radial thickness t and a tube of the same radius and thickness but with a longitudinal slit are subjected to the same twisting moment T. Compare the maximum shear stress developed in both and also compare the angular twist in these tubes.

Solution.

Mean radius $= R$

Thickness, $= t$

Closed Tubular Section

Maximum shear stress $q_1 = \dfrac{T}{(2\pi Rt)R}$ $\qquad q_1 = \dfrac{T}{2\pi R^2 t}$ ...(1)

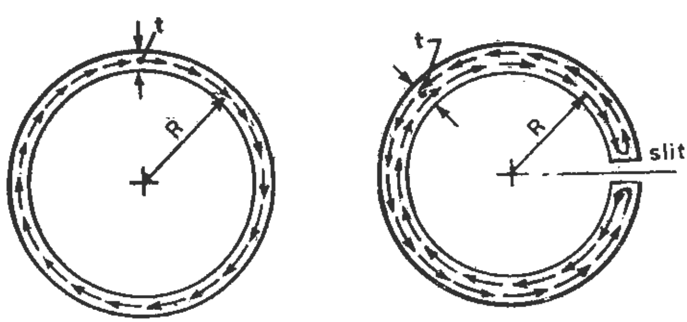


Fig. 13·43

Angular twist per unit length

$$\theta_1=\frac{T}{G\times 2\pi R^3 t} \qquad \ldots(2)$$

Tubular section with a small slit. This can be treated as a thin rectangular section of width $2\pi R$ and thickness t.

Maximum shear stress, $q_2=\dfrac{3T}{bt^2}=\dfrac{3T}{2\pi Rt^2}$

Angular per unit length, $\theta_2=\dfrac{3T}{2\pi Rt^3}$

$$\frac{q_1}{q_2}=\frac{T}{2\pi R^2 t}\times\frac{2\pi Rt^2}{3T}=\frac{t}{3R}$$

$$\frac{\theta_1}{\theta_2}=\frac{T}{G\times 2\pi R^3 t}\times\frac{2\pi Rt^3}{3T}=\frac{t^2}{3R^2}$$

$$t<<R$$

So the closed tubular section is much more stronger and stiffer than the open tubular section with a slit.

Problem 13·22. An extruded section in light alloy is in the form of a semi-circle of mean diameter 8 cm and thickness 4 mm. If a torque is applied to the section and the angle of twist is to be limited to 2° in a length of 1 metre, estimate the torque and the maximum shear stress. $G=26000$ N/mm².

Solution. The semi circular section having only one boundary can be treated as thin rectangular section of width πR and thickness t.

Width, $b=\pi R=\pi\times 40$ mm

Thickness, $t=4$ mm

Angular twist per mm, $\theta=\dfrac{4^\circ}{1000}\times\dfrac{\pi}{180}=\dfrac{4\pi}{180000}$ radian

$$\theta=\frac{3T}{Gbt^3} \quad \text{or} \quad T=\frac{G\theta\, bt^3}{3}$$

$$T=\frac{4\pi}{180000}\times\frac{26000\times(\pi\times 40)(4)^3}{3}$$

$$=4866\cdot1 \text{ Nmm}=4\cdot866 \text{ Nm}$$

Maximum shear stress, $q=\dfrac{3T}{bt^2}=\dfrac{3\times 4866\cdot1}{\pi\times 40\times(4)^2}=7\cdot26$ N/mm²

Problem 13·23. An I section with flanges 50 mm×5 mm and web 140 mm×3 mm is subjected to a twisting moment of 0·2 kNm. Find the maximum shear stress and twist per unit length neglecting stress concentration. $G=80{,}000$ N/mm².

In order to reduce the stress and the angle of twist per unit length, the I section is reinforced by welding steel plates 140 mm×5 mm as shown in the Fig. 13·44. Find the maximum stress due to the same twisting moment. What is then the value of twist per unit length.

Solution.

I section

Flanges 50×5 mm

Web 140×3 mm

$\Sigma bt^2 = 2 \times 50 \times 5^2 + 140 \times 3^2$

$= 2500 + 1260 = 3760 \text{ mm}^3$

$\Sigma bt^3 = 2 \times 50 \times 125 + 140 \times 27$

$= 12500 + 3780 \text{ mm}^4$

$= 16280 \text{ mm}^4$

Maximum shear stress

$$q = \frac{3T}{\Sigma bt^2} = \frac{3 \times 0{\cdot}2 \times 10^6}{3760}$$

$= 159{\cdot}57 \text{ N/mm}^2$

Angular twist per unit length

$$= \frac{3T}{G\Sigma bt^3} = \frac{3 \times 0{\cdot}2 \times 10^6}{80{,}000 \times 16280}$$

$= 0{\cdot}463 \times 10^{-3}$ rad/mm

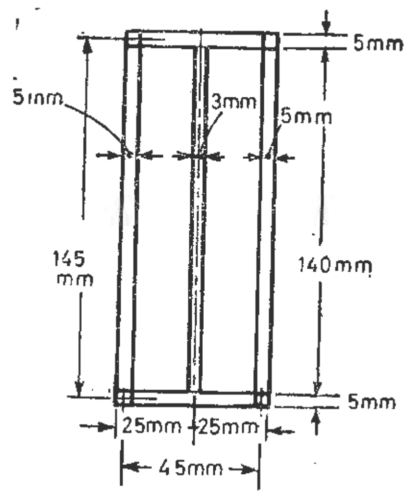


Fig. 13·44

Reinforced I section. As shown in the Fig. 13·44, there are two cells of area

$$A_1 = A_2 = 22{\cdot}5 \times 145 \text{ mm}^2 = 3262{\cdot}5$$

Line integrals
$$a_1 = a_2 = \frac{22{\cdot}5}{5} + \frac{22{\cdot}5}{5} + \frac{145}{3} + \frac{145}{5} = 86.33$$

$$a_{12} = \frac{145}{3} = 48{\cdot}33$$

Say the shear flow in cell 1 is τ_1 and shear flow in cell 2 is τ_2 Then

$$T, \text{ Torque} = 2\,\tau_1 A_1 + 2\,\tau_2 A_2$$
$$= (\tau_1 + \tau_2)(2 \times 3262{\cdot}5) = 6525\,(\tau_1 + \tau_2)$$

$$\frac{1}{2A_1G}\,(a_1\tau_1 - a_{12}\tau_2) = \frac{1}{2A_2G}\,(a_2\tau_2 - a_{12}\,\tau_1)$$

But $A_1 = A_2$

So $86{\cdot}33\,\tau_1 - 48{\cdot}33\,\tau_2 = 86{\cdot}33\,\tau_2 - 48{\cdot}33\,\tau_1$ or $\tau_1 = \tau_2$

$$\tau_1 = \tau_2 = \frac{T}{2 \times 6525} = \frac{0{\cdot}2 \times 10^6}{2 \times 6525} = 15.32 \text{ N/mm}$$

Maximum shear stress $= \dfrac{\tau_1}{t} = \dfrac{\tau_1}{5} = 3{\cdot}065 \text{ N/mm}^2$

Angular twist per unit length

$$\theta = \frac{1}{2A_1G}\,(a_1\tau_1 - a_{12}\,\tau_2)$$

$$= \frac{15{\cdot}32(86{\cdot}33 - 48{\cdot}33)}{2 \times 3262{\cdot}5 \times 80{,}000} = 0{\cdot}0011 \times 10^{-3} \text{ rad/mm}$$

$= 0{\cdot}0638 \times 10^{-3}$ degree/mm $= 0{\cdot}0638°$/metre length

Problem 13·24. A thin walled section is of two cells one closed but other having a small longitudinal slit as shown in Fig. 13·45. The section is of uniform thickness t through out and it has dimensions $2a \times a$ as shown It is subjected to a twisting moment T. Determine (*i*) Torque shared by each cell (*ii*) maximum shear stress in both the cells (*iii*) angular twist per unit length, if G is the shear modulus of the section.

Solution.

Cell I

Area, $A=a^2$

$$\oint \frac{ds}{t}=\frac{4a}{t}$$

Say shear flow in the cell is τ

Say the torque shared by cell 1 is T_1

Angular twist per unit length,

$$\theta_1=\frac{T_1}{4A^2G}\oint\frac{ds}{t}$$

$$=\frac{T_1}{4\times a^4\times G}\times\frac{4a}{t}=\frac{T_1}{Ga^3t} \quad ...(1)$$

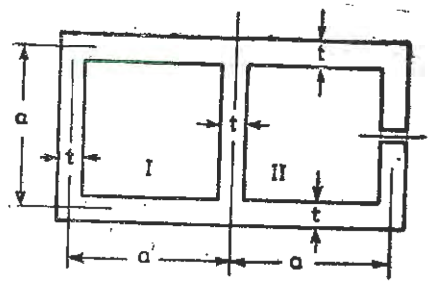


Fig. 13·45

Cell II. Open section,

Breadth, $b=a+a+\frac{a}{2}+\frac{a}{2}=3a$

Thickness, $=t$

$$\Sigma bt^3=3at^3$$

$$\Sigma bt^2=3at^2$$

Say the torque shared by cell $2=T_2$

Angular twist, $$\theta_2=\frac{3T_2}{G\Sigma bt^3}=\frac{3T_2}{G3at^3}=\frac{T_2}{Gat^3} \quad ...(2)$$

But both cells I and II are integral, for continuity $\theta_1=\theta_2$

$$\frac{T_1}{Ga^3t}=\frac{T_2}{Gat^3}$$

$$T_1at^3=T_2a^3t \quad \text{or} \quad T_1t^2=T_2a^2$$

$$\therefore \quad \frac{T_1}{T_2}=\frac{a^2}{t^2} \quad \text{or} \quad T_1=T_2\times\frac{a^2}{t^2}$$

But $$T=T_1+T_2=T_2\times\frac{a^2}{t^2}+T_2=T_2\left(\frac{a^2+t^2}{t^2}\right)$$

$$T_2=\frac{Tt^2}{(a^2+t^2)}, \text{ and } T_1=\frac{Ta^2}{(a^2+t^2)} \quad ...(3)$$

Maximum shear stress in cell 1

$$T_1=2\tau A=2\times\tau\times a^2$$

$$\tau\text{, shear flow}=\frac{T_1}{2a^2}$$

q_1, maximum shear stress,

$$=\frac{\tau}{t}=\frac{T_1}{2a^2t}=\frac{Ta^2}{2a^2t(a^2+t^2)}=\frac{T}{2t(a^2+t^2)}$$

Maximum shear stress in cell 2

$$q_2=\frac{3T_2}{\Sigma bt^2}=\frac{3\times Tt^2}{(a^2+t^2)(3at^2)}=\frac{T}{a(a^2+t^2)}$$

Angular twist per unit length

$$\theta_1=\theta_2=\frac{3T_2}{G\Sigma bt^3}=\frac{3\times Tt^2}{(a^2+t^2)\,G(3at^3)}=\frac{T}{Gat(a^2+t^2)}$$

SUMMARY

1. Torsion formula $\frac{T}{J}=\frac{q}{R}=\frac{G\theta}{l}=\frac{q_r}{r}$

where
q = maximum shear stress
T = Twisting moment
J = Polar moment of inertia $=\frac{\pi R^4}{2}$
R = Radius of solid shaft
G = Shear modulus
θ = Angular twist
l = Length of the shaft
q_r = Shear stress at any radius r

For a hollow shaft with inner dia D_1 and outer dia D_2

$$J=\frac{\pi}{32}\,(D_2^4-D_1^4)$$

Torsion formula will now be

$$\frac{32\,T}{\pi(D_2^4-D_1^4)}=\frac{2q}{D_2}$$

2. Torsional rigidity of the shaft $=GJ$

3. Modulus of rupture $q'=\frac{16\,T_{max}}{\pi D^3}$ (in a solid shaft)

$$=\frac{16\,T_{max}\,D_2}{\pi(D_2^4-D_1^4)}\text{ (in a hollow shaft)}$$

where T_{max} = maximum torque uptil failure of the shaft

4. Horse power transmitted by a shaft

$$\text{HP}=\frac{2\pi NT}{746\times 60}$$

where
N = Revolutions per minute
T = Torque in Nm

Metric HP $=\frac{2\pi NT}{4500}$ where T = Torque in kg-metre

5. If there are several shafts in series, same torque T will act on each of them,

Total angular twist $\theta=\frac{T}{G}\left(\frac{l_1}{J_1}+\frac{l_2}{J_2}+\frac{l_3}{J_3}\right)$

where l_1, l_2 and l_3 are the lengths of the shafts

6. For a compound shaft, where two shafts are co-axial or where torque acts at the junction of the two shafts.

Angular twist, $\theta_1=\theta_2$

$$\frac{T_1}{T_2}=\frac{J_1}{J_2}\times\frac{G_1}{G_2}\times\frac{l_2}{l_1}$$

Total torque, $T=T_1+T_2$

7. A shaft subjected to twisting moment T, such that the maximum shear stress is, q, then principal stresses on the surface of the shaft are $+q$, $-q$, 0

8. A shaft subjected to bending moment M and twisting moment T simultaneously.

Equivalent bending moment, $M_e=\dfrac{M+\sqrt{M^2+T^2}}{2}$

Equivalent twisting moment, $T_e=\sqrt{M^2+T^2}$

9. Strain energy in a solid shaft subjected to twisting moment

$$U=\frac{q^2}{4G}\times\text{volume of the shaft}$$

where q=maximum shear stress

10. Strain energy in a hollow shaft of diameters D_1, D_2 subjected to twisting moment

$$U=\frac{q^2}{4G}\left(\frac{D_2^2+D_1^2}{D_2^2}\right)\times\text{volume of the shaft}$$

11. Stresses in a key connecting shaft to the pulley are

Shear stress $=\dfrac{2T}{Dbl}$

Bearing stress $=\dfrac{4T}{Dtl}$

where T=Torque, b=breadth of the key,
t=thickness of the key, l=length of the key
D=Diameter of the shaft

12. Rectangular section shaft longer side b, shorter side a subjected to twisting moment T.

Maximum shear stress, $q=\dfrac{T(3b+1{\cdot}8\,a)}{8a^2b^2}$

Angular twist per unit length,

$$\theta=k\,\frac{a^2+b^2}{16a^3b^3}\times\frac{T}{G}\quad\text{where } k=3{\cdot}645-{\cdot}065\times\frac{b}{a}$$

Maximum shear stress occurs at the centre of the longer side.

13. Elliptical section shaft, major axis $2b$, minor axis $2a$.

Maximum shear stress, $q=\dfrac{2T}{\pi a^2b}$

Angular twist per unit length, $\theta=\dfrac{T}{G}\times\dfrac{a^2+b^2}{\pi a^3b^3}$

14. An equilateral triangular section shaft

Maximum shear stress, $q=\frac{15\sqrt{3}}{2a^3}\times T$

Angular twist per unit length, $\theta=\frac{15\sqrt{3}T}{Ga^4}$ where a=side of triangle

15. Membrane Analogy for shafts of non circular sections

(*i*) Slope of the membrane at any point is proportional to the magnitude of the shear stress at that point.

(*ii*) The direction of shear stress at any point is perpendicular to that of slope at that point.

(*iii*) The twisting moment is numerically equivalent to twice the volume under the membrane.

16. For a thin walled section subjected to twisting moment T

Shear flow, $\tau=\frac{T}{2A}$

where A=area enclosed by the centre line of the tube

Shear flow, τ=shear stress $q\times$thickness t

Angular twist per unit length, $\theta=\frac{T}{4A^2G}\oint\frac{ds}{t}$

where $\oint\frac{ds}{t}$ is the line integral or the ratio of length divided by thickness

17. For a section built up of thin rectangular sections such as I, T, channel etc., subjected to twisting moment T

Maximum shear stress, $q=\frac{3T}{\Sigma bt^2}$

Angular twist per unit length, $\theta=\frac{3T}{G\Sigma bt^3}$

18. Torsion of thin walled two-cell section

Shear flow $\tau_1=\tau_2+\tau_3$

Twisting moment $T=T_1+T_2$

$T_1=2\tau_1A_1 \qquad T_2=2\tau_2A_2$

A_1, A_2 are the areas enclosed by the centre line of the cells I and II respectively

τ_1=shear flow in cell I

τ_2=shear flow in cell II

τ_3=shear flow in middle web

$$\theta=\frac{1}{2A_1G}(a_1\tau_1-a_{12}\tau_2)=\frac{1}{2A_2G}(a_2\tau_2-a_{12}\tau_1)$$

where $a_1=\oint\frac{ds}{t}$ for cell I including web

$$a_2 = \oint \frac{ds}{t} \text{ for cell II including web}$$

$$a_{12} = \oint \frac{ds}{t} \text{ for the middle web.}$$

MULTIPLE CHOICE QUESTIONS

1. A solid shaft of diameter d and length l is subjected to a twisting moment T. Another shaft of the same material and diameter d and length 0·5 l is also subjected to the same twisting moment T. If the angular twist in shaft A is θ, the angular twist in shaft B is

   (*a*) 2θ (*b*) θ
   (*c*) $0·5\ \theta$ (*d*) $0·25\ \theta$

2. A solid circular shaft A, of diameter d, length l is subjected to a twisting moment T. Another shaft B of the same material, of diameter d but of length $2l$ is subjected to the same twisting moment T. If the shear angle developed on shaft A is ϕ, the shear angle developed on shaft B is

   (*a*) 2ϕ (*b*) ϕ
   (*c*) $0·5\ \phi$ (*d*) $0·25\ \phi$

3. A hollow circular shaft of inner radius 3 cm and outer radius 5 cm and length 100 cm is subjected to a twisting moment so that the angular twist is 0·01 radian. The maximum shear angle in the shaft is

   (*a*) 0·0005 radian (*b*) 0·005 radian
   (*c*) 0·003 radian (*d*) 0·0006 radian

4. Torsional rigidity of a shaft is given by

   (*a*) T/G (*b*) T/J
   (*c*) GJ (*d*) TJ

   where T=Torque, J=polar moment of inertia
   and G=Shear modulus

5. A hollow shaft of outer radius 10 cm, inner radius 4 cm is subjected to a twisting moment. The maximum shear stress developed in the shaft is 50 N/mm². The shear stress at the inner radius of the shaft is

   (*a*) 125 N/mm² (*b*) 50 N/mm²
   (*c*) 30 N/mm² (*d*) 20 N/mm²

6. A steel shaft A, of diameter d, length l is subjected to a bending moment T. Another shaft B of brass of the same diameter d, length l is also subjected to the same twisting moment T. If the shear modulus of steel is twice the shear modulus of brass, and the maximum shear stress developed in steel shaft is 50 N/mm², then the maximum shear stress developed in brass shaft is

   (*a*) 200 N/mm² (*b*) 100 N/mm²
   (*c*) 50 N/mm² (*d*) 25 N/mm²

7. A solid shaft diameter 100 mm, length 1000 mm is subjected to a twisting moment T, the maximum shear stress developed in the shaft is 60 N/mm². A hole of diameter 50 mm is drilled throughout the length of the shaft. By how much the torque T must be reduced so that the maximum shear stress developed in the hollow shaft remains the same

   (*a*) $T/2$ (*b*) $T/8$
   (*c*) $T/15$ (*d*) $T/16$

8. A stepped shaft of steel 150 cm long is subjected to a twisting moment T. For 100 cm length, the shaft diameter is 4 cm and for 50 cm length, the shaft is of diameter 2 cm. The shaft is shown in figure 13·46. If the angular twist at A is θ, then total angular twist at B is

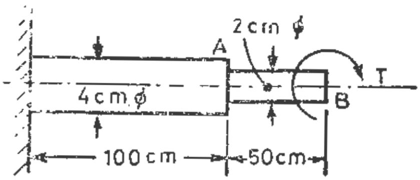


Fig. 13·46

(a) $17\ \theta$ (b) $16\ \theta$
(c) $9\ \theta$ (d) $8\ \theta$

9. A solid circular shaft is subjected to the twisting moment such that the maximum shear stress developed on this shaft is 40 N/mm². The maximum principal stress developed on the surface of the shaft is

(a) 80 N/mm² (b) 40 N/mm²
(c) 20 N/mm² (d) None of the above.

10. A solid circular shaft A of diameter d is transmitting 100 HP at 200 r.p.m. Another shaft B of the same material but hollow with outer diameter d and inner diameter $0·5\ d$ is transmitting 200 HP at 400 RPM. If the maximum shear siress developed on shaft A is 150 N/mm², the maximum shear stress developed on shaft B will be

(a) 320 N/mm² (b) 240 N/mm²
(c) 160 N/mm² (d) 150 N/mm²

11. A steel shaft A of diameter d, length l is subjected to a twisting moment T. Another shaft B of brass, diameter d, length $0·5\ l$ is also subjected to the same twisting moment T. Shear modulus of steel is 2 times the shear modulus of brass. If the angular twist in shaft A is θ, then the angular twist in shaft B is

(a) $2\ \theta$ (b) θ
(c) $0·5\ \theta$ (d) $0·25\ \theta$

12. A shaft of 100 mm diameter is keyed to a pulley trausmitting power. The breadth of the key is 20 mm and its thickness is 20 mm. If the shaft is subjected to a twisting moment of 1000 Nm, and its keyed length is 10 cm, the shear stress developed in the key is

(a) 100 N/mm² (b) 50 N/mm²
(c) 10 N/mm² (d) None of the above

13. A shaft of rectangular section $b \times a$ is subjected to a twisting moment T. The maximum shear stress occurs at

(a) the ends of diagonals (b) the centre of the longer side
(c) at the centre of the shorter side (d) none of the above

14. A shaft of elliptical section is subjected to a twisting moment. The maximum shear stress occurs at the

(a) at the centre of the ellipse (b) at the ends of minor axis
(c) at the ends of major axis (d) none of the above.

15. A stretched membrane under pressure is used to find the shear stresses in a shaft of rectangular section subjected to twisting moment. The maximum stress occurs at the point where

(a) deflection is maximum (b) deflection is minimum
(c) slope is maximum (d) slope is zero

16. A shaft of thin square section with mean perimeter 16 cm and wall thickness 0·25 cm is subjected to a twisting moment of 16 Nm. The maximum shearing stress developed in the section

(a) 2 N/mm^2 (b) 4 N/mm^2
(c) 8 N/mm^2 (d) 16 N/mm^2

17. A Tee section flange 10 cm × 1 cm and web 10 cm × 1 cm is subjected to a torque of 500 kg-cm. The maximum shear stress in the section is

(a) 25 kg/cm^2 (b) 50 kg/cm^2
(c) 75 kg/cm^2 (d) 100 kg/cm^2

ANSWERS

1. (c)	2. (b)	3. (a)	4. (c)	5. (d)
6. (c)	7. (d)	8. (c)	9. (b)	10. (c)
11. (b)	12. (c)	13. (b)	14. (b)	15. (c)
16. (a)	17. (c).			

EXERCISES

13·1. A hollow circular steel shaft revolving at 150 rpm transmits power to a crane lifting a load of 80 kN, at a speed of 2 metres/second. If the efficiency of the gearing system is 70%, determine the size of the shaft. The external diameter is 1·5 times the internal diameter and the maximum shear stress in the shaft is 80 N/mm^2. Calculate also the angular twist in the shaft over a length of 2 metres. Given G for steel = 82 kN/mm^2

[**Ans.** 90·3 mm, 60·2 mm ; 3·88 degree]

13·2. A solid circular shaft is required to transmit 250 metric horse power at 600 r.p.m. The maximum torque developed in the shaft is 1·3 times the mean torque. Determine the diameter of the shaft if the maximum shear stress is not to exceed 1200 kg/cm^2. Calculate also the angular twist per 100 cm length of the shaft. G_{steel} = 800 $tonnes/cm^2$.

[**Ans.** 5·475 cm ; 2·99°]

13·3. A hollow circular steel shaft is transmitting 150 metric horse power at 200 r.p.m. The maximum torque developed in the shaft is 40% more than the mean torque. Determine the external and internal diameters of the shaft if the maximum shearing stress is not to exceed 100 N/mm^2 and the maximum angular twist per metre length of the shaft is not to exceed 3°. The external diameter is double the internal diameter of the shaft. $G_{steel} = 82 \times 10^3$ N/mm^2.

[**Ans.** 74·2 mm, 37·1 mm]

13·4. A solid bar of a metal of diameter 2 cm and length 20 cm is tested under tension. A load of 500 kg produces an extension of 0·032 mm. At the same time the change in diameter is observed to be 0·00112 mm. Determine E, G and $1/m$ for the material.

Ans. [$0·9947 \times 10^6$ kg/cm^2, $0·368 \times 10^6$ kg/cm^2, 0·35]

13·5. A hollow marine propeller shaft with diameters ratio 0·6, is running at 150 r.p.m. It is propelling a vessel at a speed of 30 knots, at the expenditure of 8000 H.P. If the efficiency of the propeller is 85%, and the direct stress due to thrust is not to exceed 80 kg/cm^2, calculate (a) shaft diameters (b) maximum shear stress due to torque.

1 knot = 0·515 metre/second [**Ans.** 28·65 cm, 17·19 cm ; 950·4 kg/cm^2]

13·6. Compare the torques which can be transmitted by solid and hollow shafts for a given maximum shear stress, if the weight per unit length of the shafts is the same and both are made of same material. The internal diameter of the hollow shaft is 0·6 times the external diameter. [**Ans.** 0·5885]

13·7. A solid circular steel shaft is rigidly conneeted to a steel tube to make a spring as shown in the Fig. 13·47. The shaft is prevented from rotation at the end C and a torque T is applied to the tube at the end A. The useful length of the shaft and tube is 50 cm. The diameter of the shaft is 3 cm, the internal diameter of the tube is 3·5 cm and the external diameter is 4 cm. Determine the torque at the tube if the maximum shear stress is not to exceed 200 N/mm². Given $G_{steel}=80{,}000$ N/mm². Moreover calculate (*a*) the ratio of the maximum shear stresses in shaft and tube, (*b*) the total angular twist.

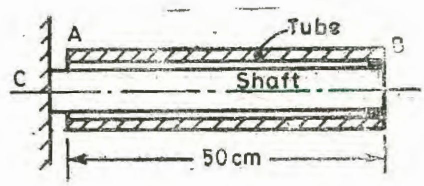


Fig. 13·47

[**Ans.** 0·353 kNm, (*a*) 3·59, (*b*) 5·76 degree]

13·8. A solid circular shaft of diameter 30 mm is surrounded by a thick copper tube of 50 mm external diameter so as to form a compound shaft. The compound shaft is subjected to a torque of $0{\cdot}8\times10^6$ Nmm. Determine

(*a*) torque in steel shaft and copper tube

(*b*) maximum shear stresses in steel and copper

(*c*) angular twist per 100 cm length of the compound shaft.

Given $G_{steel}=2\ G_{copper}=80{,}000$ N/mm²

[**Ans.** (*a*) $1{\cdot}84\times10^5$, $6{\cdot}16\times10^5$ Nmm
(*b*) 34·7, 28·8 N/mm² (*c*) 1·656 degree]

13·9. A steel shaft of diameter 6 cm runs at 250 r.p.m. This steel shaft has a 1 cm thick bronze bushing shrunk over its entire length of 5 metres. If the maximum shearing stress in steel shaft is not to exceed 700 kg/cm², find (*a*) Power of the engine (*b*) Torsional rigidity of the shaft. $G_{steel}=840$ tonnes/cm², $G_{bronze}=420$ tonnes/cm²

[**Ans.** (*a*) 215·6 horse power, (*b*) 4450 kg-m/radian]

13·10. A solid circular shaft uniformly tapered along its length is subjected to a twisting moment T. Radius at the big end is 1·2 times the radius at the small end. Determine the percent error introduced if angle of twist for a given length is calculated on the basis of the mean radius. [**Ans.** 2·72 %]

13·11. A vessel having a single propeller shaft 30 cm diameter running at 150 r.p.m. is re-engined to two propeller shafts of 20 cm diameter each running at 250 r.p.m. If the working shear stress in these shafts is 10 per cent more than the single shaft, determine the ratio of the horse powers transmitted by these two shafts with that of single shaft.

[**Ans.** 1·087]

13·12. A steel shaft of 10 mm diameter is solid for a certain part of its length and hollow for the remainder part of its length with inside diameter 30 mm. If a pure torque is applied of such a magnitude that yielding just occurs on the surface of the solid part, determine the depth of yielding in the hollow part of the shaft. Determine also the ratio of angle of twist per unit length for the two parts of the shafts. [**Ans.** 0·84 mm, 1·02]

13·13. A flanged coupling has 8 bolts of 2 cm diameter each, arranged symmetrically along the bolt circle diameter of 24 cm. If the diameter of the shaft is 8 cm and stressed upto 1 tonne/cm², calculate the shear stress in the bolts. [**Ans.** 333·33 kg/cm²]

13·14. Fig. 13·48 shows a vertical shaft with pulleys keyed to it. The shaft is rotating with a uniform angular speed of 1500 r.p.m. The belt pulls are indicated and the 3 pulleys are rigidly keyed to the shaft. If the the maximum shear stress in the shaft is not to exceed 60 N/mm², determine the necessary diameter of the shaft, which is solid. The shaft is supported in bearings near the pulleys, so that the bending of the shaft may be neglected.

[**Ans.** 17·205 mm]

13·15. A solid shaft of diameter 8 cm is transmitting 500 HP at 500 r.p.m. It is subjected to a bending moment 25000 kg-cm and an end thrust. If the maximum principal stress developed in the shaft is 1400 kg/cm², determine the magnitude of end thrust.

[**Ans.** 27·15 Tonnes]

13·16. A solid alloy shaft of diameter 500 mm is coupled to a hollow steel shaft of same external diameter. If the angular twist per unit length in steel shaft is limited to 75 % of the angular twist per unit length of solid shaft, determine the internal diameter of hollow shaft. At what speed, the shafts will transmit 150 horse power. The maximum shearing stress in steel shaft is not to exceed 100 N/mm² and that in alloy shaft it is not to exceed 50 N/mm².

[**Ans.** 41·32 mm, 858 revolutions per minute]

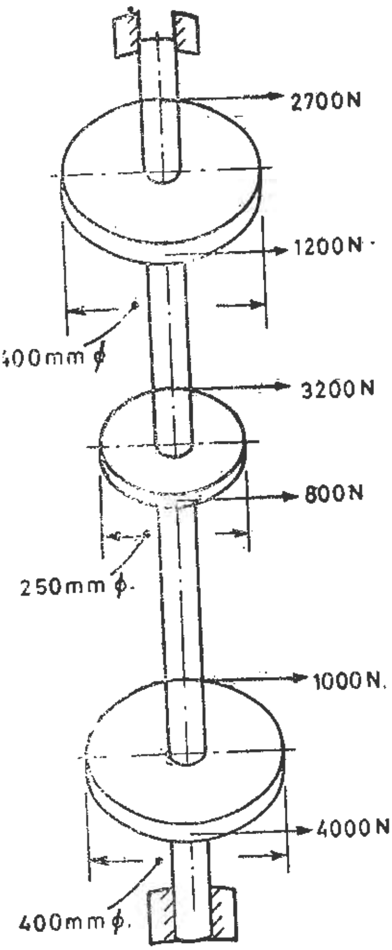


Fig. 13·48

13·17. A shaft subjected to bending and twisting moments simultaneonsly. The greater principal stress developed in the shaft is numerically 4 times the minor principal stress. Determine the ratio of bending moment and twisting moment and the angle which the plane of greater principal stress makes with the plane of bending stress. [**Ans.** 0·75, 24°18′]

13·18. A solid shaft transmits 1200 horse power at 300 revolutions per minute. The maximum torque developed in the shaft is 1·4 times the mean torque. The distance between the bearings is 1·2 metre with a flywheel weighing 2000 kg midway between the bearings. Determine the shaft diameter if (*a*) maximum permissible tensile stress is 800 kg/cm², (*b*) the maximum permissible shearing stress is 500 kg/cm². [**Ans.** (*a*) 14·36 cm (*b*) 16 cm]

13·19. A shaft of square section is subjected to twisting moment 400 Nm. If the maximum shear stress developed in the shaft is not to exceed 40 N/mm², determine the size of the shaft and angle of twist in a 2 metre length of the shaft. $G=78400$ N/mm².

[**Ans.** 36·36 mm, 2·377 degree]

13·20. A tubular section of mean radius 10 cm and thickness 1 cm having a small longitudinal slit cut into it, is subjected to a torque such that the maximum shear stress is

50 N/mm² Determine the torque on the section, and the angular twist for 1 metre length of the shaft. Compare its strength and stiffness with that of a closed tubular section of the same dimensions. $G=80{,}000$ N/mm². [**Ans.** 10472 Nm. 1·04 rad/metre length ; 1 : 30, 1 : 300]

13·21. An extruded section of brass is in the form of a semicircle of mean diameter 90 mm and thickness 4 mm. If a torque of 5 Nm is applied to the section, determine the maximum shear stress developed in the section. What is the angular twist per metre length.

[**Ans.** 6·63 N/mm², 2·435°]

13·22. An I section with flanges 50 mm×5 mm and web 65 mm×4 mm is subjected to a twisting moment of 10 kg-metre. Find the maximum shear stress and the angle of twist per metre length. $G=820{,}000$ kg/cm². Neglect stress concentrations.

In order to reduce the stress and the angle of twist per unit length, the I section is strengthened by welding steel plates 65 mm×5 mm at the ends of the flanges so as to make a section of two cells. Find the maximum stress due to the same twisting moment. What is the angular twist per unit length.

[**Ans.** 847·4 kg/cm², 0·22 radian/metre ; 31·74 kg/cm², 0·0014 radian/metre]

13·23. A thin walled box section has two compartments as shown in Fig. 13·49. The thickness of the section is constant. What is the shear stress in both the cells. Take $a=10$ cm, $t=0·8$ cm. What will be the angular twist per unit length. $G=80{,}000$ N/mm². [**Ans.** 2·484 N/mm², 0·397 N/mm², 0·036° per metre]

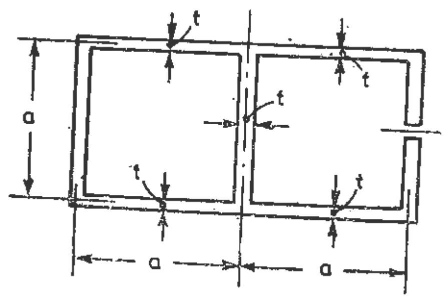


Fig. 13·49

14

Springs

The springs are commonly used to absorb the energy provided through an external force in the form of strain energy and to release the same energy as per the requirements. In the case of clockwork, the strain energy in the spiral spring is stored through winding the spring and the resumption of the spring's original shape takes place very slowly. In various mechanisms, springs are provided as a means of restoring the original configuration against the action of the external force. Springs are also used to absorb shocks such as the springs of buffers of railway rolling stock and the springs on the wheels of the vehicles and the effects of the blow on the rolling stock or the vehicles are reduced.

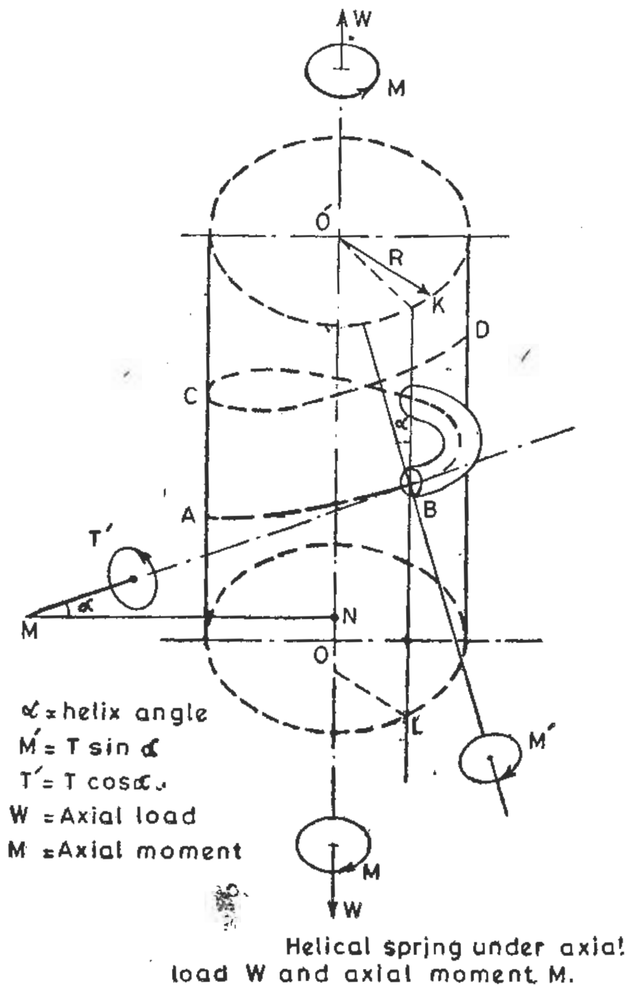


Fig. 14·1

14·1. HELICAL SPRINGS

When the axis of the wire of the spring forms an helix on the surface of a right circular cylinder or a right circular cone, a helical spring is obtained. But here we will deal only with the helical springs, when the coils of the springs form a cylindrical surface. The conical springs are not within the scope of this book.

Figure 14·1 shows coils of a helical spring subjected to axial load or axial moment. $ABCD$ is the helix described by the spring wire. The axis of the spring wire forms a right circular cylinder of radius R, $O'K=R$, the axis of the cylinder being $O'O$, which coincides with the axis of the spring. The line KBL is the generator of the cylinder. MB is the tangent to helix at the point B. MN is perpendicular to the axis of the spring. If the axis of the spring is vertical then MN is horizontal. The angle $\angle NMB$ is called the angle of helix of the spring. Let us take

R=radius of the cylinder as shown

=mean radius of the coils of the spring

α=helix angle

n=number of coils in the spring

Length of the spring wire in one coils$=\dfrac{2\pi R}{\cos\alpha}$.

Therefore length of the spring wire in n coils$=\dfrac{2\pi nR}{\cos\alpha}\simeq 2\pi nR$

If α is very small angle as in the case of close coiled helical springs.

Let us consider first the effect of axial load W only.

Close coiled helical spring. When the coils of the springs are so close to each other that they can be regarded as lying in planes at right angles to the axis of the helix, the angle α is very small and $\cos\alpha \simeq 1$.

Say the diameter of the wire$=d$

Polar moment of inertia, $J=\dfrac{\pi d^4}{32}$

Since $\cos\alpha=1$

Direct shear force on any section$=W$(axial load)

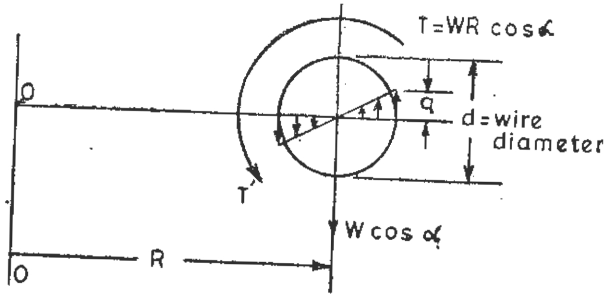


Fig. 14·2

Twisting moment on each cross section$=T'=T\cos\alpha\simeq T$

$=WR$

Direct shear stress,

$$q_0=\frac{4W}{\pi d^2}\text{, which is uniform throughout the section}$$

Maximum shear stress due to twisting moment

$$q=\frac{WR}{\pi d^4/32}\times\frac{d}{2}=\frac{16WR}{\pi d^3}$$

As is obvious from the figure 14·2, the direct shear stress q_0 is added at the inner side of the spring and substracted at the outer side *i.e.* the resultant shear stress at the inner radius of the coil will be more than the resultant shear stress at the outer radius of the coil.

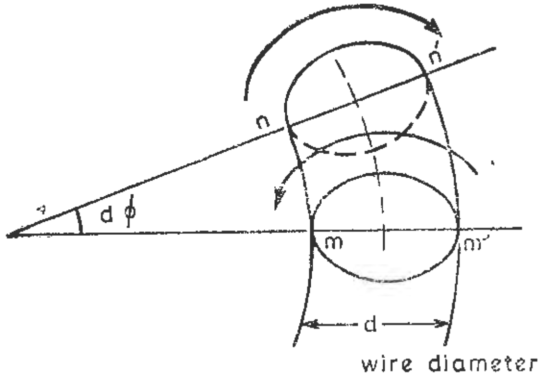


Fig. 14·3

Moreover consider a small element $mm'n'n$ of the spring wire subjected to twisting moment. The cross section mm' rotates with respect to the cross section nn'. Due to the twisting moment, the angular displacement of the point m with respect n is the same as the angular displacement of the point m' with respect to n'. But the distance mn is smaller than the distance $m'n'$. Therefore the shearing strain at the inner surface mn will be more than the shearing strain at the outer surface $m'n'$.

Taking into account, the above observation, the maximum shearing stress in the spring wire is given by the following equation by some researchers—

$$q_{max}=\frac{16WR}{\pi d^3}\left(\frac{4k'-1}{4k'-4}+\frac{0{\cdot}615}{k'}\right)$$

where $k'=\dfrac{2R}{d}$

=spring index and term in the bracket determines the correction factor

Example 14·1-1. A close coiled helical spring of mean coil radius 4 cm is made of a steel wire of diameter 8 mm. If the axial load on the spring is 10 kg, determine the maximum shearing stress developed in the spring wire.

Solution.

Spring index, $k'=\dfrac{2R}{d}$

where R=mean coil radius=4 cm

d=wire diameter =0·8 cm

$$k'=\frac{2\times4}{0\cdot8}=10$$

Axial load, $W=10$ kg

So $$q_{max}=\frac{16\times10\times4}{\pi\times0\cdot8^3}\left(\frac{40-1}{40-4}+\frac{0\cdot615}{10}\right)$$

$$=397\cdot88\ (1\cdot0833+\cdot0615)$$

Maximum shear stress $=455\cdot49$ kg/cm^3

Exercise 14·1-1. A close coiled helical spring of mean coil diameter 50 mm is made of a steel wire of 6 mm diameter. If the maximum shear stress developed is 90 N/mm², what is the axial load applied on the spring. **Ans.** [129· 8N]

14·2. CLOSE COILED HELICAL SPRING SUBJECTED TO AXIAL LOAD

Again consider a small element $mnn'm'$ of the spring wire subtending an angle $d\zeta$ at the centre of the spring. Say under the action of the twisting moment $T=WR$, the angular **twist is $\delta\theta$.**

The vertical deflection along the axis of the spring

$$=d\delta=R\delta\theta$$

Angular twist,

$$\delta\theta=\frac{TRd\zeta}{GJ}$$

because $Rd\zeta=dl=$length of the shaft considered

$$d\delta=R\delta\theta=\frac{TR^2d\zeta}{GJ}=\frac{WR^3d\zeta}{GJ}$$

because T on any section is WR

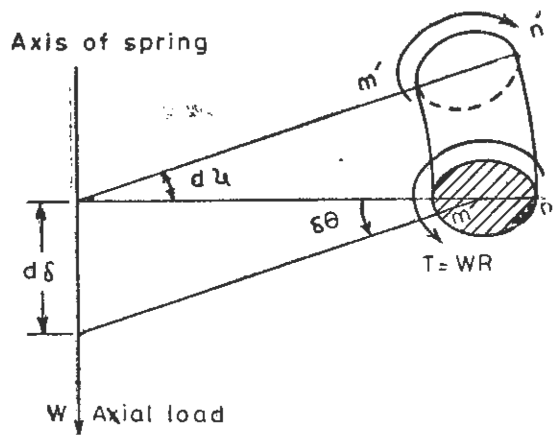


Fig. 14·4

Total axial deflection, $$\delta=\int_0^{2\pi n}\frac{WR^3}{GJ}\,d\zeta=\frac{WR^3}{GJ}\,.\,2\pi n$$

where $n=$number of turns and total angle subtended by the coils at the axis of the spring

$\zeta=2\pi n$

$J=$polar moment of inertia of spring wire section $=\dfrac{\pi d^4}{32}$

Axial deflection, $$\delta=\frac{WR^3\times2\pi n\times32}{G\times\pi d^4}=\frac{64nWR^3}{Gd^4}$$

Stiffness of the spring $=$Load per unit axial deflection$=\dfrac{W}{\delta}$

Stiffness, $k=\dfrac{Gd^4}{64nR^3}$. *i.e.*, the axial force required to extend or compress the spring by a unit axial deflection.

Example 14 2-1. A close coiled helical spring made of round steel wire is required to carry a load of 800 N for a maximum stress not to exceed 200 N/mm². Determine the wire diameter if the stiffness of the spring is 10 N/mm and the diameter of the helix is 80 mm. Calculate also the number of turns required in the spring. Neglect the correction due to the spring index. Given G for steel=80 kN/mm².

Solution.

Mean coil radius, $R=\frac{80}{2}=40$ mm

Maximum shear stress, $q=200$ N/mm²

Axial load, $W=800$ N

Stiffness, $k=10$ N/mm

Now $$q=\frac{16WR}{\pi d^3}=\frac{16\times 800\times 40}{\pi d^3}$$

$$200=\frac{12800\times 40}{\pi d^3} \quad \text{or} \quad d^3=814{\cdot}87 \text{ mm}^3$$

Wire diameter, $d=9{\cdot}34$ mm

Moreover stiffness, $$k=\frac{Gd^4}{64\, nR^3}$$

So $$10=\frac{80\times 1000\times 9{\cdot}34^4}{64\, n\times 40^3}$$

Number of coils, $n=14{\cdot}86$

Exercise 14·2-1. A close coiled helical spring made of 0·6 cm diameter steel wire carries an axial load of 40 kg. Determine the maximum shear stress in the spring wire if the mean coil radius is 3 cm and the number of turns are 8·5.

Given $G_{steel}=820$ tonnes/cm²

Calculate also the following :

(*a*) the maximum shear stress at the inner coil radius

(*b*) axial deflection. [**Ans.** 2835 kg/cm² ; (*a*) 3245 kg/cm² (*b*) 5·52 cm]

14·3. CLOSE COILED HELICAL SPRING SUBJECTED TO AXIAL MOMENT

Again consider that the spring is subjected to an axial moment M, then moment in the plane of the coil or about the axis of the coil is $M\cos\alpha$ and the twisting moment is $M\sin\alpha$ as shown in Fig. 14·5. But since α is small $M\cos\alpha \simeq M$ and $M\sin\alpha \simeq 0$.

Let us consider again that the wire is acted upon everywhere, by a bending couple which is approximately equal to M.

Say ϕ=the total angle through which one end of the spring is turned relative to the other end, when the bending couple M is applied

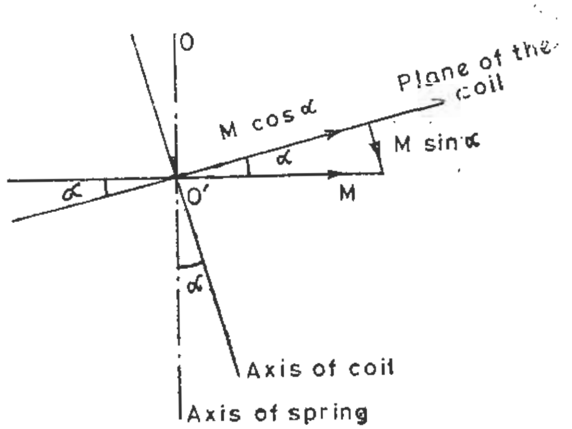


Fig. 14·5

Work done $=\frac{1}{2}M\phi$ but $\phi=\frac{Ml}{EI}$

where E=modulus of elasticity of the material

I=moment of inertia of the section

To prove that $\phi=\frac{Ml}{EI}$. Let us consider a bar of length l, initially straight, subjected to a bending moment M. After bending, the bar subtends an angle ϕ at the centre of curvature and say R is the radius of curvature.

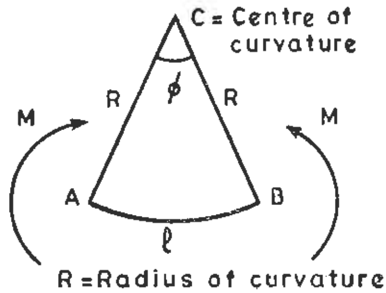


Fig. 14·6

Taking ϕ to be very small

$$R\phi=l \text{ or } \phi=\frac{l}{R}$$

But from flexure formula

$$\frac{M}{I}=\frac{E}{R}\ ; \text{ and } \frac{1}{R}=\frac{M}{EI}$$

$$\therefore \quad \phi=\frac{Ml}{EI}$$

So the work done $=\frac{1}{2} \quad M\phi=\frac{M^2l}{2EI}$

or

$$\phi=\frac{Ml}{EI}=\frac{2\pi nRM\times 64}{E\times \pi d^4}=\frac{128nRM}{Ed^4}$$

The maximum stress in the wire due to the axial couple is

$$f=\frac{Md}{2I}=\frac{32\ M}{\pi d^3}$$

Example 14·3-1. A close coiled helical spring made of round steel wire 5 mm diameter, having 10 complete turns is sxbjected to an axial moment M. Determine the magnitude of the axial couple M if the maximum bending stress in spring wire is not to exceed 2400 kg/cm². Calculate also the angle through which one end of the spring is turned relative to the other end, if the mean coil radius is 3·5 cm. E_{steel}=2000 tonnes/cm²

Solution.

Wire diameter, d=0·5 cm

Number of turns, n=10

Mean coil radius, R=3·5 cm

Maximum bending stress,

$$f=\frac{32M}{\pi d^3}=2400 \text{ kg/cm}^2$$

Axial moment $M=f\,.\,\frac{\pi d^3}{32}=\frac{2400\times\pi\times 0{\cdot}5^3}{32}$=29·45 kg-cm

The angle through which one end of the spring is turned relative to the other end,

$$\phi=\frac{128\ nRM}{Ed^4}=\frac{128\times10\times3{\cdot}5\times29{\cdot}45}{2000\times1000\times(0{\cdot}5)^4}$$

$$=1{\cdot}056\text{ radian}=60{\cdot}4^\circ$$

Exercise 14·3-2. A close coiled helical spring made of round steel wire, with mean coil radius of 4 cm and number of turns equal to 10 is snbjected to an unwinding axial couple of 4 Nm. Determine,

(*a*) wire diameter if the maximum bending stress is not to exceed 240 N/mm²

(*b*) the angle through which one end of the spring is turned relative to the other end.

$E_{steel}=208$ kN/mm² [**Ans.** 5·54 mm, 59·86°]

14·4. OPEN COILED HELICAL SPRINGS

In the case of open coiled helical springs, the effect of bending moment $WR\sin\alpha$, when the spring of mean coil radius R is subjected to axial load W cannot be neglected. Similarly when the spring is subjected to axial couple M, the effect of $M\sin\alpha$ cannot be neglected. Let us first consider that the open coiled helical spring is subjected to an axial load W only and due to this there are $WR\cos\alpha$=twisting moment and $WR\sin\alpha$=bending moment acting on the spring wire.

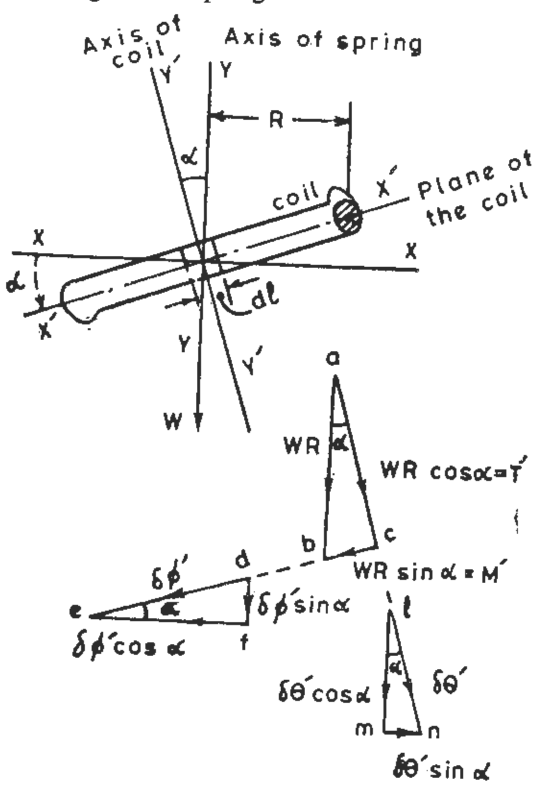


Fig. 14·7

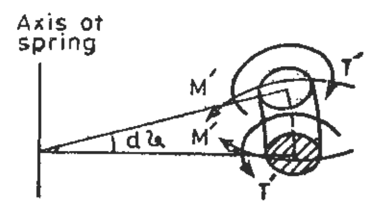


Fig. 14·8

Total moment about X-X axis (perpendicular to YY-axis of spring)$=WR$

Twisting moment about the centre of the section of the spring wire$=T'=WR\cos\alpha$

Bending moment in the plane of the coil $X'X'=M'=WR\sin\alpha$

On each element of the spring, the twisting and bending moments will act as shown in the Fig. 14·8.

Due to the twisting moment, there will be angular twist about the axis of the helix $X'X'$ and due to the bending moment one end of the spring will rotate with respect to the other end about the axis $Y'Y'$.

Considering a small element of the spring of length δl

Angular twist about axis $X'X'$,

$$\delta\theta'=\frac{T'\delta l}{GJ}=\frac{WR\cos\alpha\,\delta l}{GJ}$$

(shown by *ln* in displacement diagram *mln*)

Angular rotation about axis $Y'Y'$.

$$\delta\phi'=\frac{M\delta l}{EI}=\frac{WR\sin\alpha\,\delta l}{EI}$$

(Shown by *de* in the displacement diagram *def*)

Taking the components of angular twist and angular rotation about XX and YY axis.

$\delta\theta$,, angular twist about XX axis

$$\vec{lm}+\vec{df}=\delta\theta'\cos\alpha+\delta\phi'\sin\alpha$$
$$=\frac{WR\cos^2\alpha\,\delta l}{GJ}+\frac{WR\sin^2\alpha\,\delta l}{EI}$$

$\delta\phi$, angular rotation about YY axis

$$=\vec{mn}-\overleftarrow{fe}=\delta\theta'\sin\alpha-\delta\phi'\cos\alpha$$
$$=\frac{WR\sin\alpha\cos\alpha\,\delta l}{GJ}-\frac{WR\sin\alpha\cos\alpha\,\delta l}{EI}$$

Total angular twist about X-X axis,

$$\theta=\int_0^l\delta\theta \quad \text{where } l \text{ is the length of the spring wire}$$

$$\theta=\int_0^l\left(\frac{WR\cos^2\alpha}{GJ}+\frac{WR\sin^2\alpha}{EI}\right)\delta l$$

$$\theta=\left(\frac{WR\cos^2\alpha}{GJ}+\frac{WR\sin^2\alpha}{EI}\right)l$$

l=length of the spring wire$=2\pi nR'$

where $R'=\dfrac{R}{\cos\alpha}$, radius of the coil in its plane

So $l=2\pi nR\sec\alpha$

Angular twist, $\theta=\left(\dfrac{WR\cos^2\alpha}{GJ}+\dfrac{WR\sin^2\alpha}{EI}\right)\times2\pi Rn\sec\alpha$

Axial deflection in the spring,

$$\delta=R\theta=2\pi nR^3W\sec\alpha\left[\frac{\cos^2\alpha}{GJ}+\frac{\sin^2\alpha}{EI}\right]$$

Total angular rotation about YY axis,

$$\phi=\int_0^l \delta\phi=\int_0^l \left(\frac{WR \sin \alpha \cos \alpha}{GJ}-\frac{WR \sin \alpha \cos \alpha}{EI}\right) dl$$

$$=WR \sin \alpha \cos \alpha \left[\frac{1}{GJ}-\frac{1}{EI}\right] l.$$

where $\quad l=2\pi n\, R \sec \alpha$

So $\quad \phi=WR \sin \alpha \cos \alpha \;.\; 2\pi nR \sec \alpha \left[\frac{1}{GJ}-\frac{1}{EI}\right]$

$$=2\pi n\, WR^2 \sin \alpha \left[\frac{1}{GI}-\frac{1}{EI}\right]$$

The axial extension in the spring can also be obtained by using the principle of work done

Say $\quad \delta$=axial extension in spring under the axial load W

ϕ'=angular rotation of the spring about $Y'Y'$ axis

θ'=angular twist of the spring about $X'X'$ axis

$$\text{Work done}=\tfrac{1}{2}\, T'\theta'+\tfrac{1}{2}M'\phi'$$

$$\tfrac{1}{2}W\delta=\tfrac{1}{2}T'\theta'+\tfrac{1}{2}M'\phi'$$

$$\delta=\frac{1}{W}\Big[\ T'\ \theta'+M'\phi'\ \Big]=\left[\frac{T'^2\, l}{GJ}+\frac{M'^2\, l}{EI}\right]\frac{1}{W}$$

but $T'=WR \cos \alpha$ and $M'=WR \sin \alpha$

$$\delta=\frac{1}{W}\left[\frac{W^2R^2 \cos^2 \alpha}{GJ}+\frac{W^2R^2 \sin^2 \alpha}{EI}\right] l$$

$$=WR^2 l\left[\frac{\cos^2\alpha}{GJ}+\frac{\sin^2 \alpha}{EI}\right]$$

but $l=2\pi nR \sec \alpha$

So axial deflection, $\quad \delta=2\pi nR^3W \sec \alpha\left[\frac{\cos^2\alpha}{GJ}+\frac{\sin^2\alpha}{EI}\right].$

Example 14·4-1. An open coiled helical spring made from wire of circular cross section is required to carry a load of 10 kg. The wire diameter is 8 mm and the mean coil radius is 4 cm. Calculate (*a*) the axial deflection (*b*) angular rotation of free end with respect to the fixed end of the spring if the helix angle of the spring is 30° and the number of turns is 12. $\quad G_{steel}$=800 tonnes/cm², E_{steel}=2000 tonnes/cm²

Solution.

Wire diameter, $\quad d=0{\cdot}8$ cm

Mean coil radius, $\quad R=4$ cm

Helix angle, $\quad \alpha=30°,\ \sin \alpha=0{\cdot}5,\ \cos \alpha=0{\cdot}866$

Number of turns, $\quad n=12$

Axial load, $\quad W=10$ kg

Length of the spring wire$=2\pi n\, R \sec \alpha$

$$=\frac{2\pi \times 4\times 12}{0{\cdot}866}=348 \text{ cm}$$

$$E=2\times 10^6 \text{ kg/cm}^2,\ G=0{\cdot}8\times 10^6 \text{ kg/cm}^2$$

Axial deflection, $\delta = WR^2 l \left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI} \right]$

$$= 10 \times 4^2 \times 348 \left[\frac{0.866^2}{GJ} + \frac{0.5^2}{EI} \right]$$

Now $J = 2I = \frac{\pi d^4}{32} = \frac{\pi \times 0.8^4}{32} = 0.040 \text{ cm}^4$

$$I = 0.020 \text{ cm}^4$$

$$\delta = 16 \times 10 \times 348 \left[\frac{0.75}{0.8 \times 10^6 \times 0.04} + \frac{0.25}{2 \times 10^6 \times 0.02} \right] = 1.645 \text{ cm}$$

Angular rotation, $\phi = WR \sin\alpha \cos\alpha \, l \left[\frac{1}{GJ} - \frac{1}{EI} \right]$

$$= 10 \times 4 \times 0.866 \times 0.5 \times 348 \left[\frac{1}{0.8 \times 10^6 \times 0.04} - \frac{1}{2 \times 10^6 \times 0.02} \right]$$

$$= 0.0374 \text{ radian} = 2.14 \text{ degree.}$$

Exercise 14 4-1. An open coiled helical spring made of 1 cm diameter steel rod, 4·5 cm mean coil radius and 20° angle of helix is subjected to an axial load W. Determine the magnitude of W if the maximum shear stress in wire due to torque is limited to 1350 kg/cm². Calculate the number of turns in the spring if axial extension in the spring under the load is 4 cm.

$$G_{steel} = 800 \text{ tonnes/cm}^2, \; E_{steel} = 2000 \text{ tonnes/cm}^2$$

[**Ans.** 62·7 kg, 8·3 turns]

14·5. OPEN COILED HELICAL SPRING SUBJECTED TO AXIAL MOMENT

Consider that on open coiled helical spring of mean coil radius R, angle of helix α, number of turns n is subjected to an axial couple M. Figure 14·9 shows a winding couple applied about the axis YY of the spring which tries to wind up the spring or which tends to increase the number of coils in the spring. There are two components of M *i.e.*, $M \cos\alpha$ (in the plane of the coil acting as bending moment) and $M \sin\alpha$, as twisting moment producing angular twist about the axis $X'X'$. Considering a small element of the spring of length δl as shown.

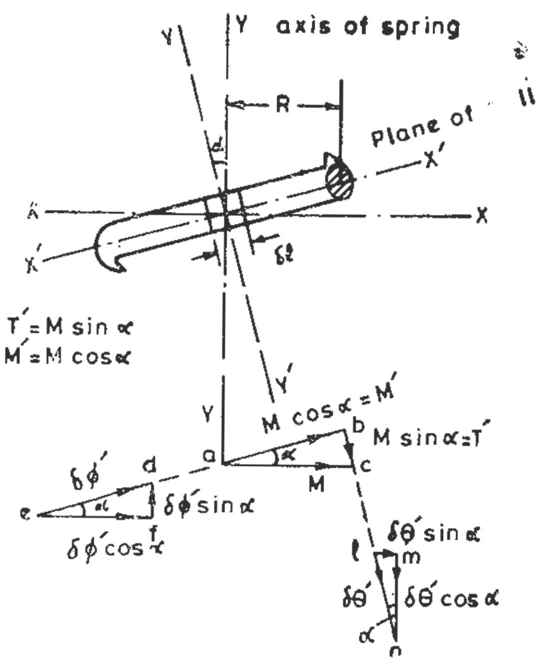


Fig. 14·9

Angular twist about the axis $X'X'$,

$$\delta\theta' = \frac{T'\delta l}{GJ} = \frac{M \sin\alpha \, \delta l}{GJ}$$

(shown by ln, in diagram lmn)

Angular rotation about the axis $Y'Y'$,

$$\delta\phi' = \frac{M'\delta l}{EI} = \frac{M \cos\alpha \, dl}{EI}$$

(shown by de in the diagram def.)

Taking the components of the angular twist and angular rotation about XX and YY axis

$\delta\theta$, angular twist about X-X axis

$$= df - mn$$

$$=\delta\phi' \sin\alpha - \delta\theta' \cos\alpha$$

$$=\frac{M \sin\alpha \cos\alpha\, \delta l}{EI} - \frac{M \sin\alpha \cos\alpha\, \delta l}{GJ}$$

$\delta\phi$, angular rotatian about YY axis,

$$=\vec{ef}+\vec{lm}=\delta\phi' \cos\alpha + \delta\theta' \sin\alpha$$

$$=\frac{M \cos^2\alpha\, \delta l}{EI} + \frac{M \sin^2\alpha\, \delta l}{GJ}$$

Total angular twist, $$\theta=\int_0^l M \sin\alpha \cos\alpha \left[\frac{1}{EI} - \frac{1}{GJ}\right] \delta l$$

$$=M \sin\alpha \cos\alpha \left(\frac{1}{EI} - \frac{1}{GJ}\right] l \text{ where } l=2\pi nR \sec\alpha$$

$$=2\pi nMR \sin\alpha \left(\frac{1}{EI} - \frac{1}{GJ}\right)$$

Total angular rotation, $$\phi=\int_0^l \left(\frac{M \cos^2\alpha}{EI} + \frac{M \sin^2\alpha}{GJ}\right) dl$$

$$=\left(\frac{M \cos^2\alpha}{EI} + \frac{M \sin^2\alpha}{GJ}\right) l$$

$$=2\pi nR \sec\alpha \cdot M \left[\frac{\cos^2\alpha}{EI} + \frac{\sin^2\alpha}{GJ}\right]$$

Axial deflection, $$\delta=R\theta=-2\pi nMR^2 \sin\alpha \left(\frac{1}{EI} - \frac{1}{GJ}\right)$$

as we have taken winding couple

Example 14·5-1. An open coiled helical spring made of steel wire 6 mm diameter and 3·6 cm mean coil radius with 65° inclination of the coils with the spring axis is subjected to an axial moment M. Determine the magnitude of M if the number of turns in the spring increase by 1/8. Calculate the change in the axial length of the spring, if the original number of turns are 10. G_{steel}=84 kN/mm², E_{steel}=210 kN/mm².

Solution.

Axial moment, $M=?$

Wire diameter, $d=0{\cdot}6$ cm$=6$ mm

Polar moment of inertia,

$$J=\frac{\pi d^4}{32}=\frac{\pi \times 6^4}{32}=127{\cdot}235 \text{ mm}^4$$

Moment of inertia, $I=\frac{J}{2}=63{\cdot}617$ mm⁴

Angle of helix, $\alpha=90-65=25°$

$$\sin\alpha=0{\cdot}423 \qquad \sin^2\alpha=0{\cdot}178$$
$$\cos\alpha=0{\cdot}907 \qquad \cos^2\alpha=0{\cdot}822$$

Number of turns $=10$

Mean coil radius, $R=36$ mm

Length of the spring wire,

$$l=2\pi nR \sec\alpha=2\pi\times10\times36\times\frac{1}{\cdot907}=2493\cdot9 \text{ mm}$$

Angular rotation $\phi=\frac{1}{8}$ turn$=\frac{1}{8}\times360°=45°=0\cdot7854$ radian

$$=Ml\left[\frac{\cos^2\alpha}{EI}+\frac{\sin^2\alpha}{GJ}\right]$$

So $$0\cdot7854=M\times2493\cdot9\left[\frac{0\cdot822}{210\times1000\times63\cdot687}+\frac{0\cdot178}{84\times1000\times127\cdot235}\right]$$

$$=\frac{M}{1000}\left[0\cdot1534+\cdot0415\right]$$

$$M=\frac{1000\times\cdot7854}{\cdot1949}=4029\cdot7 \text{ Nmm}=4\cdot029 \text{ Nm}$$

Change in axial length, $\delta=MRl\sin\alpha\cos\alpha\left(\frac{1}{GJ}-\frac{1}{EI}\right)$

$$=4029\cdot7\times36\times\cdot423\times\cdot907\left(\frac{1}{84\times1000\times127\cdot235}-\frac{1}{210\times1000\times63\cdot617}\right)l$$

Change in axial length, $\delta=4\cdot0297\times3\cdot6\times\cdot423\times\cdot307\left(\frac{1}{8\cdot4\times127\cdot35}-\frac{1}{21\times63\cdot617}\right)l$

$$=\frac{5\cdot565\times2493\cdot9}{1000}\ (0\cdot9348-0\cdot7538)=2\cdot512 \text{ mm}$$

Exercise 14·5-1. An open coiled helical spring made of 5 mm diameter steel wire, 2·5 cm mean coil radius and 23° angle of helix is subjected to an axial moment of 0 2 kg-metre. Determine (*a*) the angular rotation of one end with respect to the other end (*b*) axial deflection if number of coils in the spring is 15.

$E_{steel}=2100$ tonnes/cm², Poisson's ratio for steel$=0\cdot27$

[**Ans.** (*a*) 47·4° (*b*) 0·1928 cm]

14·6. STRESSES DEVELOPED IN SPRING WIRE OF CIRCULAR SECTION

Let us first consider an open coiled helical spring, of mean coil radius R, angle of helix α and wire diameter d, subjected to an *axial load W*.

On any section of the spring wire.

Twisting moment, $T'=WR\cos\alpha$

Bending moment, $M'=WR\sin\alpha$

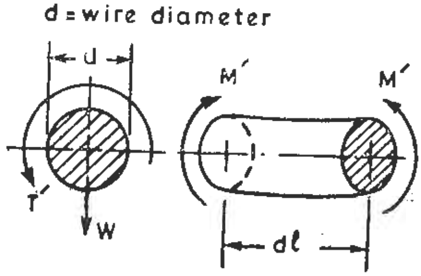


Fig. 14·10

Max. torsional shear stress on any section, $q'=\dfrac{16T'}{\pi d^3}=\dfrac{16WR\cos\alpha}{\pi d^3}$

Direct shear stress due to axial load, $q''=\dfrac{4W}{\pi d^2}$

Max. shear stress at inner coil radius $=q'+q''=q=\dfrac{16WR\cos\alpha}{\pi d^3}+\dfrac{4W}{\pi d^2}$

Minimum shear stress at outer coil radius $=q'-q''=\dfrac{16WR\cos\alpha}{\pi d^3}-\dfrac{4W}{\pi d^2}$

Maximum stress due to bending, $f=\dfrac{32M'}{\pi d^3}=\dfrac{32\,WR\sin\alpha}{\pi d^3}$

Maximum principal stress occurs at the inner coil radius,

Principal stresses $p_1, p_2=\dfrac{f}{2}\pm\sqrt{\left(\dfrac{f}{2}\right)^2+q^2}$

Let us further consider the stresses due to *axial couple M* on any section of the spring wire,

Twisting moment, $T'=M\sin\alpha$

Bending moment, $M'=M\cos\alpha$

Maximum torsional shear stress due to twisting moment,

$$q=\frac{16T'}{\pi d^3}=\frac{16M\sin\alpha}{\pi d^3}$$

Maximum stress due to bending, $f=\dfrac{32M'}{\pi d^3}=\dfrac{32M\cos\alpha}{\pi d^3}$

Principal stresses at the extreme radii (inner and outer coil radii)

$$p_1, p_2=\frac{f}{2}\pm\sqrt{\left(\frac{f}{2}\right)^2+q^2}$$

Example 14·6-1. An open coiled helical spring of wire diameter 10 mm, mean coil radius 70 mm, helix angle 20° carries an axial load of 400 N. Determine the shear stress and direct stress developed at the inner radius of the coil.

Solution. R=mean coil radius=70 mm

Wire diameter, $d=10$ mm

Axial load, $W=400$ N

Helix angle, $\alpha=20°$

$\sin\alpha=0{\cdot}342$

$\cos\alpha=0{\cdot}9397$

Twisting moment, $T'=WR\cos\alpha$

$=400\times70\times0{\cdot}9397=26{\cdot}31\times10^3$ N-mm

Bending moment, $M'=WR\sin\alpha$

$=400\times70\times0{\cdot}342=9{\cdot}576\times10^3$ Nmm.

Direct shear stress, $q'=\dfrac{4W}{\pi d^2}=\dfrac{4\times400}{\pi\times10^2}=5{\cdot}093$ N/mm^2

Torsional shear stress, $q''=\dfrac{16T'}{\pi d^3}=\dfrac{16\times26{\cdot}31\times10^3}{\pi\times10^3}$

$=134{\cdot}0$ N/mm^2

Total shear stress at the inner coil radius

$=134+5{\cdot}093=139{\cdot}093$ N/mm^2

Direct stress due to bending moment

$$=\frac{32M'}{\pi d^3}=\frac{32\times9{\cdot}576\times10^3}{\pi\times10^3}=97{\cdot}54\text{ N/mm}^2.$$

Exercise 14·6-1. An open coiled helical spring made of steel wire of 15 mm diameter, mean coil radius 9 cm with helix angle 30° is subjected to an axial moment of 40 Nm. Determine the shear and direct stresses developed in the section of the wire of the spring.

[**Ans.** 30·18 N/mm^2, 104·545 N/mm^2]

14·7. PLANE SPIRAL SPRING

The plane spiral spring consists of a uniform thin strip wound in the form of spirals as shown in the Fig. 14·11 (*a*). One end of the strip is connected to the winding spindle *A* and the other end is hinged at the point *B*.

Say R=distance of the centre of the spindle from the outer end *B*.

The spring is wound by applying a winding couple M to the winding spindle *A*. The reaction at the point *B* can be resolved into two components *i.e.*

Reaction, R_H=along the line joining the point *B* and the centre of the spindle *A*

Reaction, R_V=perpendicular to R_H.

Consider a small element *PQ* of length δs, whose co-ordinates are x and y considering the origin at the point *B* and the abscissae along the line *BA* and ordinate perpendicular to *BA*.

Say before the application of the winding couple, radius of curvature of element $PQ=r_1$.

Angle between the tangents PP' and $QQ'=\phi$.

Now after applying the winding couple, radius of curvature of the element$=r_2$.

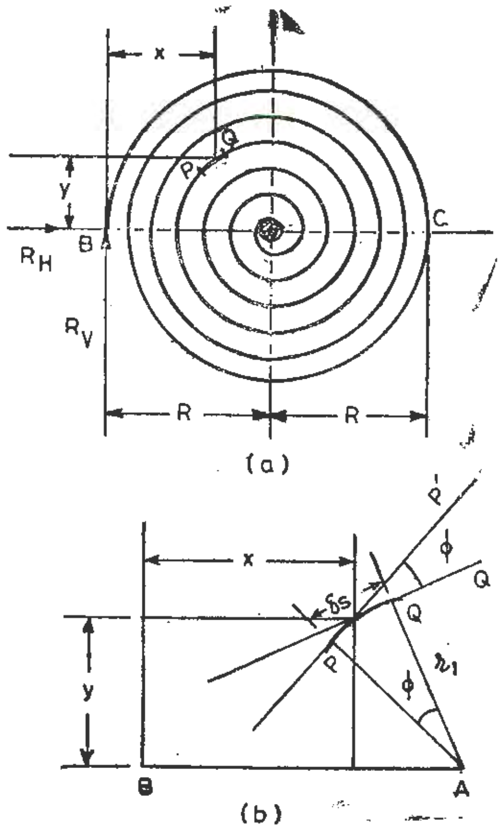


Fig. 14·11

and the angle between the tangents at P and $Q=\phi+\delta\phi$

The bending moment at the element, $M'=xR_V-yR_H$

$=EI\times$change of curvature

where $I=$moment of inertia of the strip

$$M'=EI\left[\frac{1}{r_2}-\frac{1}{r_1}\right]=EI\,\frac{\delta\phi}{\delta s} \quad \text{since } \delta\phi=\frac{\delta s}{r_2}-\frac{\delta s}{r_1}$$

or

$$\delta\phi=\frac{xR_V-yR_H}{EI}\,.\,\delta s$$

Integrating both the sides

$$\int d\phi=\int\frac{xR_V}{EI}\,ds-\int\frac{yR_H}{EI}\,ds$$

Assuming that the centroid of the spring is at the centre of the winding spindle A, then

$$\int y\,ds=0$$

and
$$\phi=\frac{R_V}{EI}\int x\,ds=\frac{R_V}{EI}\times lR$$

where l is the length of the strip forming the spring.

But $R_V \,.\, R=M$, moment applied at the spindle

or
$$\phi=\frac{Ml}{EI}$$

Energy stored, $U=\frac{1}{2}M\phi=\frac{M^2l}{2EI}$

Now bending moment at any element, $M'=xR_V-yR_H$.

This will be maximum when $y=0$, showing that maximum bending stress in one spiral occurs when the element lies along the line joining the outer point and the centre of the spindle. Moreover the maximum value of x is $2R$ *i.e.*, at the point C as shown in the Fig. 14·11.

$\therefore$ $M_{max}=2R\,.\,R_V=2\,M$

Maximum bending stress, $f_{max}=\frac{2M}{Z}$

where Z=section modulus of the strip section.

$Z=\frac{bt^2}{6}$ where b=breadth of the strip, t=thickness of the strip

$$\therefore \quad f_{max}=\frac{12\,M}{bt^2}$$

Energy stored,
$$U=\frac{M^2l}{2EI}=\left(\frac{f_{max}.bt^2}{12}\right)^2\frac{l}{2EI}$$

$$=\frac{f^2_{max}}{288}\times\frac{b^2t^4}{Ebt^3}\times 12\,l=\frac{f_{max}^2}{24E}\times btl$$

$$=\frac{f^2_{max}}{24\,E}\times\text{Volume of the strip.}$$

Example 14·7-1. A flat spiral spring is made of a strip 5 mm wide and 0·25 mm thick, 10 m long. The torque is applied at the winding spindle and 8 complete turns are given. Calculate the torque, the energy stored and the maximum stress developed at the point of greatest bending moment. $E=210$ kN/mm².

Solution.

Number of turns, $n=8$

Angle of rotation, $\phi=2\pi n=2\times\pi\times 8=16\,\pi$ radians

$=\frac{Ml}{EI}$ where $l=10000$ mm

$$E=210\times1000 \text{ N/mm}^2$$

$$I=\frac{bt^3}{12}=\frac{5(0{\cdot}25)^3}{12}=\frac{5}{768}\text{ mm}^4$$

So, $$M=210\times1000\times\frac{5}{768}\times\frac{16\pi}{10000}=6{\cdot}872 \text{ Nmm}$$

Maximum stress, $$f_{max}=\frac{12M}{bt^2}=\frac{12\times6{\cdot}872}{5\times(0{\cdot}25)^2}=263{\cdot}9 \text{ N/mm}^2$$

Work/Energy stored $$=\tfrac{1}{2}M\phi=\tfrac{1}{2}\times6{\cdot}872\times16\pi=172{\cdot}7 \text{ Nmm.}$$

Exercise 14·7-1. A plane spiral spring is made of 6 mm wide and 0·3 mm thick steel strip. The torque applied at the winding spindle is 0·1 kg-cm. Determine (*a*) the number of winding turns if the length of the strip is 250 cm. (*b*) the maximum stress developed at the point of greatest bending moment. $E=2{\cdot}1\times10^6$ kg/cm².

[**Ans.** (*a*) 1·4 turns, (*b*) 1111·1 kg/cm²]

14·8. LEAF SPRINGS

These springs are commonly used in vehicles as cars, trucks, railway wagons etc. and are termed as carriage springs also. There are two types of leaf springs (*a*) semi-elliptic and (*b*) quarter elliptic. A number of flat rectangular leaves of the same thickness and width but of different lengths are clamped together and loaded as simply supported beams and as cantilevers respectively. To arrive at a simplified theory following assumptions are made.

(*i*) The centre line of all the plates (or leaves) are initially circular arcs of the same radius R, so that the contact between the plates is only at the ends.

(*ii*) Each plate is of uniform thickness and overlaps the plate below it by an amount $p=\frac{l}{2n}$, where l is the length of the longest leaf and n is the number of leaves.

(*iii*) The overlaps are tapered in width to the triangular shape as shown in Fig. 14·12.

Since each plate is initially of the same curvature, each plate will touch the one above it only at its ends, when unloaded. After applying the central load W, if the change in curvature is uniform and is the same in all the plates, the contact will continue at the ends only.

Considering two plates only, the load at the two ends will be $W/2$ each as shown in Fig. 14·13. The bending moment varies from A to C and from B to D, but it is uniform and equal to $Wp/2$ in the portion CD.

Similarly considering next two plates, bending moment varies linearly in portions CE and DF while it is uniform in the portion EF and equal to $Wp/2$. This shows that each triangular overhanging end is loaded as a cantilever while the portion of uniform width carry uniform bending moment $\frac{Wp}{2}$ (as shown in Fig. 14·13). Over the tapered portion, M the bending moment and I, the moment of inertia are varying linearly and proprortional to the distance from the end, so M/I remains constant, while over the central portion both M and I are constant and therefore M/I is constant throughout the length of the spring.

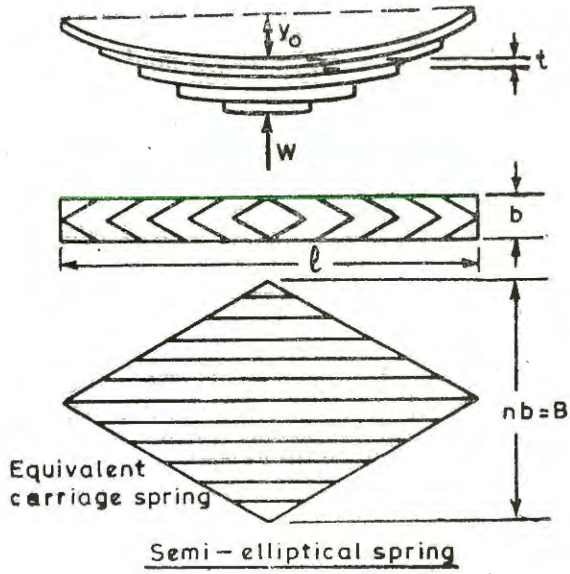


Fig. 14·12

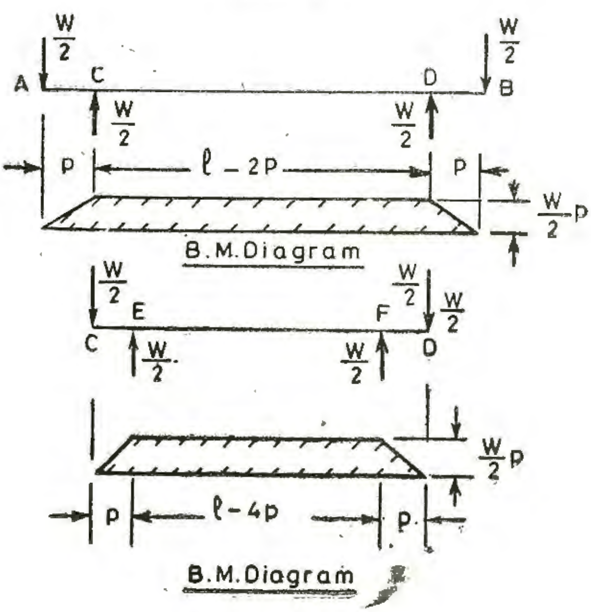


Fig. 14·13

Moreover $\dfrac{M}{EI}=\dfrac{1}{R'}-\dfrac{1}{R}$, showing that the radius of curvature R', in the strained state is also the same for each leaf and contact between the plates (leaves) continues to be at the ends only. Since the friction between the plates is negligible and each plate is of the same radius of curvature, they can be considered to be arranged side by side forming a beam of same thickness throughout but of variable width, which is maximum in the centre, equal to nb.

Maximum bending moment occurs at the centre of the spring,

$$M_{max} = \frac{Wl}{4}$$

Maximum width, $B = nb$

Moment of inertia, $I = (nb)\frac{t^3}{12}$

Bending stress, $f = \frac{Wl}{4} \times \frac{t/2}{I} = \frac{3Wl}{2nbt^2}$

Initial central deflection,

$$y_0 = \frac{l^2}{8R} \text{ (using the properties of a circle)}$$

Final radius of curvature,

$$\frac{1}{R'} = \frac{1}{R} + \frac{M}{EI} = \frac{1}{R} - \frac{Wl \times 12}{4\ Enbt^3} \text{ (curvature is reduced)}$$

$$= \frac{1}{R} - \frac{3Wl}{Enbt^3}\text{, a constant}$$

So R' is constant, showing that all the plates are bent into the circular arcs of radius R' and contact continues at ends only.

Let us determine the load W_0 which straightens all the plates *i.e.*

$$R' = \infty \quad \text{(infinity)}$$

or

$$\frac{1}{R'} = 0$$

$$\frac{1}{R} - \frac{3\ W_0\ l}{Enbt^3} = 0 \quad \text{or} \quad W_0 = \frac{Enbt^3}{3lR} \text{ (Proof load)}$$

Example 14·8-1. A carriage spring centrally loaded has 6 steel 6 mm thick and 6 cm wide. If the largest plate is 96 cm long and the load required to straighten the spring is 3 kN. Determine the following.

(*a*) initial radius of curvature
(*b*) initial central deflection provided
(*c*) the bending stress under under the proof load.

$$E = 210 \text{ kN/mm}^2$$

Solution.

Number of plates, $n = 6$
Thickness, $t = 0{\cdot}6$ cm $= 6$ mm
Width, $b = 6$ cm $= 60$ mm
Proof load, $W_0 = 3000$ N
Length $l = 96$ cm $= 960$ mm

(*a*) Radius of curvatue,

$$R = \frac{Enbt^3}{3l\ W_0} = \frac{210 \times 1000 \times 6 \times 60 \times 6^3}{3 \times 960 \times 3000}$$

$$= 1890 \text{ mm} = 1{\cdot}89 \text{ m}$$

(b) Central deflection $=\frac{l^2}{8R}=\frac{960\times960}{8\times1890}=60{\cdot}95$ mm

(e) Bending stress, $f=\frac{3W_0l}{2nbt^2}=\frac{3\times3000\times960}{2\times6\times60\times6^2}=333{\cdot}33$ N/mm²

Exercise 14·8-1. A carriage spring centrally loaded has 8 steel plates 5 mm thick and 5 cm wide. If the longest plate is 80 cm long, find the initial radius of curvature if the maximum stress is 1·5 tonnes/cm², when the plates become straight under the central load.

$E=2{\cdot}1\times10^6$ kg/cm³ [**Ans.** 350 cm]

14·9. QUARTER ELLIPTIC SPRING (CANTILEVER LEAF SPRING)

Proceeding along the same steps as in the case of semi-elliptic spring.

Bending moment $=-Wl$

Moment of inertia $=\frac{nbt^3}{12}$

Bending stress, $f=\frac{6Wl}{nbt^2}$

$$\frac{M}{EI}=\frac{1}{R'}-\frac{1}{R}$$

where R' = final radius of curvature

R = initial radius of curvature of each leaf

$$-\frac{Wl}{EI}=\frac{1}{R'}-\frac{1}{R}=\frac{2}{l^2}(y-y_0)$$

where y_0 = initial deflection

y = final deflection under the load W

$$\therefore\ y_0-y=\frac{Wl^2}{2\,EI}=\frac{6Wl^3}{nbt^3E}$$

The load W_0 required to straighten all the plates can be found by putting

$R'=\infty$ (infinity)

$$W_0=\frac{EI}{lR}=\frac{Enbt^3}{12\,lR}$$

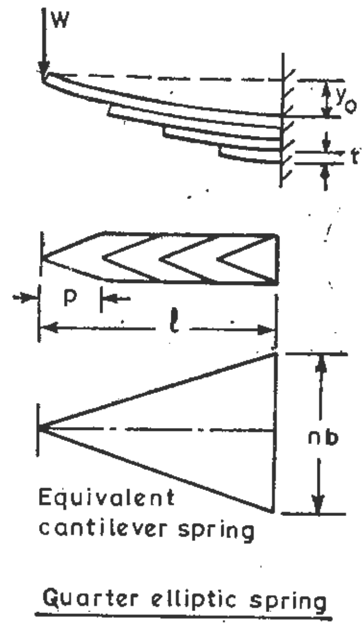


Fig. 14·14

The load required to straighten all the plates of the spring is called Proof load.

Example 14·9-1. A cantilever leaf spring of length 50 cm has 5 leaves of thickness 1 cm. If an end load of 200 kg produces a deflection of 3 cm, find the width of the leaves.

$E=2\times10^6$ kg/cm²

Solution.

Length of the spring, $l=50$ cm

Number of leaves, $n=5$

Thickness, $t=1$ cm

Breadth, $b=?$

Deflection under the load,

$$\delta=3\ \text{cm}=\frac{6Wl^3}{nbt^3E}$$

Load, $W=200$ kg

r'. $$3=\frac{6\times200\times50^3}{5\times b\times1^3\times2\times10^6}$$

Width of the leaves, $b=5$ cm

Exercise 14·9-1. A cantilever leaf spring of length 60 cm, has 6 leaves of thickness 8 mm. The width of each plate is 48 mm. If an end load of 1·5 kN is applied at its end determine the following :

(*i*) end deflection under the load

(*ii*) initial radius of curvature if the initial deflection provided is 100 mm

(*iii*) bending stress developed under the load

Given, $E=210{,}000$ N/mm²

[**Ans.** (*i*) 62·78 mm, (*ii*) 1·8 m (*iii*) 292·97 N/mm²]

Problem 14·1. Determine the stiffness of a close coiled helical spring consisting of 10 turns of 4 mm diameter steel wire coiled on a mandrel 6 cm in diameter.

Given : G for steel=840 tonnes/cm²

Solution.

Number of turns, $n=10$

Wire diameter, $d=0{\cdot}4$ cm

Inner coil diameter $=6$ cm

Outer coil diameter $=6+2\times0{\cdot}4=6{\cdot}8$ cm

Mean diameter $=6{\cdot}4$ cm

Mean coil radius, $R=3{\cdot}2$ cm

We know that $$\frac{W}{\delta}=\frac{Gd^4}{64\,nR^3}=\frac{840\times1000\times0{\cdot}4^4}{64\times10\times3{\cdot}2^3}$$

Stiffness of the spring, $k=1{\cdot}025$ kg/cm deflection.

Problem 14·2. Determine the length of 5 mm diameter wire necessary to form a close coiled helical spring with a mean coil radius of 40 mm, whose stiffness under axial load is to be 4 kg/cm. Given G for steel=820 tonnes/cm²

Solution.

Stiffness, $$k=\frac{W}{\delta}=4\ \text{kg/cm}$$

Wire diameter, $d=0{\cdot}5$ cm

Mean coil radius, $R=4$ cm

We know that $$\frac{32\,WR}{\pi d^4}=\frac{G\delta}{Rl}$$

$$l=\left(\frac{\delta}{W}\right)\frac{\pi d^4\,G}{32\,R^2}=\frac{1}{4}\times\frac{\pi\times0{\cdot}5^4\times820\times1000}{32\times4^2}$$

Length of the wire, $l=78{\cdot}6$ cm

Problem 14·3. A close coiled helical spring is made of a round wire having n turns and the mean coil radius R is 5 times the wire diameter, show that the stiffness of such a spring is $\left(\frac{R}{n}\right)\times$constant.

Determine the constant when modulus of rigidity, G of the spring wire is 82×1000 N/mm².

Such a spring is required to support a load of 1 kN with 100 mm compression and the maximum shear stress 245 N/mm². Calculate

(*i*) mean coil radius

(*ii*) number of turns

(*iii*) weight of the spring.

The material weighs 0·0078 kg/cm³

Solution.

Mean coil radius, $R=5d$

d=wire diameter, $d=0·2\ R$

Number of turns $=n$

Stiffness of the spring, $k=\frac{W}{\delta}=\frac{Gd^4}{64nR^3}$

$$=\frac{82\times1000\times(0·2\ R)^4}{64\ n\times R^3}=2·05\times\frac{R}{n}\ \text{N/mm}$$

Constant $=2·05$

W, Axial load on such a spring

$=1$ kN$=1000$ N

Axial compression, $\delta=100$ mm

Stiffness, $\frac{W}{\delta}=10$ N/mm$=2·05\ \frac{R}{n}$

or $$\frac{R}{n}=\frac{10}{2·05} \quad ...(1)$$

Moreover shear stress in wire,

$$q=\frac{16\ WR}{\pi d^3},\ \text{or}\ 245=\frac{1000\times16\times5d}{\pi d^3}$$

or $$d^2=\frac{80{,}000}{\pi\times245}=103·9376$$

Wire diameter, $d=10·2$ mm

$R=5d=51$ mm

Number of turns, $n=\frac{2·05\ R}{10}=\frac{2·05\times51}{10}=10·455$

Volume of the wire $=\frac{\pi}{4}\ d^2\times\pi nR$

$$=\frac{\pi}{4}\ (1·02)^2\times2\pi\times10·455\times5·1=273·76\ \text{cm}^3$$

Weight of the wire $=273·76\times0·0078=2·135$ kg

Problem 14·4. A close coiled helical spring is made of a round steel wire. It carries an axial load of 150 N and is to just get over a rod of 36 mm. The deflection in the spring is not to exceed 25 mm. The maximum allowable shearing stress developed in spring wire is 200 N/mm² and G for steel=80000 N/mm². Find the mean coil diameter, wire diameter and number of turns.

Solution. Say wire diameter$=d$

The spring is to just get over a rod of 36 mm *i.e.*, inner diameter of spring=36 mm

Therefore, mean coil diameter, $D=36+d$ mm

Axial load, $W=150$ N

Axial deflection, $\delta=25$ mm

Shear modulus, $C=80\times10^3$ N/mm²

Now $$\delta=\frac{8WD^3n}{Gd^4}$$

$$25=\frac{8\times150\times D^3\times n}{80\times10^3\times d^4} \quad ...(1)$$

or $$\frac{D^3n}{d^4}=\frac{25\times80\times10^3}{8\times150}=1666{\cdot}6666 \quad ...(1)$$

Shear stress developed in spring, $q=\dfrac{8WD}{\pi d^3}$

(The effect of direct shear stress and spring index has been neglected)

$$200=\frac{8\times150\times D}{\pi d^3} \quad ...(2)$$

or $$\frac{D}{d^3}=\frac{200\times\pi}{8\times150}=0{\cdot}5236 \quad ...(2)$$

or $$\frac{D^3}{d^9}=0{\cdot}1435 \quad ...(2)$$

From equations (1) and (2),

$$1666{\cdot}6666\ \frac{d^4}{n}=0{\cdot}1435\ d^9$$

or $$nd^5=11614{\cdot}40 \quad ...(3)$$

or $$n=\frac{11614{\cdot}40}{d^5}$$

Substituting this value of n in equation (1)

$$\frac{D^3}{d^4}\times\frac{11614{\cdot}40}{d^5}=1666{\cdot}6666$$

or $$\frac{D^3}{d^9}=\frac{1666{\cdot}6666}{11614{\cdot}40}=0{\cdot}1435$$

or $$\frac{D}{d^3}=\sqrt[3]{0{\cdot}1435}=0{\cdot}525$$

or $$D=d^3\ (0{\cdot}525)$$

$$(36+d)=0{\cdot}525\ d^3$$

or $$0.525\,d^3-d-36=0 \tag{4}$$

$$d=4.248 \text{ mm}$$

$$D=d^3\times 0.525=40.245$$

but $D=d+36=40.248$ (There is slight error in calculation of d)

Moreover $$n=\frac{11614.4}{d^5}=\frac{11614.4}{(4.248)^5}=8.396 \text{ turns}$$

Therefore, Wire diameter $=4.248$ mm

Mean coil diameter $=40.248$ mm

Number of turns $=8.396$.

Problem 14.5. A close coiled helical springs is to have a stiffness of 800 N/m in compression with a maximum load of 40 N and a maximum shearing stress of 105 N/mm². The solid length of the spring (*i.e.*, coils touching) is 50 mm. Find the wire diameter, mean coil radius and number of coils. $G=40{,}000$ N/mm².

Solution. Stiffness, $$k=\frac{800 \text{ N}}{m}=0.8 \text{ N/mm}$$

$$=\frac{Gd^4}{64nR^3}, \text{ or } 0.8=\frac{40{,}000\times d^4}{64\,n\,R^3}$$

but $nd=50$ mm (solid length)

$$n=\frac{50}{d}$$

Therefore, $$0.8=\frac{40000\times d^4}{64R^3\times 50}\times d, \quad \text{or } 0.064=\frac{d^5}{R^3} \quad \ldots(1)$$

Shear stress, $$q=\frac{16WR}{\pi d^3}=\frac{16\times 40\times R}{\pi d^3}=105$$

or $$\frac{R}{d^3}=\frac{105\times\pi}{640}=0.5154$$

or $$\frac{R^3}{d^9}=(0.5154)^3 \quad \ldots(2)$$

From equations (1) and (2)

$$(0.064)(0.5154)^3=\frac{1}{d^4}$$

or $$d^4=114.126 \text{ mm}^4$$

Wire diameter, $d=3.268$ mm

Number of coils, $$n=\frac{50}{d}=15.3$$

Mean coil diameter $D=2d^3\times 0.5154$ (from equation (2))

$=35.98$ mm

Problem 14.6. A close coiled helical steel spring has 25 turns, the mean radius of the coils is 6 cm while the diameter of the wire is 12 mm. Find the work done in rotating one end of the spring by 90° relative to the other end (fixed end), by a couple whose axis coincides with the axis of the spring. E for steel $=210\times 10^3$ N/mm².

Solution.

Mean coil radius, $R=60$ mm

Wire diameter, $d=12$ mm

Number of turns, $n=25$

Moment of Inertia, $I=\frac{\pi d^4}{64}=\frac{\pi\times 12^4}{64}=1017{\cdot}88\ \text{mm}^4$

Length of the wire, $l=2\pi nR=2\times\pi\times 25\times 60=9424{\cdot}8$ mm

Angular rotation, $\phi=90°=\frac{\pi}{2}$ radians

$$\phi=\frac{Ml}{EI}$$

$$M=\frac{EI\phi}{l}=\frac{\pi}{2}\times\frac{210\times 10^3\times 1017{\cdot}88}{9424{\cdot}8}$$

$$=35{\cdot}626\times 10^3\ \text{Nmm}=35{\cdot}626\ \text{Nm}$$

Work done on the spring,

$$U=\frac{1}{2}\ M\phi=\frac{1}{2}\times 35{\cdot}626\times\frac{\pi}{2}=27{\cdot}980\ \text{Nm.}$$

Problem 14·7. A weight of 250 N is dropped on to a close coiled helical spring through a height of 800 mm which produces a maximum instantaneous stress of 200 N/mm² in the spring. If the mean radius of the coil is 5 times the wire diameter determine (*a*) the instantaneous compression in the spring, (*b*) number of coils in the spring. Given wire diameter 20 mm. G for steel$=84\times 1000$ N/mm².

Solution.

The falling weight $W=250$ N

Height, $h=800$ mm

Maxm. instantaneous stress, $q=200$ N/mm³

Wire diameter, $d=20$ mm

Mean coil radius, $R=5\times 20=100$ mm.

Say W_e=equivalent static load on the spring

$$W_e\times R=\frac{\pi d^3 q}{16}$$

or

$$W_e=\frac{\pi\times 20^3\times 200}{16\times 100}=1000\,\pi\ \text{N}$$

Say the instantaneous compression in the spring$=\delta$ mm.

Then potential energy lost by falling weight=Energy stored in the spring

$$W(h+\delta)=\tfrac{1}{2}W_e\,.\,\delta$$

$$250(800+\delta)=\tfrac{1}{2}\times 1000\pi\times\delta$$

$$800+\delta=2\pi\delta$$

Instantaneous compression in spring,

$$\delta=151{\cdot}4\ \text{mm}$$

Moreover $$\delta=\frac{64nW_eR^3}{Gd^4}$$

$$151{\cdot}4=\frac{64\times n\times 1000\pi\times 100^3}{84000\times 20^4} \quad \text{where } n=\text{number of coils.}$$

$$=14{\cdot}96\ n$$

Number of coils, $n=10{\cdot}12.$

Problem 14·8. A close coiled cylindrical helical spring is of 80 mm mean coil diameter. The spring extends by 40 mm when axially loaded by a weight of 530 N. And when it is subjected to an axial couple, $M=2{\cdot}80\times10^4$ Nmm, there is an angular rotation of 60°. Determine the Poisson's ratio for the material of the spring.

Solution. Mean coil radius, $R=40$ mm

Axial load, $W=530$ N

Axial deflection, $\delta=40$ mm

Axial moment, $M=2{\cdot}8\times10^4$ Nmm

Angular rotation, $\phi=60°=\frac{\pi}{3}$ radians.

Now $$\frac{32\,WR}{\pi d^4}=\frac{G\,\delta}{Rl}$$

or $$G=\frac{32\,WR^2l}{\pi d^4\,\delta}=\frac{32\times530\times40^2\times l}{40\times\pi\,d^4}$$

$$=6{\cdot}784\times10^5\times\frac{l}{\pi d^4}$$

and $$E=\frac{Ml}{I\phi} \qquad \text{where } I=\frac{\pi d^4}{64}$$

$$=\frac{2{\cdot}8\times10^4\times l\times64\times3}{\pi d^4\times\pi}=17{\cdot}112\times10^5\times\frac{l}{\pi d^4}$$

So $$\frac{E}{G}=\frac{17{\cdot}112}{6{\cdot}784}=2{\cdot}5224=2\left(1+\frac{1}{m}\right)$$

or $\frac{1}{m}$, Poisson's ratio$=0{\cdot}2612$

Problem 14·9. Design a close helical spring to a have a mean coil diameter 120 mm and an axial deflection of 150 mm under an axial load of 4050 N, so that the maximum shear stress developed in the spring is not to exceed 320 N/mm². Steel wires are available in the following diameters :

10, 12, 14, 16 mm

Determine the most suitable diameter of the wire and the number of coils required. Calculate also the maximum shear stress developed in the spring.

G for steel $=84000$ N/mm²

Solution. Axial load, $W=4050$ N

Mean coil radius, $R=60$ mm

q, max. allowable stress$=320$ N/mm²

Say wire diameters $=d$ mm

$$d^3=\frac{16\ WR}{\pi q}=\frac{16\times 4050\times 60}{\pi\times 320}=3867{\cdot}456\ \text{mm}^3$$

$d=15{\cdot}7$ mm say 16 mm

Axial deflection, $\delta=150$ mm

Number of coils, $$n=\frac{G\delta d^4}{64WR^3}=\frac{84000\times 150\times 16^4}{64\times 4050\times 60^3}=14{\cdot}75$$

Maximum shear stress developed,

$$q=\frac{16WR}{\pi d^3}$$

(neglecting the effect of spring index and direct hear)

$$=\frac{16\times 4050\times 60}{\pi\times 16^3}=302{\cdot}145\ \text{N/mm}^2$$

Problem 14·10. In designing a valve spring it is estimated that the mass of the valve is 1 kgm and it requires an acceleration of 150 m/sec² when lifting through a height of 0·5 cm. The free length of the spring is 20 cm and the axial length of the spring is 16 cm when the valve is shut. If the total valve lift is 1 cm, determine the maximum force on the spring.

The diameter of the spring wire is 3 mm and the maximum shear stress is not to exceed 3 tonnes/cm², determine the mean coil diameter and the number of coils.

Given, G for steel$=840$ tonnes/cm²

Solution.

Mass of the valve, $m=1$ kg

Acceleration, $a=150$ m/sec²

Acceleration due to gravity,

$g=9{\cdot}8$ m/sec²

Force on the valve $$=\frac{1\times 150}{9{\cdot}8}=15{\cdot}30\ \text{kg}$$

Total valve lift $=1$ cm

Valve lift during opening and closing the valve$=0{\cdot}5$ cm
Free length of the spring$=20$ cm
Axial length when valve is shut$=16$ cm
Initial compression in the length of the spring$=4$ cm
Further compression when the valve is shut$=0{\cdot}5$ cm
δ, Total change in the length$=4{\cdot}5$ cm for a force of 15·3 kg

Stiffness of the spring, $$k=\frac{F}{\delta}=\frac{15{\cdot}3}{4{\cdot}5}=3{\cdot}4\ \text{kg/cm}$$

Maximum valve lift $=1$ cm
Maximum change in the length of the spring

$=4+1=5$ cm

W_{max}, Maximum force on the spring,

$$=\frac{15{\cdot}3}{4{\cdot}5}\times 5=17{\cdot}0 \text{ kg}$$

Wire diameter, $d=3$ mm$=0{\cdot}3$ cm

q, Maximum shear stress$=3T/\text{cm}^2=3000$ kg/cm^2

$$=\frac{16\, W_{max}.R}{\pi d^3}$$

or R, mean coil radius $=\dfrac{3000\times\pi\times{\cdot}3^3}{16\times 17}=0{\cdot}935$ cm

Mean coil diameter, $D=2R=1{\cdot}870$ cm

Number of coils, $n=\dfrac{Gd^4}{64R^3}\times\dfrac{\delta}{W}$

$$=\frac{8{\cdot}4\times 10^5\times(0{\cdot}3)^4\times 4{\cdot}5}{64\times(0{\cdot}935)^3\times 15{\cdot}3}=38{\cdot}2$$

Problem 14·11. A close coiled helical spring of 17 mm mean coil diameter and 10 turns is arranged within and concentric with an outer spring. The free length of the inner spring is 5 mm more than that of the outer. The outer spring has 12 coils of mean diameter 25 mm and wire diameter 3 mm. The spring load against which a value is opened is provided by the inner spring. The initial compression in outer spring is 5 mm when the valve is closed. Find the stiffness of the inner spring if the greatest force required to open the valve by 8 mm is 130 N. Find also the wire diameter of the inner spring. $G=80{,}000$ N/mm^2

Solution.

Initial compression in outer spring$=5$ mm

Initial compression in inner spring$=5+5=10$ mm

(Since the free length of the inner spring is 5 mm more than the free length of the outer spring.)

Say k_1=stiffness of inner spring in N/mm

k_2=stiffness of outer spring in N/mm

F_1, Initial load on valve $=10k_1+5k_2$ ...(1)

Stiffness of outer spring,

$$k_2=\frac{Cd_2^4}{8\,D_2^3 n_2} \quad \text{where} \quad d_2=3 \text{ mm}$$

$$D_2=25 \text{ mm}, \qquad n_2=12$$

$$k_2=\frac{80000\times 3^4}{8\times 25^3\times 12}=4{\cdot}32 \text{ N/mm}$$

The valve is to be opened by 8 mm, additional force required to open the valve

$$F_2=8k_1+8k_2 \qquad ...(2)$$

Total load to lift the valve by 8 mm

$$F=F_1+F_2=18\,k_1+13\,k_2=130 \text{ N} \qquad ...(3)$$

or $18\,k_1+13\times 4{\cdot}32=130$

Stiffness of inner spring,

$$k_1=\frac{130-13\times4{\cdot}32}{18}=4{\cdot}10\text{ N/mm}$$

Now $$k_1=\frac{Gd_1^4}{8D_1^3n_1}=\frac{80000\times d_1^4}{8\times17^3\times10}$$

or $$d_1^4=\frac{8\times17^3\times10}{80000}\times4{\cdot}10=20{\cdot}1433\text{ mm}^4$$

Wire diameter of inner spring,

$$d_1=2{\cdot}118\text{ mm}$$

Problem 14·12. In a compound helical spring, the inner spring is arranged within and concentric with the outer one, but is 8 mm shorter in length. The outer spring has ten coils of mean diameter 25 mm and the wire diameter 3 mm. Find the stiffness of the inner spring if an axial load of 120 N causes the outer spring to compress by 18 mm.

If the radial clearance between the springs is 1·5 mm, find the wire diameter of the inner spring, if it has 9 coils. $G=77\times10^3$ N/mm²

Solution.

Outer spring $D=25$ mm, $R=12{\cdot}5$ mm

$d=3$ mm, $n=10$

$G=77\times10^3$ N/mm²

Compression, $\delta=18$ mm

Load required, $$W=\frac{Gd^4\delta}{64\,nR^3}=\frac{77\times10^3\times3^4\times18}{64\times10\times12{\cdot}5^3}=89{\cdot}8\text{ N}$$

Inner spring

Load shared by the inner spring,

$$W'=120-89{\cdot}8,\qquad W'=30{\cdot}2\text{ N}$$

Compression, $\delta=18-8=10$ mm

Number of coils $=9$

Say the wire diameter $=d$ mm

Now mean coil radius of outer spring $=12{\cdot}5$ mm

Wire diameter $=3$ mm

Inner coil radius of outer spring $=9{\cdot}5$ mm

Radial clearance between the two springs $=1{\cdot}5$ mm

Therefore, outer radius of inner spring $=9{\cdot}5-1{\cdot}5=8$ mm

Mean coil radius, $$R'=\left(8-\frac{d}{2}\right)\text{ mm}$$

$$W'=\frac{77\times10^3\times d^4\times10}{64\times9\times\left(8-\frac{d}{2}\right)^3},\text{ or }30{\cdot}2=\frac{770000\,d^4}{576\left(8-\frac{d}{2}\right)^3}$$

$$\left(8-\frac{d}{2}\right)^3=44{\cdot}265\,d^4$$

$$(16-d)^3=8\times44{\cdot}265\,d^4,\qquad\text{or }(16-d)^3=354{\cdot}12\,d^4$$

Wire diameter, $$d=1{\cdot}696\text{ mm}$$

Problem 14·13. A composite spring has two close coiled helical springs in series. The mean coil radius of each spring is 10 cm. The wire diameter of one spring is 2·5 cm and it has 20 coils, while the number of turns in the other spring is 15. Determine the wire diameter of the other spring if the stiffness of the composite spring is 1·2 kg/cm.

Calculate the greatest axial laod which can be applied on the composite spring if the maximum shearing stress is not to exceed 3 tonnes/cm². $G=840$ tonnes/cm²

Solution. Stiffness of first spring,

$$k_1=\frac{W}{\delta_1}=\frac{G\pi d_1^4}{32R^2l}$$

$$=\frac{840\times 1000\times \pi\times 2\cdot 5^4}{32\times 10^2\times 2\pi\times 20\times 10}$$

$$=25\cdot 635 \text{ kg/cm}$$

Stiffness of second spring,

$$k_2=\frac{W}{\delta_2}=\frac{G\pi d_2^4}{32\ R^2 l_2}$$

$$=\frac{840\times 1000\times \pi d_2^4}{32\times 10^2\times 2\pi\times 15\times 10}$$

$$=0\cdot 875\ d_2^4$$

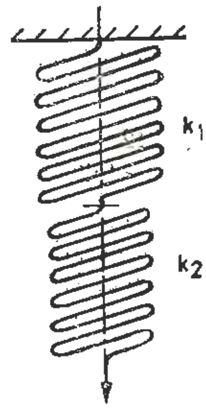


Fig. 14·15

Since the springs are connected in series as shown in the Fig. 14·15.

Stiffness of the composite spring,

$$k=\frac{k_1k_2}{k_1+k_2} \quad \text{or} \quad 1\cdot 2=\frac{25\cdot 635\times \cdot 875\ d^4{}_2}{25\cdot 635+0\cdot 875\ d_2^4}$$

or $$1\cdot 2\times 25\cdot 635+1\cdot 2\times 0\cdot 875\ d_2^4=25\cdot 635\times 0\cdot 875\ d_2^4$$

$$1\cdot 2\times 25\cdot 635=d_2^4(22\cdot 43-1\cdot 05)$$

$$d_2^4=\frac{1\cdot 2\times 25\cdot 635}{21\cdot 38}=1\cdot 4388$$

Diameter of the spring wire,

$$d_2=1\cdot 095 \text{ cm}$$

The shearing stress will be maximum in the spring of thin wire *i.e.*, in the spring with wire diameter 1·095 cm.

$$WR=\frac{\pi}{16}d_2^3\times q \quad \text{(neglecting the effect of spring index)}$$

$$W\times 10=\frac{\pi}{16}\times(1\cdot 095)^3\times 3000$$

Axial load, $$W=77\cdot 34 \text{ kg}$$

Problem 14·14. A rigid bar *AB* weighing 100 N and carrying a load *W* equal to 300 N rests on 3 springs as shown in Fig. 14·16 having the spring constants $k_1=20$ N/mm, $k_2=8$ N/mm and $k_3=10$ N/mm. If the unloaded springs were of the same length, determine the value of the distance x such that the bar remains horizontal.

Solution. Since the bar is to remain horizontal, there will be equal deflection (compression) in each spring.

Say the deflection in each spring $=\delta$

Then reactions at the springs will be $k_1\delta$, $k_2\delta$ and $k_3\delta$ respectively.

Taking moments of the forces about the point A

$$300(350-x)+100\times350=k_2\delta\times350+k_3\delta\times700$$

or $$3(350-x)+350=3{\cdot}5\ k_2\delta+7k_3\delta$$

Substituting the values of k_2 and k_3

$$1050-3x+350=3{\cdot}5\times8\delta+7\times10\ \delta$$

$$1400-3x=98\ \delta \quad \ldots(i)$$

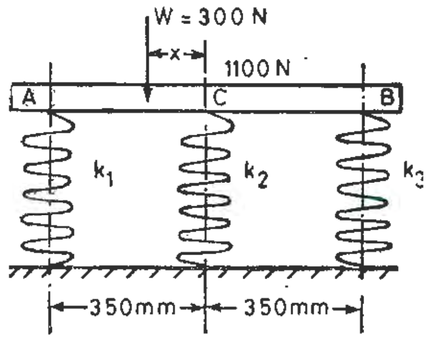


Fig. 14·16

Taking moments about B,

$$300(350+x)+100\times350=k_1\delta\times700+k_2\delta\times350$$

or $$3(350+n)+350=7k_1\delta+3{\cdot}5\ k_2\delta$$

$$1400+3x=7\times20\ \delta+3{\cdot}5\times8\ \delta=168\ \delta \quad \ldots(ii)$$

Adding the equations (i) and (ii), give

$$266\ \delta=2800$$

$$\delta=10{\cdot}5263\ \text{mm}$$

Substituting the value of δ in equation (2)

$$3x=168\times10{\cdot}5263-1400=368{\cdot}42$$

$$x=122{\cdot}80\ \text{mm}$$

Problem 14·15. Two close coiled helical springs of equal axial lengths are assembled co-axially. The wire diameter of outer springs is 1 cm and the mean coil radius is 4 cm, while the wire diameter of the inner inner spring is 0·8 cm and the mean coil radius is 3 cm. The assembly of the springs is compressed by an axial thrust of 50 kg. Calculate the maximum shear stress induced in each spring if both the springs are made of steel and the number of coils in each spring is the same.

Solution. The two springs are assembled co-axially or in other words they are parallel to each other and load will be shared by them.

Total thrust, $W=50$ kg.

Outer Spring

Say the axial thrust shared by outer spring $=W_1$

Mean coil radius, $R_1=4$ cm

Wire diameter, $d_1=1{\cdot}0$ cm

Axial compression, $=\delta_1$ (say)

$$\delta_1=\frac{64n_1W_1R_1^3}{Gd_1^4}$$

where n_1 = number of coils in the outer spring

Inner Spring

Similarly for the inner spring axial compression,

$$\delta_2=\frac{64n_2W_2R_2^3}{Gd_2^4}$$

where n_2=number of coils in inner spring
W_2=load shared by inner spring
R_2=3 cm, mean coil radius
d_2=0·8 cm, wire diameter.

But since the springs are in parallel $\delta_1=\delta_2$

$$\frac{64n_1W_1R_1^3}{Gd_1^4}=\frac{64n_2W_2R_2^3}{Gd_2^4}$$

or
$$\frac{W_1}{W_2}=\frac{R_2^3}{R_1^3}\times\frac{d_1^4}{d_2^4} \text{ as } n_1=n_2$$

$$=\frac{3^3}{4^3}\times\frac{1^4}{0{\cdot}8^4}=\frac{27}{26{\cdot}2144}=1{\cdot}03$$

$$W_1+W_2=50 \text{ kg}$$

$$1{\cdot}03\,W_2+W_2=50 \text{ kg}$$

$$W_2=24{\cdot}63 \text{ kg}$$

$$W_1=50-24{\cdot}63=25{\cdot}37 \text{ kg}$$

Stresses in the springs (neglecting the effect of spring index)

Outer spring, $q_1=\dfrac{16W_1R_1}{\pi d_1^3}=\dfrac{16\times25{\cdot}37\times4}{\pi\times1}$

Shear stress, $q_1=516{\cdot}83$ kg/cm²

Inner Spring Shear stress,

$$q_2=\frac{16W_2R_2}{\pi d_2^3}=\frac{16\times24{\cdot}63\times3}{\pi\times0{\cdot}8^3}=734{\cdot}99 \text{ kg/cm}^2.$$

Problem 14·16. The mean coil diameter of an open coiled helical spring is D, the wire diameter belng d and the coils are inclined at an angle α. Calculate the percentage error while determining the stiffness of the spring if the inclination of the coils is neglected. Given $\alpha=20°$, $E=2{\cdot}5\,G$.

Solution. Axial deflection of an open coiled helical spring,

$$\delta=2\pi nR^3W\sec\alpha\left[\frac{\cos^2\alpha}{GJ}+\frac{\sin^2\alpha}{EI}\right]$$

$$I=\frac{\pi d^4}{64},\quad J=\frac{\pi d^4}{32}$$

or
$$\delta=\frac{64nWR^3\sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G}+\frac{2\sin^2\alpha}{E}\right]$$

where n=number of coils

R=mean coil radius$=\dfrac{D}{2}$

d=wire diameter

$\alpha=20°$, $\sin\alpha=0{\cdot}342$, $\sin^2\alpha=0{\cdot}117$

$\cos\alpha=0{\cdot}94$ $\cos^2\alpha=0{\cdot}883$

But $2{\cdot}5\,G=E$

$$\frac{\delta}{W}=\frac{8D^3n}{d^4}\times\frac{1}{0{\cdot}9^4}\left[\frac{0{\cdot}883}{G}+\frac{2\times0{\cdot}117}{2{\cdot}5\,G}\right]$$

$$=\frac{8D^3n}{Gd^4}\times1{\cdot}0389$$

or $$\frac{W}{\delta}=\frac{Gd^4}{8D^3n}\times\frac{0{\cdot}962}{1}=k,\text{ stiffness of the spring}$$

Stiffness of the spring when α is neglected, $k'=\frac{GD^4}{8D^3n}$

Percentage error $$=\frac{k-k'}{k}\times100=\frac{1-0{\cdot}962}{1}\times100=3{\cdot}8\%.$$

Problem 14·17. An open coiled helical spring made of a round steel bar 1 cm diameter has 10 coils of 8 cm mean diameter and the pitch is 6 cm. If the axial moment is 40×10^3 Nmm, find the deflection and the maximum bending and shear stresses developed.

E for steel$=210\times1000$ N/mm²

Poisson's rate, $\frac{1}{m}=0{\cdot}28.$

Solution. p, Axial pitch of helix$=6$ cm

Mean coil diameter, $D=8$ cm

Say $\alpha=$helix angle

then $$\tan\alpha=\frac{p}{D}=\frac{6}{8}=0{\cdot}75$$

or $$\alpha=36°\ 51'$$

$$\sin\alpha=0{\cdot}6\qquad \sin^2\alpha=0{\cdot}36$$

$$\cos\alpha=0{\cdot}8\qquad \cos^2\alpha=0{\cdot}64$$

Number of coils, $n=10$

Mean coil radius, $R=4$ cm$=40$ mm

Axial moment, $M=40\times10^3$ Nmm

Wire diameter, $d=10$ mm

$E=210\times10^3$ N/mm²

$$G=\frac{E}{2\left(1+\frac{1}{m}\right)}=\frac{210}{2\times1{\cdot}28}\times1000$$

$$=82{\cdot}0\times10^3\ \text{N/mm}^2$$

Deflection in the spring, $\delta=2\pi nMR^2\sin\alpha\left[\frac{1}{EI}-\frac{1}{GJ}\right]$

$$\text{but } I=\frac{\pi d^4}{64},\ J=\frac{\pi d^4}{32}$$

$$\delta=\frac{64MR^2n}{d^4}\left[\frac{2}{E}-\frac{1}{G}\right]\sin\alpha$$

Substituting the values $\delta=\frac{64\times40\times10^3\times40^2\times10}{(10)^4}\left[\frac{2}{210{,}000}-\frac{1}{82{,}000}\right]\times0{\cdot}6$

$$=\frac{0{\cdot}6\times64\times40^3}{10^5}\left[\frac{2}{2{\cdot}1}-\frac{1}{0{\cdot}82}\right]=6{\cdot}56 \text{ mm.}$$

Bending Moment, $M'=M\cos\alpha=40\times0{\cdot}8\times10^3=32\times10^3$ Nmm

$$=\frac{\pi d^3}{32}\times f$$

f, maximum stress due to bending

$$=\frac{32M'}{\pi d^3}=\frac{32\times32\times10^3}{\pi\times10^3}=325{\cdot}95 \text{ N/mm}^2$$

Twisting moment, $T'=M\sin\alpha=40\times0{\cdot}6\times10^3=24\times10^3$ Nmm

$$=\frac{\pi d^3}{16}\times q$$

Maximum shear stress, $q=\frac{16\times24\times10^3}{\pi\times10^3}=122{\cdot}23$ N/mm².

Problem 14·18. In an open coiled helical spring made of steel, the stresses due to bending and twisting are 980 kg/cm² and 1050 kg/cm² respectively when the spring carries an axial load. There are 10 coils in the spring and the mean coil radius is 5 times the diameter of the wire. Determine the permissible axial load and the wire diameter if the extension in the spring is 1·6 cm.

E for steel$=2100$ tonnes/cm²

G for steel$=840$ tonnes/cm²

Solution.

Number of coils, $n=10$

Mean coil radius, $R=5\,d$ where $d=$wire diameter

Axial deflection, $\delta=1{\cdot}6$ cm

Say $W=$axial load

Torque, $T'=WR\cos\alpha$

Bending moment, $M'=WR\sin\alpha$

Now $T'=WR\cos\alpha=\frac{\pi d^3}{16}\times q$ where $q=1050$ kg/cm³

$M'=WR\sin\alpha=\frac{\pi d^3}{32}\times f$ where $f=980$ kg/cm²

So $\frac{f}{2q}=\tan\alpha$

or $\tan\alpha=\frac{980}{2\times1050}=0{\cdot}462$ or $\alpha=25°$

and $\sin\alpha=0{\cdot}423$ $\sin^2\alpha=0{\cdot}178$

$\cos\alpha=0{\cdot}907$ $\cos^2\alpha=0{\cdot}822$

Moreover, $$T'^2+M'^2=\left(\frac{\pi d^3}{32}\right)^2 [4q^2+f^2]$$

$$W^2R^2 \cos^2\alpha+W^2R^2 \sin^2\alpha=\left(\frac{\pi d^3}{32}\right)^2 [4q^2+f^2]$$

or $$WR=\frac{\pi d^3}{32}\times\sqrt{4q^2+f^2}$$

$$W\times 5d=\frac{\pi d^3}{32}\sqrt{4\times 1050^2+980^2}$$

$$W=\frac{\pi d^2}{160}\times 1000\sqrt{4{\cdot}41+{\cdot}9604}$$

$$=\frac{\pi d^2\times 1000\times 2{\cdot}3174}{160}=45{\cdot}50\ d^2$$

$$\delta=\frac{W\times 2\pi\times 10\times R^3}{0{\cdot}907}\left[\frac{0{\cdot}822}{G\pi d^4}\times 32+\frac{0{\cdot}178}{E\times\pi d^4}\times 64\right]$$

$$=\frac{20\ \pi\ W}{{\cdot}907}\times(5d)^3\left(\frac{32}{\pi d^4}\right)\left[\frac{0{\cdot}822}{840\times 1000}+\frac{0{\cdot}356}{2100\times 1000}\right]$$

$$=\frac{2500\times 32\times 45{\cdot}5\ d^2}{0{\cdot}907\ d}\left[\frac{0{\cdot}822}{0{\cdot}84}+\frac{0{\cdot}356}{2{\cdot}1}\right]\times 10^{6}$$

$$=4{\cdot}013\ d\,[0{\cdot}978+0{\cdot}169]=4{\cdot}60\ d=1{\cdot}6\text{ cm}$$

Wire diameter, $$d=\frac{1{\cdot}6}{4{\cdot}6}=0{\cdot}347\text{ cm}$$

Axial load, $$W=45{\cdot}5\ d^2=45{\cdot}5\times 0{\cdot}347^2=5{\cdot}48\text{ kg}$$

Problem 14·19. An open coiled helical spring fits loosely on two shafts and its ends are connected to the shafts which are prevented from any axial movement and are co-axial with the spring. Show that if the coils of the spring are inclined at 45° to the axis, the couple per unit angle of twist is given by

$$\frac{d^4}{64\sqrt{2}nD}\left[\frac{E}{2}+G\right]$$

where d=wire diameter

n=number of coils

D=mean coil diameter

E=Modulus of elasticity

G=Modulus of rigidity

Solution. Helix angle $\alpha=45°$

$$\sin\alpha=\cos\alpha=\frac{1}{\sqrt{2}}\quad\text{and}\quad \sin^2\alpha=\cos^2\alpha=\frac{1}{2}$$

Since the axial movement of the connecting shafts is prevented which means when an axial couple acts on the spring, its axial movement is prevented *i.e.*, its axial extension or contraction is zero *i.e.*, an axial reaction acts on the spring to make its deflection zero, Say the axial reaction $=W$

Axial couple applied $=M$

Angular rotation ϕ_1, due to M

$$=\frac{32\,MDn\sec\alpha}{d^4}\left[\frac{2\cos^2\alpha}{E}+\frac{\sin^2\alpha}{G}\right]$$

Angular rotation ϕ_2, due to W

$$=\frac{16WnD^2\sin\alpha}{d^4}\left[\frac{1}{G}-\frac{2}{E}\right]$$

But $\phi=\phi_1+\phi_2$

Axial deflection δ_1, due to M

$$=\frac{16\,MD^2n\sin\alpha}{d^4}\left[\frac{1}{G}-\frac{2}{E}\right]$$

Axial deflection δ_2, due to W

$$=\frac{8WD^3n\sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G}+\frac{2\sin^2\alpha}{E}\right]$$

But $\delta_1+\delta_2=0$ (as given in the problem)

So $\frac{8D^2n}{d^4}\left[2\,M\sin\alpha\left(\frac{1}{G}-\frac{2}{E}\right)+MD\sec\alpha\left(\frac{\cos^3\alpha}{G}+\frac{2\sin^2\alpha}{E}\right)\right]=0$

Substituting the values of $\sin\alpha$, $\cos\alpha$ etc.

$$\frac{2M}{\sqrt{2}}\left(\frac{1}{G}-\frac{2}{E}\right)+WD\sqrt{2}\left(\frac{1}{2G}+\frac{1}{E}\right)=0$$

$$M\left(\frac{E-2G}{GE}\right)+WD\left(\frac{E+2G}{2GE}\right)=0$$

or $$WD=-\frac{M(E-2G)}{E+2G}\times 2$$

$$=2M\left[\frac{2G-E}{2G+E}\right]$$

Now total angular rotation ϕ is given by

$$\phi=\frac{32\,MDn\sqrt{2}}{d^4}\left[\frac{1}{E}+\frac{1}{2G}\right]+\frac{16WnD^2}{d^4}\times\frac{1}{\sqrt{2}}\left[\frac{1}{G}-\frac{2}{E}\right]$$

Substituting the value of WD,

$$\phi=\frac{32\,MDn\sqrt{2}}{d^4}\left[\frac{1}{E}+\frac{1}{2G}\right]+\frac{16WnD^2}{\sqrt{2}d^4}$$

$$\times 2M\left[\frac{2G-E}{2G+E}\right]\left[\frac{1}{G}-\frac{2}{E}\right]$$

$$=\frac{16\,MnD\sqrt{2}}{d^4}\left[\frac{2}{E}+\frac{1}{G}+\left(\frac{1}{G}-\frac{2}{E}\right)\left(1-\frac{2E}{2G+E}\right)\right]$$

$$=\frac{16MnD\sqrt{2}}{d^4}\left[\frac{2}{E}+\frac{1}{G}+\frac{1}{G}-\frac{2}{E}-\left(\frac{1}{G}-\frac{2}{E}\right)\left(\frac{2E}{2G+E}\right)\right]$$

$$=\frac{16MnD\sqrt{2}}{d^4}\left[\frac{2}{G}-\frac{2E}{G(2G+E)}+\frac{4}{2G+E}\right]$$

$$=\frac{32MnD\sqrt{2}}{d^4}\left[\frac{2}{2G+E}+\frac{2}{2G+E}\right]$$

$$=\frac{64\,Mn\,D\sqrt{2}}{d^4}\times\frac{2}{2G+E}$$

or $$\frac{M}{\phi}=\text{couple per unit angle of twist}$$

$$=\frac{d^4(2G+E)}{2\times64\sqrt{2}Dn}=\frac{d^4\left(G+\frac{E}{2}\right)}{64\sqrt{2}Dn}$$

Problem 14·20. Determine the length of the steel strip 20 mm wide by 0·5 mm thick of a flat spiral spring to store 10,000 Nmm of energy for a maximum bending stress of 300 N/mm². Calculate also the torque required at the winding spindle and the number of turns to to wind up the spring. E for steel=210 GN/m².

Solution.

Maximum bending stress,

$$f_{max}=300\text{ N/mm}^2=\frac{M_{max}}{Z}$$

$$Z=\text{section modulus}=\frac{bt^2}{6}=\frac{20\times(0{\cdot}5)^2}{6}=\frac{5}{6}\text{ mm}^3$$

Therefore $$M_{max}=300\times\frac{5}{6}=250\text{ Nmm}$$

Twisting moment at the winding spindle,

$$M=\frac{M_{max}}{2}=125\text{ Nmm}$$

Angular rotation of strip $=\phi=2\pi n$

where n=number of winding turns

Energy stored $=\frac{1}{2}M\phi$

$$10{,}000=\tfrac{1}{2}M\times2\pi n=\pi nM=\pi\times n\times125$$

Number of winding turns, $n=\dfrac{10000}{\pi\times125}=25{\cdot}46$

Length of the strip, $l=\dfrac{\phi EI}{M}$, where $\phi=2\pi n$

$$I=\frac{bt^3}{12}=\frac{20}{12}\times\left(\frac{1}{2}\right)^3=\frac{20}{96}\text{ mm}^4$$

$$E=210\times10^9\text{ N/m}^2=210\times10^3\text{ N/mm}^2$$

So, $$l=\frac{2\pi\times25{\cdot}46\times210\times1000\times20}{125\times96}$$

$$=55989\text{ mm}=55{\cdot}989\text{ metres.}$$

Problem 14·21. A semi elliptical carriage spring made of steel leaves, 100 cm long is to support a central load of 800 kg with a maximum deflection of 6 cm and a maximum bending stress of 3200 kg/cm².

Calculate the thickness of the leaves and decide their number and breadth.

$$E=2000 \text{ tonnes/cm}^2$$

Solution.

Length of the spring, $l=100$ cm

Central load, $W=800$ kg

Central deflection, $\delta=6$ cm

Maximum bending stress $=3200$ kg/cm²

Now $$\delta=\frac{3Wl^3}{8Enbt^3}$$

$$nbt^3=\frac{3Wl^3}{8E\delta}=\frac{3\times800\times100^3}{8\times2000\times1000\times6}$$

$$=25\cdot0 \quad \text{and} \quad f=\frac{3Wl}{2nbt^2}$$

So $$nbt^2=\frac{3Wl}{2f}=\frac{3\times800\times100}{2\times3200}=37\cdot5$$

Thickness of the leaves, $t=\dfrac{25}{37\cdot5}=0\cdot666$ cm

Moreover $$nb=\frac{37\cdot5}{t^2}=\frac{37\cdot5}{(0\cdot666)^2}=84\cdot54$$

Say the number of leaves $=10$

Breadth $b=8\cdot454$ cm.

Problem 14·22. A semi elliptical laminated steel spring, length 80 cm is to carry a central load of 5000 N. Determine the number, breadth and thickness of the leaves, if the central deflection is 60 mm. Assume that breadth is 10 times the thickness. The leaves are available of breadths in multiples of 5 mm and thickness in multiples of mm. The maximum bending stress is limited to 360 N/mm². $E_{steel}=210$ kN/mm².

Solution.

Central load, $W=5000$ N

Length $l=800$ mm

$E=210\times1000$ N/mm²

Central deflection, $$\delta=60 \text{ mm}=\frac{3Wl^3}{8Enbt^3}$$

or $$nbt^3=\frac{3\times5000\times800^3}{8\times210\times1000\times60}=7\cdot619\times10^4$$

Bending stress, $$f=\frac{3Wl}{2nbt^2}$$

$$360=\frac{3\times5000\times800}{2nbt^2} \quad \text{or } nbt^2=16\cdot666\times10^3$$

So $$\frac{nbt^3}{nbt^2}=t=\frac{7\cdot169\times10^4}{16\cdot666\times10^3}=4\cdot30 \text{ mm}$$

Let the thickness, $t=5$ mm

Breadth, $b=10\ t=50$ mm

Number of leaves, $n=\dfrac{16\cdot666\times10^3}{50\times5^2}=13\cdot33$

Say number of leaves, $n=14$

Problem 14·23. A laminated carriage spring made of 12 steel plates is 1 m long. The maximum central load is 6 kN. If the maximum allowable stress in steel is 200 MN/m² and the maximum deflection is 40 mm, determine the thickness and width of the plates.

$$E_{steel}=200\ \text{GN/m}^2=200\times1000\ \text{N/mm}^2.$$

Solution.

Say, width of the plates $=b$

Thickness of the plates $=t$

Number of plates, $n=12$

Allowable stress, $f=200\ \text{MN/m}^2=200\ \text{N/mm}^2$

Length of the spring, $l=1\ \text{m}=1000$ mm

Maximum central load,

$$W_0=6\ \text{kN}=6000\ \text{N}$$

Maximum bending moment,

$$M_{max}=\frac{Wl}{4}=\frac{6000\times1000}{4}=1\cdot5\times10^6\ \text{Nmm}$$

$$M_{max}=f\times\frac{nbt^2}{6}$$

or $$1\cdot5\times10^6=\frac{200\times12}{6}\times bt^2$$

or $$bt^2=3750 \qquad \text{...(1)}$$

Maximum central deflection,

$$\delta=40\ \text{mm}=\frac{3\ W_0l^3}{8Enbt^3}$$

$$40=\frac{3\times6000\times(1000)^3}{8\times200\times1000\times12\times bt^3}$$

or $$40=\frac{18\times10^7}{192\ .\ bt^3}$$

or $$bt^3=\frac{18\times10^7}{192\times40}=2\cdot34375\times10^4 \qquad \text{...(2)}$$

From equations (1) and (2)

t, thickness of the plates $=\dfrac{2\cdot34375\times10^4}{3750}=6\cdot25$ mm

b, width of the plates $=\dfrac{3750}{(6\cdot25)^2}=96$ mm

SUMMARY

1. For a close coiled helical spring, mean coil radius R, wire diameter d subjected to axial load W

Maximum shear stress, $q_{max}=\frac{16WR}{\pi d^3}\left(\frac{4k'-1}{4k'-4}+\frac{0{\cdot}615}{k'}\right)$

where $k'=\frac{2R}{d}$, spring index.

2. Stiffness of a close coiled helical spring $\frac{W}{\delta}=k=\frac{Gd^4}{64nR^3}$

where G=Modulus of rigidity, n=number of coils,
δ=axial deflection along the load W.

3. For a close coiled helical spring subjected to axial couple M, the angular rotation of free end with respect to the fixed end

$\phi=\frac{128nRM}{Ed^4}$ where E=Modulus of elasticity.

4. For an open coiled helical spring, with helix angle α, subjected to an axial load W,
Twisting moment$=WR\cos\alpha$
Bending moment$=WR\sin\alpha$

Axial deflection, $\delta=2\pi nR^3\,W\sec\alpha\left[\frac{\cos^2\alpha}{GJ}+\frac{\sin^2\alpha}{EI}\right]$

$J=2I$=Polar moment of inertia$=\frac{\pi d^4}{32}$

Angular rotation, $\phi=2\pi nR^2\,W\sin\alpha\left[\frac{1}{GJ}-\frac{1}{EI}\right]$

5. For an open coiled helical spring, subjected to axial moment M
Twisting moment$=M\sin\alpha$
Bending moment$=M\cos\alpha$

Angular rotation, $\phi=2\pi nR\sec\alpha\;M\left[\frac{\cos}{EI}+\frac{\sin^2\alpha}{GJ}\right]$

Axial deflection, $\delta=2\pi n\,MR^2\sin\alpha\left[\frac{1}{EI}-\frac{1}{GJ}\right]$

6. Stresses developed in an open coiled helical spring, subjected to axial load W

Maximum shear stress $q=\frac{16WR\cos\alpha}{\pi d^3}+\frac{4W}{\pi d^2}$ (neglecting effect of spring index)

Maximum bending stress $f=\frac{32\,WR\sin\alpha}{\pi d^3}$

7. Stresses developed in an open coiled helical spring, subjected to axial moment M

Maximum shear stress $=\frac{16M\sin\alpha}{\pi d^3}$

Maximum bending stress$=\frac{32M\cos\alpha}{\pi d^3}$

8. A plane spiral spring made of a strip of breadth b and thickness t, and length l, subjected to a winding couple M

$$f_{max}, \text{ maximum stress}=\frac{12M}{bt^2}$$

$$\text{Energy stored} \quad =\frac{M^2 l}{2EI}=\frac{f_{max}^2}{24E}\times\text{Volume of the strip}$$

$$\text{Number of winding turns, } n=\frac{\phi}{2\pi}=\frac{Ml}{2\pi EI} \quad \text{where} \quad I=\frac{bt^3}{12}$$

9. Carriage spring of n leaves, breadth b, thickness t with the length of the longest leaf equal to l, of semi-elliptic shape.

$$\text{Initial central deflection, } y_0=\frac{l^2}{8R}, \quad R=\text{initial radius of curvature of each leaf}$$

$$\text{Proof load,} \quad W_0=\frac{Enbt^3}{3lR}$$

$$\text{Maximum stress,} \quad f_{max}=\frac{3Wl}{2nbt^2}$$

10. Cantilever leaf spring, with n leaves, breadth b, thickness t, with the length of the longest leaf equal to l, of quarter elliptic shape.

$$\text{Initial deflection, } y_0=\frac{l^2}{2R} \quad \text{where } R=\text{initial radius of curvature}$$

$$\text{Proof load,} \quad W_0=\frac{Enbt^3}{12lR}$$

$$\text{Maximum stress, } f_{max}=\frac{6Wl}{nbt^2}$$

MULTIPLE CHOICE QUESTIONS

1. Stiffness of a close coiled helical spring in terms of wire diameter d, modulus of rigidity G, number of turns n and mean coil radius R is given by

   (a) $\frac{Gd^4}{16nR^3}$ (b) $\frac{Gd^4}{32nR^3}$

   (c) $\frac{Gd^4}{64nR^3}$ (d) $\frac{Gd^4}{128nR^3}$

2. A close coiled helical spring absorbs 40 Nmm of energy while extending by 4 mm, the stiffness of the spring is

   (a) 10 N/mm (b) 8 N/mm

   (c) 6 N/mm (d) 5 N/mm

3. A close coiled helical spring of wire diameter d, coil radius R and number of turns n is subjected to an axial moment of 400 Nmm and its free end is rotated by 90° with respect to the fixed end. The energy absorbed by the spring is

   (a) 100π Nmm (b) 200π Nmm

   (c) 300π Nmm (d) 400π Nmm

4. An open coiled helical spring of wire diameter 8 mm, mean coil radius 40 mm, helix angle 45°, number of turns n, is subjected to an axial couple M. If the stress due to bending in wire section is 1200 kg/cm², then shear stress developed in the section is
   (*a*) 2400 kg/cm² (*b*) 1200 kg/cm²
   (*c*) 600 kg/cm² (*d*) 300 kg/cm².

5. A close coiled helical spring of stiffness 30 N/mm is in series with another close coiled helical spring of stiffness 60 N/mm, the stiffness of the composite spring is
   (*a*) 90 N/mm (*b*) 45 N/mm
   (*c*) 25 N/mm (*d*) 20 N/mm.

6. A flat spiral spring is made from a strip of 6 mm width and 1 mm thickness, 2 metres long. A winding couple M produces the maximum stress of 160 N/mm². The magnitude of winding couple is
   (*a*) 320 Nmm (*b*) 160 Nmm
   (*c*) 80 Nmm (*d*) 40 Nmm.

7. A flat spiral spring is made of strip of width b, thickness t and length l. If the maximum stress developed in strip is f, the energy stored in the spring is
   (*a*) $\frac{f^2}{24E} \times lbt$ (*b*) $\frac{f^2}{12E} \times lbt$
   (*c*) $\frac{f^2}{6E} \times lbt$ (*d*) $\frac{f^2}{4E} \times lbt$

8. A carriage spring is made of 6 leaves, breadth 6 cm and thickness 1 cm. Each leaf is initially bent to a radius of 2 m. If the length of the longest leaf is 80 cm, then initial central deflection provided in the spring is
   (*a*) 8 cm (*b*) 6 cm
   (*c*) 4 cm (*d*) 2·5 cm.

9. A load applied at the centre of a carriage spring to straighten all its leaves is termed as
   (*a*) Safe load (*b*) Ultimate load
   (*c*) Proof load (*d*) Yield load.

10. A close coiled helical spring is made of wire of diameter d and length l. The mean coil diameter of the spring is D and number of turns are n. The spring index is the ratio of
   (*a*) l/d (*b*) l/D
   (*c*) D/d (*d*) d/D.

ANSWERS

1. (*c*)	2. (*d*)	3. (*a*)	4. (*c*)	5. (*d*)
6. (*c*)	7. (*a*)	8. (*c*)	9. (*c*)	10. (*c*)

EXERCISES

14·1. A length of 1·2 m of 6 mm diameter steel wire is coiled to a mean coil diameter of 90 mm to make a close coiled helical spring. Determine the stiffness of the spring, G for steel=820,00 N/mm². [**Ans.** 4·29 N/mm]

14·2. Show that for a given resilience and the maximum shearing stress, the ratio of the weight of a close coiled helical spring made of tube to that one made of solid round wire is $\frac{1+\lambda^2}{\lambda^2}$, where λ is the ratio of the outside diameter of the tube to the inside diameter.

14·3. A close coiled helical spring having n turns is made of round wire such that the mean diameter of the coils D is ten times the wire diameter. This spring is required to support a load of 800 N with an extension of 100 mm and a maximum shear stress of 300 N/mm². Calculate (*i*) mean coil diameter (*ii*) number of coils (*iii*) weight of the spring, if the material weighs 7700 kg/m³.

$G=80{,}000$ N/mm² [**Ans.** (*i*) 82·4 mm (*ii*) 10·3 (*iii*) 1·09 kg]

14·4. A safety valve of 8 cm diameter is to blow off at a pressure of 12 kg/cm² by gauge. It is held in position by a close coiled helical spring of 15 cm mean coil diameter and 3 cm initial compression. Determine the diameter of the steel wire and the number of coils in the spring if the maximum shear stress in the spring is not to exceed 1200 kg/cm².

G for steel$=840$ tonnes/cm². **Ans.** [2·68 cm, 23·9 turns]

14·5. A close coiled helical spring is to have a stiffness of 0·9 kg/cm in compression and with a maximum load of 4·5 kg and a maximum shearing stress of 1200 kg/cm². The solid length of the spring (*i.e.* coils touching) is 4·5 cm. Find the wire diameter, **mean** coil diameter and number of coils.

$G=400$ tonnes/cm² **Ans.** [3·22 mm, 3·52 cm, 14 coils].

14·6. A close coiled helical steel spring having 20 turns is subjected to a couple of 2 kg-metre. The mean coil radius is 2·5 cm and the wire diameter is 0·8 cm. The axis of the couple coincides with the axis of the spring. Determine.

(*a*) Angular rotation of free end with respect to the fixed end of the spring.

(*b*) The maximum bending stress developed in spring wire.

(*c*) Work done on the spring.

$E=2100$ tonnes/cm² [**Ans.** (*a*) 85° 18′ (*b*) 3980 kg/cm². (*c*) 1·49 kg-m]

14·7. A weight of 20 kg is dropped onto a close coiled helical spring through a height of 50 cm, which instantaneously compresses the spring by 10 cm. If the mean radius of the coil is 10 cm and the diameter of the wire is 1·5 cm, determine the instantaneous stress produced and the number of coils in the spring.

G for steel$=840$ tonnes/cm². [**Ans.** (*a*) 1810 kg/cm² (*b*) 22·1 turns]

14·8. A close coiled cylindrical helical spring is of 10 cm mean coil diameter. The spring extends by 5 cm when axially loaded by a weight of 60 kg. When it is subjected to an axial couple $M=6$ kg-m, there is an angular rotation of 90°. Determine the Poisson's ratio for the material of the spring. **Ans.** [0·273]

14·9. Design a close coiled helical spring to have the following dimensions :

Mean coil diameter $=10$ cm

Number of coils $=20$

Stiffness of the spring $=25$ kg/cm

Steel wires are available of the following diameters :

12 mm, 14 mm, 16 mm, 18 mm

Determine the most suitable diameter of the wire and the maximum shear stress produced in the spring when axial deflection is 10 cm.

G for steel$=820$ tonnes/cm². [**Ans.** 16 mm, 1550 kg/cm²]

14·10. While designing a valve spring, it is estimated that the valve weighing 15 N requires an acceleration of 120 m/sec^2 when lifting through a height of 10 mm. The free length of the spring is 250 mm and axial length is 200 mm when the valve is shut. Determine the force on the spring.

Determine the mean coil diameter if it is 8 times the wire diameter if the maximum shear stress is not to exceed 250 N/mm^2. Calculate also the number of coils.

$G=800$ kN/mm^2 [**Ans.** 18·367 kg, 1·44 cm, 12·4 turns]

14·11. A close coiled helical spring of 1·8 cm mean coil diameter and 12 turns is arranged within and concentric with an outer spring. The free length of the inner spring is 6 mm more than the free length of the outer spring. The outer spring has 14 coils of mean diameter 24 mm and wire diameter 3·2 mm. The spring load against which the valve is opened is provided by the inner spring. The initial compression in the outer spring is 6 mm, when the valve is closed. Find the stiffness of the inner spring if the greatest force required to open the valve by 10 mm is 15 kg. Find also the wire diameter of the inner spring.

$G=80{,}000$ N/mm^2 [**Ans.** 5·15 kg/cm, 4·35 mm]

14·12. In a compound helical spring, the inner spring is arranged within and concentric with the outer one, but is 9 mm shorter than outer spring. The outer spring has 10 coils of mean coil diameter 24 mm and wire diameter 3 mm. Determine the stiffness of the inner spring if an axial load of 150 N causes the outer spring to compress by 18 mm.

If the radial clearance between the springs is 1·5 mm, find the wire diameter of the inner spring if it has 8 coils. $G=77000$ N/mm^2. [**Ans.** 5·33 N/mm, 2·06 mm]

14·13. A composite spring has two close coiled helical springs in series. The mean coil radius of each spring is 80 mm. The wire diameter of one spring is 2 cm and it has 16 coils, while the number of turns in the other spring is 12. Determine the wire diameter of the other spring if the stiffness of the composite spring is 4·2 N/mm.

Calculate the greatest axial load which can be applied on the composite spring if the maximum shearing stress is not to exceed 320 N/mm^2.

$G=84$ kN/mm^2

[**Ans.** 12·38 mm, 1490 N]

14·14. A rigid bar AB weighing 15 kg and carrying a load W equal to 40 kg rests on 3 springs as shown in the figure 14·17 having spring constants $k_1=20$ kg/cm, $k_2=10$ kg/cm and $k_3=12$ kg/cm. If the unloaded springs were of the same length, determine the distance x such that the bar AB remains horizontal.

[**Ans.** 10·5 cm]

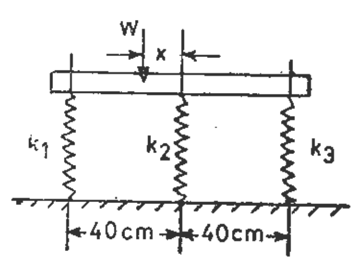


Fig. 14·17

14·15. Two close coiled helical springs of equal axial length are assembled co-axially. The wire diameter of the outer spring is 8 mm and the mean coil radius is 3·6 cm, while the wire diameter of the inner spring is 6 mm and the mean coil radius is 2·5 cm. The assembly of the springs is compressed by an axial thrust of 300 N. Calculate the maximum shear stress induced in each spring if both the springs are made of steel and the number of coils in each spring is same. [**Ans.** 47·46 N/mm^2, 65·60 N/mm^2]

14·16. The mean coil diameter of an open coiled helical spring is D and the coils are inclined at an angle of helix α. The section of the wire being a square of side a. Calculate the percentage error while determining the stiffness of the spring if the inclination of the coils is neglected. Given $\alpha=30°$. $E=2·50$ G [**Ans.** 8·7%]

14·17. An open coiled helical spring made of round steel bar 1·2 cm diameter has 15 coils of 10 cm mean coil diameter and the pitch is 6 cm. If the axial load is 70 kg, find the axial deflection and the rotation of free end relative to the fixed end of the spring.

G for steel=830 tonnes/cm^2

$1/m$ for steel=0 286 [**Ans.** 5·38 cm ; 6°21′]

14·18. In an open coiled helical spring made of steel, the stresses due to bending and twisting are 50 N/mm^2 and 60 N/mm^2 respectively, when the spring carries an axial load. There are 8 coils in the spring and mean coil radius is 6 times the wire diameter. Determine (1) angle of helix (2) permissible axial load (3) wire diameter, if the extension in the spring is 20 mm.

E for steel=210 kN/mm^2

G for steel=84 kN/mm^2

[**Ans.** (1) 22°36′, (2) 54·84 N (3) 5·078 mm]

14·19. An open coiled helical spring is subjected to an axial load and an axial couple simultaneously such that the angular rotation is completely prevented. Show that if the coils of the spring are inclined at 45° to its axis, the stiffness of the spring is given by

$$\frac{d^4\left(\frac{E}{2}+G\right)}{64\sqrt{2}nD^3}$$

where d=wire diameter,
n=number of coils and
D=mean coil diameter

14·20. A steel strip of length 15 metres, breadth 2·5 cm and thickness 0·06 cm is used to make a flat spiral spring. Determine the number of winding turns and the twisting moment at the spindle if the maximum bending stress is not to exceed 3020 kg/cm^2. Also calculate the amount of strain energy stored in the spring. E=2100 tonnes/cm^2.

[**Ans.** 5·72 turns, 2·265 kg-cm ; 40·37 cm-kg]

14·21. A semi elliptical laminated steel spring 1·2 m long is made of 10 leaves of 100 mm breadth and 10 mm thickness. What central load would produce a maximum stress of 300 N/mm^2 and what will be the corresponding central deflection.

E=210×1000 N/mm^2 [**Ans.** 16·66 kN, 51·4 mm]

14·22. A semi elliptical laminated spring of steel, 160 cm long carries a central load of 300 kg. The breadth of the leaves is 5 cm. Determine the thickness in multiples of 0·5 mm and the number of leaves if the central deflection is 8 cm and the bending stress is 2400 kg/cm^2.

Esteel=2100 tonnes/cm^2. [**Ans.** 9·5 mm, 7 leaves]

14·23. A laminated carriage spring made of 10 steel plates is 1·2 m long. The maximum central load is 400 kg. If the maximum allowable stress in steel is 2400 kg/cm^2 and the maximum deflection is 5 cm, determine the thickness and width of the plates.

E=2000 tonnes/cm^2 [**Ans.** 8·64 mm, 4·02 cm]

15

Struts and Columns

A short column when subjected to an axial compressive force fails by crushing. But when the same column becomes long, and an axial compressive force is applied, it fails by buckling before the limiting crushing stress is reached. This axial compressive force is called the buckling load and depends upon the end conditions and the ratio between length and lateral dimension. Buckling is caused by the inherent eccentricity of loading under compression and crookedness of the column. The bending moment produced due to these defects and the axial load is overcome by the resisting moment offered by the elasticity of the material. If the axial load is gradually increased, a stage comes when the bending moment due to the defects and axial load overweighs the resisting moment offered by the column and the column buckles all of a sudden. The axial load at this stage is called the Buckling load.

Any structural member in compression is called strut and when the strut takes the vertical position it is called a column or a stanchion. However the term strut is generally used for long compression members having large values of slenderness ratio (*i.e.* the ratio between the length of the column and the minimum radius of gyration of the section). Fig. 15·1 shows a strut of length l, hinged at both the ends, buckled under the axial compressive load P. Before the application of the load, the strut was straight and as the load gradually increased, the strut buckled at the load P. The bending moment at any section at a distance of x from the end A is $-Py$. This bending moment is negative in the sense that when we see from the side of the original centre line of strut, we find strut is bent showing convexity and as per the sign conventions already adopted, this B.M. is a negative bending moment.

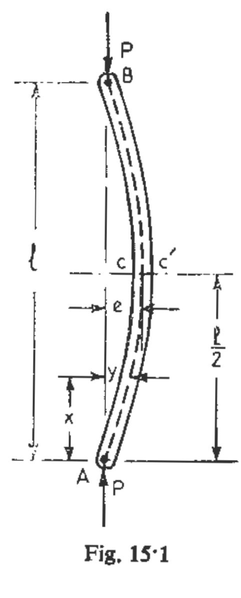


Fig. 15·1

The maximum bending moment occurs at the central section cc' of the strut and is equal to $-Pe$. The maximum and minimum stress intensities can be given by

$$f_{max}=f_0+f_b=\frac{P}{A}+\frac{Pe}{Z} \quad \text{(at the point } C\text{)}$$

$$f_{min}=f_0-f_b=\frac{P}{A}-\frac{Pe}{Z} \quad \text{(at the point } C'\text{)}$$

where f_0=direct stress due to axial load and f_b is stress due to bending ; e is the eccentricity of the load, A is the area of cross section and Z is the section modulus.

Failure of strut will occur either by f_{max} reaching the ultimate crushing strength of the material or by f_{min} reaching the ultimate tensile strength of the material.

For very long columns, Euler has developed a theory for the determination of buckling loads.

15·1. EULER'S THEORY FOR LONG COLUMNS

Following assumptions are taken while developing theory for the buckling load of very long columns—

1. The material of the column/strut is homogenous and isotropic.
2. The compressive load on the column/strut is fully axial.
3. The column/strut fails only by buckling.
4. The weight of the strut/column is neglected.
5. The column/strut is initially straight and buckles suddenly at a particular load.
6. Pin joints are frictionless and fixed ends are rigid.

Struts/columns with the following end conditions are considered.

(*i*) **Pin joint or hinged end.** The end is position fixed but direction free *i.e.* deflection $y=0$.

(*ii*) **Fixed end.** The end is position fixed as well as direction fixed *i.e.* deflection, $y=0$, slope $\frac{dy}{dx}=0$ at the fixed end.

(*iii*) **Free end.** The end is free to take any deflection and any slope.

Euler's buckling load is determined for the four cases *i.e.* 1. Both the ends are hinged 2. One end is fixed, other end is free, 3. Both the ends are fixed and 4. One end is fixed, other end is hinged. Let us take the 1st case.

1. **Both the ends are Hinged/Pin Jointed.** Fig. 15·2 shows a strut AB of length l, pin jointed or hinged at both ends. At the load P, the strut has buckled. Say EI is the flexural rigidity of the strut.

Considering a section at a distance of x from the end A, say the deflection is y.

B.M. at the section $=-Py$

or $$EI\frac{d^2y}{dx^2}=-Py$$

or $$EI\frac{d^2y}{dx^2}+Py=0 \qquad \text{...(1)}$$

The solution of this differential equation is

$$y=A\cos k'x+B\sin k'x \qquad \text{...(2)}$$

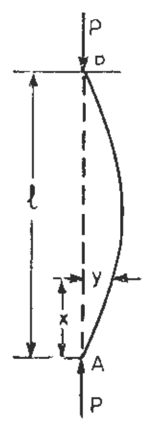


Fig. 15·2

where A and B are constants and $k'=\sqrt{\frac{P}{EI}}$

[The validity of the solution can be verified by differentiating two times the equation (2)]

So $$\frac{dy}{dx}=-Ak'\sin k'x+Bk'\cos k'x$$

$$\frac{d^2y}{dx^2}=-Ak'^2\cos k'x-Bk'^2\sin k'x=-k'^2(A\cos k'x+B\sin k'x)$$

$$=-k'^2y=-\frac{P}{EI}y$$

or $$EI\frac{d^2y}{dx^2}+y=0.$$ This shows that solution assumed is correct.

Let us determine the constants in the equation (2)

At $x=0$, at the end A ; $y=0$.

Therefore, $0=A\cos(k'\times 0)+B\sin(k'\times 0)$

$=A+0$ or $A=0$

Then $$y=B\sin k'x \quad ...(3)$$

At $x=l$, *i.e.* at end B, deflection $y=0$

Therefore, $0=B\sin k'l$

$B\neq 0$ because if we take B also equal to zero, it will lead to a condition that strut has not buckled

So $\sin k'l=0=\sin(0, \pi, 2\pi, 3\pi ... n\pi)$

The minimum significant value of angle is π

So $k'l=\pi$ or $k'^2l^2=\pi^2$

$$\frac{P}{EI}l^2=\pi^2$$

Euler's Buckling load, $$P=\frac{\pi^2EI}{l^2} \quad ...(4)$$

2. **Strut/column with one end fixed and other end free.** Fig. 15·3 shows a strut AB of length l, fixed at end A and free at the end B, buckled at the load P. Say the deflection at the free end is a. At the end A, there will be a reaction P and a fixing couple M_A. Consider a section at a distance of x from the end A.

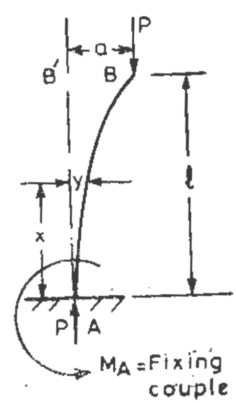


Fig. 15·3

B.M. at the section $=+P(a-y)$

The bending moment is a positive bending moment because if we see from the initial centre line AB' of the strut, we observe concavity.

Therefore, $$EI\frac{d^2y}{dx^2}=P(a-y) \quad ...(1)$$

[Note that we could have taken bending moment at the section equal to $(-M_A+P_y)$ but in this case M_A is unknown]

The solution of the differential equation (1) is

$$y=A \cos k'x+B \sin k'x+a \qquad \ldots(2)$$

where A and B are constants and $k'=\sqrt{\dfrac{P}{EI}}$

At the fixed end $x=0$, $y=0$

$$0=A+a \quad \text{or} \quad A=-a$$

So $$y=-a \cos k'x+B \sin k'x+a \qquad \ldots(2)$$

Differentiating the equation (2) we get

$$\frac{dy}{dx}=+ak \sin k'x+Bk \cos k'x$$

at $x=0$; fixed end A ; slope, $\dfrac{dy}{dx}=0$

So $$0=Bk' \cos (k' \times 0)=Bk'$$

or constant, $B=0$

and the equation of deflection becomes.

$$y=-a \cos k'x+a \qquad \ldots(3)$$

Moreover at the end B, at $x=l$, $y=a$

$$\therefore \quad a=-a \cos k'l+a$$

or $$a \cos k'l=0$$

$$\cos k'l=\cos \left[\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots\ldots \left(\frac{2n+1}{2} \right) \pi \right]$$

The minimum significant value of angle is $\pi/2$

$$\cos k'l=\cos \frac{\pi}{2} \quad \text{or} \quad k'l=\frac{\pi}{2}$$

and $$k'^2 l^2=\frac{\pi^2}{4} \quad \text{or} \quad \frac{P}{EI}.\ l^2=\frac{\pi^2}{4}$$

Euler's Buckling load, $$P=\frac{\pi^2 EI}{4l^2} \qquad \ldots(4)$$

3. **Strut/column with both the ends fixed**. Fig. 15·4 shows a column/strut AB of length l, fixed at both the ends, buckled under the axial load P. Since the ends are fixed, there will be fixing couples M_A and M_B at the ends A and B. Considering a section at a distance of x from the end A.

B.M. at the section $=M_A-P.y$

or $$EI \frac{d^2y}{dx^2}=M_A-Py \qquad \ldots(1)$$

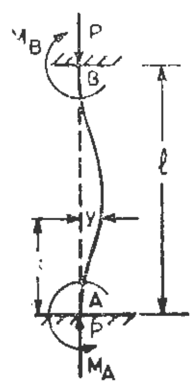


Fig. 15 4

The solution of this differential equation is

$$y=\frac{M_A}{P}+A \cos k'x+B \sin k'x \qquad \ldots(2)$$

where A and B are constants and $k'=\sqrt{\dfrac{P}{EI}}$

Now at the end A, $x=0$, $y=0$

$$\therefore \quad 0=\frac{M_A}{P}+A\cos(k'\times 0)+B\sin(k'\times 0)$$

or
$$A=-\frac{M_A}{P}$$

$$\therefore \quad y=\frac{M_A}{P}-\frac{M_A}{P}\cos k'x+B\sin k'x \qquad \ldots(3)$$

Let us differentiate equation (3)

$$\frac{dy}{dx}=+\frac{M_A k'}{P}\sin k'x+Bk'\cos k'x$$

$$\text{at } x=0\text{ ; fixed end ; } \frac{dy}{dx}=0$$

$$\therefore \quad 0=\frac{M_A k'}{P}\sin(k'\times 0)+Bk'\cos(k'\times 0)$$

or
$$B=0$$

Equation for deflection becomes

$$y=\frac{M_A}{P}-\frac{M_A}{P}\cos k'x \qquad \ldots(4)$$

Moreover at the fixed end B ; $y=0$ at $x=l$

$$0=\frac{M_A}{P}-\frac{M_A}{P}\cos k'l \quad \text{or} \quad \cos k'l=1$$

or
$$\cos k'l=\cos(0, 2\pi, 4\pi \ldots 2n\pi)$$

The minimum significant value of angle is 2π

$$\cos k'l=\cos 2\pi \quad \text{or} \quad k'l=2\pi$$

or
$$k'^2 l^2=4\pi^2 \quad \text{or} \quad \frac{P}{EI}\cdot l^2=4\pi^2$$

Eulers' buckling load, $$P=\frac{4\pi^2 EI}{l^2} \qquad \ldots(5)$$

4. **Strut/column with one end fixed and other end hinged.** Fig. 15·5 shows a strut/column AB of length l, fixed at end A and hinged at end B, buckled under the axial load P. There will be a fixing couple M_A at the end A. Since end B is position fixed but direction free, it cannot retain its position unless the hinge offers a horizontal reaction say R. Consider a section at a distance of x from end A.

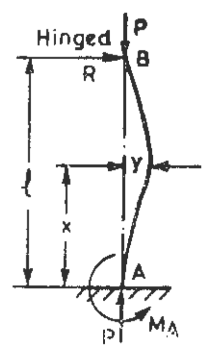


Fig. 15·5

B.M. at the section is $=-Py+R(l-x)$

[Note that if we take the B.M. as M_A-Py we will end up with a solution for the column with both the ends fixed)

Therefore, $EI\ \dfrac{d^2y}{dx^2}=-Py+R(l-x)$ ...(1)

The solution of this differential equation is

$$y=A\cos k'x+B\sin k'x+\frac{R(l-x)}{P} \quad ...(2)$$

where A and B are constants and $k'=\sqrt{\dfrac{P}{EI}}$

At the end A, $x=0$, $y=0$

$$\therefore \quad 0=A\cos(k'\times 0)+B\sin(k'\times 0)+\frac{Rl}{P}$$

or $$A=-\frac{Rl}{P}$$

$$\therefore \quad y=-\frac{Rl}{P}\cos k'x+B\sin k'x+\frac{R(l-x)}{P} \quad ...(2)$$

Differentiating the equation (2) we get

$$\frac{dy}{dx}=+\frac{Rlk'}{P}\sin k'x+Bk'\cos k'x-\frac{R}{P} \quad ...(3)$$

At the end A, $\dfrac{dy}{dx}=0$ at $x=0$

$$\therefore \quad 0=\frac{Rlk'}{P}\sin(k'\times 0)+Bk'\cos(k'\times 0)-\frac{R}{P}$$

or $$0=Bk'-\frac{R}{P} \quad \text{or} \quad B=\frac{R}{Pk'}$$

The equation for deflection becomes

$$y=-\frac{Rl}{P}\cos k'x+\frac{R}{Pk'}\sin k'x+\frac{R(l-x)}{P} \quad ...(4)$$

Now at the end B, $y=0$ at $x=l$

$$\therefore \quad 0=-\frac{Rl}{P}\cos k'l+\frac{R}{Pk'}\sin k'l+0$$

or $$\tan k'l=\frac{Rl}{P}\times\frac{Pk'}{R}=k'l$$

or $\tan\theta=\theta$. This is possible, when $\theta\simeq 4{\cdot}5$ radians.

Therefore, $\tan k'l=4{\cdot}5$ or $k'^2l^2=(4{\cdot}5)^2\simeq 2\pi^2$

$$\frac{P}{EI}\cdot l^2=2\pi^2$$

Eulers' buckling load, $P=\dfrac{2\pi^2EI}{l^2}$

In all the expressions for buckling load for cases 1 to 4, the moment of inertia is I_{min} if there are I_{max} and I_{min} for certain sections such as rectangular section, T section, I section.

Example 15·1-1. A 200 mm$\times$100 mm RSJ is used as a stanchion (strut), length 5 metres, with one end fixed other end free. Determine the Euler's buckling load. For the section.

$$I_{xx}=1696{\cdot}6\ \text{cm}^4 \quad \text{and} \quad I_{yy}=115{\cdot}4\ \text{cm}^4$$
$$E=210\ \text{kN/mm}^2$$

Solution.

End conditions are one end fixed other end free

Eulers buckling load, $P=\dfrac{\pi^2 EI}{4l^2}$

Since $I_{yy}<I_{xx}$

Therefore, buckling load$=\dfrac{\pi^2 EI_{yy}}{4l^2}$ Now, $l=5$ m

$$E=210\ \text{kN/mm}^2=210\times10^6\ \text{kN/m}^2$$
$$I_{yy}=115{\cdot}4\times10^{-8}\ \text{m}^4$$
$$P=\frac{\pi^2\times210\times10^6\times115{\cdot}4\times10^{-8}}{4\times5\times5}=\frac{\pi^2\times2{\cdot}1\times115{\cdot}4}{100}$$

Euler's buckling load $=23{\cdot}92$ kN

Example 15·1-2. A mild steel tube 3 cm internal diameter and 4 cm external diameter, length 3 metres, is used as a strut with one end fixed and the other end hinged. Calculate the Eulers buckling load. $E=2000$ tonnes/cm^2.

Solution.

End conditions : one end fixed, other end hinged

Euler's buckling load $=\dfrac{2\pi^2 EI}{l^2}$

Length, $l=3$ m$=300$ cm

$$E=2000\ \text{tonnes/cm}^2$$

The tube is hollow circular section, where

$$I_{xx}=I_{yy}=\frac{\pi}{64}(4^4-3^4)=\frac{\pi}{64}\times(256-81)=8{\cdot}59\ \text{cm}^4$$

Buckling load, $P=\dfrac{2\times\pi^2\times2000\times8{\cdot}59}{300\times300}=3{\cdot}768$ tonnes

Exercise 15·1-1. A rolled steel T section with flange 10$\times$2 cm and web 18$\times$1 cm is used as a strut with both the ends fixed. Determine the Euler's buckling load, if $E=2000$ tonnes/cm^2. Length of the strut is 5 metres. [**Ans.** 53·111 Tonnes]

Exercise 15·1-2. A round steel bar of diameter 5 cm and length 4 metres is used as a strut with both the ends hinged. Determine the Euler's buckling load if $E=210$ kN/mm^2. [**Ans.** 39·74 kN]

15·2. EQUIVALENT LENGTH

An equivalent length of a column or of a strut of a given length, given section and given end conditions is defined as the length of a column or strut of the same material, same section and having the same buckling load but having both of its ends hinged. Fig. 15·6 shows all the four cases discussed in article 15·1. The equivalent length of any strut is obtained by completing the bending curve of the column with different end conditions similar to the bending curve of a column with both the ends hinged as shown in the Fig. 15·6. *i.e.*, showing equivalent length, l_e.

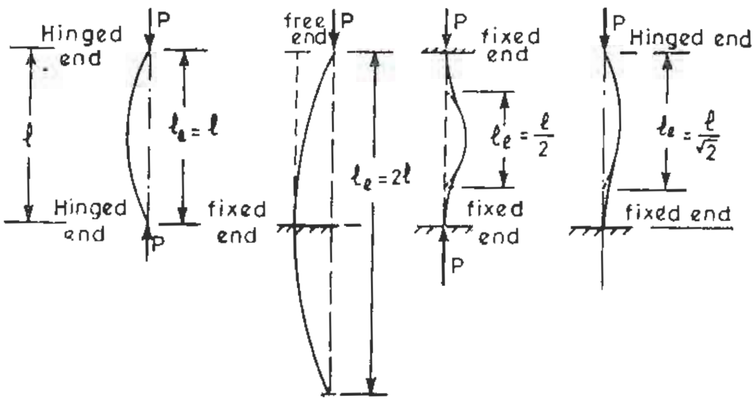


Fig. 15·6

For both the ends hinged, $l_e = l$

For one end fixed, other end free, $l_e = 2l$

For both the ends fixed, $l_e = \frac{l}{2}$

For one end fixed, other end hined, $l_e = \frac{l}{\sqrt{2}}$

The formulae for the Euler's buckling load can be modified as follows :

$$P_e = \text{Euler's Buckling load} = \frac{\pi^2 EI}{l_e^2}$$

where I is the minimum moment of inertia

and l_e=equivalent length depending upon the end conditions

Example 15·2-1. An allowable axial load for a 3 m long pin ended column of a certain elastic material is 30 kN. Three different columns made of the same material having the same cross section and length have the following end conditions.

(*i*) one end is fixed, other end is free

(*ii*) both the ends are fixed

(*iii*) one end is fixed but other end is hinged

What are the allowable loads for the three columns given above

Solution.

When the column has pin ends

Equivalent length $l_e = l$

$$\text{Allowable load} = \frac{Pe}{\text{Factor of safety } (FS)}$$

$$= \frac{\pi^2 EI}{l_e^2(FS)} \quad \text{or} \quad \frac{\pi^2 EI}{l^2(FS)} = 30 \text{ kN} \qquad ...(1)$$

(*i*) When one end fixed and other end is free, $l_e = 2l$

Allowable load $= \dfrac{\pi^2 EI}{4l^2(FS)} = \dfrac{30}{4} = 7.5$ kN

(*ii*) When both the ends are fixed, $l_e = \dfrac{l}{2}$

Allowable load $= \dfrac{4\pi^2 EI}{l^2(FS)} = 30 \times 4 = 120$ kN

(*iii*) When both one end is fixed, other end is hinged $l_e = \dfrac{l}{\sqrt{2}}$

Allowable load $= \dfrac{2\pi^2 EI}{l^2(FS)} = 30 \times 2 = 60$ kN

Exercise 15·2-1. An allowable axial load for a column of length l with both the ends fixed is 3 tonnes. Three different columns made of the same material, same length and same section have the following end conditions.

(*i*) both the ends are hinged

(*ii*) one end is fixed and other end is free

(*iii*) one end is fixed and other end is hinged

What are the allowable loads for the three columns give above.

[**Ans.** 0·75 tonnes, 0·1875 tonnes, 1·5 tonnes]

15·3. LIMITATIONS OF EULER THEORY OF BUCKLING

While deriving the expression of the buckling load for a strut or a column, we have considered that (*i*) strut has already buckled under the load P and then P is determined (*ii*) the strut is very long and the strut fails only by buckling. In other words, formula is not valid for short or medium sized struts or columns.

We know that Eulers' buckling load,

$$P_e = \frac{\pi^2 EI}{l_e^2} \quad \text{where } I = \text{minimum moment of inertia}$$

$= Ak^2$, k being the minimum radius of gyration

or
$$P_e = \frac{\pi^2 EAk^2}{l_e^2} \quad \text{or} \quad \frac{P_e}{A} = \frac{\pi^2 E}{(l_e/k)^2}$$

where $\dfrac{l_e}{k}$ is called the slenderness ratio.

This shows that Eulers buckling load is inversely proportional to the square of slenderness ratio or in other words as the length of strut decreases, buckling load goes on increasing. But when the column is short it fails by crushing. Therefore the Eulers' buckling theory is valid for a column or a strut beyond a certain value of slen derness ratio. Say f_c is the ultimate compressive strength of the material of the strut/column.

$$\frac{P_e}{A} < f_c \quad \text{(for the column to fail by buckling)}$$

or
$$\frac{\pi^2 E}{(l_e/k)^2} < f_c \quad \text{or} \quad \left(\frac{l_e}{k}\right)^2 > \frac{\pi^2 E}{f_c}$$

Slenderness ratio, $\frac{l_e}{k} > \pi\sqrt{\frac{E}{f_c}}$

Say the strut is of mild steel with $E=2100$ tonnes/cm²

Ultimate strength in compression, $f_c=3{\cdot}3$ tonnes/cm²

$$\frac{l_e}{k} > \pi \times \sqrt{\frac{2100}{3{\cdot}3}} \quad \text{or} \quad \frac{l_e}{k} > 79{\cdot}25 \quad \text{or} \quad 80$$

This shows that for a mild steel strut, Eulers theory is applicable only when the slenderness ratio is greater than 80 *i.e.* when the ends of the strut are pin ended the slenderness ratio should be more than 80 for the Euler's theory to be valid for the determination of buckling load.

Example 15·3-1. For what length of a mild steel bar 5 cm diameter used as a strut the Euler's theory is applicable if the ultimate compressive strength is 0·35 kN/mm² and $E=210$ kN/mm² for mild steel.

(*a*) when both the ends are hinged

(*b*) when one end is hinged and the other end is fixed.

Solution. For the Euler's theory to be applicable

$$\frac{l_e}{k} > \pi\sqrt{\frac{E}{f_c}} \quad \text{where } E=210 \text{ kN/mm}^2$$

$$f_c=0{\cdot}35 \text{ kN/mm}^2$$

$$> \pi \times \sqrt{\frac{210}{{\cdot}35}} > 76{\cdot}95 \quad \text{say } 77$$

Radius of gyration, $k=\sqrt{\frac{I}{A}}=\sqrt{\frac{\pi d^4}{64}\times\frac{4}{\pi d^2}}$

$$=\frac{d}{4}=\frac{5}{4}=1{\cdot}25 \text{ cm}$$

(*a*) When both the ends are hinged $l_e=l$

So $\quad l > 77\times1{\cdot}25 \quad$ where $\quad k=1{\cdot}25$ cm

Length should be greater than 96·25 cm

(*b*) When one end is hinged and other end is fixed

Equivalent length, $\quad l_e=\sqrt{2}\, l$

$$l > \sqrt{2}\times77\times1{\cdot}25 > 136{\cdot}1 \text{ cm}$$

Length of the strut should be greater than 136·1 cm

Exercise 15·3-1. For what length of a hollow steel bar 4 cm external diameter and 3 cm internal diameter used as a strut, the Eulers theory is applicable for buckling if

$$f_c=3{\cdot}3 \text{ tonnes/cm}^2 \quad \text{and} \quad E=2100 \text{ tonnes/cm}^2$$

(*a*) when both the ends are fixed

(*b*) one end is fixed, other end is free.

[**Ans.** (*a*) Length greater than 200 cm (*b*) Length greater than 50 cm]

15·4. RANKINE GORDON FORMULA

Very short columns when subjected to axial compressive load fail by crushing and crushing load is given by $P_c=f_c\,.\,A$ where f_c is the ultimate compressive strength and A is the area of cross section. Very long columns or struts fail by buckling and Euler gave the buckling load, P_e as $\pi^2 EI/l_e^2$. But there are general purpose struts and columns which can neither be classified as short nor very long struts. Such struts or columns fail by the combined effect of direct stress due to the axial load and bending stress due to the bending moment caused due to buckling. For such columns Rankine suggested an empirical relationship as follows :

$$\frac{1}{P_R}=\frac{1}{P_c}+\frac{1}{P_e} \quad \text{...(1)}$$

where P_R is the Rankine's buckling load

Curve produced by equation (1) is tangential to P_c when l/k ratio is very small and is tangential to P_e when l/k ratio is very large. The Rankine's load takes into account the direct as well as the bending stresses.

From equation (1) $$P_R=\frac{P_c \, . \, P_e}{P_c+P_e}=\frac{P_c}{1+\dfrac{P_c}{P_e}}$$

Substituting the values for P_c and P_e

$$P_R=\frac{f_c \, . \, A}{1+f_c \, . \, A\times\dfrac{l_e^2}{\pi^2 EI}}$$

where $I=I_{minimum}=Ak^2$

where k is the minimum radius of gyration of the section of the column.

So $$P_R=\frac{f_c \, . \, A}{1+\dfrac{f_c}{\pi^2 E}\left(\dfrac{l_e}{k}\right)^2}$$

where $\dfrac{f_c}{\pi^2 E}=a$,

a constant depending upon the elastic constant E and compressive strength, f_c

Rankine's load, $$P_R=\frac{f_c \, . \, A}{1+a\left(\dfrac{l_e}{k}\right)^2}$$

where l_e is the equivalent length of the strut or column.

In this formulae f_c and a are called as Rankine's constants and have been experimentally determined for various common materials, as given in Table 15·1.

TABLE 15·1

Material	*f_c in N/mm²*	*f_c in kg/cm*	*Constant a (for both the ends hinged)*
Cast Iron	550	5600	1/1600
Wrought Iron	250	2550	1/9000
Mild Steel	320	3262	1/7500
Medium Carbon Steel	500	5097	1/5000
Timber	35	357	1/3000

Example 15·4-1. A hollow cast iron column 200 mm outside diameter and 150 mm inside diameter, 8 metres long has both the ends fixed. It is subjected to an axial compressive load. Taking a factor of safety as 6, determine the safe Rankine's buckling load.

Given $f_c = 550 \text{ N/mm}^2$.

Constant, $a = \frac{1}{1600}$ for both the ends hinged.

Solution.

Outside diameter, $D = 200$ mm ; Inside diameter, $d = 150$ mm

Area of cross section of the column

$$A = \frac{\pi}{4}(200^2 - 150^2) = 1{\cdot}374 \times 10^4 \text{ mm}^2.$$

Length of the column, $l = 8 \text{ m} = 8000$ mm

End Conditions : Both the ends are fixed

Equivalent length, $l_e = \frac{l}{2} = 4000$ mm

Radius of gyration, $k = \frac{\sqrt{D^2+d^2}}{4} = \frac{\sqrt{200^2+150^2}}{4}$

or $$k^2 = \frac{6{\cdot}25 \times 10^4}{4 \times 4} \text{ mm}^2$$

$$a \cdot \frac{l_e^2}{k^2} = \frac{1}{1600} \times \frac{4000 \times 4000 \times 4 \times 4}{6{\cdot}25 \times 10^4} = 2{\cdot}56$$

Rankine's buckling load $= \dfrac{f_c \cdot A}{1 + a \cdot \dfrac{l_e^2}{k^2}} = \dfrac{550 \times 1{\cdot}374 \times 10^4}{3{\cdot}56}$

$= 212{\cdot}28 \times 10^4 \text{ N} = 2122{\cdot}8$ kN

Factor of safety $= 6$

Safe Rankine's load $= \dfrac{2122{\cdot}28}{6} = 353{\cdot}8$ kN.

Exercise 15·4-1. A cost iron column of hollow circular section, external diameter 25 cm and thickness of metal 3·5 cm has to transmit an axial compressive load P. The column is 7 m long with both the ends hinged. Take factor of safety as 8. Determine the value of P. Rankine's constants are $f_c = 5{\cdot}6 \text{ tonne/cm}^2$.

$a = \frac{1}{1600}$. **[Ans.** 26·85 Tonnes]

15·5. SPACING OF BRACES FOR BUILT UP SECTIONS

The lattice bars used for bracing are generally 50 to 80 mm wide and 8 to 10 mm thick. In order to decide about the spacing between the braces following procedure may be adopted.

Figure 15·7 shows a built up section consisting for four equal angle sections, joined by lattice bars. Say L is the length of the column of the built up section and l is the spacing between the lattice bars or braces. If the section is properly braced, the total load on the column will be equally shared by each angle. Braces are generally riveted to the angle sections therefore the end condition for each angle section between the braces, *i.e.*, for length l can be taken as hinged.

The maximum unsupported length l of the angle section so that the angle section does not buckle under the load is determined as follows

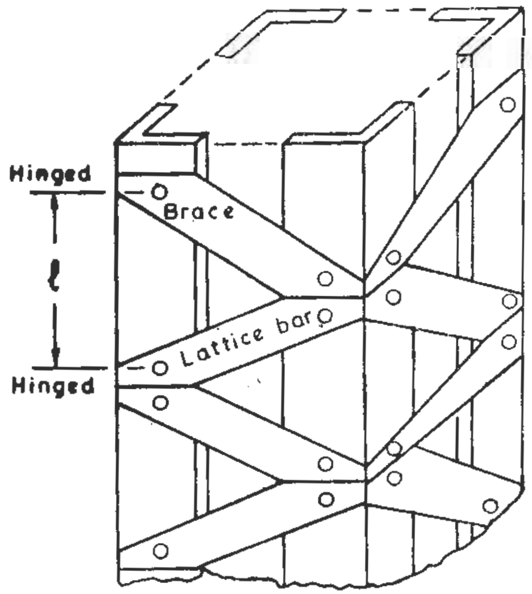


Fig. 15·7

P'=Load shared by each angle section$=\frac{P}{4}$ in this case

$$\frac{P}{4}=\frac{f_c \cdot A}{1+a \cdot \frac{l^2}{k^2}}$$

where k is the minimum radius of gyration of one angle section.
A=area of section of one angle

For the built up section

$$P=\frac{f_c \cdot 4A}{1+a \cdot \frac{L^2}{K^2}}$$

if the ends are considered hinged

or

$$\frac{P}{4}=\frac{f_c \cdot A}{1+a \cdot \frac{L^2}{K^2}}$$

where k is the minimum radius of gyration for built up section.

So

$$\frac{f_c \cdot A}{1+a \cdot \frac{L^2}{K^2}}=\frac{f_c \cdot A}{1+a \cdot \frac{l^2}{k^2}}$$

or

$$\frac{l}{k}=\frac{L}{K} \quad \text{or} \quad l=\frac{k}{K} \cdot L$$

If the end conditions for a built up section are different then L can be replaced by equivalent length L_e.

i.e., $l=\frac{k}{K} \cdot L_e$

where $L_e=\frac{L}{2}$ when both the ends are fixed

$L_e=2L$ when one end is fixed and other free

$L_e=\frac{L}{\sqrt{2}}$ when one end is fixed and other hinged.

Example 15·5-1. A braced jib of crane is built up of 4—80×80 mm angles forming a square of 40 cm overall. If the length of the jib is 12 metres and ends are fixed, calculate the safe axial load. Take factor of safety as 4. For the angle section.

Area=9·29 cm², $I_{xx}=I_{yy}=56$ cm⁴, $x=y=2·18$ cm (distance of CG from the edge)

Rankine's constants, $f_c=3·3$ tonnes/cm², $a=\frac{1}{7500}$

Determine the minimum distance between the lattice bars.

Solution. Let us first calculate the moment of inertia of the built up section. Fig. 15·8 shows 4 angle sections 80 mm×80 mm joined by lattice bars. The angles are symmetrically placed so the centroid of the section will be at G as shown.

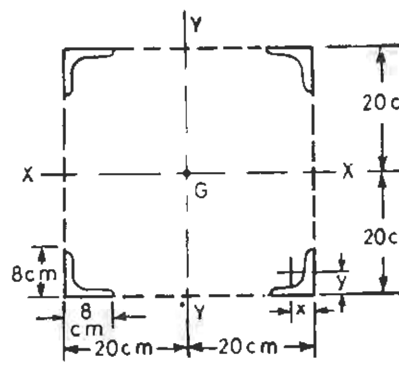


Fig. 15·8

Area of the section
$=4\times A=4\times 9·29$
$=37·16$ cm²

$I_{xx}=I_{yy}=4\times 56+4\times 9·29(20-x)^2$
$=224+37·16(20-2·18)^2$
$=224+11800·25=12024·25$ cm⁴

Radius of gyration of built up sections,

$$K^2=\frac{I_{xx}}{4A}=\frac{12024·25}{37·16}=323·58 \text{ cm}^2$$

End conditions : both the ends are fixed

Rankine's buckling load,

$$P_R=\frac{f_c \cdot \times 4A}{1+a \cdot \frac{L_e^2}{K^2}}$$

Equivalent length $L_e=\frac{L}{2}=\frac{12}{2}=6 \text{ m}=600 \text{ cm}$

$$a\frac{L_e^2}{K^2}=\frac{1}{7500}\times\frac{600\times 600}{323·58}=0·148$$

$$P_R=\frac{3·3\times 4\times 9·29}{1+0·148}=106·82 \text{ Tonnes}$$

Factor of safety $=4$

Safe axial load $=\frac{P_R}{4}=\frac{106·82}{4}=26·705$ Tonnes,

Distance between lattices bars

$$\frac{l}{k}=\frac{Le}{K} \quad \text{when } L_e=\text{equivalent length}=600 \text{ cm}$$

l=distance between lattice bars

k=minimum radius of gyration for one angle section

$$=\sqrt{\frac{56}{9{\cdot}29}}=2{\cdot}455 \text{ cm}$$

K=minimum radius of gyration of built up sections

$$=\sqrt{323{\cdot}58}=17{\cdot}99 \text{ cm}$$

$$l=\frac{k}{K}\cdot l_e=\frac{2\;455}{17{\cdot}99}\times 600=81{\cdot}18 \text{ cm.}$$

Exercise 15·5-1. A braced girder is built up of 4—100×100 mm angle sections forming a square of 45 cm overall. If the length of the girder is 16 metres and its one end is fixed and other end free, calculate the safe axial load. Take factor of safety as 5. For the angle section

Area=22·59 cm², $I_{xx}=I_{yy}=207{\cdot}0$ cm⁴, $x=y=2{\cdot}92$

(*i.e.* distance of CG from outer edges) and Rankine's constants are

$$f_c=320 \text{ N/mm}^2 \text{ and } a=\frac{1}{7500}.$$

Determine also the minimum distance between the bracings.

[**Ans.** 129·2 kN, 4·89 m]

15·6. OTHER EMPIRICAL FORMULAE FOR STRUTS AND COLUMNS

We know that Rankine's buckling load is given

$$P_R=\frac{f_c\cdot A}{1+a\cdot\frac{l_e^2}{k^2}}$$

or $$\frac{P_R}{A}=\text{working stress, } f_w=\frac{f_c}{1+a\cdot\frac{l_e^2}{k^2}}$$

or $$f_w<f_c' \text{ (allowable stress)}$$

where reduction factor is $1+a\cdot\frac{l_e^2}{k^2}$. The working stress is less than the ultimate compressive strength f_c, obtained for a short column with no buckling. The working stress is less than f_c due to the bucking effect in a column or a struct and the reduction factor is dependent on the slenderness ratio $\frac{l_e}{k}$.

In other words $$f_w=f_c'-\phi\left(\frac{l_e}{k}\right) \text{ a function of } \frac{l_e}{k}$$

Let us take $\phi\left(\frac{l_e}{k}\right)=b\cdot\frac{l_e^2}{k^2}$

$$f_w = f_c'\left[1 - b\,\frac{l_e^2}{k^2}\right]$$

This is known as the Johnson's parabolic formula, because if f_w is plotted against l_e/k, a parabolic curve is obtained.

If the function $\phi\left(\frac{l_e}{k}\right)$ is taken as $c\left(\frac{l_e}{k}\right)$, where c is a constant, then

$$f_w,\ \text{working stress} = f_c'\left[1 - c\left(\frac{l_e}{k}\right)\right]$$

a straight line relation. Now in the Johnson's parabolic formula, following values are generally taken.

$f_c' = 110$ N/mm², allowable stress in compression for mild steel

constant $b = 0{\cdot}00003$ for pinned ends
$= 0{\cdot}00002$ for fixed ends

In the straight line formula

constant $c = \frac{1}{200}$ for pinned ends for mild steel

$= \frac{1}{250}$ for fixed for ends for mild steel

For structural steel, $f_c' = 140$ N/mm² allowable stress in compression
$c = 0{\cdot}0054$ for pinned ends and
$= 0{\cdot}0038$ for riveted ends

The straight line formula is applicable for slenderness ratios greater than 90.

Gordon's formula for buckling load

$$P_G = \frac{f_c . A}{1 + a_1 . \frac{l^2}{b^2}}$$

where b is the lesser dimension of the section of the strut or column and a_1 is a constant. The value of a_1 depends upon the material and shape of the section.

Example 15·6-1. A stanchion is built up of 3—200×100 mm RSJ as shown in the Fig. 15·9. If the height of the stanchion is 6 metres, calculate the working load. The working stress being given by

$$110\left[1 - \frac{1}{200}\left(\frac{L}{k}\right)\right] \text{ N/mm}^2$$

Properties of one 200×100 mm, RSJ are

Area $= 25{\cdot}27$ cm², $I_{xx} = 1696{\cdot}6$ cm⁴, $I_{yy} = 115{\cdot}4$ cm⁴

Web thickness $= 5{\cdot}4$ mm

What factor of safety is to be used with the Rankine formula to give the same result of buckling load ?

Take $f_c = 320$ N/mm², $a = \frac{1}{7500}$

Solution. Fig. 15·9 shows the combination of 3 RS Joists of given dimensions. The sections are symmetrically arranged about G *i.e.* centroid of the whole section.

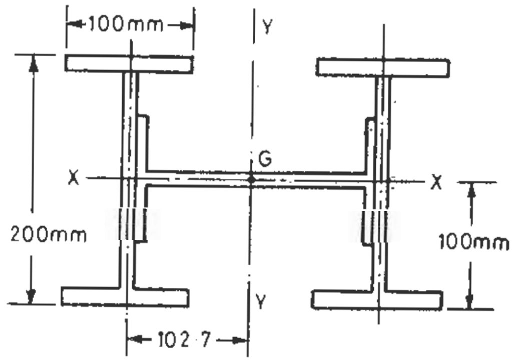


Fig. 15·9

Moment of inertia,

$$I_{xx}=2\times1696{\cdot}6+115{\cdot}4 \text{ cm}^4$$
$$=3508{\cdot}6 \text{ cm}^4$$
$$I_{yy}=1696{\cdot}6+2\times115{\cdot}4+2\times25{\cdot}27(10{\cdot}27)^2$$
$$=1696{\cdot}6+230{\cdot}8+5330{\cdot}6$$
$$=7258{\cdot}0 \text{ cm}^4$$

Now $I_{xx}<I_{yy}$

Area of the section

$$=3\times25{\cdot}27=75{\cdot}81 \text{ cm}^2$$

Radius of gyration, $k=\sqrt{\dfrac{3508{\cdot}6}{75{\cdot}81}}=6{\cdot}80$ cm

Length of the column, $L=6$ m$=600$ cm

$$\frac{L}{k}=\frac{600}{6{\cdot}8}=88{\cdot}235$$

Working stress, $f_w=110-\dfrac{110}{200}\left(\dfrac{L}{k}\right)$

$$=110-\frac{81{\cdot}235\times110}{200}=110-48{\cdot}4=61{\cdot}6 \text{ N/mm}^2$$

Working load, $P_w=f_w\times\text{area}=61{\cdot}6\times75{\cdot}81\times100$

$$=466989{\cdot}6 \text{ N}=466{\cdot}98 \text{ kN}$$

Rankine's buckling load

$$a=\frac{1}{7500},\quad a\frac{L^2}{k^2}=\frac{1}{7500}\times\frac{600\times600}{6{\cdot}8\times6{\cdot}8}=1{\cdot}038$$

$$P_R=\frac{f_c\,.\,A}{1+a\dfrac{L^2}{k^2}}=\frac{320\times75{\cdot}81\times100}{1+1{\cdot}098}$$

$$=1190343 \text{ N}=1190{\cdot}343 \text{ kN}$$

Factor of safety $=\dfrac{\text{Rankine's load}}{\text{Working load}}=\dfrac{1190{\cdot}343}{466{\cdot}98}=2{\cdot}55$

Exercise 15·6-1. A strut is built up of two 100×45 mm channels placed back to back at a distance of 100 mm apart and riveted to two flange plates each 200 mm×10 mm symmetrically. Properties of one 100×45 mm channel section are

$$\text{Area}=7{\cdot}41 \text{ cm}^2,\ I_{xx}=123{\cdot}8 \text{ cm}^4,\ I_{yy}=14{\cdot}9 \text{ cm}^4$$

$$x=1{\cdot}4 \text{ cm (distance of CG from outer edge of web)}$$

If the effective length is 5 metres calculate the working load for the strut using Johnson's parabolic formula.

$$f_w=f_c'\left[1-b\left(\frac{L}{k}\right)^2\right] \quad \text{where} \quad b=0{\cdot}00003 \text{ for pinned ends}$$

$$\text{and} \quad f_c'=110 \text{ N/mm}^2$$

What factor of safety is to be used with the Rankine's formula to give the same result ? Take Rankine's constants as

$$f_c=3\cdot3 \text{ tonnes/cm}^2, \quad a=\frac{1}{7500}$$

Hint. Convert f_c' into kg/cm². $[I_{xx}=1490\cdot93 \text{ cm}^4<1970\cdot15 \text{ cm}^4=I_{yy}]$

[**Ans.** 43·21 tonnes, 1·88]

15·7. ECCENTRIC LOADING OF LONG COLUMNS

In Chapter 9 we have studied the effect of the eccentricity of the load on short columns without buckling, and we found that the stresses produced by the bending moment due to the eccentricity of the load are added to the direct compressive stress and the working load for a column is reduced.

Now we will study the effect of eccentricity of the load on the buckling effect of long columns which fail by the combined effect of direct compressive stress and stress introdueed by the buckling or the bending of the column. Let us consider a long column *AB*, of length *l*, fixed at end *A* and free at the end *B*. Load is applied at an eccentricity *e* from the axis of the column and the column buckled at the load *P* as shown in the Fig. 15·10. Say the maximum deflection at the free end *B* is *a*. Again consider a section of the column at a distanee of *x* from the fixed end *A* and say the deflection in the column at this section is *y*.

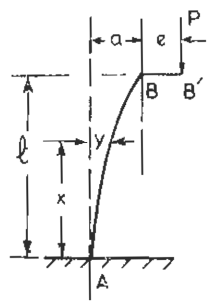


Fig. 15·10

Bending moment at the section

$$=P(a+e-y)$$

(when we see from the side of the original axis of the column we see concavity *i.e.* a positive bending moment)

$$\therefore \qquad EI\frac{d^2y}{dx^2}=P(a+e-y) \qquad ...(1)$$

The solution of the differential equation (1) is

$$y=A\cos k'x+B\sin k'x+(a+e) \qquad ...(2)$$

where A and B are constants and $k'=\sqrt{\dfrac{P}{EI}}$

At the end A, $x=0$, $y=0$

$$0=A\cos 0+B\sin 0+(a+e)$$

or

$$A=-(a+e)$$

$$\therefore \qquad y=-(a+e)\cos k'x+B\sin k'x+(a+e) \qquad ...(3)$$

Differentiating equation (3) we get

$$\frac{dy}{dx}=(a+e)k'\sin k'x+Bk'\cos k'x$$

$$\text{at } x=0\text{ ; fixed end, } \frac{dy}{dx}=0$$

$$0=(a+e)\sin 0+Bk'\cos 0$$

$k'\neq 0$ (because P will become zero)

Therefore, $B=0$

Initially the equation for deflection is

$$y=-(a+e)\cos k'x+(a+e) \qquad \ldots(4)$$

Now at the end B, free end, $x=l$, $y=a$

$\therefore$ $a=-(a+e)\cos k'l+(a+e)$

or $e=(a+e)\cos k'l$

or $(a+e)=e\sec k'l \qquad \ldots(5)$

Maximum bending moment occurs at the fixed end,

$$M_{max}=P(a+e)$$

or $M_{max}=Pe\sec k'l=Pe\sec.\sqrt{\dfrac{P}{EI}}\,.\,l$

Maximum stress at the fixed end

$$f_{max}=f_0+f_b=\frac{P}{A}+\frac{Pe\sec l\sqrt{\dfrac{P}{EI}}}{Z} \qquad \ldots(6)$$

where A=area of cross section and Z=section modulus

If both the ends are hinged the formula (6) can be modified as

$$f_{max}=f_0+f_b=\frac{P}{A}+\frac{Pe\sec\dfrac{l}{2}\sqrt{\dfrac{P}{EI}}}{Z}$$

because for a column with one end fixed and the other end free, equivaleut length $l_e=2l$

The formula in general for any end conditions can be written as

$$f_{max}=\frac{P}{A}+\frac{Pe\sec\dfrac{l_e}{2}\sqrt{\dfrac{P}{EI}}}{Z}$$

where l_e=equivalent length depending upon the end conditions.

Here we observe that in the case of short columns (with no buckling) maximum bending moment is Pe which is increased to $Pe\sec\dfrac{l_e}{2}\sqrt{\dfrac{P}{EI}}$ in the case of long columns

Example 15·7-1. A stanchion 6 m long, ends free is built up of two 40×10 cm standard channels placed 15 cm back to back with one 35 cm×1 cm plate riveted to each flange. It carries a load of 150 tonnes, which is off the axis YY in the vertical plane through the axis XX. Calculate the permissible eccentricity if the maximum permissible compressive stress is 1·2 tonnes/cm². For each channel, area of the section=63·04 cm², distance of CG from the base=2·43 cm, I_{xx}=15123·4 cm⁴, I_{yy}= 506·3 cm⁴. E=2080 tonnes/cm².

Solution. Fig. 15·11 shows this built up section with two channels and two plates. The over all dimensions are 35 cm and 42 cm. Since the section is symmetric and about the centroid G, we can easily determine I_{yy} for the section.

$$I_{yy}=2\times 506{\cdot}3+2\times 63{\cdot}04(7{\cdot}5+2{\cdot}43)^2+\frac{2\times 1\times 35^3}{12}$$

$$=1012{\cdot}6+12432{\cdot}10+7145{\cdot}83$$

$$=20590{\cdot}53\ \text{cm}^4.$$

There is no necessity of calculating I_{xx} because eccentricity is to be determined along XX axis and YY becomes the plane of bending.

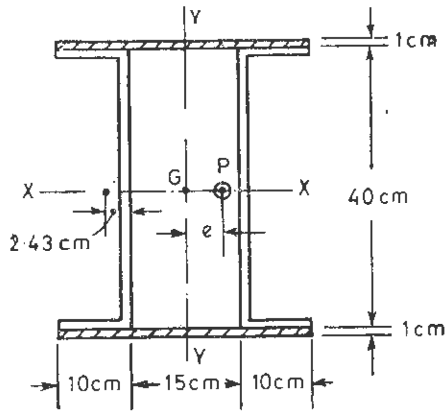


Fig. 15·11

End conditions : Both the ends are free, when both the ends are free and column buckles the equivalent length $l_e=l$ because the column takes the same shape and curvature as when the ends are pinned or hinged.

Length, $l=600$ cm, $P=150$ tonnes

$E=2080$ tonnes/cm²

So
$$\frac{P}{EI}=\frac{P}{2080\times 20590{\cdot}53}=\frac{150}{2080\times 20590{\cdot}53}$$

$$=\frac{3{\cdot}5024}{10^6}\quad\text{and}\quad\sqrt{\frac{P}{EI}}=1{\cdot}871\times 10^{-3}/\text{cm}$$

$$\sec\frac{l}{2}\sqrt{\frac{P}{EI}}=\sec.\ \frac{600}{2}\times\frac{1{\cdot}871}{1000}$$

$$=\sec\ ({\cdot}5613\ \text{radian})=\sec 32^\circ 10'=1{\cdot}181$$

Area of the section, $A=2\times 63{\cdot}04+2\times 35\times 1=196{\cdot}08\ \text{cm}^2$

Direct stress, $f_0=\dfrac{P}{A}=\dfrac{150}{196{\cdot}08}=0{\cdot}765$ tonnes/cm²

Maximum permissible stress,

$f_{max}=1{\cdot}2$ tonnes/cm²

Therefore, permissible stress in bending,

$f_b=1{\cdot}2-0{\cdot}765=0{\cdot}435$ T/cm²

Say the eccentricity $=e$ cm

Section modulus, $Z=\dfrac{I_{yy}}{17{\cdot}5}=\dfrac{20590{\cdot}53}{17{\cdot}5}=1176{\cdot}6\ \text{cm}^3$

From the secant formula

$$f_b=\frac{Pe\sec\frac{l}{2}\sqrt{\frac{P}{EI}}}{Z}=\frac{150\times e\times 1{\cdot}181}{1176{\cdot}6}$$

$$0{\cdot}435=\frac{150\times e\times 1{\cdot}181}{1176{\cdot}6},\quad\text{or } e=2{\cdot}9\ \text{cm}$$

Exercise 15·7-1. A cast iron column of hollow circular section 200 mm external diameter and 160 mm internal diameter, length 7 metres has to take a load of 200 kN at an eccentricity of 30 mm from the geometrical axis. If the ends are fixed, calculate the maximum and minimum stress intensities induced in the section, taking $E=210$ kN/mm². Moreover calculate the maximum permissible eccentricity so that no tension is induced anywhere in the section.

[Note that for finding out maximum permissible eccentricity take $f_0-f_b=0$]

[**Ans.** 31·03 N/mm² (compressive) 4·33 N/mm² (compressive) 3·97 (maximum eccentricity)]

15·8. PROF. PERRY'S FORMULA

In the last article we have learnt that f_{max}, the maximum stress developed in a long column with hinged ends is

$$f_{max}=f_0+f_b=\frac{P}{A}+\frac{P_e \sec \frac{l}{2}\sqrt{\frac{P}{EI}}}{Z} \qquad ...(1)$$

Eulers formula for bucking load,

$$P_e=\frac{\pi^2 EI}{l^2} \qquad \text{or} \qquad EI=\frac{P_e\, l^2}{\pi^2}$$

So the term

$$P.e \sec\frac{l}{2}\sqrt{\frac{P}{EI}}=Pe \sec\frac{l}{2}\times\sqrt{\frac{P}{P_e}\times\frac{\pi^2}{l^2}}$$

$$=Pe \sec\frac{l}{2}\times\frac{\pi}{l}\sqrt{\frac{P}{P_e}}=Pe \sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}$$

Prof. Perry found that the expression

$$\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}} \simeq \frac{1\cdot 2\, P_e}{P_e-P}$$

If we take stress $\quad f_e=\frac{P_e}{A}$ and $f_0=\frac{P}{A}$ then

$$\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}=\frac{1\cdot 2 f_e}{f_e-f_0} \qquad ...(2)$$

$$f_{max}=\frac{P}{A}+\frac{Pe}{Z}\cdot\frac{1\cdot 2 f_e}{f_e-f_0}$$

Moreover $Z=\frac{I}{y_c}$ where y_c=distance from the neutral axis of the extreme layer in compression.

$$=\frac{Ak^2}{y_c} \quad \text{where } k=\text{minimum radius of gyration.}$$

$$f_{max}=\frac{P}{A}+\frac{Pe}{Ak^2}\times y_c\,\frac{1\cdot 2 f_e}{f_e-f_0}=\frac{P}{A}\left[1+\frac{ey_c}{k^2}\times\frac{1\cdot 2 f_e}{f_e-f_0}\right]$$

$$=f_0\left[1+\frac{ey_c}{k^2}\times\frac{1\cdot 2 f_e}{f_e-f_0}\right]$$

Say $f_{max}=f$, allowable stress then

$$f=f_0\left[1+\frac{ey_c}{k^2}\times\frac{1\cdot2 f_e}{f_e-f_0}\right]$$

or

$$\frac{f}{f_0}-1=\frac{1\cdot2\,ey_c}{k^2}\times\left(\frac{f_e}{f_e-f_0}\right)$$

or

$$\left(\frac{f}{f_0}-1\right)\left(1-\frac{f_0}{f_e}\right)=\frac{1\cdot2\,ey_c}{k^2} \qquad ...(3)$$

This is professor Perry's approximate formula. One can work out f_0, if f (allowable stress) and e (eccentricity) are given.

Example 15·8-1. A stanchion is built up of an 25 cm × 12·5 cm RSJ section with a 15 cm × 1·2 cm plate riveted to each flange. Estimate the safe load for this stanchion, length 5 metre, ends hinged, from the Perry' formula, if the maximum compressive stress is limited to 800 kg/cm².

For the joist area of cross section = 35·53 cm²,

I_{xx} = 3717·8 cm⁴ ; I_{yy} = 193·4 cm⁴, and $E = 2\times10^6$ kg/cm²,

The eccentricity from the axis yy is 3 cm.

Solution. The Fig. 15·12 shows the built up section with an I section 25 × 12·5 cm and two plates on the flanges, each of the size 15 × 1·2 cm.

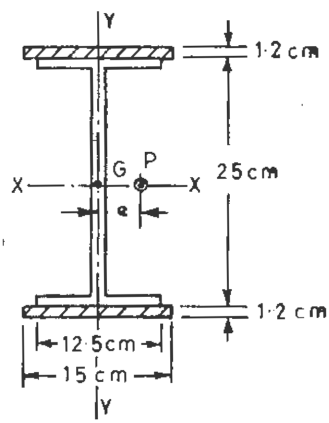


Fig. 15·12

Moment of Inertia,

$$I_{yy}=193\cdot4+2\times1\cdot2\times\frac{15^3}{12}$$

$$=193\cdot4+675\cdot0$$

$$=868\cdot4 \text{ cm}^4.$$

There is no necessity of calculating the moment of inertia I_{xx} because eccentricity is given along X-X axis as shown. The centroid of the section is at G but load is applied at a point P at a distance of 3 cm from the YY axis along the X-X axis.

Euler's buckling load,

$$P_e=\frac{\pi^2 EI}{l^2} \quad \text{(as the ends are hinged)}$$

$$l = 5 \text{ metres} = 500 \text{ cm}$$

So,

$$P_e=\frac{\pi^2\times2\times10^6\times868\cdot4}{500\times500}=68566\cdot435 \text{ kg.}$$

Area of cross section of the built up section

$$A=35\cdot53+2\times1\cdot2\times15=71\cdot53 \text{ cm}^2.$$

Stress,

$$f_e=\frac{68566\cdot435}{71\cdot53}=958\cdot57 \text{ kg/cm}^2$$

Permissible compressive stress, f = 800 kg/cm²

Now $k^2 = \frac{I_{yy}}{A} = \frac{868 \cdot 4}{71 \cdot 53} = 12 \cdot 14 \text{ cm}^2$

$y_c = 7 \cdot 5$ cm (distance of extreme layer in compression from neutral axis YY)

We know that

$$\left(\frac{f}{f_0} - 1\right)\left(1 - \frac{f_0}{f_e}\right) = \frac{1 \cdot 2\, e y_c}{k^2} \quad \text{where } f = \text{permissible stress} = 800 \text{ kg/cm}^2$$

$$\left(\frac{800}{f_0} - 1\right)\left(1 - \frac{f_0}{958 \cdot 57}\right) = \frac{1 \cdot 2 \times 3 \times 7 \cdot 5}{12 \cdot 14}$$

$$(800 - f_0)(958 \cdot 57 - f_0) = 958 \cdot 57\, f_0 \times 2 \cdot 224$$

$$800 \times 958 \cdot 57 - 800\, f_0 - 958 \cdot 57 f_0 + f_0^2 = 958 \cdot 57 \times 2 \cdot 224\, f_0$$

$$f_0^2 - 3890 \cdot 43\, f_0 + 766856 = 0$$

$$f_0 = \frac{3890 \cdot 93 - \sqrt{(3890 \cdot 93)^2 - 4 \times 766856}}{2}$$

$$= 208 \cdot 75 \text{ kg/cm}^2$$

Therefore, safe load on column

$$= f_0 \times A = 208 \cdot 75 \times 71 \cdot 53$$

$$= 14931 \cdot 8 \text{ kg} = 14 \cdot 93 \text{ Tonnes}$$

Exercise 15·8-1. A 40 cm × 14 cm RSJ is used as strut with hinged ends, having 6 metres length. Using the Perry's formula, determine the safe load if

(*a*) eccentricity along *X-X* axis is 2·4 cm.

(*b*) maximum allowable compressive stress = 75 N/mm².

(*c*) For the joist, area of cross section = 78·46 cm².

$I_{xx} = 20458 \cdot 4 \text{ cm}^4$, $I_{yy} = 622 \cdot 1 \text{ cm}^4$.

(*d*) $E = 210$ kN/mm². [**Ans.** 121·5 kN]

15·9. LONG COLUMNS WITH INITIAL CURVATURE

A column *AB* of length *l* with both the ends hinged has the initial curvature such that the deflection at the centre of the column is e', as shown in the Fig. 15·13. When this column is subjected to a gradually increasing axial load, it buckles at the load *P* as shown in the figure. Column with initial curvature is $AC'B$ and the column in the buckled state is $AC''B$. Consider a section of the column at a distance of *x* from the end *A*.

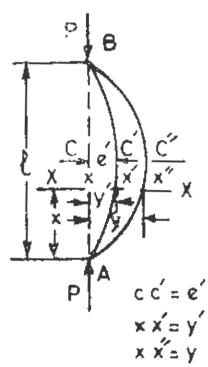


Fig. 15·13

Initial deflection at the section

$$= y' = e' \sin \frac{\pi x}{l} \quad \ldots(1)$$

where e' = maximum initial central deflection.

Final deflection ot the section = y

Change in deflection at the section

$$= y - y'$$

B.M. at the section $= -Py$

So $$EI \frac{d^2}{dx^2}(y-y') = -Py$$

$$EI \frac{d^2y}{dx^2} - EI \frac{d^2y'}{dx^2} = -Py \qquad ...(2)$$

From equation (1),

$$\frac{dy'}{dx} = e' \frac{\pi}{l} \cos \frac{\pi x}{l} \quad \text{and} \quad \frac{d^2y'}{dx^2} = -e' \frac{\pi^2}{l^2} \sin \frac{\pi x}{l}$$

Substituting in equation (2)

$$EI \frac{d^2y}{dx^2} + EI\, e' \frac{\pi^2}{l^2} \,.\, \sin \frac{\pi x}{l} = -Py$$

or $$\frac{d^2y}{dx^2} + \frac{P}{EI}\, y = -e' \frac{\pi^2}{l^2} \,.\, \sin \frac{\pi x}{l} \qquad ...(3)$$

Let us say that the solution of this differential equation is

$$y = A\, e' \sin \frac{\pi x}{l} \qquad ...(4)$$

where A is a constant.

Differentiating equation (4) two times, we get

$$\frac{d^2y}{dx^2} = -A \frac{\pi^2}{l^2}\, e' \sin \frac{\pi x}{l}$$

Substituting this value in equation (3),

$$-A \frac{\pi^2}{l^2}\, e' \;\text{in}\; \frac{\pi x}{l} + \frac{P}{EI}\, y = -e' \frac{\pi^2}{l^2} \sin \frac{\pi x}{l}$$

$$-A \frac{\pi^2}{l^2}\, e' \;\text{in}\; \frac{\pi x}{l} + \frac{P}{EI} \times A\, e' \sin \frac{\pi x}{l} = -e' \frac{\pi^2}{l^2} \,.\, \sin \frac{\pi x}{l}$$

or $$Ae'\left(\frac{P}{EI} - \frac{\pi^2}{l^2}\right) = -e' \frac{x^3}{l^2}$$

$$A\left(\frac{\pi^2}{l^2} - \frac{P}{EI}\right) = \frac{\pi^2}{l^2}$$

Dividing throughout by π^2/l^2, we get

$$A\left(1 - \frac{Pl^2}{\pi^2 EI}\right) = 1$$

$$A\left(1 - \frac{P}{P_e}\right) = 1 \qquad \text{where} \qquad P_e = \frac{\pi^2 EI}{l^2}$$

or $$A = \frac{P_e}{P_e - P}$$

The equation of the curve for the deflection y will be

$$y = \frac{P_e}{P_e - P}\, e' \sin \frac{\pi x}{l} \qquad ...(5)$$

Maximum deflection occurs at the centre *i.e.*, at $x = \frac{l}{2}$

$$y_{max} = \frac{P_e}{P_e - P} \,.\, e'$$

Maximum bending moment at the centre

$$M_{max}=P\,.\,y_{max}=\frac{P\,.\,P_e}{P_e-P}\,.\,e'$$

The maximum compressive stress at the central section of the column

$$f_{max}=f_0+\frac{M_{max}}{Z}=\frac{P}{A}+\frac{P\,.\,P_e}{(P_e-P)}\,.\,e'\,\frac{y_c}{Ak^2}$$

where k=minimum radius of gyration

A=area of cross section of the column

y_c=distance of the extreme layer in compression from the neutral axis.

or $$f_{max}=\frac{P}{A}\left[1+\frac{P_e}{P_e-P}\times e'\,\frac{y_c}{k^2}\right]=f_0\left[1+\frac{f_e}{f_e-f_0}\times\frac{e'\,y_c}{k^2}\right] \quad ...(6)$$

where $f_e=\frac{P_e}{A}$ and $f_0=\frac{P}{A}$

Equation (6) can be simplified as

$$(f_{max}-f_0)\left(\frac{f_e-f_0}{f_e\,f_0}\right)=\frac{e'y_e}{k^2}$$

$$f_0\left(\frac{f_{max}}{f_0}-1\right)\left(\frac{f_e-f_0}{f_ef_0}\right)=\frac{e'\,y_e}{k^2}$$

$$\left(\frac{f_{max}}{f_0}-1\right)\left(1-\frac{f_0}{f_e}\right)=\frac{e'y_c}{k^2} \quad ...(7)$$

Example 15·9-1. A hollow circular steel strut 4 m long, outside diameter 12 cm and inside diameter 8 cm, with both the ends hinged is initially bent. Assuming the centre line of strut as sinusoidal with maximum deviation of 6 mm, determine the maximum stress developed due to an axial load of 10 tonnes. E=2080 tonnes/cm².

Solution. Length of the strut=4m=400 cm

Maximum deviation at the centre=e'=0·6 cm

Area of cross section, $A=\frac{\pi}{4}(12^2-8^2)=62{\cdot}832$ cm²

Moment of inertia, $I=\frac{\pi}{64}(12^4-8^4)=816{\cdot}816$ cm⁴

Radius of gyration, $k^2=\frac{I}{A}=\frac{816{\cdot}816}{62{\cdot}832}=13$ cm²

Euler's load. $P_e=\frac{\pi^2\,EI}{l^2}=\frac{\pi^2\,E\,Ak^2}{l^2}$

Stress $f_e=\frac{P_e}{A}=\frac{\pi^2\,E\,k^2}{l^2}=\frac{\pi^2\times2080\times13}{400\times400}$

$=1{\cdot}668$ tonnes/cm²

Axial load, P=10 tonnes

Stress, $f_0=\frac{10}{62{\cdot}832}=0{\cdot}159$ tonne/cm²

Distance of the extreme layer in compression from the neutral axis,

$$y_c = \frac{12}{2} = 6 \text{ cm.}$$

We know that

$$\left(\frac{f_{max}}{f_0} - 1\right)\left(1 - \frac{f_0}{f_e}\right) = \frac{e' y_c}{k^2}$$

$$\therefore \left(\frac{f_{max}}{0{\cdot}159} - 1\right)\left(1 - \frac{0{\cdot}159}{1{\cdot}668}\right) = \frac{0{\cdot}6 \times 6}{13}$$

$$\frac{f_{max}}{0{\cdot}159} - 1 = 0{\cdot}306$$

$$f_{max} = 1{\cdot}306 \times 0{\cdot}159 = 0{\cdot}207 \text{ tonne/cm}^2.$$

Exercise 15·9-1. A 150 mm$\times$80 mm RS joist is used as a strut with both the ends hinged. The length of the strut is 6 metres. The strut is initially with its centre line making a sinusoidal curve with maximum deviation of 20 mm. Determine the maximum stress developed due to an axial load of 10 kN. $E = 208 \times 10^3$ N/mm². For the joist, area of the section 19·00 cm², $I_{xx} = 726{\cdot}4$ cm⁴, $I_{yy} = 52{\cdot}6$ cm⁴. [**Ans.** 27·64 N/mm²]

15·10. PERRY-ROBERTSON FORMULA

For a long column with initial curvature, the relationship between f_{max}, f_0 and f_e has been worked out as

$$\left(\frac{f_{max}}{f_0} - 1\right)\left(1 - \frac{f_0}{f_e}\right) = \frac{e' y_c}{k^2} \qquad \ldots(1)$$

where e' is the maximum deviation.

Then Prof. Perry gave the relationship between f_e, f_0 and f_{max} for a long column with eccentric loading as follows :

$$\left(\frac{f_{max}}{f_0} - 1\right)\left(1 - \frac{f_0}{f_e}\right) = \frac{1{\cdot}2\, e\, y_c}{k^2} \qquad \ldots(2)$$

Both the formulae given by equations (1) and (2) are alike and if we take

$$e_1 = 1{\cdot}2e + e'$$

where e = eccentricity of loading

e' = maximum deviation for a column with initial curvature

We can write down a relationship for a column initially bent and eccentrically loaded as follows :

$$\left(\frac{f_{max}}{f_0} - 1\right)\left(1 - \frac{f_0}{f_e}\right) = \frac{e_1 y_c}{k^2} \qquad \ldots(3)$$

Let us say f = allowable stress $= f_{max}$

and $\dfrac{e_1 y_c}{k^2} = \lambda$

$$\left(\frac{f}{f_0} - 1\right)\left(1 - \frac{f_0}{f_e}\right) = \lambda$$

or $$f_0^2 - f_0[f + f_e(1+\lambda)] + f \cdot f_e = 0 \qquad \ldots(4)$$

The solution of this quadratic equation (4) gives

$$f_0 = \frac{f+f_e(1+\lambda)}{2} - \sqrt{\left\{\frac{f+f_e(1+\lambda)}{2}\right\}^2 - f_0 f_e} \quad \ldots(5)$$

where $\lambda = \frac{e_1 y_c}{k^2}$; $f_e = \frac{\pi^2 E}{(l^2/k^2)}$ for both the ends hinged

In Rankine's and other empirical formulae, the column is assumed to be perfectly straight and the loads to be truly axial but in actual practice neither of these conditions is satisfied. The formula for f_0 given by equation (5) is Prof Perry's formula for the permissible load per unit area allowing for defects such as initial crookedness of the column and initial eccentricity of loading. In the above formula λ or e_1 is an unknown factor.

Prof. Andrew Robertson after investigating the experimental observations came to the conclusion that $\lambda = 0.003\left(\frac{l}{k}\right)$ is valid for large number of experimental observations. But he has taken f, the allowable stress in tons/in² and $f_e = \frac{\pi^2 E}{(l/k)^2}$ in tons/in².

Allowable stress, $f = 18$ tons/in² for steel columns
$= 2835$ kg/cm² $= 277.9$ N/mm².

Example 15.10-1. Two 200 mm × 70 mm mild steel channels are welded together at their toes to form a box section 200 mm × 140 mm × 6 metres long. The box section is used as a strut with both the ends hinged. Estimate the safe load for this strut from the Perry-Robertson formula using allowable stress = 250 N/mm² and $\lambda = 0.003\ (l/k)$ for each channel section, area of section = 17.77 cm², $I_{xx} = 1161.9$ cm⁴, $I_{yy} = 84.2$ cm⁴, $\bar{x} = 1.97$

$$E = 210 \times 10^3 \text{ N/mm}^2.$$

Solution. Fig. 15.14 shows the box section 200 mm × 140 mm made from two channels.

Area of the section

$A = 2 \times 17.77 = 35.54$ cm²

CG of the section will be at G as shown

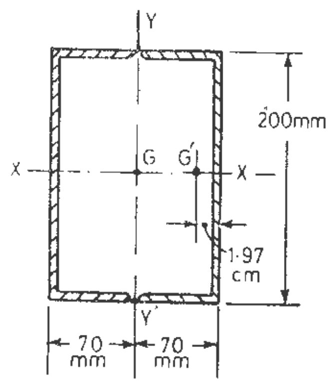


Fig. 15.14

Moment of inertia

$I_{xx} = 2 \times 1161.9 = 2323.8$ cm⁴

$I_{yy} = 2 \times 84.2 + 2 \times 17.77(7 - 1.97)^2$

$= 168.4 + 899.2$

$= 1067.6$ cm⁴

So $I_{yy} < I_{xx}$

$$k^2 = \frac{I_{yy}}{35.54} = \frac{1067.6}{35.54} = 30.04 \text{ cm}^2$$

$k = 5.48$ cm

Length of the strut $= 6$ metres $= 600$ cm

$$\lambda = 0.003\left(\frac{l}{k}\right) \text{ for both the ends hinged}$$

$$= 0.003 \times \frac{600}{5.48} = 0.328$$

Allowable stress, $f=250$ N/mm²

Stress due to Euler's load,

$$f_e=\frac{\pi^2 E k^2}{l^2}=\pi^2\times 210\times 1000\times\left(\frac{5{\cdot}48}{600}\right)^2=172{\cdot}89 \text{ N/mm}^2$$

$$f_e(1+\lambda)=172{\cdot}89(1+0{\cdot}328)=229{\cdot}6 \text{ N/mm}^2$$

$$\frac{f+f_e(1+\lambda)}{2}=\frac{250+229{\cdot}6}{2}=239{\cdot}8 \text{ N/mm}^2$$

Stress,

$$f_0=\frac{f+f_e(1+\lambda)}{2}-\sqrt{\left\{\frac{f+f_e(1+\lambda)}{2}\right\}^2-f\cdot f_e}$$

$$=239{\cdot}8-\sqrt{(239{\cdot}8)^2-250\times 172{\cdot}89}=239{\cdot}8-119{\cdot}5$$

$$f_0=120{\cdot}3 \text{ N/mm}^2$$

Safe load $=2\times 17{\cdot}77\times 100\times 120{\cdot}3$ N

$=427{\cdot}546$ kN.

Exercise 15·10-1. A stanchion is built up of an 500 mm × 180 mm RS section with 200 mm−20 mm plate riveted to each flange. Estimate the safe load for this stanchion, length 5 metres, ends hinged from Perry Robertson formula taking allowable stress $f=2800$ kg/cm², $\lambda=0{\cdot}003\ (l/k)$. For the joist area of cross section $=95{\cdot}50$ cm², $I_{xx}=38579$ cm⁴, $I_{yy}=1063{\cdot}9$ cm⁴, $E=2\times 10^6$ kg/cm². [**Ans.** 218 Tonnes]

15·11. LATERAL LOADING ON STRUTS

If a strut or a long column carries lateral loading, perpendicular to its axis in addition to the axial thrust, the section of the strut will have to resist the effect of axial thrust and bending moment due to lateral loading. Lateral loading produces deflection in the strut and axial thrust produces additional bending moment due to the deflection. The bending stress at any section will be the algebraic sum of the stress produced by the lateral loads and the stress produced by the eccentricity (due to the deflection) of the longitudinal strut.

Consider a column AB of length l, with hinged ends A, B carrying a transverse load, W at its centre and a longitudinal thrust P. The reactions due to W at C, at the ends A and B are $\frac{W}{2}$ each. Taking a section X-X at a distance of x from the end A. B M. at the section is

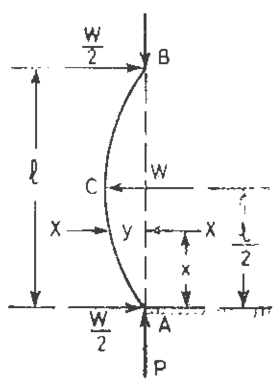


Fig. 15 15

$$M=-Py-\frac{W}{2}x$$

or $$EI\frac{d^2y}{dx^2}=-Py-\frac{W}{2}x$$

or $$EI\frac{d^2y}{dx^2}+Py=-\frac{W}{2}x \qquad \ldots(1)$$

The solution of this differential equation is

$$y=A\cos k'x+B\sin k'x-\frac{Wx}{2P} \qquad \ldots(2)$$

where $$k'=\sqrt{\frac{P}{EI}}$$

At the end A ; $y=0$; $x=0$

Putting the values in equation (2),

$$0=A+B\times 0-0 \quad \text{or} \quad A=0$$

Therefore, $$y=B\sin k'x-\frac{Wx}{2P} \qquad ...(3)$$

Differentiation equation (3)

$$\frac{dy}{dx}=Bk'\cos k'x-\frac{W}{2P}$$

Now at the centre C ; $\frac{dy}{dx}=0$, $x=\frac{l}{2}$ because the strut is symmetrically loaded about its centre C.

So $$0=Bk'\cos k'\frac{l}{2}-\frac{W}{2P}$$

or $$B=\frac{W}{2Pk'}\sec k'\frac{l}{2}$$

Equation of the deflection becomes

$$y=\frac{W}{2Pk'}\times\sec\frac{k'l}{2}\,.\,\sin k'x-\frac{Wx}{2P}$$

Deflection is maximum at the centre *i.e.* at $x=l/2$, we get

$$y_{max}=\frac{W}{2Pk'}\sec\frac{k'l}{2}\sin\frac{k'l}{2}-\frac{Wl}{4P}$$

$$=\frac{W}{2Pk'}\tan\frac{k'l}{2}-\frac{Wl}{4P}$$

Maximum bending moment occurs at the centre

$$M_{max}=Py_{max}+\frac{W}{2}\times\frac{l}{2}$$

$$=\frac{W}{2k'}\tan\frac{k'l}{2}-\frac{Wl}{4}+\frac{Wl}{4}=\frac{W}{2k'}\tan\frac{k'l}{2}$$

$$M_{max}=\frac{W}{2}\sqrt{\frac{EI}{P}}\tan\frac{k'l}{2}$$

Section modulus, $$Z=\frac{Ak^2}{y_c}$$

where A=area of cross section of the section

k=minimum radius of gyration

y_c=distance of the extreme layer in layer in compression from the neutral axis.

Stress due to bending, $$f_b=\frac{M_{max}}{Z}=\frac{Wy_c}{2Ak^2}\sqrt{\frac{EI}{P}}\tan\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}$$

Direct stress, $f_0=\frac{P}{A}$

Maximum stress, $f_{max}=f_0+f_b=\frac{P}{A}+\frac{Wy_c}{2Ak^2}\sqrt{\frac{EI}{P}}\tan\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}$

Example 15·11-1. A circular steel strut of 25 mm diameter 1 metre long is subjected to an axial thrust of 12 kN. In addition, a lateral load W acts at the centre of the strut. If the strut is to fail at $f_{max}=320$ N/mm², determine the magnitude of W. Given,

$$E=210 \text{ kN/mm}^2.$$

Solution. Diameter, $d=25$ cm

Area of cross section, $A=\frac{\pi}{4}\times d^2$

$$=\frac{\pi}{4}\times 25^2=490{\cdot}875 \text{ mm}^2$$

Moment of Inertia, $I=\frac{\pi d^4}{64}=19{\cdot}175\times 10^3 \text{ mm}^4.$

Length, $l=1000$ mm

Axial load, $P=12$ kN

Direct stress, $f_0=\frac{P}{A}=\frac{12\times 1000}{490{\cdot}875}=24{\cdot}45 \text{ N/mm}^2$

$$f_{max}=320 \text{ N/mm}^2$$

$$f_b=320-24{\cdot}45=295{\cdot}55 \text{ N/mm}^2$$

Now $\frac{P}{EI}=\frac{12000}{210\times 1000\times 19{\cdot}175\times 10^3}=2{\cdot}98\times 10^{-6}$

$$\sqrt{\frac{P}{EI}}=1{\cdot}726\times 10^{-3}$$

$$\sqrt{\frac{EI}{P}}=0{\cdot}579\times 10^3$$

Moreover $\sqrt{\frac{P}{EI}}\times\frac{l}{2}=1{\cdot}726\times 10^{-3}\times 500=0{\cdot}863$ radian

$$=49{\cdot}45°$$

$$\tan\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}=1{\cdot}17$$

Radius of gyration, $k^2=\frac{d^2}{16}=\frac{25^2}{16}=39{\cdot}06 \text{ mm}^2$

and $y_c=12{\cdot}5$ mm.

Substituting the values in the expression for f_b

$$f_b=\frac{Wy_c}{2Ak^2}\sqrt{\frac{EI}{P}}\tan\sqrt{\frac{P}{EI}}\frac{l}{2}$$

$$295{\cdot}55=\frac{W\times 12{\cdot}5\times 0{\cdot}579\times 10^3\times 1{\cdot}17}{2\times 490{\cdot}875\times 39{\cdot}06}\times 1{\cdot}17$$

$$=0{\cdot}22\,W$$

or $W=1343{\cdot}4 \text{ N}=1{\cdot}343 \text{ kN}.$

Exercise 15·11-1. A horizontal pin ended strut 4 m long is formed from a standard T section 15 cm × 10 cm × 1·25 cm. The axial compressive load is 6 tonnes. A lateral concentrated load of 0·6 tonne acts at the centre of the strut. Find the maximum stress if the X-X axis is horizontal and the table of the Tee forms the compressive face. The centroid is 2·4 cm below the top. $I_{xx}=250$ cm^4, $A=31$ cm^2, $E=2000$ tonnes/cm^2.

[**Ans.** 0·8835 tonnes/cm^2]

15·12. STRUT WITH UNIFORMLY DISTRIBUTED LATERAL LOAD

Fig. 15·16 shows a strut AB of length l subjected to longitudinal thrust P and carrying a uniformly distributed load w per unit length throughout its length. Its ends A and B are hinged. Consider a section at a distance of x from the end A.

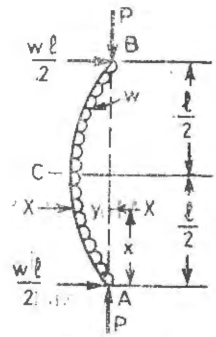


Fig. 15·16

B.M. at the section

$$=-Py-\frac{wl}{2}x+\frac{wx^2}{2}$$

$$EI\frac{d^2y}{dx^2}=-Py-\frac{wx}{2}(l-x)$$

or
$$\frac{d^2y}{dx^2}+\frac{P}{EI}y=-\frac{wx}{2EI}(l-x) \quad \ldots(1)$$

The solution of the differential equation is

$$y=\text{Complementary function}+\text{Particular integral}$$

Complementary function $=A\cos k'x+B\sin k'x$

where A and are constants and $k'=\sqrt{\dfrac{P}{EI}}$

Particular integral $=+\dfrac{wx^2}{2P}-\dfrac{wlx}{2P}-\dfrac{wEI}{P^2}$

So
$$y=A\cos k'x+B\sin k'x+\frac{wx^2}{2P}-\frac{wlx}{2P}-\frac{wEI}{P^2} \quad \ldots(2)$$

At $x=0$, $y=0$ at end A. Therefore,

$$0=A\cos 0°+B\sin 0°-\frac{wEI}{P^2} \quad \text{or} \quad A=\frac{wEI}{P^2}$$

So
$$y=\frac{wEI}{P^2}\cos k'x+B\sin k'x+\frac{wx^2}{2P}-\frac{wlx}{2P}-\frac{wEI}{P^2}$$

Differentiating
$$\frac{dy}{dx}=-\frac{wEI}{P^2}\times k'\sin k'x+Bk'\cos k'x+\frac{wx}{P}-\frac{wl}{2P} \quad \ldots(3)$$

But
$$\frac{dy}{dx}=0$$

at the centre of the strut because strut is symmetrically loaded about its centre C

So at $x=\dfrac{l}{2}$, $\dfrac{dy}{dx}=0$, substituting in equation (3)

$$0=-\frac{wEI}{P^2}\,.\,k'\sin k'\frac{l}{2}+Bk'\cos k'\frac{l}{2}+\frac{wl}{2P}-\frac{wl}{2P}$$

$$B=\frac{wEI}{P^2}\tan k'\frac{l}{2}=\frac{wEI}{P^2}\tan\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}$$

The equation for deflection will now be

$$y=\frac{wEI}{P^2}.\cos k'x+\frac{wEI}{P^2}\tan k'\frac{l}{2}\times\sin k'x$$

$$+\frac{wx^2}{2P}-\frac{wlx}{2P}-\frac{wEI}{P^2}$$

Maximum deflection occurs at the centre *i e.* at $x=l/2$

$$\therefore \quad y_{max}=\frac{wEI}{P^2}\cos k'\frac{l}{2}+\frac{wEI}{P^2}\tan k'\frac{l}{2}\sin k'\frac{l}{2}$$

$$+\frac{wl^2}{8P}-\frac{wl^2}{4P}-\frac{wEI}{P^2}$$

$$=\frac{wEI}{P^2}\left[\cos k'\frac{l}{2}+\tan k'\frac{l}{2}\times\sin k'\frac{l}{2}\right]-\frac{wl^2}{8P}-\frac{wEI}{P^2}$$

$$=\frac{wEI}{P^2}\sec k'\frac{l}{2}-\frac{wl^2}{8P}-\frac{wEI}{P^2}$$

$$=\frac{wEI}{P^2}\sec\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}-\frac{wl^2}{8P}-\frac{wEI}{P^2}$$

Maximum bending moment occurs at the centre *i.e.* at $x=l/2$

$$M_{max}=-P.y_{max}-\frac{wl}{2}\cdot\frac{l}{2}+\frac{wl^2}{8}$$

$$=-\frac{wEI}{P}\sec\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}+\frac{wl^2}{8}+\frac{wEI}{P}-\frac{wl^2}{4}+\frac{wl^2}{8}$$

$$=-\frac{wEI}{P}\left[\sec\sqrt{\frac{P}{EI}}\times\frac{l}{2}-1\right]$$

Now maximum stress, $f_{max}=f_0+f_b$

$$f_0=\frac{P}{A},\ f_b=\frac{M_{max}}{Ak^2}\times y_c$$

$$f_{max}=\frac{P}{A}+\frac{wEI}{P}\times\frac{y_c}{Ak^2}\left[\sec\sqrt{\frac{P}{EI}}\times\frac{l}{2}-1\right]$$

$$=\frac{P}{A}+\frac{wEy_c}{P}\left[\sec\sqrt{\frac{P}{EI}}\times\frac{l}{2}-1\right]\text{ since } I=Ak^2$$

Example 15·12-1. A rod of rectangular section 80 mm×40 mm is supported horizontally through pin joints at its ends and carries a vertical load of 3300 N/m length and an axial thrust of 100 kN. If its length is 2·0 m, estimate the maximum stress induced.

$E=208\times1000$ N/mm^2

Solution.

Axial thrust, $P=100\text{ kN}=100\times10^3\text{ N}$

Lateral load, $w=3000\text{ N/m}=3\text{ N/mm}$

$E=208\times1000\text{ N/mm}^2$

Minimum $I=\dfrac{80\times40^3}{12}=42{\cdot}66\times10^4\text{ mm}^4$

Area $A=80\times40=3200\text{ mm}^2$

$$\sqrt{\frac{P}{EI}}=\sqrt{\frac{100\times10^3}{208\times10^3\times42{\cdot}66\times10^4}}=0{\cdot}106\times10^{-2}$$

Length, $l=2{\cdot}0\text{ m}=2000\text{ mm}$

$$\sqrt{\frac{P}{EI}}\times\frac{l}{2}=0{\cdot}106\times10^{-2}\times1000=1{\cdot}06\text{ radian}=60{\cdot}7^\circ$$

$$\sec 60{\cdot}7^\circ=2{\cdot}043 \qquad y_c=20\text{ mm}$$

$$f_{max}=\frac{100{,}000}{3200}+\frac{3\times208\times1000\times20}{100{,}000}(2{\cdot}043-1)$$

$$=31{\cdot}25+130{\cdot}16=161{\cdot}41\text{ N/mm}^2$$

Exercise 15·12-1. A circular rod of diameter 50 mm is supported horizontally through pin joints at its ends and carries a uniformly distributed lateral load of 200 kg/m run throughout its length and an axial thrust of 5 tonnes. If its length is 2·4 m, estimate the maximum stress induced. $E=2\times10^6\text{ kg/cm}^2$. [**Ans.** 2521·65 kg/cm²]

Problem 15·1. A straight bar of steel 2·4 m long, of rectangular section 3 cm × 1·6 cm is used as a strut with both the ends hinged. Assuming that the Euler's formula is applicable and the material attains its yield strength at the time of buckling, determine the central deflection. $E=210\text{ kN/mm}^2$, yield strength $=290\text{ N/mm}^2$

Solution.

Length of strut, $l=2{\cdot}4\text{ m}=2{\cdot}4\times10^3\text{ mm}$

Breadth of the section, $b=3\text{ cm}=30\text{ mm}$

Thickness of the section, $t=1{\cdot}6\text{ cm}=16\text{ cm}$

Area of cross section $=30\times16=480\text{ mm}^2$

Minimum moment of inertia,

$$I_{min}=\frac{bt^3}{12}=\frac{30\times16^3}{12}=10240\text{ mm}^4$$

$$E=210\times10^3\text{ N/mm}^2$$

End conditions : both the ends hinged

P_e, Eulers' buckling load $=\dfrac{\pi^2EI}{l^2}=\dfrac{\pi^2\times210\times10^3\times10240}{2400\times2400}=3684{\cdot}67\text{ N}$

Maximum stress developed $=290\text{ N/mm}^2$

Section modulus, $Z=\dfrac{bt^2}{6}=\dfrac{30\times16^2}{6}=1280\text{ mm}^3$

Say the central deflection $=e$

(Refer to article 15·1)

$$f_{max}=\frac{P_e}{A}+\frac{P_e \cdot e}{Z}$$

$$290=\frac{3684·67}{480}+\frac{3684·67\times e}{1280}$$

$$290-7·676=2·878\ e$$

Central deflection, $e=98·09$ mm$=9·809$ cm

Problem 15·2. A 2·5 m length of tube has a crippling load of 110 kg when used as a strut with pin jointed ends. Calculate the crippling load for a 3 m length of the same tube when used as a strut if

(*i*) both the ends are fixed

(*ii*) one end is fixed and the other end is hinged.

Solution.

Length of the strut, $l=2·5$ m$=250$ cm

Crippling load, $P=110$ kg

End conditions : pin jointed ends

Eulers' Buckling load,

$$P_e=\frac{\pi^2\times EI}{l^2}$$

$$110=\frac{\pi^2\times EI}{250\times 250}$$

or $$EI=\frac{110\times 250\times 250}{\pi^2}=\frac{6875\times 10^3}{\pi^2}\ \text{kg cm}^2$$

(*i*) a 3 m length of the same tube is used as a strut with both ends fixed, *i.e.* EI remains the same.

Eulers buckling load, $P_e'=\dfrac{4\pi^2 EI}{300\times 300}=\dfrac{4\pi^2\times 6875\times 10^3}{\pi^2\times 300\times 300}=305·55$ kg.

(*ii*) a 3 m length of the same tube is used as a strut with one end fixed and other end hinged.

Eulers' buckling load, $P_e''=\dfrac{2\pi^2\times EI}{300\times 300}=\dfrac{2\pi^2\times 6875\times 10^3}{\pi^2\times 300\times 300}=152·77$ kg.

Problem 15·3. A round vertical bar is clamped at the lower end and is free at the other. The effective length is 2 metres. If a horizontal force of 40 kg at the top produces a horizontal deflection of 1·5 cm, what is the buckling load for the bar under the given conditions ?

Solution.

Horizontal force, $W=40$ kg

Length $l=2$ m$=200$ cm

Horizontal deflection, $\delta=1·5$ cm

For a cantilever loaded at the free end

Deflection, $\delta=\dfrac{Wl^3}{3EI}$

or $$EI=\frac{Wl^3}{3\delta}=\frac{40\times 200^3}{3\times 1\cdot 5}=71\cdot 11\times 10^6 \text{ kg cm}^2$$

End conditions for buckling : one end fixed, other end free

Buckling load, $$P_e=\frac{\pi^2 EI}{4l^2}=\frac{\pi^2\times 71\cdot 11\times 10^6}{4\times 200\times 200}$$

$$=4\cdot 38\times 10^3 \text{ kg}=4\cdot 38 \text{ Tonnes}$$

Problem 15·4. A thin vertical strut of uniform section and length l is rigidly fixed at its bottom end and its top end is free. At the top there is a horizontal force H and a vertical load P acting through the centroid of the section. Prove that the horizontal deflection at the top is

$$\frac{H}{P}\left(\frac{\tan \mu l}{\mu}-l\right) \text{ where } \mu^2=\frac{P}{EI}$$

Solution. Fig. 15·17 shows a vertical strut AB, of length l, fixed at A and free at B, carrying a horizontal force H at B and vertical load P at B. Assuming deflection at B to be a.

Let us consider a section at a distance of x from the end A.

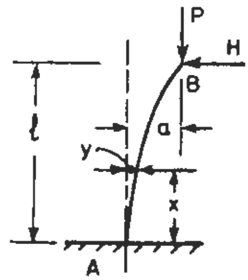


Fig. 15·17

Bending moment at section

$$=P(a-y)-H(l-x)$$

or $$EI\frac{d^2y}{dx^2}=P(a-y)-H(l-x)$$

$$EI\frac{d^2y}{dx^2}+P\,.\,y=P\,.\,a-H(l-x) \qquad \ldots(1)$$

The solution of the differential equation (1) is

$$y=A\cos\mu x+B\sin\mu x+a-\frac{H(l-x)}{P} \qquad \ldots(2)$$

where $$\mu=\sqrt{\frac{P}{EI}}$$

at the fixed end $x=0$, $y=0$

So $$0=A+0+a-\frac{Hl}{P}$$

or $$A=\left(\frac{Hl}{P}-a\right)$$

Differentiating the equation (2)

$$\frac{dy}{dx}=-A\mu\sin\mu x+B\mu\cos\mu x+\frac{H}{P}$$

Again at $x=0$, fixed end, $\frac{dy}{dx}=0$

Therefore $$0=-A\mu \sin 0+B\mu \cos 0+\frac{H}{P}$$

or $$B=-\frac{H}{\mu P}$$

Equation for deflection becomes

$$y=\left(\frac{Hl}{P}-a\right)\cos \mu x-\frac{H}{\mu P}\sin \mu x+a-\frac{H(l-x)}{P}$$

But deflection at the end B, $y=a$

$\therefore$ $$a=\left(\frac{Hl}{P}-a\right)\cos \mu l-\frac{H}{\mu P}\sin \mu l+a-0$$

or $$\left(\frac{Hl}{P}-a\right)\cos \mu l=\frac{H}{\mu P}\sin \mu l$$

$$\left(\frac{Hl}{P}-a\right)=\frac{H}{\mu P}\tan \mu l$$

or $$a=\frac{Hl}{P}-\frac{H}{\mu P}\tan \mu l=\frac{H}{P}\left(l-\frac{\tan \mu l}{\mu}\right)$$

Deflection at top, $$a=-\frac{H}{P}\left(\frac{\tan \mu l}{\mu}-l\right)$$

Problem 15·5. A strut of length l is fixed at its lower end, its upper end is elastically supported against a lateral deflection so that the resisting force is k times the end deflection. Show that the crippling load P is given by

$$1-\frac{P}{kl}=\frac{\tan \mu l}{\mu l} \quad \text{where} \quad \mu=\sqrt{\frac{P}{EI}}$$

Solution. Fig. 15·18 shows a strut AB of length l, fixed at end A and free at end B. At the end B there is a crippling load P and a horizontal reaction H. Say the deflection at the free end is a, then horizontal reaction $H=ka$, as given in the problem. Consider a section X-X at a distance of x from the end A.

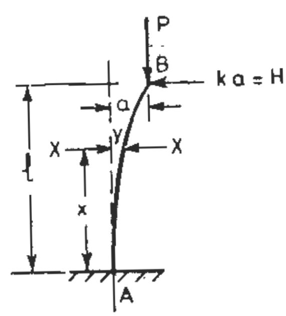


Fig. 15·18

B.M. at the section

$=P(a-y)-ka(l-x)$

where y=deflection at the section

or $$EI\frac{d^2y}{dx^2}=P(a-y)-ka\,(l-x) \quad \text{...(1)}$$

The solution of the differential equation (1) is

$$y=A\cos \mu x+B\sin \mu x+a-\frac{ka(l-x)}{P} \quad \text{...(2)}$$

where $$\mu=\sqrt{\frac{P}{EI}}$$

At the fixed end $y=0$, $x=0$.

So $0=A \cos 0+B \sin 0+a-\frac{kal}{P}$

or $$A=\frac{kal}{P}-a=a\left(\frac{kl}{P}-1\right)$$

Differentiating equation (2)

$$\frac{dy}{dx}=-A\mu \sin \mu x+B \mu \cos \mu x+\frac{ka}{P}$$

at the fixed end A, $\frac{dy}{dx}=0$ at $x=0$.

So $$0=-A\mu \sin 0+B\mu \cos 0+\frac{ka}{P}$$

or $$B=-\frac{ka}{\mu P}$$

The equation for the deflection becomes

$$y=a\left(\frac{kl}{P}-1\right)\cos \mu x-\frac{ka}{\mu P}\sin \mu x+a-\frac{ka\,(l-x)}{P}$$

But at the end B, $y=a$, $x=l$

So $$a=a\left(\frac{kl}{P}-1\right)\cos \mu l-\frac{ka}{\mu P}\sin \mu l+a-0$$

or $$\frac{ka}{\mu P}\sin \mu l=a\left(\frac{kl}{P}-1\right)\cos \mu l$$

$$\frac{k}{\mu P}\tan \mu l=\left(\frac{kl}{P}-1\right)$$

or $$\frac{k}{\mu P}\times\frac{P}{kl}\tan \mu l=1-\frac{P}{kl}$$

$$\frac{\tan \mu l}{\mu l}=1-\frac{P}{kl}$$

Problem 15·6. A 325×165 mm RS Joist is used as a strut, 6 m long, one end fixed and other hinged. Calculate the crippling load by Rankine's formula. Compare this with the load obtained by Euler's formula. For what length of this strut will the two formulae give the same crippling load ?

For the joist area of the section, $A=54{\cdot}90$ cm²

$I_{xx}=9874{\cdot}6$ cm⁴

$I_{yy}=510{\cdot}8$ cm⁴

For the steel $E=2100$ tonnes/cm²

$f_c=3{\cdot}3$ tonnes/cm²

a (for both the ends hinged)$=\frac{1}{7500}$.

Solution. End conditions : one end fixed, other hinged

Length of the strut, $l=6\text{m}=600$ cm

Equivalent length, $l_e=\dfrac{l}{\sqrt{2}}=\dfrac{600}{\sqrt{2}}$ cm

$$I_{min}=I_{yy}=510{\cdot}8 \text{ cm}^4$$

Area of cross section, $A=54{\cdot}90$ cm²

Radius of gyration, $k^2=\dfrac{I_{yy}}{A}=\dfrac{510{\cdot}80}{54{\cdot}9}=9{\cdot}3$ cm²

$$a\,.\,\frac{l_e^2}{k^2}=\frac{1}{7500}\times\frac{1}{9{\cdot}30}\left(\frac{600}{\sqrt{2}}\right)^2=2{\cdot}580$$

Rankine's load, $P_R=\dfrac{f_c\,.\,A}{1+a\,.\,\dfrac{l_e^2}{k^2}}=\dfrac{3{\cdot}3\times54{\cdot}90}{1+2{\cdot}58}=50{\cdot}60$ Tonnes.

Euler's buckling load, $P_e=\dfrac{2\pi^2 EI}{l^2}$ (for the given end conditions)

$$=\frac{2\times\pi^2\times2100\times510{\cdot}8}{600\times600}=58{\cdot}8 \text{ tonnes}$$

Say length is l for which Rankine's and Euler's buckling load are the same.

$$\frac{f_c\,.\,A}{1+a\,.\,\dfrac{l^2}{2k^2}}=\frac{2\pi^2EI}{}\quad\text{since}\quad l_e=\frac{l}{\sqrt{2}}$$

or
$$f_c\,.\,A\,.\,l^2=2\pi^2EI+a\,.\,\frac{l^2}{2k^2}\times2\pi^2\,EI$$

$$=2\pi^2EI+al^2\pi^2EA\qquad\text{as } I=Ak^2$$

or
$$l^2[\,f_c\,.\,A-a\pi^2EA]=2\pi^2EI\qquad\text{where } I=I_{yy}=I_{min}$$

Substituting the values

$$l^2\left[\,3{\cdot}3\times54{\cdot}9-\frac{1}{750}\quad\pi^2\times2100\times54{\cdot}9\,\right]=2\pi^2\times2100\times510{\cdot}8$$

$$l^2[181{\cdot}170-151{\cdot}716 \;=21\;\;{\cdot}39\times10^4$$

$$l^2=\frac{2117\;\;9}{29{\cdot}454}\times10^4$$

$$l=847 \text{ cm.}$$

Problem 15·7. A hollow cast iron column of external diameter 200 mm, length 4 m with both the ends fixed, supports an axial load 800 kN. Find the thickness of the metal required. Use Rankine's constants, $a=\dfrac{1}{6400}$ and a working stress=80 N/mm².

Solution.

Working stress, $f_w=80\times10^6$ N/m²

External diameter $=0{\cdot}2$ m

Say internal diameter $=d$ metre

Area of cross section, $A=\dfrac{\pi}{4}\,(0{\cdot}2^2-d^2)=0{\cdot}7854\,(0{\cdot}04-d^2)$ m²

Radius of gration, $k^2=\dfrac{0{\cdot}2^2+d^2}{16}=\dfrac{0{\cdot}04+d^2}{16}$

Length $=4$ m

Axial load $=0{\cdot}8\times10^6$ N

Rankine's load $P_R=\dfrac{f_w\,.\,A}{1+a\,.\,\dfrac{l^2}{k^2}}$

$$0{\cdot}8\times10^6=\frac{80\times10^6\times0{\cdot}7854(0{\cdot}04-d^2)}{1+\dfrac{1}{6400}\times\dfrac{4\times4\times16}{(0{\cdot}04+d^2)}}$$

$$0{\cdot}8+\frac{0{\cdot}032}{(0{\cdot}04+d^2)}=2{\cdot}51328-62{\cdot}832$$

$$0{\cdot}032+0{\cdot}8d^2+0{\cdot}032=0{\cdot}1-2{\cdot}51328d^2+2{\cdot}51328d^2-62{\cdot}832d^4$$

$$62{\cdot}832d^4+0{\cdot}8d^2-0{\cdot}036=0$$

$$_2=\frac{-0{\cdot}8+\sqrt{0{\cdot}64+4\times0{\cdot}036\times62{\cdot}832}}{2\times62{\cdot}832}$$

$$=\frac{-0{\cdot}8+3{\cdot}112}{2\times62{\cdot}832}=0{\cdot}0184$$

$$d=0{\cdot}1356\text{ m}=13{\cdot}56\text{ cm}=135{\cdot}6\text{ mm}$$

Thickness of the metal $=\dfrac{200-135{\cdot}6}{2}=32{\cdot}2$ mm.

Problem 15·8. A short length of tube 3 cm internal and 4 cm external diameter failed under a compressive load of 18 tonnes. When a 2 m length of the same tube is tested as a strut with both the ends hinged, the buckling load was 4·08 tonnes. Assuming that f_c for the Rankine's formula is given by first test, determine the constant take a for the Rankine's formula.

Estimate the crippling load for a piece of the tube 3 m long when used as a strut with fixed ends.

Solution. Inner dia $=3$ cm
Outer dia $=4$ cm

A, Area of cross-section $=\dfrac{\pi}{4}\,(4^2-3^2)=5{\cdot}50\text{ cm}^2$

Radius of gyration, $k^2=\dfrac{4^2+3^2}{16}=\dfrac{25}{16}\text{ cm}^2$

Ultimate compressive load $=18$ tonnes$=f_c\,.\,A$

End conditions : both the ends hinged.

Length of the strut $=2$ m$=200$ cm

Rankine's buckling load$=4{\cdot}08$ tonnes$=\dfrac{f_c\,.\,A}{1+a\left(\dfrac{l^2}{k^2}\right)}$

or
$$1+a\cdot\left(\frac{l^2}{k^2}\right)=\frac{18}{4\cdot08}$$
$$a\left(\frac{l^2}{k^2}\right)=3\cdot411 \quad \text{or} \quad a\left(\frac{200\times200\times16}{25}\right)=3\cdot411$$
$$a=\frac{3\cdot411\times25}{16\times200\times200}=\frac{1}{7505}.$$

Crippling load for another strut

Length $=3$ m $=300$ cm

$$k^2=\frac{25}{16}$$

End conditions = Both the ends are fixed

Equivalent length $=\frac{l}{2}=150$ cm

$$\text{Rankines' crippling load}=\frac{f_c\,.\,A}{1+a\,.\left(\frac{150^2}{k^2}\right)}=\frac{18}{1+\frac{1}{7505}\times\frac{150\times150\times16}{25}}$$
$$=\frac{18}{1+1\cdot9187}=6\cdot167 \text{ Tonnes.}$$

Problem 15·9. A strut is built up of the T-sections 8 cm×16 cm×1 cm riveted back to back so as to form a section of a cross. Using Gordon's formula, determine the safe load for the strut, length 4 metres with both the ends fixed. Constants in the Gordon's formula may be taken as

$$f_c=3\cdot2 \text{ tonnes/cm}^2$$
$$a_1=\frac{1}{250}. \quad \text{Use a factor of safety of 4}$$

Solution.

$$\text{Gordon's buckling load, } P_G=\frac{f_c\,A}{1+a_1\,.\,\frac{l^2}{b^2}}$$

where b is the lesser over all dimension.

Fig. 15·19 shows the built up section with overall dimensions 16 cm×16 cm. So $b=16$ cm.

Area of cross section, A
$=2\times16\times1+2\times7\times1$
$=32+14=46$ cm²

Constant, $a_1=\frac{1}{250}$

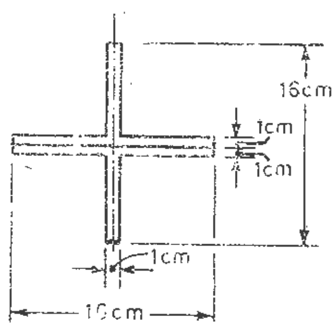


Fig. 15·19

Length of the column $=4$ m

End conditions : Both the ends fixed

Equivalent length.

$l_e=2$ m $=200$ cm

Gordon's buckling load,

$$P_G=\frac{3\cdot2\times46}{1+\frac{1}{250}\left(\frac{200}{16}\right)^2}$$

$$=\frac{3{\cdot}2\times46}{1+0{\cdot}625}=90{\cdot}58 \text{ Tonnes}$$

Factor of safety $=4$

Safe load $=\dfrac{90{\cdot}58}{4}=22{\cdot}645$ Tonnes.

Problem 15·10. A hollow circular steel pipe 6 cm outside diameter, 5 cm inside diameter, 120 cm long is fixed at both the ends, so as to prevent any expansion in its length. The pipe is unstressed at the normal temperature. Calculate the temperature stress in the pipe if its temperature rises by 40°C and the factor of safety against failure as a strut. Use Rankine's formula for buckling.

$$f_c=3{\cdot}3 \text{ tonnes/cm}^2,\quad a=\frac{1}{7500} \text{ (for both the ends hinged)}$$

$$E=2080 \text{ tonnes/cm}^2,\ \alpha=11{\cdot}1\times10^{-6}/°\text{C}.$$

Solution. Outside diameter $=6$ cm

Inside diameter $=5$ cm

Area of cross section, $A=\dfrac{\pi}{4}(36-25)=8{\cdot}639 \text{ cm}^2$

Radius of gyration, $k^2=\dfrac{6^2+5^2}{16}=3{\cdot}81 \text{ cm}^2$

Coefficient of thermal expansion, $\alpha=11{\cdot}1\times10^{-6}/°\text{C}$

Temperature rise, $T=40°\text{C}$

$E=2080 \text{ tonnes/cm}^2$

Expansion in the pipe is prevented by the fixed ends

$\therefore$ f, Temperature stress in pipe $=\alpha TE$

$=11{\cdot}1\times10^{-6}\times40\times2080$

$=0{\cdot}9235 \text{ tonne/cm}^2$ (a compressive stress)

Compressive Axial load, $P=f\times A=0{\cdot}9235\times8{\cdot}639=7{\cdot}978$ tonnes.

End conditions : Both the ends are fixed

Length of the pipe as strut $=120$ cm

Equivalent length, $l_e=60$ cm

Rankine's buckling load $P_R=\dfrac{f_c\,.\,A}{1+a\dfrac{l_e^2}{k^2}}=\dfrac{3{\cdot}3\times8{\cdot}639}{1+\dfrac{1}{7500}\times\dfrac{60\times60}{3{\cdot}81}}$

$$=\frac{3{\cdot}3\times8{\cdot}639}{1+0{\cdot}126}=25{\cdot}32 \text{ Tonnes.}$$

Factor of safety as a strut $=\dfrac{25{\cdot}32}{7{\cdot}978}=3{\cdot}17.$

Problem 15·11. A long strut AB of length L is of uniform section throughout. A thrust P is applied at the ends eccentrically on the same side of the centre line, with eccentricity at the end B twice that at the end A. Show that the maximum bending moment occurs at a distance x from the end A, where

$$\tan kx=\left(\frac{2-\cos kL}{\sin kL}\right) \quad\text{and}\quad k=\sqrt{\frac{P}{EI}}$$

Solution. Fig. 15·20 shows a strut of length l, having eccentricity in loading e at end A and $2e$ at end B The strut is buckled under the thrust P. Since the eccentricities at the two ends are different there will be equal and opposite lateral reaction at the ends, say the reaction is F as shown.

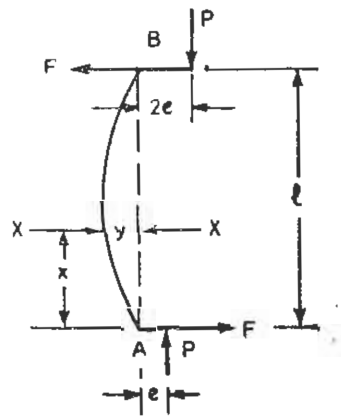


Fig. 15·20

Let us first calculate the magnitude of F.

Taking moments about the point A

$$2Pe-Pe-Fl=0$$

$$F=\frac{Pe}{l} \qquad ...(i)$$

Now consider a section X-X at a distance of x from the end A.

B.M. at the section$=-P(2e+y)+F(l-x)$

or $$EI\frac{d^2y}{dx^2}=-2Pe-Py+\frac{Pe}{l}(l-x) \qquad ...(i)$$

or $$\frac{d^2y}{dx^2}+\frac{P}{EI}y=-\frac{2Pe}{EI}+\frac{Pe}{lEI}(l-x) \qquad ...(ii)$$

$$=-\frac{2Pe}{EI}-\frac{Pex}{lEI}+\frac{Pe}{EI}=-\frac{Pe}{EI}-\frac{Pex}{lEI} \qquad ...(i)$$

The solution of the differential equation (1) is

$$y=A\cos kx+B\sin kx-e-\frac{ex}{l} \qquad ...(2)$$

where A and B are constants of integration.

$$k=\sqrt{\frac{P}{EI}}$$

At the end A, $x=0$, $y=0$

So $$0=A\cos 0+B\sin 0-e-0$$

or $$A=e$$

At the end B, $x=l$, $y=0$.

So $$0=A\cos kl+B\sin kl-2e$$
$$=e\cos kl+B\sin kl-2e$$

or $$B=\frac{2e-e\cos kl}{\sin kl}$$

So $$y=e\cos kx+\left(\frac{2e-e\cos kl}{\sin kl}\right).\sin kx-e-\frac{ex}{l}$$

B.M. at the section.

$$M_x=-2Pe-Pe\cos kx-P\left(\frac{2e-e\cos kl}{\sin kl}\right)\sin kx+Pe+\frac{Pex}{l}+\frac{Pel}{l}-\frac{Pex}{l}$$

$$M_x = -Pe\cos kx - Pe\ \frac{(2-\cos kl)}{\sin kl}\ \sin kx$$

For the bending moment to be maximum let us put

$$\frac{dM_x}{dx} = 0 = +Pe\,k\sin kx - Pek\ \frac{(2-\cos kl)}{\sin kl}\ \cos kx$$

or $$\sin kx = \left(\frac{2-\cos kl}{\sin kl}\right)\cos kx$$

or $$\tan kx = \frac{2-\cos kl}{\sin kl} \qquad \text{where} \quad k = \sqrt{\frac{P}{EI}}$$

Problem 15·12. A steel tube 80 mm outer diameter and 60 mm inner diameter, 2·8 m long is used as a strut with both the ends hinged. The load is parallel to the axis of the strut but is eccentric. Find the maximum value of the eccentricity so that crippling load on strut is equal to 60% of the Euler's crippling load. Given yield strength=320 N/mm². $E=210$ kN/mm².

Solution.

Length, $l = 2{\cdot}8\ \text{m} = 2800\ \text{mm}$

Outside diameter, $D = 80\ \text{mm}$

Inside diameter, $d = 60\ \text{mm}$

Area of cross section $A = \frac{\pi}{4}\ (80^2 - 60^2) = 21{\cdot}99 \times 10^2\ \text{mm}^2$

Moment of Inertia, $I = \frac{\pi}{64}\ (80^4 - 60^4) = 137{\cdot}445 \times 10^4\ \text{mm}^4$

Radius of gyration, $k^2 = \frac{I}{A} = \frac{137{\cdot}445 \times 10^4}{21{\cdot}99 \times 10^2} = 6{\cdot}25 \times 10^2\ \text{mm}^2$

End conditions : Both ends hinged

Euler's buckling load, $P_e = \frac{\pi^2 EI}{l^2}$

$$= \frac{\pi^2 \times 210 \times 10^3 \times 137{\cdot}445 \times 10^4}{2800 \times 2800} = 363{\cdot}35 \times 10^3\ \text{N}$$

Stress due to Euler's crippling load,

$$f_e = \frac{363{\cdot}35 \times 10^3}{21{\cdot}99 \times 10^2} = 165{\cdot}23\ \text{N/mm}^2$$

Crippling load, $P = 0{\cdot}6\ P_e = 0{\cdot}6 \times 363{\cdot}35 \times 10^3\ \text{N} = 218{\cdot}01 \times 10^3\ \text{N}$

Stress, $f_0 = \frac{P}{A} = \frac{218{\cdot}01 \times 10^3}{21{\cdot}99 \times 10^2} = 99{\cdot}14\ \text{N/mm}^2$

Stress, $f_{max} = f_0 + f_b = 320$

$$f_b = 320 - 99{\cdot}14 = 220{\cdot}86\ \text{N/mm}^2$$

$$\sec \frac{l}{2}\sqrt{\frac{P}{EI}} = \sec \frac{\pi}{2}\sqrt{\frac{P}{P_e}} = \sec \frac{\pi}{2}\sqrt{0{\cdot}6}$$

$\sec\ (1{\cdot}216\ \text{radian}) = \sec\ (69{\cdot}7^\circ) = 2{\cdot}883$

Stress due to bending $f_b = \frac{P}{A} \cdot \frac{ey_c}{k^2} \sec \frac{l}{2}\sqrt{\frac{P}{EI}}$

$$290{\cdot}86 = f_0 \cdot \frac{e \cdot y_c}{625} \times 2{\cdot}883$$

$$f_0 = 99{\cdot}14 \text{ N/mm}^2 \quad y_c = 40 \text{ mm}$$

$$e = \frac{220{\cdot}86 \times 625}{99{\cdot}14 \times 40 \times 2{\cdot}883}$$

Maximum eccentricity, $e = 12{\cdot}073$ mm.

Problem 15·13. A double angle strut is made up of two 100 mm×65 mm×6 mm angles riveted together along the longer legs with a 1·5 cm separator in between. Calculate the safe axial load for the strut, 3 metres long, ends fixed. The safe load in kg/cm² being $1080 - \frac{1}{45}\left(\frac{l}{k}\right)^2$ for fixed ends. For each angle, area of section $= 9{\cdot}55$ cm², $\bar{x} = 1{\cdot}47$ cm, $\bar{y} = 3{\cdot}19$ cm, $I_{xx} = 96{\cdot}7$ cm⁴ $I_{yy} = 32{\cdot}4$ cm⁴.

Solution. Fig. 15·21 shows the built up section used for the strut.

Area of cross section

$$= 2 \times 9{\cdot}55 + 10 \times 1{\cdot}5$$
$$= 34{\cdot}1 \text{ cm}^2.$$

Length of the strut, $l = 300$ cm

The section is symmetrical about YY-axis. Let us determine the position of the X-X axis. Choose the lower edge AB as the reference line

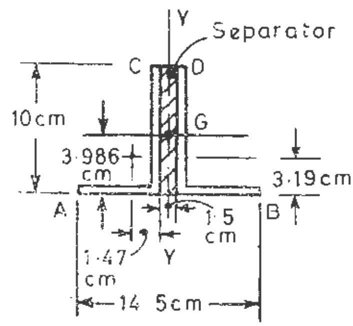


Fig. 15·21

$$\bar{y} = \frac{2 \times 9{\cdot}55 \times 3{\cdot}19 + 1{\cdot}5 \times 10 \times 5}{34{\cdot}1} = 3{\cdot}986 \text{ cm}$$

Moment of inertia, $I_{xx} = 2 \times 96{\cdot}7 + 2 \times 9{\cdot}55(3{\cdot}986 - 3{\cdot}19)^2 + 1{\cdot}5 \times \frac{10^3}{12} + 15(5 - 3{\cdot}986)^2$

$$= 193{\cdot}4 + 12{\cdot}10 + 125 + 15{\cdot}423 = 345{\cdot}923 \text{ cm}^4$$

$$I_{yy} = \frac{10 \times 1{\cdot}5^3}{12} + 2 \times 32{\cdot}4 + 2 \times 9{\cdot}55(0{\cdot}75 + 1{\cdot}47)^2$$

$$= 2{\cdot}8125 + 64{\cdot}8 + 94{\cdot}1324 = 161{\cdot}75 \text{ cm}^4$$

So $\qquad I_{yy} < I_{xx}$.

Minimum radius of gyration, $k^2 = \frac{I_{yy}}{A} = \frac{161{\cdot}75}{34{\cdot}1} = 4{\cdot}743$ cm²

Safe stress $= 1080 - \frac{1}{45}\left(\frac{l}{k}\right)^2$

End conditions. Both the ends are fixed and above formula is given for these end conditions.

Safe stress $=1080-\frac{1}{45}\times\frac{300\times 300}{4{\cdot}743}=1080-421{\cdot}67$

$=658{\cdot}33$ kg/cm²

Safe load $=658{\cdot}33\times 34{\cdot}1$ kg $=22449$ kg

$=22{\cdot}449$ tonnes.

Problem 15 14. A strut has initial curvature in the form of a parabolic arc and is hinged at both the ends. Show that the maximum compressive stress produced by a load P is

$$\frac{P}{A}\left[1+\frac{e'y_c}{k^2}\cdot\frac{8P_e}{\pi^2 P}\left(\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}-1\right)\right]$$

where A = cross sectional area

e' = initial central deflection

P_e = Eulerian buckling load

k = least radius of gyration

y_c = distance of the extreme fibre in compression from the neutral axis.

Solution. Fig. 15·22 shows a strut *ACB* with initial curvature which is parabolic subjected to thrust P.

Say $y'=\dfrac{4e'\,x(l-x)}{l^2}$ ...(1)

Consider a section X-X at a distance of x from the end A.

B.M. at the section $=-Py$

Change in deflection (as shown)

$=y-y'$

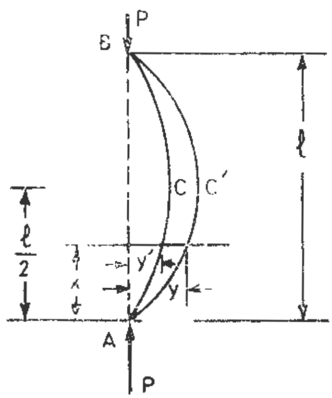


Fig. 15·22

So $$EI\frac{d^2}{dx^2}(y-y')=-Py \quad ...(2)$$

From equation (1) $\dfrac{dy'}{dx}=\dfrac{4e'}{l^2}(l-2x)$

$$\frac{d^2y'}{dx^2}=-\frac{8e'}{l^2}$$

So $EI\dfrac{d^2y}{dx^2}+EI\dfrac{8e'}{l^2}=-Py$ (putting values in equation (2))

or $$EI\frac{d^2y}{dx^2}+Py=-\frac{8e'EI}{l^2} \quad ...(3)$$

The solution of the differential equation (3) is

$$y=A\cos k'x+B\sin k'x-\frac{8e'EI}{Pl^2} \qquad \text{where} \quad k'=\sqrt{\frac{P}{IE}}$$

at $x=0$, $y=0$ at end A

$$0=A\cos 0+B\sin 0-\frac{8e'EI}{Pl^2}$$

or
$$A=\frac{8e'EI}{Pl^2}=K \quad (a \text{ constant})$$

$\therefore$
$$y=K\cos k'x+B\sin k'x-K \qquad \ldots(4)$$

At the centre of the strut, $\frac{dy}{dx}=\text{slope}=0$ because the strut carries the load symmetrically about its centre

From equation (4)

$$\frac{dy}{dx}=-Kk'\sin k'x+Bk'\cos k'x$$
$$=0 \quad \text{at} \quad x=l/2$$

or
$$-Kk'\sin\frac{k'l}{2}+Bk'\cos\frac{k'l}{2}=0$$

$$B=K\tan\frac{k'l}{2}$$

$$y=K\cos k'x+K\tan\frac{k'l}{2}\sin k'x-K$$

Maximum deflection takes place at the centre *i.e.* at $x=l/2$

$$y_{max}=K\cos\frac{k'l}{2}+K\tan\frac{k'l}{2}\sin\frac{k'l}{2}-K$$

$$=K\left[\cos\frac{k'l}{2}+\frac{\sin^2\frac{k'l}{2}}{\cos\frac{k'l}{2}}\right]-K$$

$$y_{max}=K\sec\frac{k'l}{2}-K$$

Now
$$\frac{k'l}{2}=\sqrt{\frac{P}{EI}}\times\frac{l}{2}=\sqrt{\frac{Pl^2}{4EI}}$$

$$=\sqrt{\frac{P}{4}\times\frac{l^2}{\pi^2EI}\times\pi^2}=\frac{\pi}{2}\sqrt{\frac{P}{P_e}}$$

$$y_{max}=K\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}-K$$

Maximum bending moment

$$=M_{max}=P.y_{max}=PK\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}-PK$$

Direct stress,
$$f_0=\frac{P}{A}$$

f_b, maximum stress due to bending$=\frac{M_{max}}{Ak^2}\times y_o$

$$=\frac{Py_oK}{Ak^2}\left[\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}-1\right]$$

Maximum compressive stress$=f_0+f_b$

$$=\frac{P}{A}+\frac{Py_c}{Ak^2}\times\frac{8e'EI}{Pl^2}\left[\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}-1\right]$$

$$=\frac{P}{A}\left[1+\frac{e'y_c}{k^2}\times\frac{8EI}{Pl^2}\left(\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}-1\right)\right]$$

but $P_e=\frac{\pi^2EI}{l^2}$ - Euler's load

$\frac{EI}{l^2}=\frac{Pe}{\pi^2}$ substituting above

Maximum compressive stress

$$=\frac{P}{A}\left[1+\frac{e'y_e}{k^2}\cdot\frac{8P_e}{\pi P}\left(\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}-1\right)\right]$$

SUMMARY

1. Maximum stress developed in the section of a long column or strut

$f_{max}=\frac{P}{A}-\frac{P_e}{Z}$ where P=Buckling load,

e=maximum deflection in column,
Z=section modulus
A=area of cross section

2. Eulers Buckling load

$P_e=\frac{\pi^2EI}{l_e^2}$ where E=Young's modulus

I=minimum moment of inertia of the section
l_e=equivalent length

$l_e=l$ (for both the ends hinged or both the ends free)

$=2l$ (for one end fixed, other end free)

$=\frac{l}{2}$ (for both the ends fixed)

$=\frac{l}{\sqrt{2}}$ (for one end fixed and other end hinged)

3. Euler's buckling load is applicable for long columns or struts

if $\frac{l_e}{k}>80$ (for mild steel struts) where k=minimum radius of gyration

4. Rankine's buckling/crippling load,

$$P_R = \frac{f_c \cdot A}{1 + a \cdot \frac{l_e^2}{k^2}}$$

where f_c=Rankine's constant for ultimate compressive strength
a=Rankine's constant (connecting f_c and E)
l_e=equivalent length (as explained above)
k=minimum radius of gyration of the section
A=area of cross section

5. For braced girders, minimum distance between the lattice bars,

$$l = L_e \cdot \frac{k}{K}$$

where L_e=equivalent length of column or strut
k=minimum radius of gyration of a member (a girder) used for built up section
K=minimum radius of gyration of the built up section of the column/stanchion

6. Straight line formula for working stress,

$$f_w = f_c' \left[1 - c \frac{l_e}{k}\right]$$

f_c'=allowable stress in compression,
=110 N/mm^2 (for mild steel)

$c = \frac{1}{200}$ for pinned ends (mild steel)

$= \frac{1}{250}$ for fixed ends (mild steel)

7. Johnson's parabolic formula

Working stress, $f_w = f_c' \left[1 - b\left(\frac{l_e}{k}\right)^2\right]$

where $b = 3 \times 10^{-5}$ for pinned ends, mild steel strut
$= 2 \times 10^{-5}$ for fixed ends, mild steel strut

8. Gordon's formula for buckling load

$$P_G = \frac{f_c \cdot A}{1 + a_1 \cdot \frac{l^2}{b^2}}$$

where a_1=constant,
b=lesser overall dimension of the section
l=length of the column

9. If the axial thrust P on a long column acts with an eccentricity e, maximum stress developed is

$$f_{max} = \frac{P}{A} + \frac{Pe \sec \frac{l_e}{2}\sqrt{\frac{P}{EI}}}{Z}$$

where l_e=equivalent length of the column

Z=section modulus

A=area of cross section

10. Prof. Perry's formula for long columns with eccentric load P

$$\left(\frac{f}{f_0}-1\right)\left(1-\frac{f_0}{f_e}\right)=\frac{1\cdot 2\, e\, y_c}{k^2}$$

where f_e = Eulers' load per unit area

e = eccentricity

y_c = distance of extreme layer in compression from the neutral axis

k = minimum radius of gyration

$f_0=\frac{P}{A}$ = Axial thrust per unit area

f = allowable stress in compression

11. For long columns having initial curvature (of sinusoidal shape)

$$\left(\frac{f_{max}}{f_0}-1\right)\left(1-\frac{f_0}{f_e}\right)=\frac{e' y_c}{k^2}$$

f_{max} = maximum allowable stress

e' = initial central deflection in strut/column

12. Perry-Robertson formula taking into account the inherrent crookedness of the column and eccentricity in loading

$$\frac{P}{A}=f_0=\frac{f+f_e(1+\lambda)}{2}-\sqrt{\left\{\frac{f+f\,(1+\lambda)}{2}\right\}^2-f\,.\,f_e}$$

where f = allowable stress in compression

f_e = Eulers' buckling load per unit area

$\lambda=\frac{e_1 . y_c}{k^2}=0\cdot 003\left(\frac{l}{k}\right)$

valid for large number of experimental observations

f = 2835 kg/cm^2, 278 N/mm^2 for steel columns

$e_1=1\cdot 2\, e+e'$

$=1\cdot 2\times$eccentricity in loading+central deflection due to initial crookedness in the column

13. For a long column/strut carrying axial thrust P and a central lateral load W,

$$f_{max}=\frac{P}{A}+\frac{W y_c}{2Ak^2}\sqrt{\frac{EI}{P}}\tan\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}$$

where l = length of the strut

A = area of cross section

k = minimum radius of gyration

y_c = distance of the extreme layer in compression from the neutral axis

14. For a long strut/column carrying axial thrust P and a lateral uniformly distributed load w per unit length

$$f_{max}=\frac{P}{A}+\frac{w\ .\ E\ y_c}{P}\left[\sec\sqrt{\frac{P}{EI}}\ .\ \frac{l}{2}-1\right]$$

MULTIPLE CHOICE QUESTIONS

1. A column of length 240 cm, area of cross section 20 cm^2, moment of inertia $I_{xx}=720$ cm^4 and $I_{yy}=80$ cm^4 is subjected to a buckling load. The slenderness ratio of the column is

   (*a*) 40   (*b*) 80
   (*c*) 120   (*c*) 160.

2. A strut of length l is fixed at one end and free at the other end. The Euler's buckling load for this strut is 10 kN. If both the ends of the strut are now fixed, its Euler's buckling loud will be

   (*a*) 160 kN   (*b*) 120 kN
   (*c*) 80 kN   (*d*) 40 kN.

3. A long column of length l is fixed at one end and hinged at the other end. If EI is the flexural rigidity of the column then Euler's buckling load for the column is

   (*a*) π^2EI/l^2 (*b*) $\pi^2EI/4l^2$
   (*c*) $2\pi^2EI/l^2$ (*d*) $4\pi^2EI/l^2$.

4. Rankine's constant for the compressive strength of a cast iron column is generally taken as

   (*a*) 200 N/mm^2 (*b*) 250 N/mm^2
   (*c*) 320 N/mm^2 (*d*) 550 N/mm^2.

5. A braced girder 4 m long both ends hinged is made up of 4 angle sections braced by lattice. The minimum radius of gyration of the built up section is 17·5 cm while the minimum radius of gyration of one angle section is 3·5 cm. The minimum distance between the bracings is

   (*a*) 1·6 m   (*b*) 1 m
   (*c*) 0·8 m   (*d*) 0·4 m.

6. Euler's buckling load formula is applicable for

   (*a*) Short columns   (*b*) Columns of medium length
   (*c*) Long columns   (*d*) None of the above.

7. The ratio of equivalent length of a column with both its ends fixed to its own length is

   (*a*) 2·0   (*b*) 1·414
   (*c*) 1·00   (*d*) 0·50.

8. A hollow circular section with outer diameter 8 cm and inner diameter 6 cm is subjected to buckling. The radius of gyration of the section is

   (*a*) 2·5 cm   (*b*) 2·0 cm
   (*c*) 1·5 cm   (*d*) 1·00 cm

9. A long column of length l is subjected to a buckling load P at an eccentricity e. One end of the column is fixed and other end is free. The flexural rigidity of the column is EI and its area of cross section is A and Z is the section modulus. The maximum stress developed in column section due to bending is

(a) $\frac{Pe}{Z} \cdot \sec \frac{l}{2} \sqrt{\frac{P}{EI}}$ (b) $\frac{Pe}{Z} \cdot \sec l \sqrt{\frac{P}{EI}}$

(c) $\frac{Pe}{Z} \sec \frac{l}{4} \sqrt{\frac{P}{EI}}$ (d) $\frac{Pe}{Z} \sec 2l \sqrt{\frac{P}{EI}}$.

10 The cross section of a strut is rectangular with dreadth 6 cm and thickness 1 cm. If the length of the columu is 1 metre with both the ends fixed, and $E=1000$ tonnes/cm², the Euler's buckling load for the strut is

(a) 0·25 π^2 Tonnes (b) 0·20 π^2 Tonnes

(c) 0·10 π^2 Tonnes (d) 0·05 π^2 Tonnes.

ANSWERS

1. (c)	2. (a)	3. (c)	4. (d)	5. (c)
6. (c)	7. (d)	8. (a)	9. (b)	10. (b)

EXERCISES

15·1. A straight bar of an aluminium alloy 1 m long of circular section, diameter 3 cm is used as a strut with both the ends hinged. Assuming the Euler's formula to apply and that the material attains its yields strength at the time of buckling, estimate the central deflection. $E=0{\cdot}75\times10^6$ kg/cm², yield strength$=3090$ kg/cm² [**Ans.** 12·459 cm]

15·2. A tube 1·8 m long has a crippling load of 8 kN when used as a strut with fixed ends. Calculate the crippling load for a 3 m length of the same tube when used as a strut if

(a) Both the ends are pin jointed.

(b) One end is fixed and the other end is free. [**Ans.** (i) 0·72 kN (ii) 0·18 kN]

15·3. An alloy tube 5 m long extends by 3 mm under a tensile load of 60 kN. Calculate the buckling load for the tube when used as a strut with pin jointed ends. The tube diameters are 40 mm and 3 . mm. [**Ans.** 6·311 kN]

15·4. A thin vertical strut of circular section 6 cm diameter and length 2 m is rigidly fixed at the bottom and its top end is free. At the top there is a horizontal load 200 kg and a vertical load 1000 kg acting through the centroid of the section. Determine the maximum stress developed in the section of the strut. $E=2080$ tonnes/cm². [**Ans.** 1·706 tonnes/cm²]

15·5. A 200×100 mm Rolled steel joist is used as a strut, 5 metres long, both the ends fixed. Calculate the buckling load by Rankine's formula. Compare this with the load obtained by Euler's formula. For what length of the strut will the two formulae give the same buckling load ? For the joist area of the section$=25\ 27$ cm², $I_{xx}=1696{\cdot}6$ cm⁴ $I_{yy}=115{\cdot}4$ cm⁴. $E=210$ kN/mm²

Rankine's constants are $f_c=320$ N/mm²,

$$a=\frac{1}{7500} \text{ (if both the ends are hinged)}$$

[**Ans.** 286·04 kN, 382·69 kN, 9·31 m]

15·6. A hollow cast iron column with hinged ends supports an axial load of 100 tonnes. If the column is 6 m long and has an external diameter of 30 cm, find the thickness of the metal required.

Use the Rankine's formula, taking constant

$a=\frac{1}{1600}$ (for both the ends hinged) and a working stress of

800 kg/cm² [**Ans.** 23·05 mm]

15·7. A short length of tube 36 mm outside diameter and 24 mm inside diameter is tested under compression. It failed at a load of 181 kN. When a 2·5 m length of the same tube is tested as a strut with fixed ends, the failing load is 65·11 kN. Determine Rankine's constants.

Estimate the crippling load for a piece of tube 2 m long, when used as a strut with one end fixed and the other end hinged. $\left[\textbf{Ans.}\ \ 320\ \text{N/mm}^2.\ \frac{1}{7503},\ 56{\cdot}07\ \text{kN}\right]$

15·8. A double angle strut is made up of two 90 mm × 60 mm × 8 mm angles riveted back to back along the longer side. Using Gordon's formula, determine the safe load for the strut, 5 metres long with both the ends hinged. Constants in Gordon's formula may be taken as

$$f_c=320\ \text{N/mm}^2,\ \ a_1=\frac{1}{250}$$

Use factor of safety as 3. For an angle, area of the section = 11·37 cm².

[**Ans.** 18·18 kN]

15·9. A hollow circular steel pipe 45 mm inside diameter and 60 mm outside diameter, 2 metres long is fixed at both the ends so as to prevent any expansion in its length. The pipe is unstresed at the normal temperature conditions. Calculate the thermal stress in the pipe if its temperature rises by 60°C, and the factor of safety against failure as a strut. Use Rankine's formula

$E=210\ \text{kN/mm}^2$, $\alpha=11{\cdot}1\times10^{-6}/°\text{C}$

$f_c=320\ \text{N/mm}^2$, $a=\frac{1}{7500}$ (for both the ends hinged)

[**Ans.** 139·86 N/mm², 1·66]

15·10. A long strut, initially straight, 1 metre long and 30 mm diameter is subjected to a thrust $P=40$ kN. Load P is applied eccentrically with an eccentricity e at one end and $2e$ at the other end. Calculate the eccentricity e, if the maximum stress developed in the strut is 250 N/mm². (Refer to Problem 15·11) $E=210\times10^3$ N/mm². [**Ans.** $e=3{\cdot}9$ mm]

15·11. A steel tube 66 mm outer diameter and 50 mm inner diameter, 4 m long is used as a strut with both the ends fixed. The load is parallel to the axis of the strut but is eccentric. Find the maximum value of eccentricity so that the crippling load on the strut is equal to 70 % of the Euler's crippling load. Given yield strength = 320 N/mm²

$E=208000$ N/mm². [**Ans.** $e=3{\cdot}55$ mm]

15·12. A strut is having a built up section made by riveting a 150 mm × 80 mm RSJ to a 100 × 45 mm channel section so as to make a T-shape. Calculate the safe axial load for the strut, 2 metres long both the ends hinged. The safe load, in N/mm² is $110-(l/k)^2/110$ for both the ends hinged. For each RSJ, area of the section = 18·08 cm², $I_{xx}=688{\cdot}2$ cm⁴, $I_{yy}=55{\cdot}2$ cm⁴. For the channel section. Area = 7·41 cm², $I_{xx}=123{\cdot}8$ cm⁴, $I_{yy}=14{\cdot}9$ cm⁴, $\bar{x}=1{\cdot}4$ cm, web thickness = 3·0 mm. [**Ans.** 148·4 kN]

15·13. A circular strut 1·2 m long, diameter 3·6 cm has initial curvature and its centre line forms a parabolic curve with a central deflection of 5 mm. The strut is hinged at both the ends. It is subjected to a thrust of 2 tonnes. Determine the maximum compressive stress produced in the section. $E=2100$ tonnes/cm². [**Ans.** 0·415 tonne/cm²]

16

Strain Energy Methods

In various chapters we have studied about the strain energy stored in a body or a structure subjected to external forces and couples. Then in chapter 11, we have studied about the deflection of beams and cantilevers subjected to various types of transverse loads. In this chapter we will study about the use of strain energy for the determination of displacement or deflection at a particular point, and in a particular direction. The entire body or the structure and the forces acting on it are considered. There are displacements due to externally applied forces and the work done by these forces is stored as strain energy in the members. It is assumed that external forces are gradually applied. The strain energy stored in a body is utilised to determine the displacement along a certain applied force or to determine the angular rotation due to a certain applied bending moment or to determine the angular twist due to a certain applied torque.

16·1. CASTIGLIANO'S FIRST THEOREM

If a body is acted upon by forces $F_1, F_2, F_3 \ldots F_n$ and U is the strain energy stored in the body, then partial derivative of the strain energy with respect to a force F_i gives the displacement of the body in the direction of F_i, or the displacement $\delta_i = \dfrac{\partial U}{\partial F_i}$. This theorem is extremely useful in determining the displacements of complicated structures. To prove this

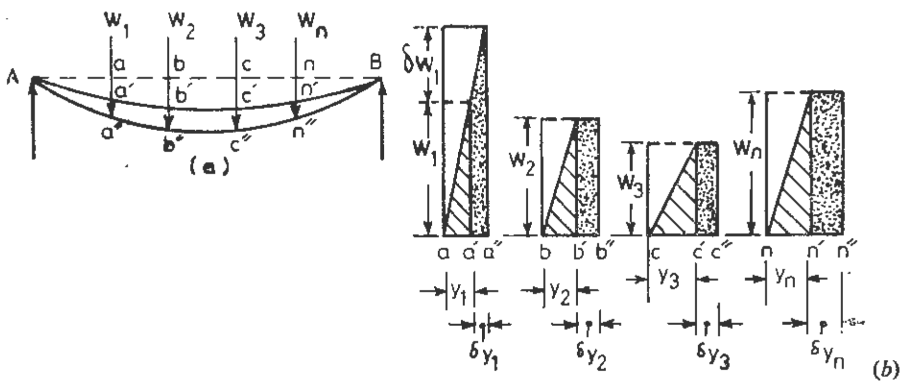


Fig. 16·1

Theorem, let us consider a beam AB of length l, simply supported at the ends and this beam is initially straight. Say a number of transverse loads W_1, W_2, W_3 ...n are gradually applied on the beam and under these loads say the deflections are y_1, y_2, y_3 ...y_n respectively. Then strain energy stored in the beam is

$$U=\frac{1}{2}\ [W_1 y_1+W_2 y_2+W_3 y_3 \ldots W_n y_n] \qquad \ldots(1)$$

i.e. area covered under triangles shown in the figure 16·1 (*b*).

Let us say that load W_1 is increased by δW_1 and due to this additional load δW_1, the deflections increase *i.e.* y_1 increases to $y_1+\delta y_1$; y_2 increases to $y_2+\delta y_2$; y_3 increases to $y_3+\delta y_3$ and so on, y_n increases to $y_n+\delta y_n$.

Additional energy,

$$\delta U=\tfrac{1}{2}\delta W_1\delta y_1+W_1\delta y_1+W_2\delta y_2+W_3\delta y_3 \ldots\ldots W_n\delta y_n \qquad \ldots(2)$$

(*i.e.*, shaded areas shown in the figure)

Differentiating partially the equation (1) with respect to load W_1

$$2\frac{\partial U}{\partial W_1}=y_1+W_1\frac{\partial y_1}{\partial W_1}+y_2\frac{\partial W_2}{\partial W_1}+W_2\frac{\partial y_2}{\partial W_1}$$

$$+y_3\frac{\partial W_3}{\partial W_1}+W_3\frac{\partial y_3}{\partial W_1}\ \ldots\ldots y_n\frac{\partial W_n}{\partial W_1}+W_n\frac{\partial y_n}{\partial W_1}$$

but loads W_1, W_2, W_3......W_n are constants

So
$$2\frac{\partial U}{\partial W_1}=y_1+W_1\frac{\partial y_1}{\partial W_1}+W_2\frac{\partial y_2}{\partial W_1}+W_3\frac{\partial y_3}{\partial W_1}\ \ldots\ldots W_n\frac{\partial y_n}{\partial W_1} \qquad \ldots(3)$$

From equation (2) neglecting product of small quantities

$$\delta U=W_1\delta y_1+W_2\delta y_2+W_3\delta y_3\ldots\ldots W_n\delta y_n$$

Dividing throughout by δW_1 we get

$$\frac{\delta U}{\delta W_1}=W_1\frac{\delta y_1}{\delta W_1}+W_2\frac{\delta y_2}{\delta W_1}+W_3\frac{\delta y_3}{\delta W_1}\ldots\ldots W_n\frac{\delta y_n}{\delta W_1}$$

or in the limits

$$\frac{\partial U}{\partial W_1}=W_1\frac{\partial y_1}{\partial W_1}+W_2\frac{\partial y_2}{\partial W_1}+W_3\frac{\partial y_3}{\partial W_1}\ \ldots\ldots W_n\frac{\partial y_n}{\partial W_1} \qquad \ldots(4)$$

Substracting equation (4) from equation (3) we get

$$\frac{\partial U}{\partial W_1}=y_1$$

Similarly it can be proved that

$$\frac{\partial U}{\partial W_2}=y_2,\ \frac{\partial U}{\partial W_3}=y_3\ldots\ldots\ \frac{\partial U}{\partial W_n}=y_n.$$

If a system of forces F_1, F_2...F_n, bending moments M_1, M_2, M_3...M_n and twisting moments T_1, T_2, T_3...T_n are simultaneously acting on a body, then Castigliano's theorem can be extended to find angular rotation due to bending moments and angular twist due to twisting moments also.

i.e.,
$$\frac{\partial U}{\partial M_i}=\phi_i\text{, angular rotation}$$

$$\frac{\partial U}{\partial T_i}=\theta_i\text{, angular twist.}$$

16·2. ELASTIC STRAIN ENERGY DUE TO AXIAL FORCE

Consider a bar of length l, cross sectional area A and young's modulus E, subjected to an axial tensile force P as shown in the figure 16·2.

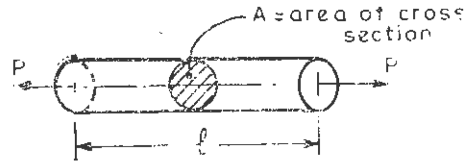


Fig. 16·2

Stress in the bar, $f=\dfrac{P}{A}$

Strain energy per unit volume, $u=\dfrac{f^2}{2E}$

Total strain energy, $U=\dfrac{f^2}{2E}\times\text{Volume}=\dfrac{f^2}{2E}\times Al=\dfrac{P^2}{2A^2E}\times Al=\dfrac{P^2l}{2AE}$

Displacement along the force $P=\dfrac{\partial U}{\partial P}=\dfrac{Pl}{AE}=$axial extension.

Example 16·2-1. A structure is shown in Fig. 16·3 (a). The area of cross section of each member is A and Young's modulus of elasticity is E. Load P is applied at the joint C of the structure. Determine the deflection under the load P.

Solution. Let us first determine the forces in each member so that strain energy can be calculated. Give Bow's notations to spaces in and around the structure say E, F, G, H, I, as shown. Draw a line ef parallel to force P (represented by Bow's notations EF) and to some suitable scale. From e, draw a line ei parallel to EI and a line fi parallel to FI. Similarly consider the forces at the joint B and complete the force diagram for the structure as shown by Fig. 16·3 (b).

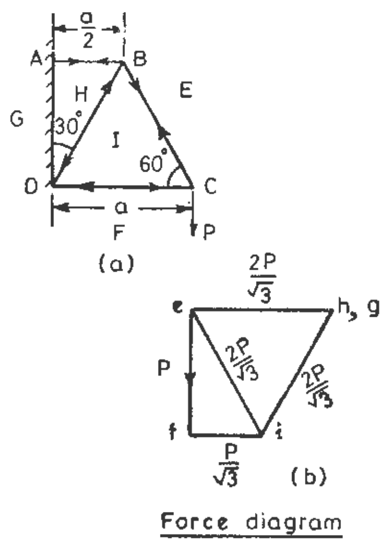


Fig. 16 3

Members	*Bow's rotations*	*Force, F*	*Length l,*	F^2l
AB	EH	$+\dfrac{2P}{\sqrt{3}}$ (tensile)	$\dfrac{a}{2}$	$\dfrac{2P^2a}{3}$
BC	EI	$+\dfrac{2P}{\sqrt{3}}$	a	$\dfrac{4P^2a}{3}$
CD	FI	$-\dfrac{P}{\sqrt{3}}$	a	$\dfrac{P^2a}{3}$
DB	HI	$-\dfrac{2P}{\sqrt{3}}$ (compressive)	a	$\dfrac{4P^2a}{3}$

$$\Sigma F^2 l = \frac{11P^2 a}{3}$$

Strain energy, $$U = \frac{11P^2 a}{2 \times 3AE} = \frac{11P^2 a}{6AE}$$

Deflection under the load, $$\delta_P = \frac{\partial U}{\partial P} = \frac{11Pa}{6AE}.$$

Exercise 16·2-1. A structure is shown in the Fig. 16·4. The cross section of the members AB, BC and CA is each 5 cm². Determine the deflection under the load P if $P = 5$ tonnes

$E = 2000$ tonnes/cm².

[**Ans.** 1·91 mm]

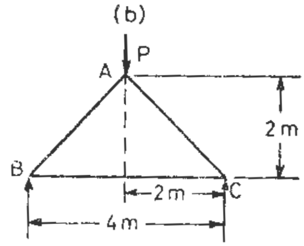


Fig. 16·4

16·3. ELASTIC STRAIN ENERGY DUE TO SHEAR STRESS

Consider a rectangular block of dimensions $l \times b \times h$ as shown in the Fig. 16·5, fixed at lower end subjected to a tangential force Q at the top face. The block is distorted under this shear force. Say the displacement of the top face along the direction of Q is δs.

Work done on the block

$= \frac{1}{2} Q \, . \, ds$

$=$ Energy stored

Energy stored,

$= \frac{1}{2} Q \, . \, \delta s$

$$= \frac{1}{2} \frac{Q}{lbh} \times \delta s \; lbh$$

$$= \frac{1}{2} \left(\frac{Q}{lb}\right) \times \left(\frac{\delta s}{h}\right) lbh$$

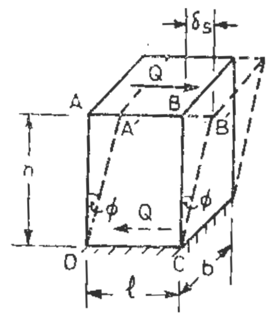


Fig. 16·5

where $$\frac{Q}{lb} = q, \text{ shear stress}$$

$$\frac{\delta s}{h} = \phi, \text{ shear strain as the angle is very small, } \tan \phi = \phi$$

Therefore, $$U = \frac{1}{2} q . \phi \; lbh$$

where shear strain, $\phi = \frac{q}{G}$, G = shear modulus

$$U = \frac{1}{2} \frac{q^2}{G} \times lbh$$

$$= \frac{q^2}{2G} \times (\text{Volume of the body})$$

Shear strain energy per unit volume, $u_s = \dfrac{q^2}{2G}$

We have seen in chapter on distribution of shear stresses in beams, that the distribution of shear stress across a section is complicated, therefore shear strain energy must be integrated over the whole section of a body and may not be taken as a constant. The shear strain energy due to shear deformation is very small and many a times ignored. Therefore the error caused by assuming uniform distribution of the shear force across the section will be very small.

Example 16·3-1. A beam of rectangular cross section breadth b, depth d and of length l is simply supported at its ends. It carries a concentrated load W at its centre. Determine the shear strain energy in the beam and find the deflection due to shear.

G=Modulus of rigidity for the beam.

Solution. Fig. 16·6 (a) shows a beam AB of length l, simply supported at the ends and carrying a concentrated load W at its centre C. Shear force between A to C is $+W/2$ and between C to B, shear force is $-W/2$.

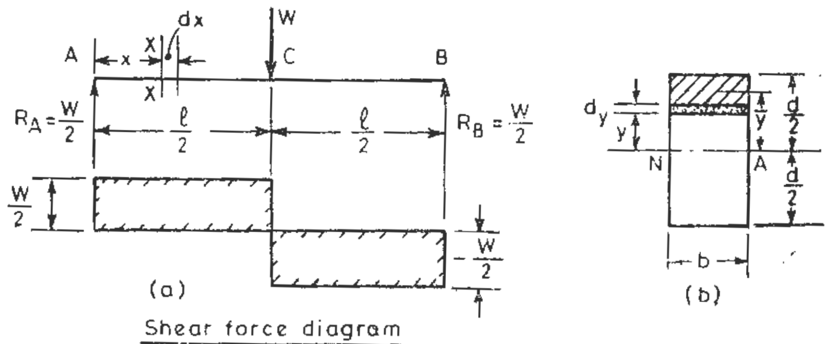


Fig. 16·6

Consider a section X-X at a distance of x from the end A.

Shear force $\quad F_x = +\dfrac{W}{2}$

Now consider a small length dx. Let us determine shear strain energy for the portion AC. Fig. 16·6 (b) shows the section of the beam. Consider a layer of thickness dy at a distance of y from the neutral axis.

Shear stress q at the layer$=\dfrac{Fa\bar{y}}{Ib}$

where $\quad F$=shear force at the section$=W/2$

$a\bar{y}$=first moment of the area above the layer about neutral axis

I=moment of inertia$=\dfrac{bd^3}{12}$

b=breadth

Shear stress, $\quad q = \dfrac{W}{2} \times \dfrac{12}{bd^3} \times \dfrac{b}{b}\left(\dfrac{d}{2} - y\right)\left(y + \dfrac{d/2 - y}{2}\right)$

$$=\frac{6W}{bd^3}\left(\frac{d}{2}-y\right)\left(\frac{d/2+y}{2}\right)$$

$$=\frac{3W}{bd^3}\left(\frac{d^2}{4}-y^2\right)$$

$$q^2=\frac{9W^2}{b^2d^6}\left(\frac{d^4}{16}+y^4-\frac{d^2y^2}{2}\right)$$

Volume of the layer $=b.dx\ dy$

Shear strain energy in the layer

$$=\frac{q^2}{2G}\ b.dx.dy$$

$$=\frac{1}{2G}\ b\times\frac{9W^2}{b^2d^6}\left(\frac{d^4}{16}+y^4-\frac{d^2y^2}{2}\right)dy.dx$$

Total shear strain energy for the portion AC

$$U_s'=\frac{1}{2G}\int_0^{l/2}\int_{-d/2}^{+d/2}\frac{9W^2}{bd^6}\left(\frac{d^4}{16}+y^4-\frac{d^2y^2}{2}\right)dy\ dx$$

$$=\frac{1}{2G}\int_0^{l/2}\frac{9W^2}{bd^6}\left[\frac{d^4y}{16}+\frac{y^5}{5}-\frac{d^2y^3}{6}\right]_{-d/2}^{d/2}dx$$

$$=\frac{1}{2G}\int_0^{l/2}\frac{9W^2}{bd^6}\left[\left(\frac{d^5}{32}+\frac{d^5}{32}\right)+\left(\frac{d^5}{160}+\frac{d^5}{160}\right)-\left(\frac{d^5}{48}+\frac{d^5}{48}\right)\right]dx$$

$$=\frac{1}{2G}\int_0^{l/2}\frac{9W^2}{bd^6}\left(\frac{d^5}{16}+\frac{d^5}{80}-\frac{d^5}{24}\right)dx$$

$$=\frac{1}{2G}\int_0^{l/2}\left(\frac{9W^2}{bd^6}\times\frac{d^5}{30}\right)dx=\frac{1}{2G}\int_0^{l/2}\frac{3W^2}{30\ bd}\ dx$$

$$=\frac{1}{2G}\times\frac{9W^2}{30bd}\times\frac{l}{2}=\frac{2W^2l}{40Gbd}$$

Since the beam is symmetrically loaded, shear force in the portion CB is the same *i.e.* $-W/2$. Shear strain energy for the beam

$$U_s=2U_s'=\frac{3W^2l}{20Gbd}$$

Deflection at the centre due to shear

$$=\frac{\partial U_s}{\partial W}=\frac{3Wl}{10Gbd}$$

Exercise 16·3-1. A beam 8 metres long carries loads of 40 kN each at a distance of 2 metres and 6 metres from one end. The beam is simply supported at the ends. The beam is of rectangular section with breadth b and depth d. If $d=2b$, and the shear stress is not to exceed 75 N/mm². Determine (*i*) size of the beam, (*ii*) shear strain energy in the beam, (*iii*) deflection due to shear under the load of 40 kN.

Given, $G=80{,}000$ N/mm². Note $q_{max}=\dfrac{3F}{2bd}$

[**Ans.** (*i*) 20 mm, 40 mm, (*ii*) 30 Nm, (*iii*) 0·75 mm]

16·4. STRAIN ENERGY DUE TO BENDING

Consider a bar of length l, initially straight subjected to a gradually increasing bending moment. As the bending moment increases, curvature in the bar increases or the angular rotation ϕ goes on increasing. Say at a particular instant.

Bending moment $=M$

Angular rotation $=\phi$

Radius of curvature $=R$

Work done on the bar $=\frac{1}{2}M\phi$

But $\phi=l/R$ since ϕ is very small and the stress in bar remains within the elastic limit.

Work done = Energy stored in the bar,

$$U=\tfrac{1}{2}M\phi=\frac{Ml}{2R}$$

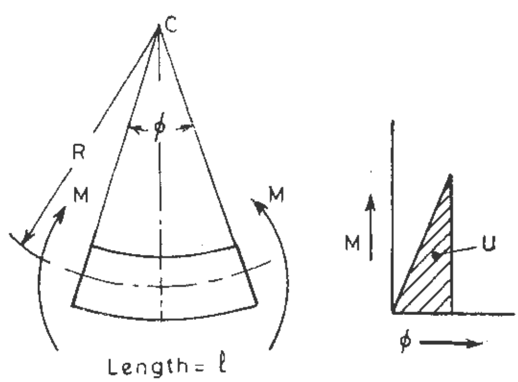


Fig. 16·7

But from the flexure formula,

$$\frac{M}{I}=\frac{E}{R}, \quad \text{or} \quad \frac{1}{R}=\frac{M}{EI}$$

Therefore, strain energy,

$$U=\frac{M^2l}{2EI}$$

If we consider a beam subjected to transverse loads W_1, W_2,..., w etc., where the radius of curvature goes on changing from one section to the other.

Strain energy due to bending,

$$U=\int\frac{M_x^2\,dx}{2EI} \qquad ...(1)$$

Where M_x is the bending moment at any section X-X and dx is the small length along the axis of the beam.

Say δ_1 is the deflection under the load W_1. Then

$$\delta_1=\frac{\partial U}{\partial W_1}=\int\frac{M_x}{EI}\cdot\frac{\partial M_x}{dW_1}\times dx \qquad ...(ii)$$

Example 16·4-1. A circular cantilever of length l, free at one, fixed at the other end, with diameter d for half of its length and diameter $2d$ for the rest of its length carries a concentrated load W at the free end. If E is the Young's modulus of the material determine the deflection and slope at the free end.

Solution Fig. 16·8 shows a cantilever ABC, fixed at end C and free at end A with diameter d for half of its length AB and diameter $2d$ for next half of its length BC. Since we have to find out the slope at free end A, let us apply a fictitious moment $M=0$ at the free end.

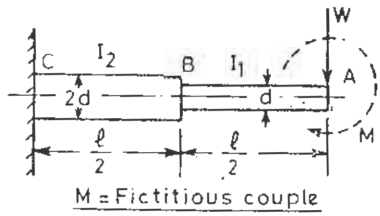


Fig. 16·8

Portion AB

$M_x = M + Wx$ where $x=0$ to $\frac{l}{2}$ taking origin at A.

Portion BC

$M_x = M + W\left(x+\frac{l}{2}\right)$, where $x=0$ to $\frac{l}{2}$, origin at B

$$\text{Strain energy, } U = \int_0^{l/2} \frac{(M+Wx)^2}{2EI_1}\,dx + \int_0^{l/2} \frac{[M+W(x+l/2)]^2}{2EI_2}\,dx$$

$$\text{where} \qquad I_1 = \frac{\pi d^4}{64}, \quad I_2 = \frac{\pi(2d)^4}{64} = \frac{\pi d^4}{4}$$

$$\text{Deflection at } A, \quad \delta = \frac{\partial U}{\partial W} = \int_0^{l/2} \frac{(M+Wx)(x)}{EI_1}\,dx + \int_0^{l/2} \left(\frac{M+W(x+l/2)}{EI_2}\right) \times \left(x+\frac{l}{2}\right) dx$$

$$= \frac{1}{EI_1}\left|\frac{Mx^2}{2} + \frac{Wx^3}{3}\right|_0^{l/2} + \frac{1}{EI_2}\left|\frac{Mx^2}{2} + \frac{Mlx}{2} + \frac{Wlx^2}{2} + \frac{Wx^3}{3} + \frac{Wl^2x}{4}\right|_0^{l/2}$$

But $M=0$.

$$\delta = \frac{1}{EI_1} \times \frac{Wl^3}{24} + \frac{1}{EI_2}\left(\frac{Wl^3}{8} + \frac{Wl^3}{24} + \frac{Wl^3}{8}\right)$$

$$= \frac{1}{EI_1} \times \frac{Wl^3}{24} + \frac{1}{EI_2} \times \frac{7\,Wl^3}{24}$$

Substituting the values of I_1 and I_2,

$$\delta = \frac{Wl^3}{24E}\left[\frac{64}{\pi d^4} + \frac{7\times 4}{\pi d^4}\right] = \frac{Wl^3}{24E} \times \frac{92}{\pi d^4}$$

$$= \frac{23Wl^3}{6E\pi d^4}$$

Slope at end A, $\phi = \frac{\partial U}{\partial M} = \int_0^{l/2} \frac{M+Wx}{EI_1} \times (1)dx + \int_0^{l/2} \frac{(M+W(x+l/2)(1)dx}{EI_2}$

$$\phi = \frac{1}{EI_1}\left| Mx + \frac{Wx^2}{2} \right|_0^{l/2} + \frac{1}{EI_2}\left| Mx + \frac{Wx^2}{2} + \frac{Wl}{2} . x \right|_0^{l/2}$$

But $M=0$

$$\phi = \frac{1}{EI_1} \times \frac{Wl^2}{8} + \frac{1}{EI_2} \times \left(\frac{Wl^2}{8} + \frac{Wl^2}{4} \right)$$

$$= \frac{1}{EI_1} \times \frac{Wl^2}{8} + \frac{1}{EI_2} \times \frac{3}{8} Wl^2$$

Substituting the values of I_1 and I_2.

$$\phi = \frac{Wl^2}{8E}\left[\frac{6^4}{\pi d^4} + \frac{3 \times 4}{\pi d^4} \right] = \frac{Wl^2 \times 76}{8E\pi d^4}$$

$$= \frac{19Wl^2}{2\pi Ed^4}$$

Example 16·4-2. A circular ring of mean radius R and second moment of area of its cross section I, with a slit at one section is shown in Fig. 16·9. Points A and B are subjected to forces P as shown.

Determine the relative displacemets between the points A and B. Only the strain energy due to bending is to be taken into account.

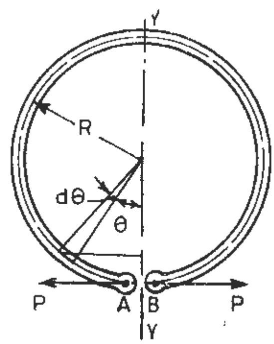


Fig. 16·9

Solution.

Consider an element of length $dS = R\,d\theta$ at an angle θ from the vertical axis.

Bending moment of the force P on the element,

$$M_x = P(R - R\cos\theta) = PR(1-\cos\theta)$$

Strain energy, $U = 2\int_0^{\pi} \frac{[PR(1-\cos\theta)]^2}{2EI} . R\,d\theta$

$$= \frac{P^2R^3}{EI}\int_0^{\pi} (1+\cos^2\theta + 2\cos\theta)\; d\theta$$

$$= \frac{P^2R^3}{EI}\int_0^{\pi}\left[1 + \left(\frac{1+\cos 2\theta}{2} \right) - 2\cos\theta \right] d\theta$$

$$= \frac{P^2R^3}{EI}\left| \frac{3}{2}\,\theta + \frac{\sin 2\theta}{4} - 2\sin\theta \right|_0^{\pi}$$

$$= \frac{3\pi}{2} \times \frac{P^2.R^3}{EI}$$

Relative displacement between the points A and B

$$\delta = \frac{\partial U}{\partial P} = \frac{3\pi}{2} \times \frac{2PR^3}{EI} = 3\pi\,\frac{PR^3}{EI}$$

Example 16·4-3. A beam ABC of length l, hinged at both the ends A and C is subjected to a couple M applied at B, at a distance of $l/3$ from one end. If EI is the flexural rigidity of the beam, determine the rotation of the point C.

Solution. Fig. 16·10 shows the beam of length l subjected to couple M as given in the problem.

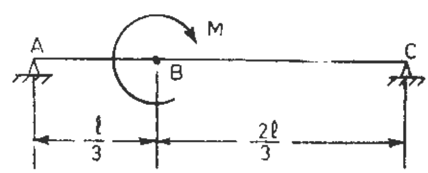


Fig. 16·10

Reactions at A and $C = \pm \frac{M}{l}$

Strain energy due to bending

Portion AB. origin at A

$$M_x = -\frac{M}{l}\,x$$

Strain energy,
$$U_1 = \int_0^{l/3} \left(\frac{M^2x^2}{l^2} \right) \frac{dx}{2EI} = \frac{M^2}{2EI\,l^2} \left| \frac{x^3}{3} \right|_0^{l/3}$$

$$= \frac{M^2}{2EI\,l^2} \times \frac{l^3}{81} = \frac{M^2l^3}{162\,EI\,l^2} = \frac{M^2l}{162\,EI}$$

Portion CB, origin at C

$$M_x = +\frac{Mx}{l}$$

Strain energy,
$$U_2 = \int_0^{2l/3} \frac{M^2x^2}{l^2} \times \frac{dx}{2EI} = \frac{M^2}{2EI\,l^2} \left| \frac{x^3}{3} \right|_0^{2l/3}$$

$$= \frac{M^2}{2EI\,l^2} \times \frac{8l^3}{81} = \frac{8M^2l^3}{162\,EI\,l^2} = \frac{8}{162}\;\frac{M^2l}{EI}$$

Total strain energy of the beam,

$$U = U_1 + U_2 = \frac{9}{162}\;\frac{M^2l}{EI} = \frac{M^2l}{18EI}$$

Rotation at the point C,

$$\frac{\partial U}{\partial M} = \phi = \frac{2Ml}{18EI} = \frac{Ml}{9EI}$$

Exercise 16·4-1. A circular beam of length l simply supported at its ends carries a concentrated load at its centre. The diameter of the beam is $2d$ for half of its length and d for another half. If E is the young's modulus of elasticity of the beam, determine deflection at the centre.

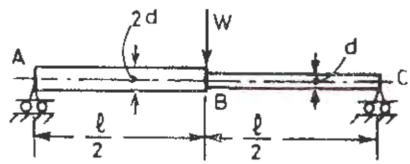


Fig 16·11

$$\left[\textbf{Ans.}\ \frac{17\ Wl^3}{24\ E\pi\ d^4}\right]$$

Exercise 16·4-2. A steel ring 2 cm diameter is bent into a quadrant of 1·5 m radius. One end is rigidly fixed in the ground and at the other end a vertical load P is applied. Determine the value of P so that the vertical deflection at the point of loading is 1·6 cm.

$E=208000\ \text{N/mm}^2$ [**Ans.** 9·86 kg]

Example 16·4-3. A beam of length l, hinged at both the ends is subjected to a couple M applied in a vertical plane at its centre. If EI is the flexural rigidity of the beam, determine the angular rotation of the centre of the beam due to M.

$$\left[\textbf{Ans.}\ \frac{Ml}{12\ EI}\right]$$

16·5. STRAIN ENERGY DUE TO TWISTING MOMENT

Fig. 16·12 shows a shaft of diameter d and length l subjected to a gradually increasing twisting moment. As the twisting moment increases gradually, the angular twist also increases gradually.

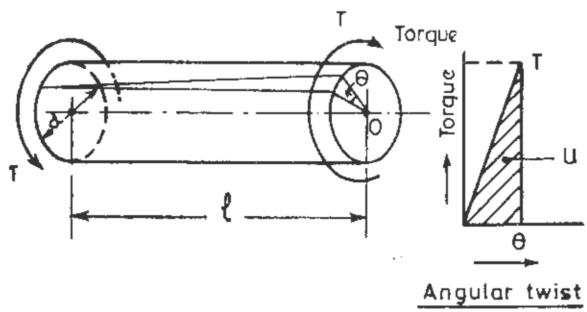


Fig. 16·12

At a particular stage say, the torque applied is T and angular twist in the shaft is θ

Work done on the shaft$=\frac{1}{2}\ T\theta$

$=U$, strain energy stored in the shaft

From tension formula,

$$\frac{T}{J}=\frac{G\theta}{l} \text{ or } \theta=\frac{Tl}{GJ}$$

Strain energy due to twisting moment,

$$U=\frac{1}{2}\times\frac{T^2l}{GJ}$$

where $\quad J=$ Polar moment of inertia of the shaft

$$=\frac{\pi d^4}{32} \text{ (for a solid shaft)}$$

$$U=\frac{T^2 l}{2GJ}$$

If the torque varies or the section varies along the length the shaft

$$U=\frac{1}{2G}\int_0^l \frac{T^2 dl}{J}$$

Angular twist due to the twisting moment,

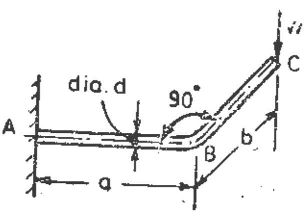
$$\theta=\frac{\partial U}{\partial T}.$$

Example 16·5-1. A circular bar of diameter d is bent at right angle. It is fixed at one end and a load W is applied at the other end as shown in the Fig. 16·13. Determine the deflection under the load W if E=Young's modulus and G=Shear modulus of the material.

Solution. Let us calculate the strain energy.

Portion BC

$$U_1=\int_0^b \frac{(Wx)^2}{2EI}\,dx=\frac{W^2b^3}{6EI}$$



ig. 16·13

Portion AB

U_2=Strain energy due to bending

$$=\int_0^a \frac{(Wx)^2}{2EI}\,dx=\frac{W^2a^3}{6EI}$$

Straight portion AB is also subjected to a twisting moment $T=Wb$

U_3=Strain energy due to twisting moment

$$=\frac{(Wb)^2\,a}{2GJ}=\frac{W^2ab^2}{2GJ}$$

where $\quad I=\frac{\pi d^4}{64}$ and $J=\frac{\pi d^4}{32}$

Total strain energy $\quad U=U_1+U_2+U_3=\frac{W^2b^3}{6EI}+\frac{W^2a^3}{6EI}+\frac{W^2ab^2}{2GJ}$

Deflection under the load,

$$\delta=\frac{\partial U}{\partial W}=\frac{Wb^3}{3EI}+\frac{Wa^3}{3EI}+\frac{Wab^2}{GJ}$$

$$=\frac{W\times 64}{3E\times\pi d^4}\,(a^3+b^3)+\frac{W\times 32}{G\times\pi d^4}\,(ab^2)$$

$$=\frac{32\,W}{\pi d^4}\left[\frac{2(a^3+b^3)}{3E}+\frac{ab^2}{G}\right].$$

Exercise 16 5-1. A circular bar 2 cm diameter, 20 cm long is bent at right angle at the centre. It is fixed at one end and at the other end a load 200 N is applied. If

$$E=2,08,000 \text{ N/mm}^2 \quad \text{and} \quad G=82,000 \text{ N/mm}^2.$$

Determine the deflection under the load. [**Ans.** 6·845 mm]

Problem 16·1. A structure of horizontal length $2a$ and vertical height a carries a load P at its end as shown in the Fig. 16·14. The area of cross section of each member is A, and E is the Young's modulus. Determine the deflection of the structure at the point 1.

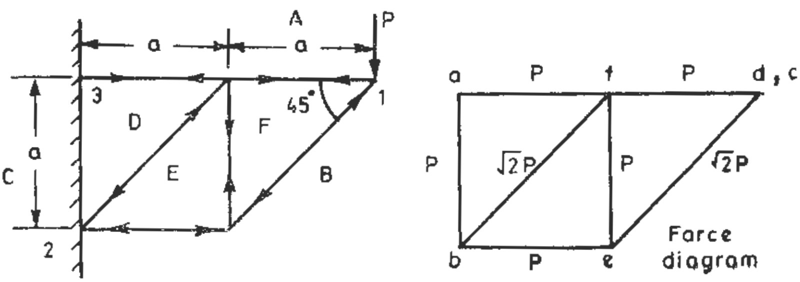


Fig. 16·14

Solution. Let us give Bow's notations to the spaces in and around the structure as shown. Taking $ab=P$, the vertical load, and then drawing lines af parallel to member AF and fb parallel to member FB. Force in member AF is P (tensile) and in FB it is $\sqrt{2}\,P$ (compressive) since the angle $\angle afb=45°$.

Similarly the complete force diagram for the structure is drawn. Following table gives the forces in members :

Members	*Length*	*Force, F*	F^2l
AD	a	$+2P$	$4P^2a$
AF	a	$+P$	P^2a
FB	$\sqrt{2}a$	$-\sqrt{2}P$	$2P^2\times\sqrt{2}a$
EF	a	$+P$	P^2a
BE	a	$-P$	P^2a
DE	$\sqrt{2}a$	$-\sqrt{2}P$	$2\sqrt{2}P^2a$

$$\Sigma F^2l=7P^2a+4\sqrt{2}P^2a$$

Strain energy, $$U=\frac{\Sigma F^2l}{2AE}=\frac{7P^2a+4\sqrt{2}P^2a}{2AE}$$

where A, area of cross section of each member is the same

Deflection at the point 1,

$$\frac{\partial U}{\partial P}=\delta=\frac{2Pa}{2AE}(7+4\sqrt{2})$$

$$=\frac{Pa}{AE}(7+4\sqrt{2}).$$

Problem 16·2. For a centilever made of steel, length l, breadth b and depth d, show that it y_b and y_s are the deflections due to bending and shear at the free end due to the concentrated load load W at the free end

$$\frac{y_s}{y_b}=K\cdot\left(\frac{d}{l}\right)^2 \text{ where } K \text{ is a constant.}$$

Determine the value of K for steel and the least value of $\frac{l}{d}$ if the deflection due to shear is not to exceed 1·5% of the total deflection.

Take $\qquad \frac{E}{G}=2{\cdot}6.$

Solution. Length of the cantilever$=l$

Section is $b\times d$

Moment of inertia about neutral axis

$$I_{xx}=\frac{bd^3}{12}$$

Say the concentrated load at the free end$=W$

Deflection due to bending,

$$y_b=\frac{Wl^3}{3EI}$$

$$=\frac{Wl^3}{3E\times bd^3}\times 12=\frac{4Wl^3}{Ebd^3} \quad \ldots(1)$$

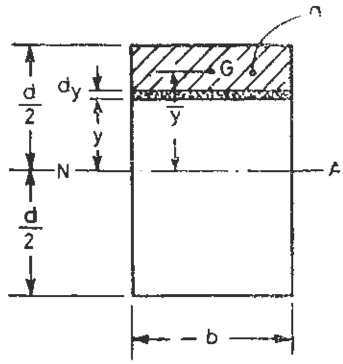


Fig. 16·15

Deflection due to shear. Shear stress in any layer at a distance y from neutral axis,

$$q=\frac{F\,a\bar{y}}{Ib}$$

In this case, Shear Force is constant throughout the length of the cantilever $F=W$

$$a\bar{y}=b\left(\frac{d}{2}-y\right)\left(y+\frac{d/2-y}{2}\right) \text{ (See Fig. 16·15)}$$

$$=\frac{b}{2}\left(\frac{d^2}{4}-y^2\right)$$

$$q=\frac{Wb}{b\times 2}\left(\frac{d^2}{4}-y^2\right)\times\frac{12}{bd^3}=\frac{6W}{bd^3}\left(\frac{d^2}{4}-y^2\right)$$

$$q^2=\frac{36\,W^2}{b^2d^6}\left(\frac{d^4}{16}-\frac{d^2y^2}{2}+y^4\right)$$

Shear strain energy per unit volume$=\frac{q^2}{2G}$

Shear strain energy for the cantilever

$$=\frac{36W^2}{2b^2d^6G}\int_0^l 2\int_0^{d/2}\left(\frac{d^4}{16}-\frac{d^2y^2}{2}+y^4\right)dy\,dx$$

$$=\frac{36\;W^2lb}{Gb^2d^6}\left|\frac{d^4}{16}y-\frac{d^2y^3}{6}+\frac{y^5}{5}\right|_0^{d/2}$$

$$=\frac{36\;W^2l}{Gbd^6}\left[\frac{d^5}{32}-\frac{d^5}{48}+\frac{d^5}{160}\right]$$

$$U_s=\frac{36\;W^2\;l}{Gbd^6}\times\frac{d^5}{60}=\frac{6}{10}\;\frac{W^2l}{Gbd} \qquad ...(2)$$

Deflection due to shear at the free end

$$y_s=\frac{\partial U_s}{\partial W}=\frac{12}{10}\times\frac{Wl}{Gbd}$$

Now $$\frac{y_s}{y_b}=\frac{12}{90}\times\frac{Wl}{Gbd}\times\frac{Ebd^3}{Wl^3}=\frac{12}{10}\times\frac{d^2}{l^2}\times\frac{E}{G}$$

$$=2{\cdot}6\times1{\cdot}2\times\left(\frac{d}{l}\right)^2$$

So constant $$K=2{\cdot}6\times1{\cdot}2=3{\cdot}12.$$

$$\frac{y_s}{y_b}=3{\cdot}12\left(\frac{d}{l}\right)^2$$

In this case, y_s is negligible, therefore, $y_s+y_b\simeq y_b$

$$y_s=y_b\times3{\cdot}12\left(\frac{d}{l}\right)^2$$

$$y_s=1{\cdot}5\%\;y_b=0{\cdot}015$$

So $$\cdot015=3{\cdot}12\times\left(\frac{d^2}{l^2}\right)$$

or $$\frac{l^2}{d^2}=\frac{3{\cdot}12}{0{\cdot}015}=208$$

$$\frac{l}{d}=14{\cdot}42.$$

Problem 16·3. A simply supported beam of *I* section is loaded as shown in the figure 16·16. Determine the shear strain energy in the beam. Given $G=80{,}000$ N/mm². Find the deflection due to shear under one load.

Solution. Fig. 16·16 shows a beam $ABCD$, 10 metres long carrying 12 kN at C and D, 2 meters from each end.

The shear force diagram is shown below. Reactions $R_A=R_B=12$ kN. SF is constant in portions AC and DB and equal to $\pm$ 12 kN. The beam is of I section. 15 cm$\times$20 cm with flanges 1 cm thick and web 1 cm thick.

Moment of inertia,

$$I_{xx}=\frac{15\times20^3}{12}-\frac{14\times18^3}{12}$$
$$=3196 \text{ cm}^4$$

Say shear force is F in portion AC

Shear stress in flange,

$$q_1=\frac{F\times15(10-y)\left(y+\frac{10-y}{2}\right)}{I\times15}$$
$$=\frac{F}{2I}\times(100-y^2)$$
$$q_1^2=\frac{F^2}{4I^2}(10{,}000-200\,y^2+y^4)$$

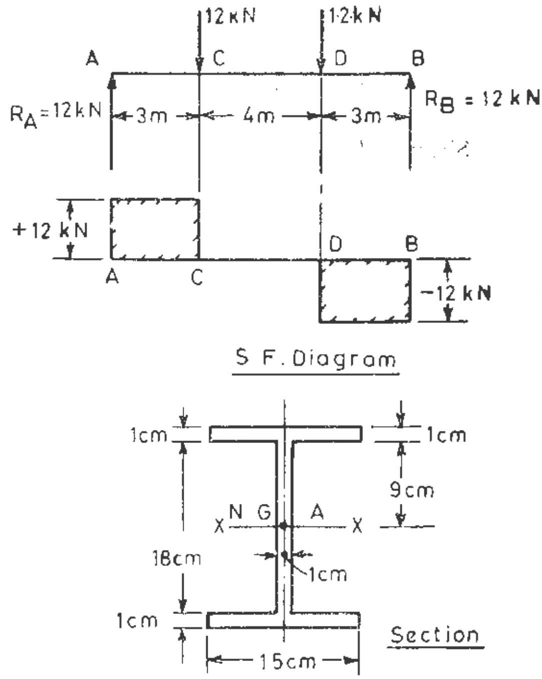


Fig. 16·16

Shear strain energy in portion AC

$$(a)\text{ Shear energy in flanges}=2\int_0^{300}\int_9^{10}\frac{q_1^2}{2G}\times B\,dy\,dx$$

$$=\frac{BF^2}{2I^2G}\int_0^{300}\left[10000\,y-\frac{200\,y^3}{3}+\frac{y^5}{5}\right]_9^{10}dx$$

$B=15$ cm

$$=\frac{15\times F^2}{2I^2G}\int_0^{300}\left[10000(10-9)-\frac{200}{3}(1000-729)+\frac{1}{5}(100000-59049)\right]dx$$

$$=\frac{15F^2}{2I^2G}\times\left\{10000-\frac{200}{3}\times271+\frac{40951}{5}\right\}\times300$$

$$=\frac{15\times300\,F^2}{2\,I^2\,G}\times\,123\cdot54=\frac{277965\,F^2}{I^2G}=\frac{2\cdot78\times10^5\,F^2}{I^2G}\qquad\ldots(1)$$

Shear stress in the web,

$$q_2=\frac{F}{Ib}\left[15\times9\cdot5+1(9-y)\left(y+\frac{9-y}{2}\right)\right]$$

where $b=1$ cm

$$q_2=\frac{F}{I}\times\left[\frac{15\times19+81-y^2}{2}\right]=\frac{F}{2I}\ [366-y^2]$$

$$q_2^2=\frac{F^2}{4I^2}\times[133596-732\ y^2+y^4]$$

Shear strain energy in the web

$$=2\int_0^{300}\int_0^{9}\frac{q_2^2}{2G}\times b\ dy\ dx$$

$$=\frac{F^2}{4I^2G}\int_0^{300}[133596-732\ y^2+y^4]\ dy\ dx$$

$$=\frac{F^2}{4I^2G}\times300\left[133596\ y-\frac{732\ y^3}{3}+\frac{y^5}{5}\right]$$

$$=\frac{75\ F^2}{I^2G}\left[133596\times9-\frac{732\times9^3}{3}+\frac{9^5}{5}\right]$$

$$=\frac{75\ F^2}{I^2G}\ [1205604-177876+11809{\cdot}8]$$

$$=\frac{75\times1039537{\cdot}8\ F^2}{I^2G}=\frac{779{\cdot}65\times10^5\ F^2}{I^2G}$$

Total shear strain energy in portion AC

$$=\frac{(2{\cdot}78+779{\cdot}65)\times10^5\ F^2}{I^3G}=\frac{(782{\cdot}43)\times10^5\ F^2}{I^2G}$$

Total shear strain energy of the beam (as the beam is symmetrically loaded)

$$U=\frac{2\times782{\cdot}43\times10^5\times F^2}{I^2G}$$

Deflection under the load F,

$$\frac{\partial U}{\partial F}=\frac{4\times782{\cdot}43\times10^5}{I^2G}\times F\quad(\text{as } F=12\text{ kN})$$

Substituting the values $F=12000$ N

$G=80{,}000\ \text{N/mm}^2=8\times10^6\ \text{N/cm}^2$

$I=3196\ \text{cm}^4$

$$U=\frac{2\times782{\cdot}43\times10^5\times12000\times12000}{3196\times3196\times8\times10^6}\ \text{N cm}$$

$=275{\cdot}76$ N cm$=2{\cdot}75$ N m

Deflection due to shear $\delta=\dfrac{\partial U}{\partial F}=\dfrac{4\times782{\cdot}43\times12000}{3196\times3196\times8\times10^6}$

$=0{\cdot}046$ cm$=0{\cdot}46$ mm.

Problem 16·4. A 24 cm × 16 cm, I section beam with web 1 cm thick and flanges 2 cm thick is simply supported over a span of 5 metres. A concentrated load of 2 tonnes acts at a distance of 2 metres from one end of the beam. Assuming that the shearing force is carried by the web only and the shearing stress is uniformly distributed over the web, determine the total deflection produced under the concentrated load.

Given $E=2080$ tonnes/cm²

$G=800$ tonnes/cm².

Solution. I_{xx} of the section

$$=\frac{16\times 24^3}{12}-\frac{15\times 20^3}{12}$$

$$=32\times 24\times 24-25\times 400=8432 \text{ cm}^4$$

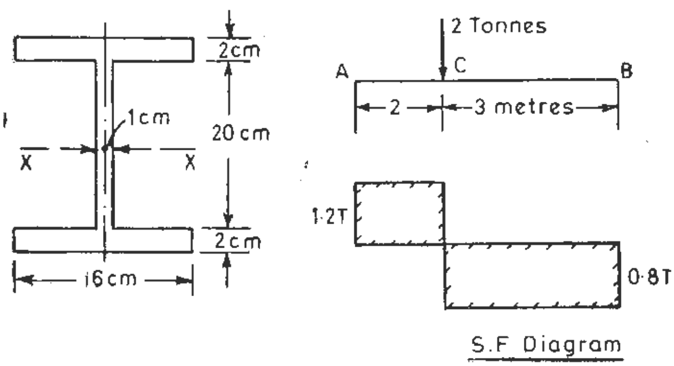


Fig. 16·17

Deflection due to bending at the point of load,

$$y_b=\frac{Wa^2b^2}{3EIl}$$

where $W=2000$ kg, $a=200$ cm

$b=300$ cm, $E=2080\times 1000$ kg/cm²

$l=500$ cm, $I=8432$ cm⁴

Therefore, $$y_b=\frac{2000\times(200)^2\times(300)^2}{3\times 2080\times 1000\times 8432\times 500}=0{\cdot}2735 \text{ cm.}$$

Deflection due to shear

Area of cross section of the web $=20$ cm²

Shear force in the portion AC, $F_1=+1200$ kg

Shear stress in the portion AC, $q_1=\frac{1200}{20}=60$ kg/cm²

Shear force in the portion CB, $F_2=800$ kg

Shear stress in the portion CB, $q_2=\frac{800}{20}=40$ kg/cm²

Shear strain energy in the portion AC,

$$=\int_0^{200}\int_{-10}^{+10}\frac{q_1^2}{2G}\,.\,b\,dy\,dx,=\int_0^{200}\int_{-10}^{+10}\frac{3600}{2G}\times 1\,dy\,.\,dx$$

$$=\int_0^{200}\left|\frac{1800}{G}\,.\,y\right|_{-10}^{+10}dx$$

$$=\frac{1800}{G}\times 20\times 200 \text{ cm kg}$$

$$=\frac{72\times 10^5}{G}\text{ cm-kg}$$

Shear strain energy in the portion CB,

$$=\int_0^{300}\int_{-10}^{+10}\frac{q_2^2}{2G}\,b\,dy\,dx,=\int_0^{300}\left(\frac{1600}{2G}\times 20\right)dx$$

$$=\frac{800}{G}\times 20\times 300=\frac{48\times 10^5}{G}\text{ cm-kg}$$

Total shear strain energy,

$$U_s=\frac{(72+48)\times 10^5}{G}=\frac{120\times 10^5}{800\times 1000}=15 \text{ cm-kg.}$$

If y_s is the deflection due to shear, then

$$\frac{1}{2}Wy_s=15,\quad\text{or } y_s=\frac{15\times 2}{2000}=0{\cdot}015 \text{ cm.}$$

Total deflection under the concentrated load, y

$$=0\,2735+0{\cdot}015=0{\cdot}2885 \text{ cm.}$$

Problem 16·5. A bar $ABCD$ of rectangular section having uniform width b throughout but thickness varying as $3t$ for $3a$ length, $2t$ for length $2a$ and t for the length a is of the shape shown in the Fig. 16·18. A load W is applied at the end D. Determine the deflection under the load. Given E=Young's modulus of the material. (Consider only the strain energy due to bending).

Solution. Let us determine the strain energy due to bending

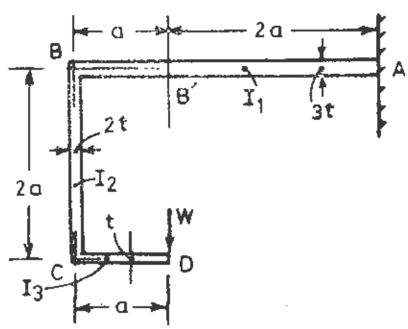


Fig. 16·18

Portion AB

$$U_1=\int_0^{2a}\frac{(Wx)^2}{2EI_1}\,dx+\int_0^{a}\frac{(Wx)^2}{2EI_1}\,.\,dx$$

Taking the origin for x at B' as shown

$$U_1=\frac{8W^2a^3}{6EI_1}+\frac{W^2a^3}{6EI_1}$$

$$=\frac{9}{6}\,\frac{W^2a^3}{EI_1}=\frac{3}{2}\,\frac{W^2a^3}{EI_1}.$$

Portion BC

Bending moment is constant Wa

$$U_2=\int_0^{2a}\frac{(W^2a^2)\,dx}{2EI_2}=\frac{W^2a^3}{EI_2}.$$

Portion CD

Bending moment $=Wx$ (taking origin for x at D)

$$U_3=\int_0^{a}\frac{W^2x^2\,dx}{2EI_3}=\frac{W^2a^3}{6EI_3}$$

Total strain energy, $U=U_1+U_2+U_3=\dfrac{W^2a^3}{E}\left[\dfrac{3}{2I_1}+\dfrac{1}{I_2}+\dfrac{1}{6I_3}\right]$

$$\frac{\partial U}{\partial W}=\text{vertical deflection at } D=\frac{2Wa^3}{E}\left[\frac{3}{2I_1}+\frac{1}{I_2}+\frac{1}{6I_3}\right]$$

where $I_1=\dfrac{b}{12}(3t)^3=\dfrac{27}{12}bt^3$; $I_2=\dfrac{b}{12}(2t)^3=\dfrac{8}{12}bt^3$; $I_3=\dfrac{bt^3}{12}$

$$\delta_D=\frac{2Wa^3}{E}\left[\frac{3}{2}\times\frac{12}{27bt^3}+\frac{12}{8bt^3}+\frac{12}{6bt^3}\right]$$

$$=\frac{24Wa^3}{Ebt^3}\left[\frac{3}{54}+\frac{1}{8}+\frac{1}{6}\right]=\frac{24Wa^3}{Ebt^3}\times\frac{25}{72}$$

$$=\frac{25Wa^3}{3Ebt^3}.$$

Problem 16·6. A cantilever of length l, fixed at one end and propped at the other end carries a concentrated load W at its centre and a uniformly distributed load, w per unit length from the centre upto the fixed end. If EI is the flexural rigidity of the cantilever determine the reaction at the prop.

Solution. Fig. 16·19 shows a cantilever ABC of length l, fixed at end C and propped at end A, carrying loads as given in the problem. Let us first determine the strain energy due to bending. Say the reaction at the prop$=R$.

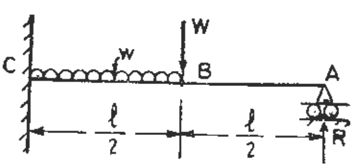


Fig. 16·19

Portion AB. (Origin at A)

$$U_1=\int_0^{l/2}\frac{(Rx)^2dx}{2EI}=\left|\frac{R^2x^3}{6EI}\right|_0^{l/2}=\frac{R^2l^3}{48EI}.$$

Portion BC. (Taking origin at B)

M_x, bending moment $= R\left(x+\frac{l}{2}\right) - Wx - \frac{wx^2}{2}$

$$U_2 = \int_0^{l/2} \frac{(M_x^2)\,dx}{2EI}$$

Total strain energy $= U_1 + U_2$

$$\frac{\partial U}{\partial R} = 0 \text{ at the propped end}$$

So

$$\frac{\partial U}{\partial R} = \frac{2Rl^3}{48EI} + \int_0^{l/2} \left(\frac{M_x}{EI} \cdot \frac{dM_x}{dR} \right) dx$$

$$= \frac{Rl^3}{24EI} + \int_0^{l/2} \left[R\left(x+\frac{l}{2}\right) - Wx - \frac{wx^2}{2} \right] \left(x+\frac{l}{2}\right) dx$$

$$= \frac{Rl^3}{24EI} + \frac{1}{EI} \int_0^{l/2} \left[R\left(x^2+\frac{l^2}{4}+lx\right) - W\left(x^2+\frac{xl}{2}\right) - \frac{wx^2}{2}\left(x+\frac{l}{2}\right) \right] dx$$

$$0 = \frac{Rl^3}{24EI} + \frac{1}{EI} \left[\frac{Rx^3}{3} + \frac{Rl^2}{4}x + \frac{Rlx^2}{2} - \frac{Wx^3}{3} - \frac{Wx^2 l}{4} - \frac{wx^4}{8} - \frac{wx^3 l}{12} \right]_0^{l/2}$$

or

$$\frac{Rl^3}{24} + \frac{Rl^3}{24} + \frac{Rl^3}{8} + \frac{Rl^3}{8} - \frac{Wl^3}{24} - \frac{Wl^3}{16} - \frac{wl^4}{128} - \frac{wl^4}{24} = 0$$

$$\frac{Rl^3}{12} + \frac{Rl^3}{4} - \frac{5Wl^3}{48} - \frac{19wl^4}{384} = 0$$

$$\frac{R}{3} - \frac{5W}{48} - \frac{19wl}{384} = 0$$

Reaction at the prop, $R = \frac{5W}{16} + \frac{19wl}{128}$.

Problem 16·7. A bar is bent in the shape shown in Fig. 16·20 with radius of the bend R and length of the straight portion l. Determine the horizontal deflection due to the force P applied at the end A, if EI is the flexural rigidity of the bar. Consider only the strain energy due to bending.

Solution.

Portion AC

Consider an element of length

$R d\theta = ds$

The bending moment on ds

$= PR \sin \theta$

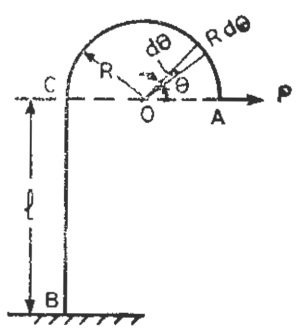


Fig. 16·20

Strain energy, $$U_1=\int_0^{\pi} \frac{(PR \sin \theta)^2}{2EI} \,.\, Rd\theta=\frac{P^2R^3}{2EI}\int_0^{\pi} \sin^2\theta \,.\, d\theta$$

$$=\frac{P^2R^3}{2EI}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta=\frac{P^2R^3}{2EI}\left|\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right|_0^{\pi}$$

$$=\frac{P^2R^3}{2EI}\times\frac{\pi}{2}=\frac{\pi P^2R^3}{4EI} \qquad ...(1)$$

Portion CB. (origin at C)

M_x, bending moment $=Px$

U_2, strain energy $$=\int_0^{l}\frac{P^2x^2}{2EI}\,dx=\frac{P^2l^3}{6EI}$$

Total strain energy, $$U=U_1+U_2=\frac{\pi P^2R^3}{4EI}+\frac{P^2l^3}{6EI}$$

Horizontal deflection at A,

$$\delta_A=\frac{\partial U}{\partial P}=\frac{\pi PR^3}{2EI}+\frac{Pl^3}{3EI}=\frac{P}{EI}\left[\frac{\pi R^3}{2}+\frac{l^3}{3}\right]$$

Problem 16·8. Fig. 16·21 shows a steel rod bent into the form of three quarters of a circle of radius r. End A is fixed while end B of the rod is constrained to move vertically. If a load W is applied at the end B, determine the vertical deflection at the end B. Given EI is the flexural rigidity of the rod.

Solution. Fig. 16·21 shows a rod bent into the form of three quarters of a circle. Since the end B is constrained to move only vertically, a horizontal reaction R will be offered by the constraint.

Consider an element of length

$ds=rd\theta$

at an angle θ to the vertical

Bending moment,

$$M_\theta=W\times r\sin\theta-R\times(r-r\cos\theta)$$

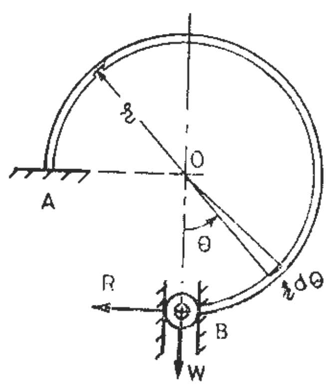


Fig. 16·21

Strain energy, $$U=\int_0^{3\pi/2}\frac{M\theta^2}{2EI}\,.\,rd\theta=\int_0^{3\pi/2}\frac{[Wr\sin\theta-Rr(1-\cos\theta)]^2}{2EI}\,.\,rd\theta$$

Since there is no horizontal deflection, $\frac{\partial U}{\partial R}=0$

$$0 = -\frac{1}{2EI}\int_0^{3\pi/2} 2[(Wr\sin\theta - Rr(1-\cos\theta)](1-\cos\theta)\, r^2 .\, d\theta$$

$$= -\frac{r^3}{EI}\int_0^{3\pi/2} [W\sin\theta - R + R\cos\theta](1-\cos\theta)\, d\theta$$

$$= -\frac{r^3}{EI}\int_0^{3\pi/2} [W\sin\theta - W\sin\theta\cos\theta - R + R\cos\theta + R\cos\theta - R\cos^2\theta]\, d\theta$$

$$= -\frac{r^3}{EI}\int_0^{3\pi/2} \left[W\sin\theta - \frac{W\sin 2\theta}{2} - R + 2R\cos\theta - \frac{R(1+\cos 2\theta)}{2}\right] d\theta$$

$$= -\frac{r^3}{EI}\int_0^{3\pi/2} \left[W\sin\theta - \frac{W\sin 2\theta}{2} - \frac{3R}{2} + 2R\cos\theta - \frac{R\cos 2\theta}{2}\right]_0^{3\pi/2}$$

or $$0 = \left[-W\cos\theta + \frac{W\cos 2\theta}{4} - \frac{3R}{2}\theta + 2R\sin\theta - \frac{R\sin 2\theta}{4}\right]_0^{3\pi/2}$$

or $$-W\left(\cos\frac{3\pi}{2} - \cos 0^\circ\right) + \frac{W}{4}(\cos 3\pi - \cos 0) - \frac{3R}{2}\times\frac{3\pi}{2}$$

$$+ 2R\left(\sin\frac{3\pi}{2} - \sin 0^\circ\right) - \frac{R}{4}(\sin 3\pi - \sin 0) = 0$$

or $$W + \frac{W}{4}(-2) - \frac{9R\pi}{4} + 2R(-1) = 0$$

$$\frac{W}{2} - \frac{9\pi}{4}R - 2R = 0$$

$$W = \frac{9\pi}{2}R + 4R$$

$$R = \frac{W}{9\pi/2 + 4} \quad \ldots(1)$$

Vertical deflection, $$\delta_B = \frac{\partial U}{\partial W} = \int_0^{3\pi/2} \frac{[Wr\sin\theta - Rr(1-\cos\theta)]\, r\sin\theta\, .\, r\, d\theta}{EI}$$

$$= \frac{r^3}{EI}\int_0^{3\pi/2} [W\sin^2\theta - R\sin\theta + R\sin\theta\cos\theta]\, d\theta$$

$$= \frac{r^3}{EI}\int_0^{3\pi/2} \left[\frac{W}{2}(1-\cos 2\theta) - R\sin\theta + \frac{R\sin 2\theta}{2}\right] d\theta$$

$$=\frac{r^3}{EI}\left|\frac{W}{2}\ \theta-\frac{W}{4}\ \sin 2\theta+R\ \cos\theta-\frac{R}{4}\ \cos 2\theta\right|_0^{3\pi/2}$$

$$=\frac{r^3}{EI}\left[\frac{W}{2}\times\frac{3\pi}{2}-\frac{W}{4}\ (\sin 3\pi-\sin 0)\right.$$

$$\left.+R\left(\cos\frac{3\pi}{2}-\cos 0\right)-\frac{R}{4}\ (\cos 3\pi-\cos 0)\right]$$

$$=\frac{r^3}{EI}\left[\frac{3\pi W}{4}-R+\frac{R}{2}\right]=\frac{r^3}{EI}\left[\frac{3\pi W}{4}-\frac{R}{2}\right]$$

$$=\frac{r^3}{EI}\left[\frac{3\pi W}{4}-\frac{W}{9\pi+8}\right] \text{ putting the value of reaction, } R$$

$$=\frac{Wr^3}{EI}\left[\frac{3\pi}{4}-\frac{1}{9\pi+8}\right]$$

Problem 16·9. A circular bar of diameter d is bent in the shape of U of radius R and straight portion of length l as shown in the Fig. 16·22. Equal loads P are applied at the ends A and E. Determine the relative shift between the points A and E.

E=Young's modulus of the matertial

Solution. Let us first determine the strain energy due to bending.

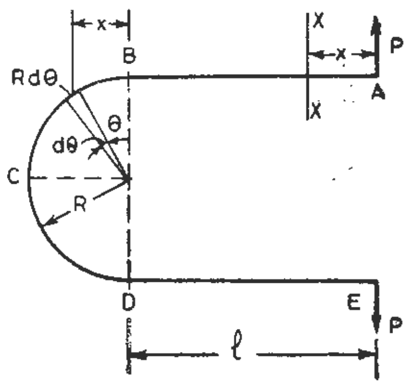


Fig. 16·22

Portion AB

Bending moment $M_x=Px$

Strain energy,

$$U_1=\int_0^l \frac{M_x^2dx}{2EI}=\int_0^l\frac{P^2x^2dx}{2EI}$$

where $I=\frac{\pi d^4}{64}$ (moment of inertia)

$$=\frac{P^2l^3}{6EI}$$

Portion BC. Taking origin at B,

$x=R\sin\theta$ where θ varies from 0 to $\pi/2$

Length of the element considered, $ds=Rd\theta$

Bending moment, $M_x=P(l+R\sin\theta)$

Strain energy, $$U_2=\int_0^{\pi/2}\frac{[P(l+R\sin\theta)]^2}{2EI}Rd\theta$$

$$=\frac{P^2}{2EI}\int_0^{\pi/2}(l^2R+R^3\sin^2\theta+2lR^2\sin\theta)\,d\theta$$

$$=\frac{P^2}{2EI}\int_0^{\pi/2}\left(l^2R+\frac{R^3(1-\cos 2\theta)}{2}+2lR^2\sin\theta\right)d\theta$$

$$=\frac{P^2}{2EI}\left[l^2R+\frac{R^3}{2}\right]_0^{\pi/2}-\frac{P^2R^3}{4EI}\left|\frac{\sin 2\theta}{2}\right|_0^{\pi/2}+\frac{P^2lR^2}{2EI}\left|-2\cos\theta\right|_0^{\pi/2}$$

$$=\frac{P^2}{2EI}\times\frac{\pi}{2}\left(l^2R+\frac{R^3}{2}\right)-0+\frac{2P^2lR^2}{2EI}$$

Total strain energy, $U=2U_1+2U_2$

$$=\frac{P^2l^3}{3EI}+\frac{P^2\pi}{2EI}\left(l^2R+\frac{R^3}{2}\right)+\frac{2P^2lR^2}{EI}$$

$$=\frac{4P^2l^3+6P^2\pi l^2R+3P^2\pi R^3+24\,P^2lR^2}{12\,EI}$$

Relative shift between the points A and E

$$\delta=\frac{\partial U}{\partial P}=\frac{8Pl^3+12P\pi l^2R+6P\pi R^3+48\,PlR^2}{12\,EI}$$

$$=\frac{2Pl^3+3P\pi l^2P+1{\cdot}5\,P\pi R^3+12\,PlR^2}{3\,EI}$$

But $I=\frac{\pi d^4}{64}$

So $\delta=\frac{64\,(2Pl^3+3P\pi l^2R+1{\cdot}5\,P\pi R^3+12PlR^2)}{3\,E\,\pi d^4}$

$$=\frac{32\,P}{3E\,\pi d^4}\,(4l^3+6\pi\,l^2R+3\pi R^3+24\,lR^3)$$

SUMMARY

1. If U is the strain energy in a system due to the applied forces, couples and twisting moments, then

$\frac{\partial U}{\partial W_i}=\delta_i$, displacement along a particular load W_i

$\frac{\partial U}{\partial M_i}=\phi_i$, angular rotation at a point where a particular couple M_i is applied

$\frac{\partial U}{\partial T_i}=\theta_i$, angular twist at a point where a particular twisting moment T_i is applied

2. Strain energy due to direct force P on a bar of length l, area of cross section A, and Young's modulus E

$U=\frac{P^2l}{2AE}$, $\quad u=\frac{f^2}{2E}$ (strain energy per unit volume)

3. Strain energy per unit volume due to shear stress, q

$$U=\frac{1}{2G}\times q^2 \text{ where } G=\text{shear modulus}$$

4. Strain energy due to bending, $U=\int_0^l \frac{(M_x)^2dx}{2EI}$

where l=length of the beam or cantilever

M_x=bending moment at any section

EI=Flexural rigidity of the beam or the cantilever

5. Strain energy due to twisting moment on a shaft

$$U=\frac{T^2l}{GJ}$$

where T=twisting moment

l=length of the shaft

G=shear modulus

J=Polar moment of inertia of shaft section

MULTIPLE CHOICE QUESTIONS

1. A beam of length l, simply supported at the ends carries a concentrated load W at its centre. If EI is the flexural rigidity of the beam, strain energy due to bending is

(a) $\frac{W^2l^3}{96EI}$ (b) $\frac{W^2l^3}{48EI}$

(c) $\frac{W^2l^3}{24EI}$ (d) $\frac{W^2l^2}{12EI}$

2. A shaft of length l, polar moment of inertia J is subjected to a twisting moment T. If G is the shear modulus, the strain energy stored in the shaft is

(a) $\frac{Tl^2}{2GJ}$ (b) $\frac{T^2l}{2GJ}$

(c) $\frac{Tl^2}{GJ}$ (d) $\frac{T^2l}{4GJ}$

3. A body is subjected to a direct force F, a twisting moment T and a bending moment M. The energy stored in the body is u. Displacement in the direction of F is given by

(a) $\frac{\partial U}{\partial M}+\frac{\partial U}{\partial T}+\frac{\partial U}{\partial F}$ (b) $\frac{\partial U}{\partial M}+\frac{\partial U}{\partial F}$

(c) $\frac{\partial U}{\partial T}+\frac{\partial U}{\partial F}$ (d) $\frac{\partial U}{\partial F}$

4. A shaft of diameter 20 mm is subjected to a torque of 10 k Nm. The length of the shaft is 1000 mm and angular twist produced by the torque is 1/100 radian. The strain energy stored in the shaft is

(a) 200 Nm (b) 100 Nm

(c) 50 Nm (d) None of the above.

5. A cantilever of length l is fixed at one end and at the other end a couple M is applied so as to bend the cantilever. If EI is the flexural rigidity of the cantilever, then slope at the free end of the cantilever is

(a) $\frac{Ml}{2EI}$ (b) $\frac{Ml}{EI}$

(c) $\frac{2Ml}{EI}$ (d) $\frac{4Ml}{EI}$

ANSWERS

1. (a) 2. (b) 3. (d) 4. (c) 5. (b)

EXERCISES

16·1. A structure $ABCDE$ shown in the Fig. 16·23 carries a concentrated load W at C. The area of cross section of each member is A and the modulus of elasticity is E. Determine deflection under the load.

$$\left[\textbf{Ans. } \frac{43\ Wa}{6\ AE}\right]$$

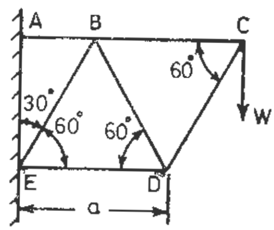


Fig. 16·23

16·2. An aluminium strip loaded as cantilever 30 cm long cross section 2 cm×5 cm deep carries a concentrated load at the free end. Show that the deflection will be under estimated by 0·387 per cent if the deflection due to shear is neglected.

16·3. A beam of length 2 metres is of I section shown in the Fig. 16·24. It is supported at both the ends. The beam carries a concentrated load of 1 tonne at the mid span. Determine the central deflection due to shear and bending.

$G=840$ tonnes/cm²
$E=2100$ tonnes/cm²

[**Ans.** 0·012 cm (due to shear)
0·019 cm (due to bending)]

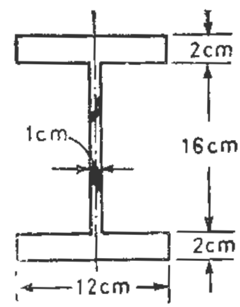


Fig. 16·24

16·4. Fig. 16·25 shows a frame. At the ends A and B, two equal and opposite forces P are applied. Considering only the the strain energy due to bending, determine the relative displacement between the points A and B. EI is the flexural rigidity of the frame.

$$\left[\textbf{Ans. } \frac{Pb^3}{6EI}+\frac{Pb^2a}{2EI}\right]$$

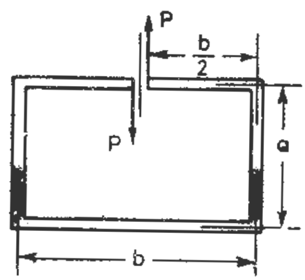


Fig. 16·25

16·5. A cantilever of length l, fixed at one end and propped at the other end carries a concentrated load W at a distance of $\frac{l}{3}$ from the propped end. If EI is the flexural rigidity of the cantilever, determine the reaction at the prop. $\left[\textbf{Ans.} \quad \frac{14}{27} W\right]$

16·6. A bar is bent in the shape shown in the figure 16·26. With radius of the bend R and length of the straight portion l. Determine the vertical deflection due to the load W at the end A, if EI is the flexural rigidity of the bar. Consider only the strain energy due to bending.

$$\left[\textbf{Ans.} \quad \frac{WR^2}{EI}\left(\frac{3}{2}\pi R+4l\right)\right]$$

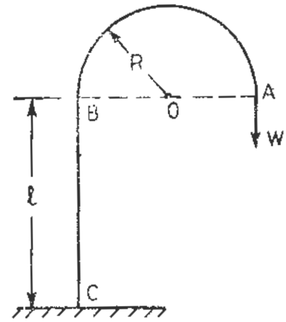


Fig. 16·26

16·7. Fig. 16·27 shows a steel rod of diameter 20 mm bent into the form of three quarters of a circle of radius 200 mm. End A is fixed while end B of the rod is constrained to move vertically only. If a load 50 N is applied at the end B, determine (1) reaction offered by the constraint (2) vertical deflection at the end B.

$E=208{,}000$ N/mm²

[**Ans.** 2·756 N, 0·57 mm]

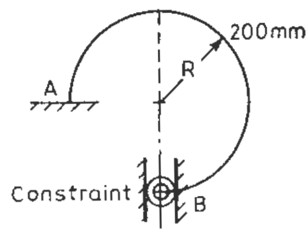


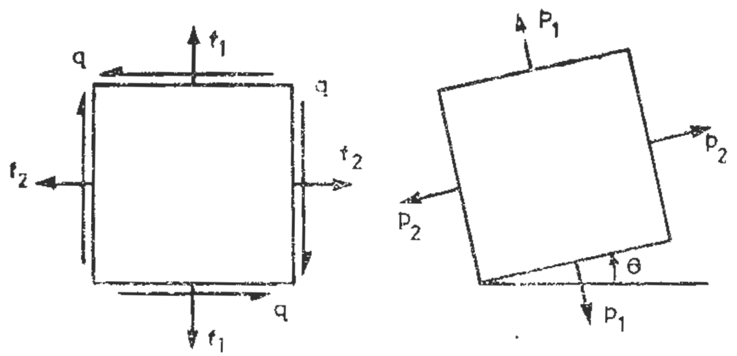
Fig. 16·27

16·8. A circular bar of diameter 2 cm is bent in the shape of U with radius of the curved part equal to 3 cm and length of the straight part equal to 10 cm. How much load can be safely applied at the ends so that the relative shift between the ends is not to exceed 1 cm. $E=2\times10^6$ kg/cm². [**Ans.** 240 kg]

17

Theories of Failure

We have learnt about principal stresses in Chapter 3 and have observed that there always exists a set of 3 principal stresses at a point. Out of these 3 principal stresses acting on principal planes, one is the maximum stress, other is the minimum stress and third one is of some intermediate value. If we know the magnitude and direction of applied forces on a body, we can find out the stresses f_1, f_2 and q acting on the body and from this stress system, principal stresses can be determined using the formulae derived in chapter 3, [as shown in Fig. 17·1 (*a*) and (*b*)].



Stress system at a point with the known forces.

(a)

Equivalent system with principal stresses p_1 and p_2.

(b)

Fig. 17·1

Various theories of failure based on the physical behaviour of the materials have been developed. In each theory a relationship is developed between principal stresses and the failure stress (or the yield point stress) in a simple tensile test on a specimen of the material.

When a tensile test is performed on a standard specimen of a material and a graph is plotted between tensile load and extension, a yield point is observed on the graph. At the yield point, there is considerable extension, Hooke's law is not obeyed and the stress no longer remains proportional to the strain after the yield point. Fig. 17·2 (*a*) show a tensile test specimen and 17·2 (*b*) shows the load-extension graph of a most common structural material *i.e.*, mild steel. If the test piece is unloaded after the stage of yield point, there remains permanent deformation in the material, rendering the material useless for further application.

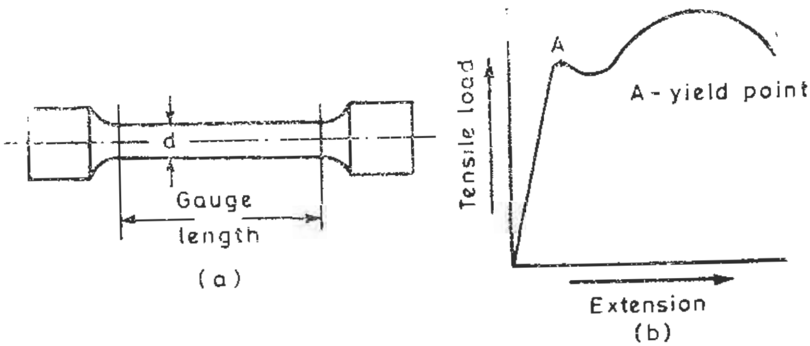


Fig. 17·2

In a simple tensile test we can easily determine the stress at which yielding has occured in a material but in the case of machine members subjected to various combinations of loads, practically it is impossible to know where and at what stage yielding has started, rendering the material useless, but certainly the principal stresses at a critical point can be known. In various theories of failure, the principal stresses have been expressed in terms of the yield stress in the simple tension or compression test and assuming that the stresses developed in the material are proportional to the applied loads, a limit is worked out such that if this limit, is exceeded, yielding is assumed to begin in the material.

The materials generally fail by fracture or by excessive deformation at yielding. In ductile materials, failure by yielding is the usual basis while in brittle materials like cast iron, concrete etc. failure by fracture or by ultimate stress is the criteria because yield point does not exist for a brittle materials.

17·1. THE MAXIMUM PRINCIPAL STRESS THEORY (Rankine's Theory)

This theory is based on the assumption that failure of the material or the yielding of the material is governed by the maximum principal stress and is not influenced by other stresses present at right angles. According to this theory, yielding in the material occurs when the applied loads are such that the maximum principal stress reaches the value of f_{yp}, *i.e.*, yield point stress in simple tensile test.

Say $p_1 > p_2 > p_3$ are the principal stresses

then $\quad p_1 \leqslant \pm f_{yp}$.

Consider the case of a thin cylindrical shell subjected to an internal pressure p. The principal stresses at any point are (see chapter 5) $\frac{pD}{2t}$, $\frac{pD}{4t}$ and $-p$ where D is the internal diameter of the cylinder and t is the radial wall thickness.

Principal stress, $\quad p_1 = \frac{pD}{2t}$.

According to the maximum principal stress theory $\frac{pD}{2t}$ should be less than f_{yp} or at the most it can be equal to f_{yp}, so that failure of the material can be avoided.

$$p_1 \leqslant f_{yp}$$

$$\frac{pD}{2t} \leqslant f_{yp} \quad \text{or} \quad p \leqslant \frac{2f_{yp} \cdot t}{D}$$

In a two dimensional case when $p_3=0$ this theory can be explained graphically as shown in Fig. 17·3.

Taking $x=\frac{p_1}{f_{yp}}$ and $y=\frac{p_2}{f_{yp}}$ and plotting the graph. Boundary of the figure *ABCD* defines the beginning of yielding. Experimental results show that this theory is acceptable when the principal stresses are of the same sign. It is unsafe to design a machine member on the basis of this theory if the principal stresses p_1 and p_2 are of the opposite sign. The theory is commonly used for brittle materials.

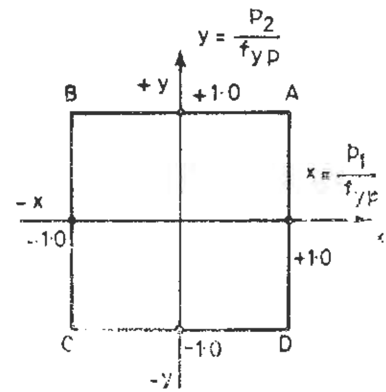


Fig. 17·3

Example 17·1-1. A solid circular shaft of diameter d is subjected to a pure torque of 20 Nm. Determine the diameter of the shaft according to the maximum principal stress theory, taking the factor of safety as 2 ; if the yield strength of the material is 310 N/mm².

Solution. f_{yp}, yield strength of material $=310$ N/mm²

Factor of safety $=2$

Allowable maximum principal stress

$$=\frac{f_{yp}}{2}=\frac{310}{2}=155 \text{ N/mm}^2$$

Torque on the shaft $=20$ Nm $=20\times10^3$ N mm

Say the diameter of the shaft $=d$ mm

Maximum shear stress developed in shaft, $q=\frac{16T}{\pi d^3}$.

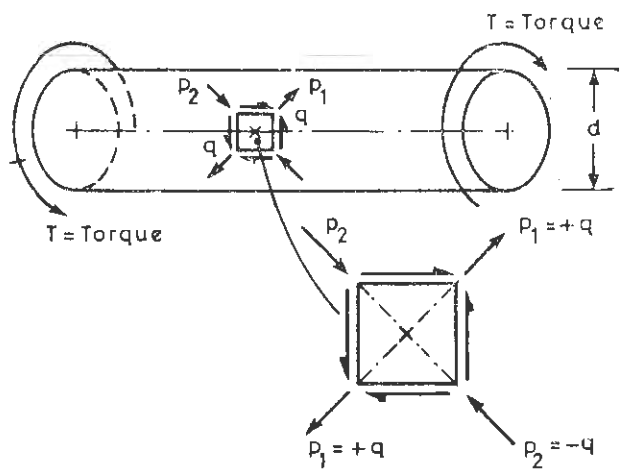


Fig. 17·4

The state of stress in a shaft subjected to pure torsion is shown in the Fig. 17·4. If q is the maximum shear stress on the surface of the shaft, then principal stresses at a point on the surface are p_1, p_2 and p_3 where $p_1=+q$, $p_2=-q$ $p_3=0$.

So the maximum principal stress or the maximum shear stress in the shaft is not to exceed 155 N/mm²

$$\frac{16T}{\pi d^3} \leqslant 155$$

or
$$d^3 \geqslant \frac{16\times 20\times 10^3}{\pi \times 155}$$

Shaft diameter, $d \geqslant 8{\cdot}7$ mm.

Exercise 17·1-1. A thin cylindrical shell of diameter 0·5 m and thickness t is subjected to an internal pressure of 2 N/mm². Determine the thickness of the cylinder according to the maximum principal stress theory taking the factor of safety as 3, if the yield strength of the material is 285 N/mm². [**Ans.** 5·26 mm]

17·2. MAXIMUM SHEAR STRESS THEORY (Coulomb's theory)

This theory assumes that the yielding of the material subjected to combined stresses is governed by the maximum shear stress developed in the material and in order to avoid yielding the maximum shear stress developed should not exceed the maximum shear stress at yield point in a simple tensile or compression test on the material. Fig. 17·5 shows three dimensional Mohr's stress circle at a point having principal stresses p_1, p_2, p_3.

Say $p_1 > p_2 > p_3$

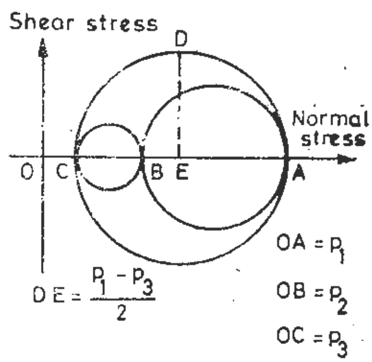
In the figure

$OA=p_1$; $OB=p_2$; $OC=p_3$

Maximum shear stress,

$$q_{max}=\frac{p_1-p_3}{2}$$

(represented by DE in the figure)



Three dimensional Mohr's stress circle.

Fig. 17·5

Maximum shear stress in a simple tension case at stress f_{yp} is $\frac{f_{yp}}{2}$ (as the principal stresses are f_{yp}, 0, 0)

As per this theory $\frac{p_1-p_3}{2} \leqslant \frac{f_{yp}}{2}$

$$(p_1-p_3) \leqslant f_{yp}.$$

Considering the example of thin cylindrical shell subjected to internal fluid pressure, having principal stresses $\frac{pD}{2t}$, $\frac{pD}{4t}$, $-p$. As per this theory $\frac{1}{2}\left(\frac{pD}{2t}+p\right) \leqslant f_{yp}$

Internal fluid pressure, $p \leqslant f_{yp} \cdot \dfrac{2t}{D+2t}$

where D=diameter of thin cylindrical shell

t=wall thickness of shell

i.e., if the internal pressure in the cylinder exceeds the value of $f_{yp} \cdot \dfrac{2t}{D+2t}$, yielding will take place in the material.

In a two dimensional case where $p_3=0$ and $p_1>p_2$ then as per the maximum shear stress theory

$$\frac{p_1-p_2}{2} \leqslant \frac{f_{yp}}{2} \text{ if } p_1 \text{ and } p_2 \text{ are of opposite sign}$$

and $$\frac{p_1}{2} \leqslant \frac{f_{yp}}{2} \text{ if } p_1 \text{ and } p_2 \text{ are of the same sign}$$

or $p_1 \leqslant f_{yp}$ is the governing equation to determine the onset of yielding. Under these conditions

The maximum shear stress theory coincides with the maximum principal stress theory.

Graphically this theory can be illustrated by Fig. 17·6.

In the quadrants I and III, when p_1 and p_2 are of the same sign, the maximum shear stress theory coincides with the maximum principal stress theory, *i.e.*, lines AB, BC ; DE and EF. In the quadrants II and IV when the principal stresses are of the opposite sign

Line CD shows

$p_1-p_2=-f_{yp}$

Line AF shows

$p_1-p_2=+f_{yp}$

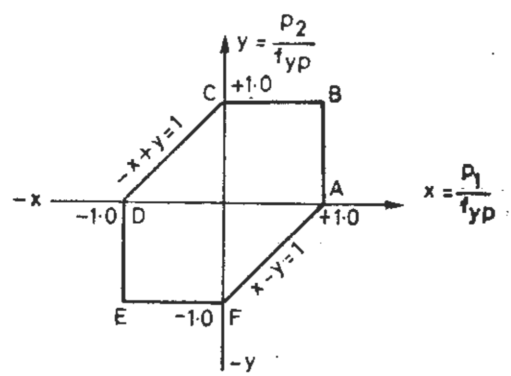


Fig. 17·6

The designers very often use this theory to design the machine components made of ductile materials such as mild steel.

Example 17·2-1. A thick cylinder of internal diameter 10 cm is subjected to an internal pressure of 500 kg/cm². Determine the thickness of the cylinder according to the maximum shear stress theory if the yield strength of the material of the cylinder is 2800 kg/cm², taking a factor of safety of 2.

Solution. In the case of thick cylinders subjected to internal pressure, maximum radial and circumferential stresses occur at the inner radius and axial stress is uniform.

Inner radius of the cylinder, $R_1=5$ cm

Say outer radius of the cylinder $=R_2$

At Inner Radius. Circumferential stress,

$$f_c=p \cdot \frac{R_2^2+R_1^2}{R_2^2-R_1^2} \text{ tensile}$$

where p=radial pressure

Axial stress, $f_a = \dfrac{p\ R_1^2}{R_2 - R_1^2}$

Principal stresses are $f_c, f_a, -p$.

Maximum shear stress $= \dfrac{f_c+p}{2} = \dfrac{p}{2} \cdot \dfrac{R_2^2+R_1^2}{R_2^2-R_1^2} + \dfrac{p}{2} = \dfrac{p}{2} \cdot \dfrac{2R_2^2}{R_2^2-R_1^2}$

or $$\frac{p}{2} \times \frac{2R_2^2}{R_2^2-R_1^2} \leqslant \frac{f_{vp}}{2\times FS}$$

Factor of safety, $FS=2$

or $$\frac{500}{2} \times \frac{2R_2^2}{R_2^2-R_1^2} \leqslant \frac{2800}{2\times 2}$$

or $$\frac{R_2^2}{R_2^2-R_1^2} = 1{\cdot}4, \text{ or } R_2^2 = 1{\cdot}4\ R_2^2 - 1{\cdot}4\ R_1^2$$

$$R_2^2 = 3{\cdot}5\ R_1^2 = 3{\cdot}5 \times 5^2$$

$$R_2 = 9{\cdot}354 \text{ cm}$$

Thickness of the cylinder $= 9{\cdot}354 - 5 = 4{\cdot}354$ cm.

Exercise 17·2-1. A thick steel cylinder of internal radius 40 mm and external radius 60 mm is subjected to an internal fluid pressure of intensity p. Determine the limiting value of p according to the following theories

(*i*) Maximum principal stress theory.

(*ii*) Maximum shear stress theory.

Given yield stress of steel=280 N/mm². [**Ans.** (*i*) 107·7 N/mm² (*ii*) 77·77 N/mm²]

17·3. MAXIMUM PRINCIPAL STRAIN THEORY (St. Venant's Theory)

In this theory, it is assumed that failure by yielding takes place in a material, when the maximum principal strain in the material subjected to combined stresses is equal to the strain at the yield point in a simple tensile or compression test on the material.

If $p_1 > p_2 > p_3$ are the principal stresses, the maximum principal strain is

$$\epsilon_1 = \frac{1}{E}\left(p_1 - \frac{p_2}{m} - \frac{p_3}{m}\right)$$

Strain at the yield point in a simple tensile or compression test is f_{vp}/E, then as per this theory

$$\epsilon_1 \leqslant \frac{f_{vp}}{E}$$

or $$\frac{1}{E}\left(p_1 - \frac{p_2+p_3}{m}\right) \leqslant \frac{f_{vp}}{E}$$

or $$\left(p_1 - \frac{p_2+p_3}{m}\right) \leqslant f_{vp}$$

Considering the case of thin cylinder subjected to internal pressure p again, where the principal stresses are $\dfrac{pD}{2t}$, $\dfrac{pD}{4t}$, $-p$ at the inner radius of the cylinder, we can write that

$$\left(\frac{pD}{2t} - \frac{pD}{4tm} + \frac{p}{m}\right) \leqslant f_{vp}$$

or $$p \leqslant f_{yp} \cdot \frac{4tm}{2Dm-D+4t} \quad \text{to avoid yielding in cylinder}$$

Now in a biaxial stress system, where $p_3=0$

Principle strains, $$\epsilon_1 = \frac{p_1}{E} - \frac{p_2}{mE}$$

$$\epsilon_2 = \frac{p_2}{E} - \frac{p_1}{mE}$$

Strain at the yield point in a simple tensile or compression test,

$$\epsilon_{yp} = \frac{f_{yp}}{E}$$

So $$\epsilon_1 \leqslant \epsilon_{yp}$$

$$\left(p_1 - \frac{p_2}{m}\right) \leqslant f_{yp}$$

$$\frac{p_1}{f_{yp}} - \frac{p_2}{m\,f_{yp}} \leqslant 1$$

or $$x - \frac{y}{m} \leqslant 1 \quad \text{if} \quad \frac{p_1}{f_{yp}} = x, \quad \text{and} \quad \frac{p_2}{f_{yp}} = y$$

Similarly $$\epsilon_2 \leqslant \epsilon_{yp}$$

or $$\left(p_2 - \frac{p_1}{m}\right) \leqslant f_{yp}$$

$$\frac{p_2}{f_{yp}} - \frac{p_1}{mf_{yp}} \leqslant 1 \quad \text{or} \quad y - \frac{x}{m} \leqslant 1$$

The above relationships between p_1, p_2 and f_{yp} can be shown through a graph as in Fig. 17·7.

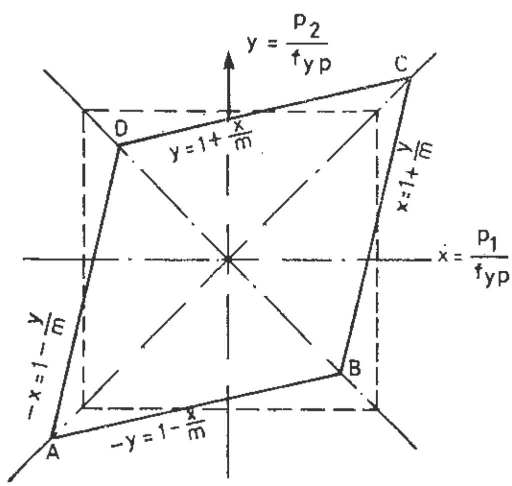


Fig. 17·7

To plot a graph showing the relationship between p_1 and p_2 and f_{yp} so that yielding just begins, the equation will be

$$x=1+\frac{y}{m} \quad \text{and} \quad y=1+\frac{x}{m}$$

where $1/m$ is the Poisson's ratio. This theory assumes that the material obeys Hook's law. Experimental results have shown that this theory is not quite acceptable.

Taking the case of biaxial tension *i.e.* p_1 and p_2 both are positive, then as per this theory,

$$p_1 \leqslant f_{yp}+\frac{p_2}{m}$$

showing thereby that principal stress p_1 can be greater than f_{yp}, which is not acceptable to designers.

Example 17·3-1. A certain type of steel has yield strength of 270 N/mm². At a point in the strained region the principal stresses are +120 N/mm², +80 N/mm² and −30 N/mm². Determine the factor of safety according to the maximum principal strain theory. Given : $1/m$ for steel=0·285.

Solution.

Principal stresses are $p_1=+120$ N/mm²,

$p_2=80$ N/mm² and $p_3=-30$ N/mm²

Maximum principal strain,

$$\epsilon_1=\frac{p_1}{E}-\frac{p_2}{mE}-\frac{p_3}{mE}$$

$$=\frac{1}{E}[120-0{\cdot}285\times 80+0{\cdot}285\times 30]=\frac{105{\cdot}75}{E}$$

According to the maximum principal strain theory

$$\epsilon_1 \leqslant \frac{f_{yp}}{E\times F.S}$$

$$\frac{105{\cdot}75}{E} \leqslant \frac{270}{E}\times\frac{1}{\text{(Factor of safety)}}$$

or Factor of safety, $FS=\dfrac{270}{105{\cdot}75}=2{\cdot}55$

Example 17·3-2. A shaft is simultaneously subjected to a bending moment of 20 kg-metre and a twisting moment of 15 kg-metre. Design the diameter of the shaft according to the maximum principal strain theory.

Given yield strength of the material=2100 kg/cm²

$1/m$, Poisson's ratio =0·3

Factor of safety =2

Solution.

Bending moment, $M=20$ kg-metre=2000 kg-cm

Twisting moment, $T=15$ kg-metre=1500 kg-cm

Say diameter of the shaft $=d$

Stress due to bending, $f=\dfrac{32\,M}{\pi d^3}$

Shear stress due to twisting, $q=\dfrac{16\,T}{\pi d^3}$

Principal stresses

$$p_1=\frac{f}{2}+\sqrt{\left(\frac{f}{2}\right)^2+q^2}$$

$$=\frac{16\,M}{\pi d^3}+\sqrt{\left(\frac{16\,M}{\pi d^3}\right)^2+\left(\frac{16\,T}{\pi d^3}\right)^2}$$

$$=\frac{16}{\pi d^3}\left[\,M+\sqrt{M^2+T^2}\,\right]$$

$$p_2=\frac{f}{2}-\sqrt{\left(\frac{f}{2}\right)^2+q^2}$$

$$=\frac{16}{\pi d^3}\left[\,M-\sqrt{M^2+T^2}\,\right]$$

Substituting the values of M and T we get,

$$p_1=\frac{16\times 4500}{\pi d^3}, \qquad p_2=-\frac{16\times 500}{\pi d^3}$$

Now according to the maximum principal strain theory

$$\frac{p_1}{E}-\frac{p_2}{mE}\leqslant\frac{f_{yp}}{E\times\text{Factor of Safety}}$$

or

$$\frac{1}{\pi d^3}\,(72000+0{\cdot}3\times 8000)\leqslant\frac{2100}{\text{F . S}}$$

$$74400=\frac{2100}{2}\times\pi d^3$$

$$d^3=22{\cdot}554476$$

Shaft diameter $d=2{\cdot}825$ cm

Exercise 17·3-1. Determine the thickness of a thin steel cylinder of diameter 600 mm subjected to an internal pressure of 3 N/mm² according to

(*a*) maximum shear stress theory

(*b*) maximum principal stress theory

(*c*) maximum principal strain theory

Take factor of safety of 2. Yield strength of steel=280 N/mm², $1/m$ for steel=0·28

[**Ans.** (*a*) 6·56 mm, (*b*) 6·42 mm, (*c*) 5·56 mm]

Exercise 17·3-2. A hollow circular shaft of internal diameter 3 cm and external diameter 5 cm is subjected to a torque of 12000 kg-cm. Determine the factor of safty according to the maximum principal strain theory.

Given, yield stress of steel=2700 kg/cm²

1/m for steel=0·29 [**Ans.** 3·72]

17·4. STRAIN ENERGY THEORY (Beltrami, Haigh)

This theory is based on the assumption that failure or yielding of the material occurs when the strain energy stored in a unit volume due to the principal stresses developed in the machine component is equal to the strain energy stored in a unit volume at the yield point stress in a simple tensile test performed on a specimen of the same material. The principal stresses at a point in a strained machine member are p_1, p_2 and p_3 as shown in the Fig. 17·8. The principal strains will be

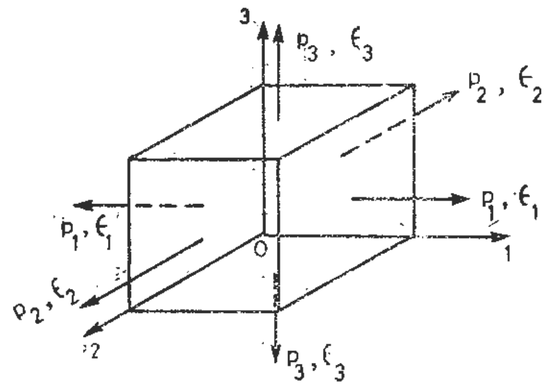


Fig. 17·8

$$\epsilon_1 = \frac{p_1}{E} - \frac{p_2+p_3}{mE}$$

$$\epsilon_2 = \frac{p_2}{E} - \frac{p_1+p_3}{mE}$$

$$\epsilon_3 = \frac{p_3}{E} - \frac{p_1+p_2}{mE}$$

strain energy per unit volume,

$$u = \frac{1}{2} p_1 \epsilon_1 + \frac{1}{2} p_2 \epsilon_2 + \frac{1}{2} p_3 \epsilon_3$$

$$= \frac{1}{2E}\left[p_1\left(p_1 - \frac{p_2+p_3}{m}\right) + p_2\left(p_2 - \frac{p_1+p_3}{mE}\right) + p_3\left(p_3 - \frac{p_1+p_2}{mE}\right)\right]$$

$$= \frac{1}{2E}\left[\left(p_1^2+p_2^2+p_3^2\right) - \frac{2}{m}\left(p_1p_2+p_2p_3+p_3p_1\right)\right]$$

Strain energy at yield point in a simple tensile test

$$u' = \frac{f_{yp}^2}{2E}$$

According to this theory $u \leqslant u'$

$$\left[(p_1^2+p_2^2+p_3^2) - \frac{2}{m}(p_1p_2+p_2p_3+p_3p_1)\right] \leqslant f_{yp}^2$$

Again taking the example of a thin cylindrical shell subjected to internal pressure p; where the principal stresses are $\frac{pD}{2t}$, $\frac{pD}{4t}$, $-p$, we can write that

$$\left(\frac{pD}{2t}\right)^2+\left(\frac{pD}{4t}\right)^2+(-p)^2-\frac{2}{m}\left[\frac{pD}{2t}\times\frac{pD}{4t}-\frac{pD}{4t}\times p-\frac{pD}{2t}\times p\right]\leqslant f_{yp}^2$$

$$p^2\left[\frac{D^2}{t^2}\left(\frac{5}{16}-\frac{1}{4m}\right)+1+\frac{3D}{2tm}\right]\leqslant f_{yp}^2$$

$$p\leqslant\frac{f_{yp}}{\sqrt{\frac{D^2}{t^2}\left(\frac{5}{16}-\frac{1}{4m}\right)+1+\frac{3D}{2tm}}}$$

Showing that yielding in the cylinder will begin if the magnitude of internal fluid pressure exceeds the value given by the expression above.

In a two dimensional case, when $p_3=0$, according to this theory

$$\left(p_1^2+p_2^2-\frac{2}{m}p_1p_2\right)\leqslant f_{yp}^2$$

or
$$\frac{p_1^2}{f_{yp}^2}+\frac{p_2^2}{f_{yp}^2}-\frac{2}{m}\frac{p_1p_2}{f_{yp}^2}\leqslant 1$$

or
$$x^2+y^2-\frac{2}{m}xy=1$$

where
$$x=\frac{p_1}{f_{yp}},\text{ and } y=\frac{p_2}{f_{yp}}$$

Let us take $\frac{1}{m}=0{\cdot}25$

The above equation will be

$$x^2+y^2-0{\cdot}5xy=1$$

which is shown in the Fig. 17·9, plotted on the rectangular co-ordinate system as an ellipse.

Experimental evidence available so far does not permit the use of this theory for design purposes.

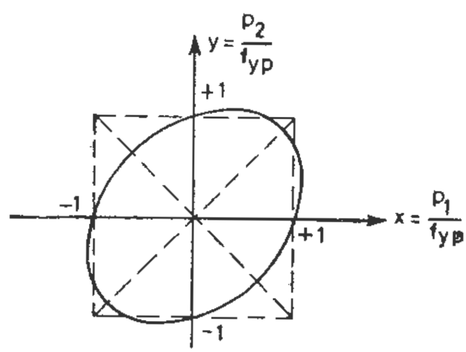


Fig. 17·9

Example 17·4-1. A thick cylinder of internal diameters 200 mm and external diameter 300 mm is subjected to an internal pressure p. Determine the maximum value of p according to the strain energy theory if the yield point stress of the material is 180 N/mm², taking a factor of safety of 2. Given $1/m=0{\cdot}32$.

Solution. Inner radius, $R_1=100$ mm
Outer radius $R_2=150$ mm
Internal pressure $=p$ in N/mm².

Maximum circumferential stress,

$$f_c=p \cdot \frac{R_2^2-R_1^2}{R_2^2-R_1^2}=p\times\frac{150^2+100^2}{150^2-100^2}=2\cdot6\,p$$

Axial stress,

$$f_a=p \cdot \frac{R_1^2}{R_2^2-R_1^2}=p\times\frac{100^2}{150^2-100^2}=0\cdot8\,p$$

The principal stresses at the inner radius are

$2\cdot6\,p,\ 0\cdot8\,p,\ -p$ N/mm²

Yield point stress $=180$ N/mm²

Factor of safety $=2$

Allowable stress, $f_{yp}'=\frac{180}{2}=90$ N/mm²

Applying the strain energy theory

$$(2\cdot6\,p)^2+(0\cdot8\,p)^2+(-p)^2-\frac{2}{m}\,(2\cdot6\times0\cdot8\,p^2-2\cdot6\,p^2-0\cdot8\,p^2)\leqslant 90^2$$

$$p^2[6\cdot76+0\cdot64+1-2\times0\cdot32\,(2\cdot08-2\cdot6-0\cdot8)]\leqslant 90^2$$

$$p^2[8\cdot4+0\cdot8448]\leqslant 90^2$$

Internal pressure, $p=\sqrt{\frac{8100}{9\cdot2448}}=29\cdot6$ N/mm²

Exercise 17·4-1. A pipe 15 cm in diameter is subjected to a pressure of 50 kg/cm² and an axial compressive force of 800 kg. Determine the wall thickness using the strain energy theory. The design stress is 1200 kg/cm² and the pipe is to be considered closed at the ends.

[**Ans.** 3·1 mm]

17·5. SHEAR STRAIN ENERGY (DISTORTION ENERGY) THEORY (Von Mises)

In this theory, it is assumed that failure by yielding occurs when the energy which is used in changing the shape of a unit volume of a component is equal to the distortion energy (or the shear strain energy) per unit volume at the yield stress of a specimen subjected to a simple tensile or compression test. Total strain energy at a point consists of two components *i.e.*,

(*i*) Volumetric strain energy causing the change in volume.

(*ii*) Shear strain energy causing the change in shape of the body.

In the previous article we have determined the total strain energy

$$u=\frac{1}{2E}\left[p_1^2+p_2^2+p_3^2-\frac{2}{m}\,(p_1p_2+p_2p_3+p_3p_1)\right]$$

$$=u_v+u_s$$

$=$Volumetric strain energy$+$shear strain energy

Volumetric strain energy can be determined by the volumetric

Stress component $p_m=\frac{p_1+p_2+p_3}{3}$ which is equal in all the directions. Principal stresses (not accompanied by any shear stress) can be represented by a tensor as follows

$$\begin{vmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{vmatrix} = \begin{vmatrix} p_m & 0 & 0 \\ 0 & p_m & 0 \\ 0 & 0 & p_m \end{vmatrix} + \begin{vmatrix} (p_1-p_m) & 0 & 0 \\ 0 & (p_2-p_m) & 0 \\ 0 & 0 & (p_3-p_m) \end{vmatrix}$$

Strain energy determined by 3 principal stresses at a point, *i.e.* p_m, p_m, p_m gives the volumetric strain energy, where p_m is the mean stress.

Volumetric strain energy, $u_v = 3\times\frac{1}{2}\, p_m\, \epsilon_m$

where strain, $$\epsilon_m = \frac{p_m}{E} - \frac{p_m+p_m}{mE} = \frac{p_m}{E}\left(1-\frac{2}{m}\right)$$

$$u_v = \frac{3}{2}\, p_m \times \frac{p_m}{E}\left(1-\frac{2}{m}\right) = \frac{3p_m^2}{2E}\left(1-\frac{2}{m}\right)$$

$$= \frac{3}{2E}\left(\frac{p_1+p_2+p_3}{3}\right)^2\left(1-\frac{2}{m}\right)$$

$$= \frac{1}{6E}\left(1-\frac{2}{m}\right)(p_1^2+p_2^2+p_3^2+2p_1p_2+2p_2p_3+2p_3p_1)$$

Strain energy given by (p_1-p_m), (p_2-p_m) and (p_3-p_m) is the shear strain energy required for changing the shape of a unit volume of a member

$$u_s = u - u_v$$

$$= \frac{1}{2E}\left[p_1^2+p_2^2+p_3^2-\frac{2}{m}(p_1p_2+p_2p_3+p_3p_1)\right]$$

$$-\frac{1}{6E}\left(1-\frac{2}{m}\right)(p_1^2+p_2^2+p_3^2+2p_1p_2+2p_2p_3+2p_3p_1)$$

$$= \frac{p_1^2+p_2^2+p_3^2}{3E}\left(1+\frac{1}{m}\right) - \frac{p_1p_2+p_2p_3+p_3p_1}{3E}\left(1+\frac{1}{m}\right)$$

$$= \left(1+\frac{1}{m}\right)\frac{1}{3E}\,[p_1^2+p_2^2+p_3^2-p_1p_2-p_2p_3-p_3p_1]$$

We know that $$E = 2G\left(1+\frac{1}{m}\right)$$ where E = Modulus of elasticity, G = Modulus of rigidity

or $$\left(1+\frac{1}{m}\right) = \frac{E}{2G}$$

$\therefore$ $$u_s = \frac{1}{12G}\,[(p_1-p_2)^2+(p_2-p_3)^2+(p_3-p_1)^2]$$

In a simple tensile test, at the yield point of the material, the principal stresses are f_{yp}, 0, 0

Shear strain energy per unit volume,

$$u_s' = \frac{1}{12G}\,(f_{yp}^2+f_{yp}^2) = \frac{f_{yp}^2}{6G}$$

According to this theory failure by yielding occurs if

$$\frac{1}{12G}\,[(p_1-p_2)^2+(p_2-p_3)^2+(p_3-p_1)^2] \leqslant \frac{f_{yp}^2}{6G}$$

or $$[(p_1-p_2)^2+(p_2-p_3)^2+(p_3-p_1)^2 \leqslant 2f_{yp}^2$$

Let us again consider the example of a thin cylindrical shell subjected to internal pressure p, where the principal stresses are $\frac{pD}{2t}$, $\frac{pD}{4t}$ and $-p$, and apply this theory, we can write

$$\left(\frac{pD}{2t}-\frac{pD}{4t}\right)^2+\left(\frac{pD}{4t}+p\right)^2+\left(-p-\frac{pD}{2t}\right)^2=2f_{yp}^2$$

$$p^2\left[\frac{D^2}{16t^2}+\frac{D^2}{16t^3}+\frac{D}{2t}+1+1+\frac{D^2}{4t^2}+\frac{D}{t}\right]\leqslant 2f_{yp}^2$$

$$p^2\left[2+\frac{3D^2}{8t^2}+\frac{3D}{2t}\right]\leqslant 2f_{yp}^2$$

$$p\leqslant\frac{f_{yp}}{\sqrt{1+\frac{3}{16}\frac{D^2}{t^2}+\frac{3}{4}\frac{D}{t}}}$$

i.e., yielding in the cylinder begins if the internal pressure is of the value given above

In a two dimensional stress system when $p_3=0$, according to this theory

$$[p_1^2+p_2^2-2p_1p_2+p_2^2+p_1^2]=2f_{yp}^2$$

$$p_1^2+p_2^2-p_1p_2=f_{yp}^2$$

or

$$\frac{p_1^2}{f_{yp}^2}+\frac{p_2^2}{f_{yp}^2}-\frac{p_1p_2}{f_{yp}^2}=1$$

$$x^2+y^2-xy=1$$

where $x=\frac{p_1}{f_{yp}}$, and $y=\frac{p_2}{f_{yp}}$

This is the equation of ellipse, shown plotted in Fig. 17·10. In this theory, it is assumed that the material obeys Hooke's law. For ductile materials, this theory is in good agreement with experimental results and is being used for design purposes.

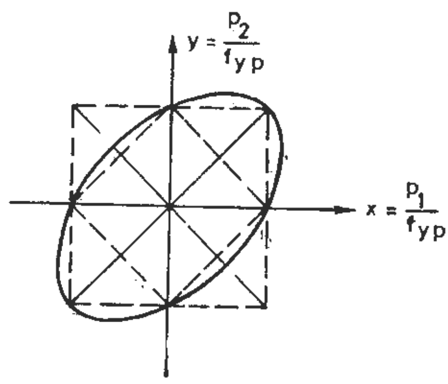


Fig. 17·10

Example 17·5-1. A thin spherical shell has a diameter of 400 mm and a thickness of t mm. It is subjected to an internal fluid pressure of 5 N/mm². The material has a yield strength of 265 N/mm². Determine the thickness of the shell according to the distortion energy theory, taking a factor of safely of 3.

Solution. Circumferential stress,

$$f_c=\frac{pD}{4t}=\frac{5\times 400}{4\times t}=\frac{500}{t}\ \text{N/mm}^2$$

Principal stresses at the inner radius of thin spherical shell are

$f_c, f_c, -p$ or $\frac{500}{t}$, $\frac{500}{t}$, -5 N/mm² in this case, Applying the distortion energy theory

$$\left[\left(\frac{500}{t}-\frac{500}{t}\right)^2+\left(\frac{500}{t}+5\right)^2+\left(-5-\frac{500}{t}\right)^2\right]=2f_{yp}'$$

where $$f_{yp}'=\frac{f_{yp}}{\text{factor of safety}}=\frac{265}{3}=88{\cdot}33\ \text{N/mm}^2$$

or $$2\left(\frac{250000}{t^2}+25+\frac{5000}{t}\right)=2(f_{yp}')^2$$

$$\frac{250000}{t^2}+25+\frac{5000}{t}=(88{\cdot}33)^2=7802{\cdot}8$$

$$250000+25t^2+5000t=7802{\cdot}8t^2$$

$$7777{\cdot}8t^2-5000t-250000=0$$

$$t^2-0{\cdot}6428t-32{\cdot}143=0$$

$$t=\frac{0{\cdot}6428+\sqrt{0{\cdot}4132+123{\cdot}572}}{2}=\frac{0{\cdot}6428+11{\cdot}3572}{2}$$

Thickness, $t=6$ mm.

Exercise 17-5-1. A thick cylindrical shell 15 cm internal radius and 20 cm external radius is subjected to an internal pressure of 300 kg/cm². Determine the factor of safety according to the distortion energy theory of failure if the yield stress of the material is 2800 kg/cm². Consider also the influence of axial stress. [**Ans.** 2·38]

Problem 17 1. What combination of principal stresses will give the same factor of safety for failure by yielding according to the maximum shear stress theory and the distortion energy theory ? Consider only a two dimensional stress system.

Solution. Say p_1 and p_2 are the principal stresses at a point.

(*a*) when p_1 and p_2 are of the same sign and

$$p_1>p_2 \quad \text{and} \quad p_3=0$$

Then according to maximum shear stress theory

$$\frac{p_1}{2}=\frac{f_{yp}}{2\,.\,F.S}$$

Factor of safety, $$FS=\frac{f_{yp}}{p_1} \qquad \ldots(1)$$

According to the distortion energy theory

$$\{(p_1-p_2)^2+p_1^2+p_2^2\}=2\left(\frac{f_{yp}}{FS}\right)^2$$

Factor of safety, $$FS=\frac{f_{yp}}{\sqrt{p_1^2+p_2^2-p_1p_2}} \qquad \ldots(2)$$

As per the condition given

$$\frac{f_{yp}}{p_1}=\frac{f_{yp}}{\sqrt{p_1^2+p_2^2-p_1p_2}}$$

or $$p_1^2+p_2^2-p_1p_2=p_1^2 \quad \text{or} \quad p_2(p_2-p_1)=0$$

or $$p_1=p_2$$ Since $p_2\neq 0$

(*b*) when p_1 and p_2 are of opposite sign and $p_1 > p_2$

According to maximum shear stress theory

$$\frac{p_1+p_2}{2}=\frac{f_{yp}}{2(FS)}$$

Factor of safety $$=\frac{f_{yp}}{p_1+p_2} \quad ...(3)$$

According to the distortion energy theory

$$\{(p_1+p_2)^2+p_1^2+p_2^2\}=2\left(\frac{f_{yp}}{FS}\right)^2$$

or Factor of safety $$=\frac{f_{yp}}{\sqrt{{}_1^2+p_2^2+p_1p_2}}$$

As per the condition given

$$\frac{f_{yp}}{p_1+p_2}=\frac{f_{yp}}{\sqrt{p_1^2+p_2^2+p_1p_2}}$$

or $$(p_1+p_2)^2=p_1^2+p_2^2+p_1p_2$$

or $$p_1p_2=0$$

i.e., either $p_1=0$ or $p_2=0$.

Problem 17·2. A shaft is subjected to a bending moment and a twisting moment simultaneously and at a particular section the bending moment is M and twisting moment is T. Show that the strain energy per unit volume is

$$u=\frac{1}{2E}\left\{f^2+2q^2\left(\frac{m+1}{m}\right)\right\}$$

where f is the maximum bending stress and q is the maximum shear stress and $1/m$ is the Poisson's ratio.

Solution.

Bending moment $=M$

Twisting moment $=T$

Maxm. bending stress, $f=\dfrac{32\,M}{\pi d^3}$

Maxm. shearing stress, $q=\dfrac{16\,T}{\pi d^3}$

Principal stresses on the surface of the shaft

$$p_1=\frac{f}{2}+\sqrt{\left(\frac{f}{2}\right)^2+q^2}$$

$$p_2=\frac{f}{2}-\sqrt{\left(\frac{f}{2}\right)^2+q^2}$$

Strain energy per unit volume

$$u=\frac{1}{2E}\left\{p_1^2+p_2^2-\frac{2}{m}\,p_1p_2\right\}$$

$$=\frac{1}{2E}\left\{\frac{f^2}{4}+\frac{f^2}{4}+q^2+f\sqrt{\left(\frac{f}{2}\right)^2+q^2}+\frac{f^2}{4}+\frac{f^2}{4}+q^2\right.$$

$$-f\sqrt{\left(\frac{f}{2}\right)^2+q^2}-\frac{2}{m}\left\{\left[\frac{f}{2}+\sqrt{\left(\frac{f}{2}\right)^2+q^2}\right]\right.$$

$$\left.\left.\times\left[\frac{f}{2}-\sqrt{\left(\frac{f}{2}\right)^2+q^2}\right]\right\}\right\}$$

$$=\frac{1}{2E}\left\{f^2+2q^2-\frac{2}{m}\left(\frac{f^2}{4}-\frac{f^2}{4}-q^2\right)\right\}$$

$$=\frac{1}{2E}\left\{f^2+2q^2+\frac{2}{m}q^2\right\}$$

$$=\frac{1}{2E}\left\{f^2+2q^2\left(1+\frac{1}{m}\right)\right\}$$

$$=\frac{1}{2E}\left\{f^2+2q^2\left(\frac{m+1}{m}\right)\right\}.$$

Problem 17·3. The internal pressure in a steel drum is 10 N/mm². The maximum circumferential stress is 85 N/mm² and longitudinal stress is 22 N/mm². Find the equivalent tensile stress in a simple tensile test according to each of the theories. Take Poisson's ratio =0·3.

Solution. The principal stresses at the critical point are

$$+85,\ +22,\ -10\ \text{N/mm}^2$$

Say the equivalent tensile stress in a simple tensile test is f.

(*a*) Maximum principal stress theory $f=p_1=85$ N/mm²

(*b*) Maximum shear stress theory

$$\frac{f}{2}=\frac{p_1-p_3}{2}=\frac{85+10}{2}$$

$$f=95\ \text{N/mm}^2$$

(*c*) Maximum principal strain theory

$$\frac{f}{E}=\frac{1}{E}\left(85-\frac{22}{m}+\frac{10}{m}\right)$$

$$f=85-0{\cdot}3\times22+0{\cdot}3\times10=81{\cdot}4\ \text{N/mm}^2$$

(*d*) Strain energy theory

$$\frac{f^2}{2E}=\frac{1}{2E}\left\{85^2+22^2+(-10)^2-\frac{2}{m}(85\times22-22\times10-10\times85)\right\}$$

$$f^2=\{7225+484+100-0{\cdot}6(1870-220-850)\}$$

$$=(7225+584-480)=7329$$

$$f=85{\cdot}6\ \text{N/mm}^2$$

(*e*) Distortion energy theory

$$2f^2=(85-22)^2+(22+10)^2+(-10-85)^2$$

$$=3969+1024+9025$$

$$f=83{\cdot}72\ \text{N/mm}^2.$$

Problem 17·4. The load on a bolt consists of an axial thrust of 800 kg together with a transverse shear force of 400 kg. Calculate the diameter of the bolt according to

(*a*) maximum principal stress theory

(*b*) maximum shear stress theory

(*c*) strain energy theory. Take 3 as factor of safety.

Yield strength of the material of the bolt$=2850$ kg/cm²

Poisson's ratio$=0{\cdot}3$

Solution.

Say the diameter of the bolt $=d$

Area of cross section, $A=\frac{\pi}{4}\,d^2$

Axial compressive stress on bolt, $f=\frac{800}{A}$

Shear stress on bolt, $q=\frac{400}{A}$

Maximum principal stress

$$p_1=\frac{f}{2}+\sqrt{\left(\frac{f}{2}\right)^2+q^2}$$

$$=\frac{400}{A}+\sqrt{\left(\frac{400}{A}\right)^2+\left(\frac{400}{A}\right)^2}$$

$$=\frac{965{\cdot}6}{A}\text{ kg/cm}^2$$

Minimum principal stress

$$p_2=\frac{400}{A}-\sqrt{\left(\frac{400}{A}\right)^2+\left(\frac{400}{A}\right)^2}$$

$$=-\frac{165{\cdot}6}{A}\text{ kg/cm}^2$$

Yield strength $=2850$ kg/cm²

Factor of safety $=3$

Allowable $f_{yp}'=\frac{2850}{3}=950$ kg/cm²

(*a*) Maximum principal stress theory

$$\frac{965{\cdot}6}{A}\leqslant 950$$

$$A=1{\cdot}01642=\frac{\pi}{4}\,d^2$$

$$d=1{\cdot}137\text{ cm}$$

(*b*) Maximum shear stress theory

$$\frac{p_1-p_2}{2}\leqslant\frac{f_{yp}'}{2}$$

$$\frac{965 \cdot 6}{A}+\frac{165 \cdot 6}{A}=950$$

$$A=\frac{1131 \cdot 2}{950}=1 \cdot 190 \text{ cm}^2=\frac{\pi}{4} d^2$$

$$d=1 \cdot 23 \text{ cm}$$

(c) Strain energy theory

$$\left(\frac{965 \cdot 6}{A}\right)^2+\left(-\frac{165 \cdot 6}{A}\right)^2+\frac{2}{m}\left(\frac{965 \cdot 6}{A} \times \frac{165 \cdot 6}{A}\right)=f_{y'p}'^2$$

$$932383 \cdot 36+27423 \cdot 36+95942 \cdot 016=f^2_{y'p} \times A^2$$

Putting the value of f'_{yp}

$$A^2=1 \cdot 0331+0 \cdot 0304+0 \cdot 1063=1 \cdot 1698$$

$$A=1 \cdot 081573=\frac{\pi}{4} \times d^2$$

$$d=1 \cdot 173 \text{ cm.}$$

Problem 17·5. A thin aluminium alloy tube has a mean diameter of 20 cm and wall thickness 2 mm. The tube is subjected to an internal pressure of 20 kg/cm² and a torque of 12000 cm-kg. If the yield strength of the material is 2400 kg/cm² and Poisson's ratio is **0·33**, determine factor of safety according to

(a) Maximum shear stress theory
(b) Maximum principal strain theory
(c) Strain energy theory.

Solution.

External diameter of the tube, $D=20 \cdot 2$ cm

Internal diameter of the tube, $d=19 \cdot 8$ cm

Polar moment of inertia.

$$J=\frac{\pi}{32}\left(20 \cdot 2^4-19 \cdot 8^4\right)$$

$$=\frac{\pi}{32}(166496 \cdot 64-153695 \cdot 36)=\mathbf{1256 \cdot 76 \text{ cm}^4}$$

Maximum stresses occur at the inner radius of the tube.

So, shear stress due to twisting moment, at the inner radius

$$q=\frac{12000}{1256 \cdot 76} \times \frac{19 \cdot 8}{2}=94 \cdot 53 \text{ kg/cm}^2$$

Axial stress due to internal pressure

$$f_a=\frac{pD'}{4t} \quad \text{where } D'=\text{mean diameter}$$

$$=\frac{20 \times 20}{4 \times 0 \cdot 2}=500 \text{ kg/cm}^2$$

Circumferential stress due to internal pressure

$$f_c = \frac{pD'}{2t} = \frac{20\times 20}{2\times 0{\cdot}2} = 1000 \text{ kg/cm}^2$$

The stresses at a point (at the inner radius) are f_c, f_a, q and $-p$ (radial stress)

Let us first determine the principal stresses, because f_c and f_a are no longer principal stresses since they are accompanied by shear stress q in this case. The stress system is shown in Fig. 17·11.

Maximum principal stress

$$p_1 = \frac{1000+500}{} + \sqrt{\left(\frac{1000-500)^2}{2}\right) + (94{\cdot}53)^2}$$

$$= 750 + 100\sqrt{6{\cdot}25 + 0{\cdot}8936}$$

$$= 750 + 267{\cdot}27$$

$$= 1017{\cdot}27 \text{ kg/cm}^2$$

Other principal stress

$$p_2 = \frac{1000+500}{2} - \sqrt{\left(\frac{1000-500}{2}\right)^2 + (94{\cdot}53)^2}$$

$$= 750 - 267{\cdot}27$$

$$= 482{\cdot}73 \text{ kg/cm}^2$$

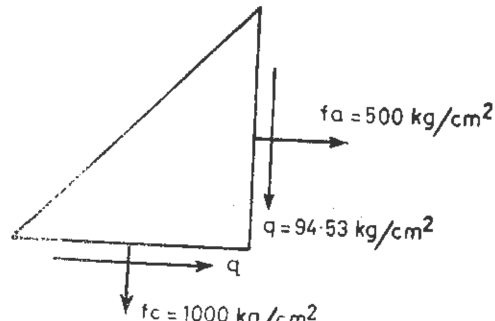


Fig. 17·11

Now the principal stresses at the inner radius of the tube are

$+1017{\cdot}27$ kg/cm² $+482{\cdot}73$ kg/cm² and -20 kg/cm².

(a) Maximum shear stress theory

$$\frac{1017{\cdot}27+20}{2} = \frac{2400}{2\times F.S.}$$

or Factor of safety $= \dfrac{2400}{1037{\cdot}27} = 2{\cdot}313.$

(b) Maximum principal strain theory

$$\frac{p_1}{E} - \frac{p_2}{mE} - \frac{p_3}{mE} = \frac{f_{yp}}{E\times FS}$$

$$1017{\cdot}27 - 0{\cdot}33\times 482{\cdot}73 + 0{\cdot}33\times 20 = \frac{2400}{F.S} \quad \text{or} \quad 1017{\cdot}27 - 152{\cdot}70 = \frac{2400}{F.S}$$

Factor of safety $= \dfrac{2400}{864{\cdot}57} = 2{\cdot}776.$

(c) Strain energy theory

$$(1017{\cdot}27)^2 + (482{\cdot}73)^2 + (-20)^2 - 2\times 0{\cdot}33(1017{\cdot}27\times 482{\cdot}73$$
$$-482{\cdot}73\times 20 - 1017{\cdot}27\times 20) = \left(\frac{2400}{F.S}\right)^2$$

$$1034838{\cdot}2+233028{\cdot}25+400-0{\cdot}66(491066{\cdot}74-96546-203454)$$

$$=\frac{2400\times2400}{(F.S)^2}$$

or $$1268266{\cdot}4-0{\cdot}66(191066{\cdot}74)=\frac{2400\times2400}{(FS)^2}$$

$$1268266{\cdot}4-126104{\cdot}04=\frac{2400\times2400}{(FS)^2}$$

$$1142162{\cdot}4=\frac{2400\times2400}{(FS)^2}$$

Factor of safety $=\sqrt{5{\cdot}043}=2{\cdot}246.$

Problem 17·16. A hollow shaft 30 mm internal diameter and 50 mm external diameter is subjected to a twisting moment of 800 Nm and an axial compressive force of 40 kN. Determine the factor of safety according to each of 5 theories of failure if the tensile and compressive yield strength of the material is 280 N/mm² and Poisson's ratio, $1/m=0{\cdot}3$.

Solution.

Internal diameter of shaft = 30 mm

External diameter of shaft = 50 mm

Area of cross section, $A=\pi(50^2-30^2)/4=12{\cdot}5664\times10^4$ mm²

Polar moment of inertia,

$$J=\pi\,(50^4-30^4)/32=53{\cdot}40\times10^4\ \text{mm}^4$$

Maximum shear stress at the outer surface of the shaft

$$q=\frac{T}{J}\times\frac{D}{2}\quad\text{where } T=800\times10^3\ \text{N mm}$$

$$=\frac{800\times10^3}{53{\cdot}40\times10^4}\times25=37{\cdot}45\ \text{N/mm}^2$$

Axial compressive stress, $f=\dfrac{P}{A}$ where $P=40$ kN

$$=\frac{40\times1000}{12{\cdot}5664\times10^2}=31.830\ \text{N/mm}^2.$$

On the outer surface of the shaft

The stresses on any element are shown in Fig. 17·12.

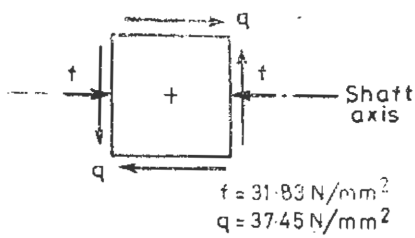


Fig. 17·12

Principal stresses at the point are

$$p_1=\frac{f}{2}+\sqrt{\left(\frac{f}{2}\right)+q^2}$$

$$=\frac{31{\cdot}83}{2}+\sqrt{\left(\frac{31{\cdot}83}{2}\right)^2+(37{\cdot}45)^2}$$

$$=15{\cdot}915+\sqrt{253{\cdot}287+1402{\cdot}50}$$
$$=15{\cdot}915+40{\cdot}691 \text{ (compressive)}$$
$$=56{\cdot}606 \text{ N/mm}^2 \text{ (compressive)}$$
$$p_2=\frac{f}{2}-\sqrt{\left(\frac{f}{2}\right)+q^2}$$
$$=15{\cdot}915-40{\cdot}691$$
$$=-24{\cdot}776 \text{ N/mm}^2 \text{ (tensile)}$$
$$f_{yp}=280 \text{ N/mm}^2$$

Poisson's ratio $=1/m=0{\cdot}3.$

(*a*) **Maximum principal stress theory**

Factor of safety, $FS=\dfrac{f_{yp}}{p_1}=\dfrac{280}{56{\cdot}606}=4{\cdot}9464.$

(*b*) **Maximum shear stress theory**

p_1 and p_2 are of opposite sign. Therefore

$$\frac{p_1-p_2}{2}=\frac{f_{yp}}{2\times FS}$$
$$FS=\frac{280}{56{\cdot}606+24{\cdot}776}=\frac{280}{81{\cdot}382}$$

Factor of safety $=3{\cdot}44$

(*c*) **Maximum principal strain theory**

$$\frac{p_1}{E}-\frac{p_2}{mE}=\frac{f_{yp}}{E\,(FS)}$$
$$56{\cdot}606+0{\cdot}3\times24{\cdot}776=\frac{280}{F.S.}$$

Factor of safety, $FS=\dfrac{280}{64{\cdot}0388}=4{\cdot}372.$

(*d*) **Strain energy theory**

$$(56{\cdot}606)^2+(-24{\cdot}776)^2+\frac{2}{m}(56{\cdot}606\times24{\cdot}776)=\left(\frac{f_{yp}}{FS}\right)^2$$
$$3204{\cdot}24+613{\cdot}85+841{\cdot}48=\frac{280\times280}{(FS)^2} \quad \text{since } \frac{1}{m}=0{\cdot}3$$

Factor of safety, $FS=\sqrt{\dfrac{280\times280}{4659{\cdot}57}}=\sqrt{16{\cdot}8255}=4{\cdot}10.$

(*e*) **Shear strain energy theory**

$$(56{\cdot}606+24{\cdot}776)^2+(-24{\cdot}776)^2+(-56{\cdot}606)^2=2\times\left(\frac{280}{FS}\right)^2$$
$$6623{\cdot}03+613{\cdot}85+3204{\cdot}24=\frac{2\times280\times28\text{[illegible]}}{(FS)^2}$$

$$(FS)^2=\frac{2\times 280\times 280}{10441\cdot 12}=15\cdot 017546$$

Factor of safety $=3\cdot 875$.

Problem 17·7. A 50 mm diamcter mild steel shaft when subjected to pure torsion ceases to be elastic when the torque reaches 4 kNm. A similar shaft is subjected to a torque 2·4 kNm and a bending moment M kNm. If maximum strain energy is the criterion for elastic failure, find the value of M. Poisson's ratio=0·28.

Solution.

Torque $T'=4 \text{ kNm}=4\times 10^6$ Nmm

Shaft diameter $=50$ mm

Maximum shear stress,

$$q'=\frac{16\,T'}{\pi d^3}=\frac{16\times 4\times 10^6}{\pi(50)^3}=162\cdot 97 \text{ N/mm}^2$$

Principal stresses on the surface of the shaft 162·97, −162·97, 0 N/mm²

Strain energy per unit volume at which the shaft ceases to be elastic

$$u'=\frac{1}{2E}\left[(162\cdot 97)^2+(-162\cdot 97)^2-\frac{2}{m}(162\cdot 97)(-162\cdot 97)\right]$$

$$=\frac{1}{2}\times(162\cdot 97)^2\left[2+\frac{2}{m}\right]$$

$$=\frac{1}{2E}(162\cdot 97)^2(2\cdot 56) \qquad \ldots(1)$$

Shaft subjected to M and T

Maximum shear stress due to T,

$$q=\frac{16\,T}{\pi d^3} \qquad \text{where } T=2\cdot 4 \text{ kNm}=2\cdot 4\times 10^6 \text{ Nmm}$$

Maximum bending stress due to M,

$$f=\frac{32\,M}{\pi d^3}$$

Principal stresses are $p_1=\frac{f}{2}+\sqrt{\left(\frac{f}{2}\right)^2+q^2}$

$$=\frac{16\,M}{\pi d^3}+\sqrt{\left(\frac{16\,M}{\pi d^3}\right)^2+\left(\frac{16\,T}{\pi d^3}\right)^2}$$

$$=\frac{16}{\pi d^3}\left[M+\sqrt{M^2+T^2}\right]$$

$$p_2=\frac{f}{2}-\sqrt{\left(\frac{f}{2}\right)^2+q^2}$$

$$=\frac{16}{\pi d^3}\left[M-\sqrt{M^2+T^2}\right]$$

Strain energy per unit volume

$$u=\frac{1}{2E}\left[p_1^2+p_2^2-\frac{2}{m}(p_1p_2)\right]$$

$$=\frac{1}{2E}\left[M^2+M^2+T^2+2M\sqrt{M^2+T^2}+M^2+M^2+T^2\right.$$

$$\left.-2M\sqrt{M^2+T^2}-\frac{2}{m}(M^2-M^2-T^2)\right]\times\left(\frac{16}{\pi d^3}\right)^2$$

$$=\left(\frac{16}{\pi d^3}\right)^2\times\frac{1}{2E}\left[4M^2+2T^2+\frac{2}{m}T^2\right]$$

But as per the strain energy theory $u'=u$

$$\left(\frac{16}{\pi d^3}\right)^2[4M^2+(2\cdot56)\;T^2]=(162\cdot97)^2(2\cdot56) \quad \text{as} \quad \frac{1}{m}=0\cdot28$$

$$4M^2+2\cdot56\;T^2=(162\cdot97)^2(2\cdot56)\left(\frac{\pi d^3}{16}\right)^2$$

$$=162\cdot97\times162\cdot97\times2\cdot56\times\frac{\pi^2\times50^6}{256}$$

$$=2\cdot66\times10^4\times\pi^2\times15625\times10^4$$

$$=41\cdot020\times10^{12}$$

$$4M^2=41\cdot020\times10^{12}-2\cdot56\times(2\cdot4\times10^6)^2$$

$$=41\cdot020\times10^{12}-14\cdot7456\times10^{12}=26\cdot2744\times10^{12}$$

$$M^2=6\cdot5686\times10^{12}$$

$$M=2\cdot563\times10^6\text{ Nmm}=2\cdot563\text{ kNm.}$$

Problem 17·8. A hollow circular steel shaft is subjected to a twisting moment of 80 kg-metre and a bending moment of 120 kg-metre. The internal diameter of the shaft is 60% of the external diameter. Determine the external diameter of the shaft according to (*a*) Maximum principal stress theory (*b*) Maximum shear stress theory (*c*) Shear strain energy theory. Take

Factor of safety	$=2$
Yield strength of steel	$=2700$ kg/cm^2

Solution.

Say external diameter $=D$

Then, internal diameter $=0\cdot6\;D=d$

Area of cross section, $A=\frac{\pi}{4}(D^2-0\cdot36\;D^2)$

Polar moment of inertia,

$$J=\frac{\pi}{32}(D^4-d^4)=\frac{\pi}{32}(D^4-0\cdot6^4\times D^4)$$

$$=\frac{\pi D^4}{32}(0\cdot8704)=0\cdot08545\times D^4$$

Moment of inertia, $I=\frac{J}{2}=0\cdot042725\;D^4$

Maximum stresses will be developed on the outer surface of the shaft.

$$M=120 \text{ kg-m}, \qquad T=80 \text{ kg-m}$$

Maximum direct stress due to bending moment

$$f=\frac{M}{I}\times\frac{D}{2}=\frac{120\times10^2}{0\cdot042725\ D^4}\times\frac{D}{2}$$

$$=\frac{14\cdot04\times10^4}{D^3}\ \text{kg/cm}^2$$

Maximum shearing stress due to twisting moment

$$q=\frac{T}{J}\times\frac{D}{2}$$

$$=\frac{80\times10^2}{0\cdot08545\ D^4}\times\frac{D}{2}=\frac{4\cdot68\times10^4}{D^3}\ \text{kg/cm}^2$$

Principal stresses

$$p_1=\frac{f}{2}+\sqrt{\left(\frac{f}{2}\right)^2+q^2}$$

$$=\frac{7\cdot02\times10^4}{D^3}+\sqrt{\left(\frac{7\cdot02\times10^4}{D^3}\right)^2+\left(\frac{4\cdot68\times10^4}{D^3}\right)^2}$$

$$=\frac{7\cdot02\times10^4}{D^3}+\frac{10^4}{D^3}\sqrt{49\cdot28+21\cdot90}$$

$$=\frac{7\cdot02\times10^4}{D^3}+\frac{8\cdot437\times10^4}{D^3}=\frac{15\cdot457}{D^3}\times10^4$$

$$p_2=\frac{f}{2}-\sqrt{\left(\frac{f}{2}\right)^2+q^2}$$

$$=\frac{7\cdot02\times10^4}{D^3}-\frac{8\cdot437\times10^4}{D^3}=-\frac{1\cdot417\times10^4}{D^3}$$

(a) Maximum Principal Stress Theory

Yield point stress, $f_{yp}=2700$ kg/cm²

Factor of safety $=2$

Allowable $f_{yp}'=\dfrac{2700}{2}=1350$ kg/cm²

$$p_1=f_{yp}'$$

$$\frac{15\cdot457\times10^4}{D^3}=1350$$

$$D^3=\frac{154570}{1350}=114\cdot496$$

Shaft diameter $=4\cdot855$ cm

(*b*) **Maximum shear stress theory**

Since p_1 and p_2 are of opposite sign

$$\frac{p_1-p_2}{2}=\frac{f_{yp}'}{2}$$

or
$$\frac{15{\cdot}457}{D^3}\times10^4+\frac{1{\cdot}417\times10^4}{D^3}=1350$$

$$D^3=\frac{16{\cdot}874\times10^4}{1350}=124{\cdot}9925$$

Shaft diameter, $D=5$ cm

(*c*) **Shear strain energy theory**

$$(p_1-p_2)^2+p_1^2+p_2^2=2f_{yp}'^2$$

$$(16{\cdot}874\times10^4)^2+(15{\cdot}457\times10^4)^2+(1{\cdot}417\times10^4)^2=(1350)^2\times2\times D^6$$

$$\frac{10^8}{1350\times2700}[(16{\cdot}874)^2+(15{\cdot}457)^2+(1{\cdot}417)^2]=D^6$$

$$D^6=27{\cdot}43\ [284{\cdot}73+238{\cdot}92+2{\cdot}00]=14418{\cdot}5$$

$$D^3=120{\cdot}077$$

Shaft diameter, $D=4{\cdot}934$ cm

SUMMARY

If at a point in a strained body p_1, p_2 and p_3 are principal stresses such that $p_1>p_2>p_3$ and f_{yp} is the yield point stress of the material when tested in simple tension or compression test.

(*a*) Maximum principal stress theory $p_1\leqslant f_{yp}$

(*b*) Maximum shear stress theory

(*i*) $\frac{p_1-p_3}{2}\leqslant\frac{f_{yp}}{2}$

(*ii*) If p_3 is equal to zero and p_1 and p_2 are of opposite sign. Then

$$\frac{p_1-p_2}{2}\leqslant\frac{f_{yp}}{2}$$

(*c*) Maximum principal strain theory

$$\frac{1}{E}\left[p_1-\frac{p_2+p_3}{m}\right]\leqslant\frac{f_{yp}}{E}$$

where $\frac{1}{m}$ = Poisson's ratio, E = Young's modulus

(*d*) Strain energy theory

$$\frac{1}{2E}\left[p_1^2+p_2^2+p_3^3-\frac{2}{m}(p_1p_2+p_2p_3+p_3p_1)\right]\leqslant\frac{f_{yp}^2}{2E}$$

(*e*) Shear strain energy theory

$$[(p_1-p_2)^2+(p_2-p_3)^2+(p_3-p_1)^2]\leqslant2f_{yp}^2$$

MULTIPLE CHOICE QUESTIONS

1. The elastic limit stress for a material is 270 N/mm². A machine member of circular section subjected to a uniaxial stress is to be designed. If the diameter obtained by using the maximum principal stress theory is 40 mm, then by using the maximum shear stress theory, the diameter of the machine member will be

   (*a*) $40\sqrt{2}$ mm (*b*) 40 mm

   (*c*) $40/\sqrt{2}$ mm (*d*) None of the above

2. The principal stresses at a point are 70 N/mm², 60 N/mm² and −18 N/mm². Say f_{yp} is the stress at the yield point of the material. Using the maximum principal stress theory we get factor of safety of 4. What is the factor of safety if maximum principal strain theory is used. Poisson's ratio for the material is 1/3

   (*a*) 4 (*b*) 4·5

   (*c*) 5·0 (*d*) 5·5

3. A thin cylindrical shell with D/t ratio equal to 40 is subjected to internal fluid pressure of 2 N/mm². The yield point stress of the material is 210 N/mm². Using the maximum shear stress theory for designing the thin shell, the factor of safety is

   (*a*) 5·75 (*b*) 5·50

   (*c*) 5·25 (*d*) 5·00

4. The principal stresses developed at a point are +60 N/mm², −60 N/mm² and 0·0 N/mm². Using the shear strain energy theory, the factor of safety obtained is $\sqrt{3}$. The yield point stress of the material is

   (*a*) $60\sqrt{6}$ N/mm² (*b*) $60\times\sqrt{3}$ N/mm³

   (*c*) 60 N/mm² (*d*) None of the above.

5. A shaft subjected to pure torsion is to be designed. The yield point stress of the material is 280 N/mm² and Poisson's ratio=0·3. Which of the following theories gives the largest diameter of the shaft

   (*a*) Maximum principal stress theory (*b*) Maximum shear stress theory

   (*c*) Maximum principal strain theory (*d*) Shear strain energy theory.

6. A shaft subjected to pure torsion is to be designed. The yield point stress of the material is 2700 kg/cm² and Poisson's ratio is 0·3. Which of the following theories of failure gives the smallest diameter of the shaft

   (*a*) Maximum principal stress theory (*b*) Maximum principal strain theory

   (*c*) Strain energy theory (*d*) Shear strain energy theory.

7. A thick cylinder of internal diameter 10 cm and external diameter 20 cm is subjected to internal fluid pressure p. The yield strength of the material is 240 N/mm². Taking a factor of safety of 2 and using the maximum principal stress theory of failure, the maximum value of internal pressure p is

   (*a*) 120 N/mm² (*b*) 90 N/mm²

   (*c*) 72 N/mm² (*d*) N/mm²

8. A thick cylinder of internal diameter 20 cm and external diameter 30 cm is subjected to internal fluid pressure p. The yield strength of the material is 270 N/mm². Taking a factor of safety of 3 and using the maximum shear stress theory, the maximum allowable value of internal pressure p is

   (*a*) 50 N/mm² (*b*) 32·5 N/mm²

   (*c*) 25 N/mm² (*d*) N/mm²

9. At a point in a strained material the principal stresses are p_1, p_2 and zero. What combination of principal stresses will give the same factor of safety by yielding according to the maximum shear stress theory and the distortion energy theory of failure ?

(a) $p_1=-p_2$ (b) $p_1=\frac{1}{2}p_2$
(c) $p_1=p_2$ (d) $p_1=-2p_2$

10. A shaft is subjected to bending moment M and a twisting moment T simultaneously (at a particular section). The maximum shear stress due to T is 30 N/mm² and maximum bending stress due to M is 80 N/mm². The yield point stress of the material is 280 N/mm². The maximum shear stress theory is used to design the shaft, the factor of safety obtained is

(a) 3·5 (b) 3·0
(c) 2·8 (d) 2·5

ANSWERS

1. (b)	2. (c)	3. (d)	4. (a)	5. (b)
6 (a)	7. (c)	8. (c)	9. (c)	10. (c)

EXERCISES

17·1. A shaft is subjected to a bending moment M and a twisting moment T. The ratio of $M/T=0·5$. Show that strain energy per unit volume is

$$u=\frac{512M^2}{E\pi^2d^6}\left(2+\frac{1}{m}\right)$$

where d is the diameter of the shaft and $1/m$ is the Poisson's ratio of the material and E is the Young's modulus of the material.

17·2. Considering the principal stresses in a steam boiler as p, $0·5\,p$, 0. Poisson's ratio, $1/m=0·28$, and the equivalent stress in a simple tensile test as f, find p in terms of each of the theories. [**Ans.** f, f, 1·162 f, 1·016 f, 1·154 f]

17·3. The load on a bolt consists of an axial pull of 10 kN together with a transverse shear force of 6 kN. Calculate the diameter of the bolt according to

(a) Maximum principal strain theory
(b) Shear strain energy theory.
Take factor of safety as 2.
Given yield strength of the material = 310 N/mm² [**Ans.** 10·53 mm, 10·9 mm]
Poisson's ratio = 0·27

17·4. A thin copper alloy tube has a mean diameter of 200 mm and a wall thickness of 3 mm The tube is subjected to an internal pressure of 2N/mm² and a bending moment of 2×10^6 N mm. If the yield strength of the material is 180 N/mm² and Poisson's ratio is 0·35, determine the factor of safety according to (a) maximum principal stress theory (b) shear strain energy theory. [**Ans.** (a) 2·70 (b) 2·92]

17·5. A hollow circular steel shaft is subjected to a twisting moment of 60 kg-metre and a bending moment of 50 kg-metre. The internal diameter of the shaft is half the external diameter. Determine the external diameter of the shaft according to

(a) Maximum principal strain theory.
(b) Shear strain energy theory.
Take factor of safety of 2.
Yield strength of steel = 2700 kg/cm²
Poisson's ratio = 0·3. [**Ans.** (a) 3·8 cm (b) 3·872 cm]

17·6. A 5 cm diameter mild steel shaft when subjected to pure torsion ceases to be elastic when the torque reaches 42000 kg-cm. A similar shaft is subjected to a torque 28000 kg-cm and a bending moment M kg-cm. If the maximum strain energy is the criterion for elastic failure, find the value of M. Poisson's ratio = 0·3. [**Ans.** 24940 kg-cm]

18

Rotating Discs and Cylinders

The problem of determining the stresses developed in bodies like shafts and discs rotating at high speeds is of considerable interest. Due to their high speeds of rotation, steam turbine shafts and discs experience large magnitudes of centrifugal forces. The stresses caused by these centrifugal forces are distributed symmetrically about their axis of rotation.

18·1. ROTATING RINGS

Let us consider a thin ring of mean radius R, rotating about its axis at angular speed ω radians/second. Say t is the radial thickness and b is the axial width of the ring. The axial width, b and radial thickness t both are small and it is assumed that there is no variation of stress along the thickness b or t. It is further assumed that there is no stress in the axial direction since thickness b is very small.

ρ=weight density of the material.

Now consider a small element $abcd$ subtending an angle $\delta\theta$ at the centre, at an angle θ from the horizontal axis X-X.

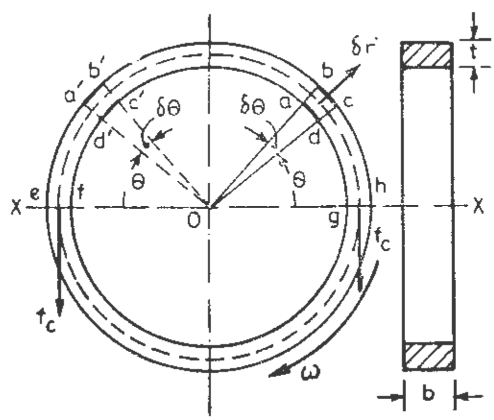


Fig. 18·1

Volume of the small element $= R\delta\theta \cdot b \cdot t$

Weight of the small element $= \rho R\delta\theta \cdot b \cdot t$

δF, centrifugal force acting on the small element

$$= \left(\frac{\rho Rbt\ \delta\theta}{g}\right)\omega^2 R$$

where g=acceleration due to gravity

$$\text{Vertical component of } \delta F = \left(\frac{\omega^2\rho R^2 bt}{g}\right)\delta\theta \cdot \sin\theta$$

$$\text{Horizontal component of } \delta F = \left(\frac{\rho\omega^2 R^2 bt}{g}\right)\delta\theta \cos\theta.$$

The horizontal component of δF will be cancelled when we consider another small element $a'b'c'd'$ in 2nd quadrant at an angle θ, but the vertical component of δF will be added.

Total vertical component or the bursting force across the horizontal diameter X-X

$$=\int_0^{\pi}\left(\frac{\rho\omega^2R^2bt}{g}\right)\sin\theta\ d\theta=\frac{\rho\omega^2R^2bt}{g}\Big|-\cos\theta\Big|_0^{\pi}$$

$$=\frac{2\rho\omega^2R^2bt}{g}$$

Say f_c is the hoop stress developed along the horizontal section.

Area of cross section $=2\times b\times t$ (as shown by ef and gh)

Resisting force $=f_c\times 2\times b\times t$

For equilibrium, $2f_c\ .\ bt=\dfrac{2\rho\omega^2R^2bt}{g}$

Circumferential stress, $f_c=\dfrac{\rho\omega^2R^2}{g}$

$=\dfrac{\rho V^2}{g}$ where V=linear velocity of the ring$=\omega R$.

Example 18·1-1. Find the safe number of revolutions per minute for a thin ring 2 metres in diameter if the stress is not to exceed 150 N/mm².

Given, weight density $=7·8$ cm³ (gram force/cm³)

Solution. Weight density,

$$\rho=0·0078\ \text{kg/cm}^3$$
$$=0·0078\times 9·8\ \text{N/cm}^3=0·07644\ \text{N/cm}^3$$
$$=76·44\times 10^3\ \text{N/m}^3$$

Allowable stress, $f_c=150\ \text{N/mm}^2=150\times 10^6\ \text{N/m}^2$

Now $f_c=\dfrac{\rho V^2}{g}$ and $g=9·8$ m/sec²

$$V^2=\frac{f_c\ .\ g}{\rho}=\frac{150\times 10^6\times 9·8}{76·44\times 10^3}=1·923\times 10^4$$

$V=138·675$ metres/sec

But $R=1$ metre

Angular velocity, $\omega=\dfrac{V}{R}=\dfrac{138·675}{1}=138·675$ radian/second

$$\text{R.P.M.}=\frac{138·675\times 60}{2\pi}=1324·246 \text{ resolutions per minute.}$$

Exercise 18·1-1. Calculate the stress in the rim of a pulley when linear velocity of the rim is 80 metres/second. What will be the stress if the speed is increased by 20 %?

Specific weight $=0·0078$ kg/cm³

Acceleration due to gravity,

$g=981$ cm/sec². [**Ans.** 508·87 kg/cm², 732·77 kg/cm²]

18·2. ROTATING THIN DISC

Let us consider a thin disc of inner radius R_1 and outer radius R_2 rotating at angular speed ω about its axis O. The thickness t of the disc is small and it is assumed that stresses

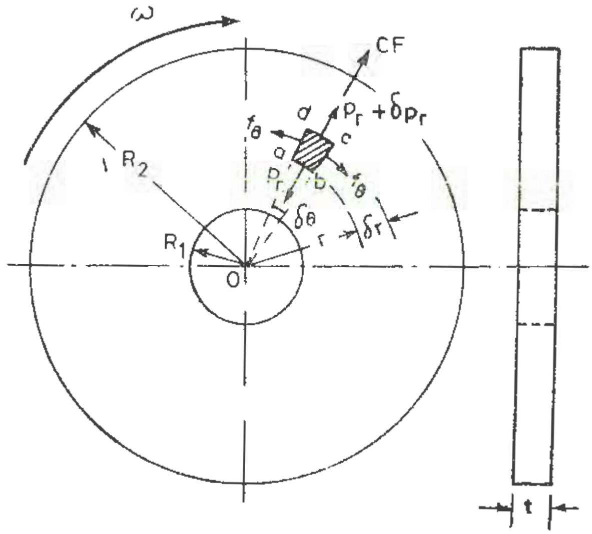


Fig. 18·2

do not vary across the thickness and there is no axial stress in the disc. Consider a small element *abcd* at radius r from the axis and subtending an angle $\delta\theta$ at the centre. Say the radial thickness of the small elemens is δr as shown in the Fig. 18·2.

When the disc is rotating at high speed, let us say that radius r changes to $r+u$ and radius $r+\delta r$ changes to $r+\delta r+u+\delta u$

In other words change in radius $r=u$

Change in radial thickness $\delta r=\delta u$

Moreover say the circumferential stress developed $=f_\theta$

(This stress varies with the radius)

Radial stress at radius r is p_r

(This stress also varies with the radius)

and Radial stress at radius $r+\delta r$ is $p_r+\delta p_r$

Weight of the small element

Considered $=(r\delta\theta)(\delta r)(t)\rho$

where ρ = weight density of the material.

Centrifugal force on the small element along the radial direction *eo*

$$=\left(\frac{\rho\, rt\ \delta r\ \delta\theta}{g}\right)\omega^2 r$$

$$=\frac{\rho\omega^2 r^2 t\,\delta r\,\delta\theta}{g}$$

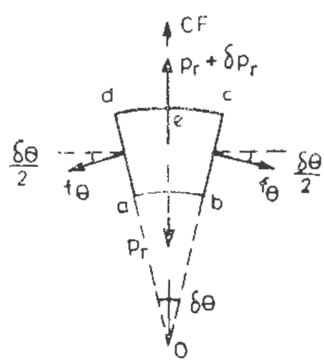


Fig. 18·3

Circumferential force on faces ad and $bc = f_\theta.\delta r.t$

Radial force on face $ab = (p_r.r\delta\theta t)$

Radial force on face $cd = (p_r + \delta p_r)(r + \delta r)\,\delta\theta t$

Resolving the forces in the radial direction co as shown in the figure and considering the equilibrium of the forces.

$$p_r.r.\delta\theta.t + 2f_\theta \,.\, \sin\frac{\delta\theta}{2} \,.\, \delta r.t$$

$$= (p_r + \delta p_r)(r + \delta r)\,\delta\theta\, t + \frac{\rho\omega^2\, r^2\, t\, \delta r\, \delta\theta}{2}$$

Since $\delta\theta$ is very small, $\sin\frac{\delta\theta}{2} \simeq \frac{\delta\theta}{2}$

Now $t.\delta\theta$ is commpn on both the sides, the above expression can be simplified as

$$p_r.r + f_\theta\, \delta r = p_r.r + r\,\delta p_r + p_r\,\delta r + \delta p_r\,\delta r + \frac{\rho\omega^2 r^2 \delta r}{g}$$

Neglecting the term $\delta p_r\, \delta r$, the expression can be further simplified as

$$f_\theta.\delta r = r\,\delta p_r + p_r\,\delta r + \frac{\rho\omega^2 r^2 \delta r}{g}$$

Dividing throughout δr we get

$$f_\theta = r\,\frac{\delta p_r}{\delta r} + p_r + \frac{\rho\omega^2 r^2}{g}$$

$$f_\theta - p_r = r\,\frac{\delta p_r}{\delta r} + \frac{\rho\omega^2 r^2}{g} \qquad \ldots(1)$$

Now consider the circumferential and radial strains

Circumferential strain, $\epsilon_c = \dfrac{r+u-r}{r} = \dfrac{f_\theta}{E} - \dfrac{p_r}{mE}$

$$\frac{u}{r} = \frac{1}{E}\left[f_\theta - \frac{p_r}{m}\right] \qquad \ldots(2)$$

Radial strain, $\epsilon_r = \dfrac{\delta r + \delta u - \delta r}{\delta r} = \dfrac{\delta u}{\delta r}$

$$= \frac{p_r}{E} - \frac{f_\theta}{mE}$$

or
$$\frac{\delta u}{\delta r} = \frac{1}{E}\left[p_r - \frac{f_\theta}{m}\right] \qquad \ldots(3)$$

or In the limits
$$\frac{du}{dr} = \frac{1}{E}\left[p_r - \frac{f_\theta}{m}\right] \qquad \ldots(3)$$

Now differentiating the equation (2) with respect to r

$$\frac{du}{dr} = \frac{1}{E}\left(f_\theta - \frac{p_r}{m}\right) + \frac{r}{E}\left(\frac{df_\theta}{dr} - \frac{1}{m}\,\frac{dp_r}{dr}\right) \qquad \ldots(4)$$

Equating equations (3) and (4)

$$\frac{1}{E}\left(p_r - \frac{f_\theta}{m}\right) = \frac{1}{E}\left(f_\theta - \frac{p_r}{m}\right) + \frac{r}{E}\left(\frac{df_\theta}{dr} - \frac{1}{m}\,\frac{dp_r}{dr}\right)$$

$$\left(1+\frac{1}{m}\right)\left(p_r-f_\theta\right)=r\left(\frac{df_\theta}{dr}-\frac{1}{m}\frac{dp_r}{dr}\right) \qquad \ldots(5)$$

or
$$\left(1+\frac{1}{m}\right)\left(f_\theta-p_r\right)=r\left(\frac{1}{m}\frac{dp_r}{dr}-\frac{df_\theta}{dr}\right)$$

Substituting the value of $(f_\theta-p_r)$ from equation (1), we get

$$\left(1+\frac{1}{m}\right)\left(r\frac{dp_r}{dr}+\frac{\rho\omega^2r^2}{g}\right)=\frac{r}{m}\frac{dp_r}{dr}-\frac{r\,df_\theta}{dr}$$

or
$$r\frac{dp_r}{dr}+\frac{\rho\omega^2r^2}{g}+\frac{\rho\omega^2r^2}{mg}=-r\frac{df_\theta}{dr}$$

or
$$r\left(\frac{dp_r}{dr}+\frac{df_\theta}{dr}\right)=-\left(1+\frac{1}{m}\right)\frac{\rho\omega^2r^2}{g}$$

or
$$\frac{d}{dr}\left(p_r+f_\theta\right)=-\left(\frac{m+1}{m}\right)\frac{\rho\omega^2r}{g} \qquad \ldots(6)$$

Integrating the equation (6) we get

$$p_r+f_\theta=-\left(\frac{m+1}{m}\right)\frac{\rho\omega^2r^2}{2g}+A \qquad \ldots(7)$$

where A is the constant of integration

But from equation (1)

$$f_\theta-p_r=r\frac{dp_r}{dr}+\frac{\rho\omega^2r^2}{g} \qquad \ldots(1)$$

Substracting equation (1) from equation (7)

$$2p_r=A-\left(\frac{m+1}{m}\right)\frac{\rho\omega^2r^2}{2g}-\frac{\rho\omega^2r^2}{g}-r\frac{dp_r}{dr}$$

$$2p_r+r\frac{dp_r}{dr}=A-\frac{3m+1}{2m}\cdot\frac{\rho\omega^2r^2}{g} \qquad \ldots(8)$$

Multiplying equation (8) throughout by r

$$2rp_r+r^2\frac{dp_r}{dr}=Ar-\frac{3m+1}{2m}\cdot\frac{\rho\omega^2r^3}{g} \qquad \ldots(9)$$

Integrating equation (9) we get

$$r^2p_r=\frac{Ar^2}{2}-\frac{3m+1}{8m}\cdot\frac{\rho\omega^2r^4}{g}+B$$

where B is another constant of integration

or
$$p_r=\frac{A}{2}+\frac{B}{r^2}-\frac{3m+1}{8m}\cdot\frac{\rho\omega^2r^2}{g} \qquad \ldots(10)$$

Now substituting the value of p_r in equation (1)

$$f_\theta=A-\left(\frac{m+1}{m}\right)\frac{\rho w^2r^2}{2g}-\frac{A}{2}-\frac{B}{r^2}+\frac{3m+1}{8m}\cdot\frac{\rho\omega^2r^2}{g}$$

$$=\frac{A}{2}-\frac{B}{r^2}-\frac{m+3}{8m}\times\frac{\rho\omega^2r^2}{g}$$

Let us put $\frac{3m+1}{8m}=k_1$, a constant for the material

and $\frac{m+3}{8m}=k_2$, another constant for the material

The expressions for radial and circumferential stresses can be written as

$$p_r=\frac{A}{2}+\frac{B}{r^2}-k_1\frac{\rho\omega^2r^2}{g}$$

$$f_\theta=\frac{A}{2}-\frac{B}{r^2}-k_2\frac{\rho\omega^2r^2}{g}$$

The constants A and B can be evaluated by using the boundary conditions.

Solid Disc

The inner radius $R_1=0$, say outer radius $=R$

At the centre $r=0$ and stresses cannot be infinite at the centre of the disc, therefore the constant $B=0$. Expressions for stresses will now be

$$p_r=\frac{A}{2}-k_1\frac{\rho\omega^2r^2}{g}$$

$$f_\theta=\frac{A}{2}-k_2\frac{\rho\omega^2r^2}{g}$$

Now at the outer radius, $r=R$, radial stress $p_r=0$

$$\therefore \qquad 0=\frac{A}{2}-k_1\frac{\rho\omega^2R^2}{g}$$

or $$\frac{A}{2}=k_1\frac{\rho\omega^2R^2}{g}$$

Radial stress at any radius r

$$p_r=k_1\frac{\rho\omega^2}{g}(R^2-r^2) \qquad \text{where} \quad k_1=\frac{3m+1}{8m}$$

Circumferential stress at any radius r

$$f_\theta=\frac{\rho\omega^2}{g}(k_1R^2-k_2r^2) \qquad \text{where} \quad k_2=\frac{m+3}{8m}$$

Hollow Disc

Boundary conditions are that radial stress p_r is zero at the inner radius R_1 and at the outer radius R_2.

Therefore $$0=\frac{A}{2}+\frac{B}{R_1^2}-k_1\frac{\rho\omega^2R_1^2}{g}$$

$$0=\frac{A}{2}+\frac{B}{R_2^2}-k_1\frac{\rho\omega^2R_2^2}{g}$$

From these equations

$$\frac{B}{R_1^2}-\frac{B}{R_2^2}=-k_1\frac{\rho\omega^2R_2^2}{g}+k_1\frac{\rho\omega^2R_1^2}{g}=-k_1\frac{\rho\omega^2}{g}(R_2^2-R_1^2)$$

or $$B=-k_1\frac{\rho\omega^2R_1^2R_2^2}{g}$$

Then $$\frac{A}{2}=-\frac{B}{R_1^2}+k_1\frac{\rho\omega^2R_1^2}{g}=+k_1\frac{\rho\omega^2R_1^2R_2^2}{R_1^2\times g}+k_1\frac{\rho\omega^2R_1^2}{g}$$

$$=k_1\frac{\rho\omega^2}{g}(R_1^2+R_2^2)$$

Radial Stress

$$p_r=k_1\frac{\rho\omega^2}{g}(R_1^2+R_2^2)-k_1\frac{\rho\omega^2R_1^2R_2^2}{gr^2}-k_1\frac{\rho\omega^2r^2}{g}$$

For the maximum value

$$\frac{dp_r}{dr}=0=0+k_1^2\cdot\frac{2\rho\omega^2R_1^2R_2^2}{gr^3}-2k_1\frac{\rho\omega^2r}{g}$$

or $$r^4=R_1^2R_2^2$$

$$r=\sqrt{R_1R_2}$$

$$p_{r\,max}=k_1\frac{\rho\omega^2}{g}(R_1^2+R_2^2)-k_1\frac{\rho\omega^2}{g}\times\frac{R_1^2R_2^2}{R_1R_2}-k_1\frac{\rho\omega^2}{g}R_1R_2$$

$$=k_1\frac{\rho\omega^2}{g}[R_1^2+R_2^2-R_1R_2-R_1R_2]=k_1\frac{\rho\omega^2}{g}(R_2-R_1)^2$$

Circumferential stress

$$f_\theta=k_1\frac{\rho\omega^2}{g}(R_1^2+R_2^2)+k_1\frac{\rho\omega^2R_1^2R_2^2}{gr^2}-k_2\frac{\rho\omega^2r^2}{g}$$

Obviously maximum value of f_θ occurs where r is minimum *i.e.* at $r=R_1$

$$f_{\theta\,max}=k_1\frac{\rho\omega^2}{g}(R_1^2+R_1^2)+k_1\frac{\rho\omega^2R_2^2}{g}-k_2\frac{\rho\omega^2R_1^2}{g}$$

$$=\frac{\rho\omega^2}{g}[k_1(2R_2^2+R_1^2)-k_2R_1^2]$$

Example 18·2-1. A thin uniform steel disc of diameter 40 cm is rotating about its axis at 1800 r.p.m. Calculate the maximum principal stress and maximum shearing stress in the disc.

Draw the circumferential and radial stress distribution along the radius of the disc.

Density $=7700$ kg/m³

Poisson's ratio, $\frac{1}{m}=0{\cdot}3$

Solution. $R=20$ cm $=200$ mm

Density $=7700\times9{\cdot}8\times10^{-9}$ N/mm³ $=7{\cdot}456\times10^{-5}$ N/mm³

Constant, $$k_1=\left(\frac{3m+1}{8m}\right)=\frac{1}{8}\left(3+\frac{1}{m}\right)=\frac{3{\cdot}3}{8}$$

$$k_2=\frac{m+3}{8m}=\frac{1}{8}\left(1+\frac{3}{m}\right)=\frac{1{\cdot}9}{8}$$

Radial stress, $$p_r=k_1\frac{\rho\omega^2}{g}(R^2-r^2)$$

where $$g = 981 \times 10 \text{ mm/sec}^2 = 9{\cdot}81 \times 10^3 \text{ mm/sec}^2$$

$$\omega = \text{angular velocity}$$

$$= \frac{2 \times \pi \times 1800}{60} = 188{\cdot}496 \text{ rad/sec}$$

Now $$\frac{\rho\omega^2}{g} k_1 = \frac{3{\cdot}3}{8} \times \frac{7{\cdot}456 \times 10^{-5} \times (188{\cdot}496)^2}{9{\cdot}81 \times 10^3} = 0{\cdot}11 \times 10^{-3} \text{ N/mm}^4$$

Radial stress

$$p_r = 0{\cdot}11 \times 10^{-3}\,(200^2 - r^2) = 4{\cdot}4 \text{ N/mm}^2 \quad \text{at } r=0, \textit{ i.e. } \text{at the centre}$$
$$= 0{\cdot}11 \times 10^{-3}\,(200^2 - 50^2) = 4{\cdot}125 \text{ N/mm}^2 \quad \text{at } r = 50 \text{ mm}$$
$$= 0{\cdot}11 \times 10^{-3}\,(200^2 - 100^2) = 3{\cdot}30 \text{ N/mm}^2 \quad \text{at } r = 100 \text{ mm}$$
$$= 0{\cdot}11 \times 10^{-3}\,(200^2 - 150^2) = 1{\cdot}925 \text{ N/mm}^2 \quad \text{at } r = 150 \text{ mm}$$
$$= 0{\cdot}11 \times 10^{-3}\,(200^2 - 200^2) = 0 \quad \text{at } r = 200 \text{ mm}$$

Hoop stress

$$f_\theta = \frac{\rho\omega^2}{g} k_1 R^2 - \frac{\rho\omega^2}{g} \times k_2 r^2$$
$$= 4{\cdot}4 - \frac{\rho\omega^2 k_2}{g} \times r^2$$

Now $$\frac{\rho\omega^2 k_2}{g} = \frac{1{\cdot}9}{8} \times \frac{7{\cdot}456 \times 10^{-5} \times (188{\cdot}496)^2}{9{\cdot}81 \times 10^3} = 0{\cdot}064 \times 10^{-3} \text{ N/mm}^4$$

$$f_\theta = 4{\cdot}4 - 0{\cdot}064 \times 10^{-3}\, r^2 = 4{\cdot}4 \text{ N/mm}^2 \text{ at } r=0 \textit{ i.e. } \text{at the centre}$$
$$= 4{\cdot}4 - 0{\cdot}064 \times 10^{-3} \times 50^2 = 4{\cdot}24 \text{ N/mm}^2 \text{ at } r = 50 \text{ mm}$$
$$= 4{\cdot}4 - 0{\cdot}064 \times 10^{-3} \times 100^2 = 3{\cdot}76 \text{ N/mm}^2 \text{ at } r = 100 \text{ mm}$$
$$= 4{\cdot}4 - 0{\cdot}064 \times 10^{-3} \times 150^2 = 2{\cdot}96 \text{ N/mm}^2 \text{ at } r = 150 \text{ mm}$$
$$= 4{\cdot}4 - 0{\cdot}064 \times 10^{-3} \times 200^2 = 1{\cdot}84 \text{ N/mm}^2 \text{ at } r = 200 \text{ mm}$$

Fig. 18·4 shows the distribution of circumferential and radial stresses along the radius of the disc. At the centre of the disc, there are maximum stresses 4·4 N/mm², 4·4 N/mm² as f_θ and p_r. So the principal stresses at the centre of the disc are 4·4 N/mm², 4·4 N/mm² and 0·0 N/mm². Since the principal stress p_1 and p_2 are of the same sign.

Maximum shear stress

$$= \frac{4{\cdot}4}{2} = 2{\cdot}2 \text{ N/mm}^2$$

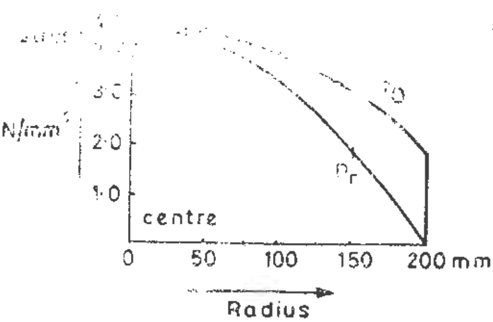


Fig. 18·4

Example 18·2-2 A thin uniform disc of inner radius 5 cm and outer radius 20 cm is rotating at 6000 revolutions per minute about its axis. Draw the circumferential and radial stress distribution along the radius of the disc.

Calculate the maximum principal stress and maximum shear stress in the disc.

Given weight density $=7800$ kg/m^3 and Poisson's ratio $=0{\cdot}28$

$$g=981 \text{ cm/sec}^2$$

Solution. Inner radius $R_1=5$ cm

Outer radius $R_2=20$ cm

Constants
$$k_1=\frac{3m+1}{8m}=\frac{3+0{\cdot}28}{8}=0{\cdot}41$$

$$k_2=\frac{m+3}{8m}=\frac{1{\cdot}84}{8}=0{\cdot}23$$

Density, $\rho=7800 \text{ kg/m}^3=0{\cdot}0078 \text{ kg/cm}^3$

Angular velocity, $\omega=\dfrac{2\times\pi\times 6000}{60}=628{\cdot}32$ radians/sec.

$$\frac{k_1\rho\omega^2}{g}=\frac{0{\cdot}41\times 0{\cdot}0078\times(628{\cdot}32)^2}{981}=1{\cdot}287 \text{ kg/cm}^4$$

$$\frac{k_2\rho\omega^2}{g}=\frac{0{\cdot}23\times 0{\cdot}0078\times(628{\cdot}32)^2}{981}=0{\cdot}722 \text{ kg/cm}^4$$

Moreover let us calculate

$$k_1\,\frac{\rho\omega^2}{g}\,(R_1^2+R_2^2)=1{\cdot}287\,(5^2+20^2)=546{\cdot}97 \text{ kg/cm}^2$$

$$k_1\,\frac{\rho\omega^2}{g}\times R_1^2R_2^2=1{\cdot}287\times 5^2\times 20^2=1{\cdot}287\times 10^4 \text{ kg.}$$

Radial stress, p_r

$$p_r=k_1\,.\,\frac{\rho\omega^2}{g}(R_1^2+R_2^2)-k_1\,\frac{\rho\omega^2}{g}\,\frac{R_1^2R_2^2}{r^2}-k_1\,\frac{\rho\omega^2r^2}{g}$$

$$=546{\cdot}97-\frac{1{\cdot}287\times 10^4}{r^2}-1{\cdot}287\times r^2$$

$$=546{\cdot}97-\frac{1{\cdot}287\times 10^4}{5^2}-1{\cdot}287\times 5^2$$

$$=546{\cdot}97-514{\cdot}80-32{\cdot}17=\mathbf{0} \qquad \text{at } r=5 \text{ cm}$$

$$=546{\cdot}97-\frac{1{\cdot}287\times 10^4}{10^2}-1{\cdot}287\times 10^2$$

$$=546{\cdot}97-128{\cdot}7=\mathbf{289{\cdot}57} \text{ kg/cm}^2 \qquad \text{at } r=10 \text{ cm}$$

$$=546{\cdot}97-\frac{1{\cdot}287\times 10^4}{15^2}-1{\cdot}287\times 15^2$$

$$=546{\cdot}97-57{\cdot}2-289{\cdot}575=\mathbf{200{\cdot}195} \text{ kg/cm}^2 \qquad \text{at } r=15 \text{ cm}$$

$$=546{\cdot}97-\frac{1{\cdot}287\times 10^4}{20^2}-1{\cdot}287\times 20^2=546{\cdot}97-32{\cdot}17-514{\cdot}80=0$$

$$\text{at } r=20 \text{ cm}$$

Maximum p_r occurs at $r=\sqrt{R_1R_2}=\sqrt{5\times 20}=10$ cm radius.

Circumferential stress, f_θ

$$f_\theta=k_1\,\frac{\rho\omega^2}{g}\,(R_1^2+R_2^2)+k_1\,\frac{\rho\omega^2}{g}\,\frac{R_1^2R_2^2}{r^2}-k_2\,\frac{\rho\omega^2r^2}{g}$$

$$=546{\cdot}97+\frac{1{\cdot}287\times10^4}{r^2}-0{\cdot}722\,r^2$$

$$=546{\cdot}97+\frac{1{\cdot}287\times10^4}{5^2}-0{\cdot}722\times5^2=546{\cdot}97+514{\cdot}80-18{\cdot}05$$

$$=1043{\cdot}72 \text{ kg/cm}^2 \quad \text{at } r=5 \text{ cm}$$

$$=546{\cdot}97+\frac{1{\cdot}287\times10^4}{10^2}-0{\cdot}722\times10^2=546{\cdot}97+128{\cdot}7-72{\cdot}2$$

$$=603{\cdot}47 \text{ kg/cm}^2 \quad \text{at } r=10 \text{ cm}$$

$$=546{\cdot}97+\frac{1{\cdot}287\times10^4}{15^2}-0{\cdot}722\times15^2=546{\cdot}97+57{\cdot}2-162{\cdot}45$$

$$=441{\cdot}72 \text{ kg/cm}^2 \quad \text{at } r=15 \text{ cm}$$

$$=546{\cdot}97+\frac{1{\cdot}287\times10^4}{20^2}-0{\cdot}722\times20^2=546{\cdot}97+32{\cdot}17-288{\cdot}8$$

$$=290{\cdot}34 \text{ kg/cm}^2 \quad \text{at } r=20 \text{ cm}$$

Figure 18·5 shows the radial and circumferential stress distribution along the radius of the disc

Maximum stresses occur at the inner radius, where circumferential stress f_θ is maximum.

Therefore maximum principal stress

$=1043{\cdot}72$ kg/cm²

Again at the inner radius the principal stresses are

1043·72 kg/cm², 0, 0

So the maximum shear stress

$$=\frac{1043{\cdot}72}{2}=521{\cdot}86 \text{ kg/cm}^2$$

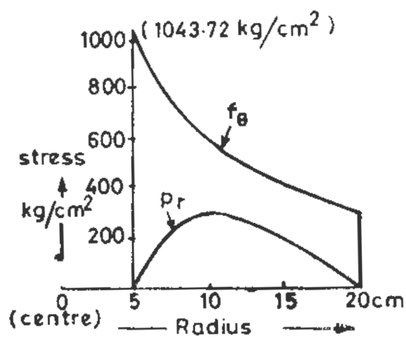


Fig. 18·5

Exercise 18·2-1. A thin uniform steel disc of radius 30 cm is rotating about its axis at 3000 r.p.m. Draw the radial and circumferential stress distribution diagram along the radius of the disc.

What are the maximum and minimum values of circumferential and radial stresses.

$\rho=0{\cdot}0078$ kg/cm³

Poisson's ratio$=0{\cdot}3$

acceleration due to gravity$=981$ cm/sec².

[**Ans.** 291·336 kg/cm² and 123·598 kg/cm², 291·336 kg/cm² and 0·0 kg/cm²]

Exercise 18·2-2. A thin uniform disc of inner diameter 5 cm and outer diameter 25 cm is rotating at 10,000 R.P.M. Calculate the maximum and minimum values of circumferential and radial stresses.

Draw the radial and circumferential stress distribution diagrams along the radius of the disc.

Given density $= 8830$ kg/m³ and Poisson's ratio $= 0{\cdot}33$

Acceleration due to gravity, $g = 9{\cdot}81$ m/sec².

[**Ans.** 1291·8 kg/cm², 309·06 kg/cm² ; 910 kg/cm² and 0·0]

18·3. DISC OF UNIFORM STRENGTH

In the last article we have observed that radial and circumferential stresses in a rotating disc of uniform thickness vary along the radius of the disc and the circumferential stress is maximum at the inner radius of the rotating disc. There are many components in industry such as the rotor of a steam turbine which rotate at very high speeds and consequently high stresses are developed. For such applications, rotors having constant strength throughout the radius have been designed by varying their thickness.

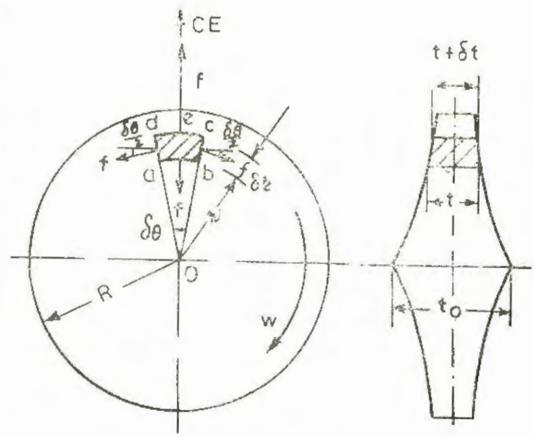


Fig. 18·6

Consider a disc of radius R, of variable thickness, rotating at an angular speed ω radians/second about its axis O. Say the thickness at the centre$=t_0$. Take a small element *abcd* of radial thickness δr, at a radius r subtending an angle $\delta\theta$ at the centre. The disc considered is of uniform strength *i.e.*, radial stress on faces *ab* and *cd* is f, and circumferential stress on faces *bc* and *da* is also f. The thickness at the radius r is t and say the thickness at the radius $r+\delta r$ is $t+\delta t$.

Volume of the element $=(r\delta\theta\ .\ t\ .\ \delta r)$

Say weight density $=\rho$

Mass of the element $=\dfrac{\rho(rt\ \delta\theta\ .\ \delta r)}{g}$

Centrifugal force on the element,

$$CF=\frac{\rho(rt\ \delta\theta\ \delta r)\ w^2 r}{g}$$

$$=\frac{\rho\omega^2\ r^2\ \delta\theta\ \delta r\ t}{g}$$

Radial force on face $ab=r\delta\theta\ .\ t\ .\ f$

Radial force on face $cd=(r+\delta r)(t+\delta t)\ f\ .\ \delta\theta$

Circumferential force on faces *bc* and *da*

$=f\ .\ t\ .\ \delta_r$.

(Inclined at an angle $\delta\theta/2$ to the radial direction *eo*).

Resolving all the forces along the radial direction *eo* and considering equilibrium, we get

$$\frac{\rho\omega^2 r^2}{g}\,(t\delta r\delta\theta)+f(r+\delta r)(t+\delta t)\delta\theta=fr\ t\delta\theta$$

$$+2\ .\ f\ .\ t\ .\ \delta r\sin\frac{\delta\theta}{2} \qquad \ldots(1)$$

but $\delta\theta$ is very small, so $\sin\frac{\delta\theta}{2}=\frac{\delta\theta}{2}$

The above equation (1) can be simplified as follows (taking $\delta\theta$ on both the sides)

$$\frac{\rho\omega^2r^2}{g}\,t\delta r+frt+fdrt+fr\,.\,\delta t+f\,\delta r\delta t$$
$$=frt+f\,.\,t\;\delta r$$

Neglecting the product of small quantities

$$\frac{\rho\omega^2r^2t}{g}\,\delta r+fr\delta t=0$$

or $$f\,\frac{\delta t}{t}=-\frac{\rho\omega^2r}{g}\,\delta r$$

or $$\frac{\delta t}{t}=-\frac{\rho\omega^2r}{fg}\,\delta r$$

Integrating both the sides

$$\ln t=-\frac{\rho\omega^2r^2}{2fg}+\ln A \quad \text{where } \ln A \text{ is a constant}$$

$$\ln\frac{t}{A}=-\frac{\rho\omega^2r^2}{2fg}$$

or $$\frac{t}{A}=e^{-\frac{\rho\omega^2r^2}{2fg}}$$

at $$r=0 \quad \text{Thickness, } t=t_0$$
$$t_0=A$$

So thickness at any radius, $t=t_0\,e^{-\frac{\rho\omega^2r^2}{2fg}}$.

Example 18·3-1. A steel disc of a turbine is to be designed so that the radial and circumferential stresses are to be the same and constant throughout and equal to 80 N/mm², when running at 3500 rpm. If the axial thickness at the centre is 1·5 cm what is the thickness at a radius of 50 cm

$$\rho \text{ for steel}=0{\cdot}0078 \text{ kg/cm}^3,\ g=981 \text{ cm/sec}^2$$
$$\rho=0{\cdot}0078\times9{\cdot}8\times10^{-3}\text{ N/mm}^3=0{\cdot}07644\times10^{-3}\text{ N/mm}^3$$

Solution. Radius $r=500$ mm

Thickness $t_0=15$ mm

Angular velocity $\omega=\frac{2\times\pi\times3500}{60}=366{\cdot}52$ rad/sec

Constant strength, $f=80$ N/mm²

So $$\frac{\rho\omega^2r^2}{2fg}=\frac{0{\cdot}07644\times10^{-3}\times(366{\cdot}52)^2\times(500)^2}{2\times80\times9810}=1{\cdot}635562$$

$$e^{-1{\cdot}6355}=0{\cdot}195$$

$$t=t_0\,e^{1{\cdot}6355}=15\times0{\cdot}195=2{\cdot}925 \text{ mm.}$$

Exercise 18·3-1. A steel rotor of a steams turbine is rotating at 10,000 r.p.m. At the blade ring its diameter is 60 cm and its axial thickness is 9 cm. at the centre. Calculate the thickness of the rotor at the blade ring, if it has uniform distribution of stress equal to 1600 kg/cm². $\rho=0\ 0078$ kg/cm³

g=acceleration due to gravity=9·81 m/sec². [**Ans.** 7·76 mm]

18·4. ROTATING LONG CYLINDERS

The analysis is similar to that of a thin disc. The only difference is that the length of the cylinder along the axis is large as compared to the radius and axial stress is considered along the length of the cylinder *i.e.*, at any radius the stresses are f_θ (circumferential stress), p_r (radial stress) and f_a (axial stress). Following assumptions are made while developing theory for long cylinders

(1) Transverse sections of the cylinder remain plane at high speeds of rotation. This is true only for sections away from the ends.

(2) At the central cross section of the cylinder, shear stress is zero due to symmetry and there are only 3 principal stresses *i.e.*, f_θ, p_r, and f_a.

If E is the Young's modulus

Hoop strain, $$\epsilon_\theta=\frac{f_\theta}{E}-\frac{p_r}{mE}-\frac{f_a}{mE}$$

Radial strain, $$\epsilon_r=\frac{p_r}{E}-\frac{f_0}{mE}-\frac{f_a}{mE}=\frac{1}{E}\left[p_r-\frac{1}{m}(f_\theta+f_a)\right]$$

Axial strain, $$\epsilon_a=\frac{f_a}{E}-\frac{p_r}{mE}-\frac{f_\theta}{mE}=\frac{1}{E}\left[f_a-\frac{1}{m}(p_r+f_\theta)\right]$$

Refer to the article 18·2, considering a small element of rotating cylinder subtending an angle $\delta\theta$ at the centre. The element is at a distance r and of radial thickness δr. The equation of equilibrium obtained is

$$f_\theta-p_r=r\frac{\delta p_r}{dr}+\frac{\rho\omega^2r^2}{g} \qquad ...(1)$$

Again we have considered

Circumferential strain $=\dfrac{u}{r}$

and radial strain $=\dfrac{\delta u}{\delta r}$

So $$\frac{u}{r}=\frac{1}{E}\left[f_\theta-\frac{1}{m}(p_r+f_a)\right] \qquad ...(2)$$

and $$\frac{\delta u}{\delta r}=\frac{1}{E}\left[p_r-\frac{1}{m}(f_\theta+f_a)\right] \qquad ...(3)$$

Differentiating equation (2) with respect to r

$$\frac{du}{dr}=\frac{1}{E}\left[f_\theta-\frac{1}{m}(p_r+f_a)\right]+\frac{1}{E}\left[r\frac{df_\theta}{dr}-\frac{r}{m}\left(\frac{dp_r}{dr}+\frac{df_a}{dr}\right)\right] \qquad ...(4)$$

Equating equations (3) and (4)

$$p_r - \frac{1}{m} f_\theta - \frac{f_a}{m} = f_\theta - \frac{p_r}{m} - \frac{f_a}{m} + \frac{r df_\theta}{dr} - \frac{r dp_r}{m dr} - \frac{r df_a}{m dr}$$

$$p_r \left(1 + \frac{1}{m}\right) - f_\theta \left(1 + \frac{1}{m}\right) - r \frac{df_\theta}{dr} + r \frac{dp_r}{m dr} + r \frac{df_a}{m dr} = 0 \qquad \ldots(5)$$

Now as per the first assumptions that the transverse sections remain plane even after the long cylinder rotates at high speed, it is implied that axial strain is constant *i.e.*

$$\epsilon_a = \frac{1}{E}\left[f_a - \frac{1}{m}(f_\theta + p_r)\right] = \text{constant}$$

or
$$f_a - \frac{1}{m}(f_\theta + p_r) = \text{constant, since } E \text{ is constant}$$

Differentiating this expression with respect to r we get

$$\frac{df_a}{dr} - \frac{1}{m}\frac{df_\theta}{dr} - \frac{1}{m}\frac{dp_r}{dr} = 0$$

or
$$r \frac{df_a}{dr} = \frac{r}{m}\left(\frac{df_\theta}{dr} + \frac{dp_r}{dr}\right)$$

Substituting the value of $r \frac{df_a}{dr}$ in equation (5),

$$p_r \left(1 + \frac{1}{m}\right) - f_\theta \left(1 + \frac{1}{m}\right) - r \frac{df_\theta}{dr} + \frac{r}{m}\frac{dp_r}{dr} + \frac{r}{m^2}\frac{df_\theta}{dr} + \frac{r}{m^2}\frac{dp_r}{dr} = 0$$

or
$$-(f_\theta - p_r)\left(1 + \frac{1}{m}\right) - r\left(1 - \frac{1}{m^2}\right)\frac{df_\theta}{dr} + \frac{r}{m}\left(1 + \frac{1}{m}\right)\frac{dp_r}{dr} = 0$$

or
$$f_\theta - p_r + r\left(1 - \frac{1}{m}\right)\frac{df_\theta}{dr} - \frac{r}{m}\frac{dp_r}{dr} = 0 \qquad \ldots(6)$$

From equation (1) $f_\theta - p_r = \frac{r\, dp_r}{dr} + \frac{\rho\omega^2 r^2}{g}$

Substituting in equation (6),

$$\frac{r\, dp_r}{dr} + \frac{\rho\omega^2 r^2}{g} + r\left(1 - \frac{1}{m}\right)\frac{df_\theta}{dr} - \frac{r}{m}\frac{dp_r}{dr} = 0$$

$$\left(1 - \frac{1}{m}\right)\left(\frac{df_\theta}{dr} + \frac{dp_r}{dr}\right) + \frac{\rho\omega^2 r}{g} = 0$$

$$\frac{df_\theta}{dr} + \frac{dp_r}{dr} = -\left(\frac{m}{m-1}\right)\frac{\rho\omega^2 r}{g} \qquad \ldots(7)$$

Integrating equation (7),

$$f_\theta + p_r = -\left(\frac{m}{m-1}\right)\frac{\rho\omega^2 r^2}{2g} + A \qquad \ldots(8)$$

where A is the constant of integration.

Subtracting equation (1) from equation (8),

$$2p_r = A - \left(\frac{m}{m-1}\right)\frac{\rho\omega^2 r^2}{2g} - r\frac{dp_r}{dr} - \frac{\rho\omega^2 r^2}{g}$$

$$2p_r + r\frac{dp_r}{dr} = A - \frac{\rho\omega^2 r^2}{2g}\left(\frac{m}{m-1} + 2\right)$$

$$= A - \frac{\rho\omega^2 r^2}{2g}\left(\frac{3m-2}{m-1}\right)$$

or $$2p_r.r + r^2\frac{dp_r}{dr} = A - \frac{\rho\omega^2 r^3}{2g}\left(\frac{3m-2}{m-1}\right)$$

Integrating both the sides

$$pr.r^2 = \frac{Ar^2}{2} - \frac{\rho\omega^2 r^4}{8g}\left(\frac{3m-2}{m-1}\right) + B$$

or Radial stress $$p_r = \frac{A}{2} + \frac{B}{r^2} - \frac{\rho\omega^2 r^2}{8g}\left(\frac{3m-2}{m-1}\right) \quad \text{...(9)}$$

But from equation (8)

Hoop stress, $$f_\theta = -\frac{A}{2} - \frac{B}{r^2} + \frac{\rho\omega^2 r^2}{8g}\left(\frac{3m-2}{m-1}\right) - \frac{m}{m-1}\left(\frac{\rho\omega^2 r^2}{2g}\right) + A$$

$$= \frac{A}{2} - \frac{B}{r^2} + \frac{\rho\omega^2 r^2}{8g}\left(\frac{3m-2-4m}{m-1}\right)$$

$$= \frac{A}{2} - \frac{B}{r^2} - \frac{\rho\omega^2 r^2}{8g}\left(\frac{m+2}{m-1}\right) \quad \text{...(10)}$$

Solid Cylinders

The stresses can not be infinite at the centre, therefore constant $B=0$. The expressions for stresses will now be

$$p_r = \frac{A}{2} - \frac{\rho\omega^2 r^2}{8g}\left(\frac{3m-2}{m-1}\right)$$

$$f_\theta = \frac{A}{2} - \frac{\rho\omega^2 r^2}{8g}\left(\frac{m+2}{m-1}\right)$$

Say the radius of the solid cylinder is R

Radial stress $p_r = 0$ at $r = R$

So $$0 = \frac{A}{2} - \frac{\rho\omega^2 R^2}{8g}\left(\frac{3m-2}{m-1}\right)$$

or $$\frac{A}{2} = \frac{\rho\omega^2 R^2}{8g}\left(\frac{3m-2}{m-1}\right)$$

$$p_r = \frac{\rho\omega^2}{8g}\left(\frac{3m-2}{m-1}\right)(R^2 - r^2)$$

Radial stress is maximum at $r=0$

$$p_{r\,max} = \frac{\rho\omega^2 R^2}{8g}\left(\frac{3m-2}{m-1}\right)$$

Hoop stress, $$f_\theta = \frac{\rho\omega^2 R^2}{8g}\left(\frac{3m-2}{m-1}\right) - \frac{\rho\omega^2 r^2}{8g}\left(\frac{m+2}{m-1}\right)$$

$$= \frac{\rho\omega^2}{8g}\left[R^2\left(\frac{3m-2}{m-1}\right) - r^2\left(\frac{m+2}{m-1}\right)\right]$$

If we put $\frac{3m-2}{8(m-1)}=k_3$ and $\frac{m+2}{8(m-1)}=k_4$

Then $$p_r=\frac{\rho\omega^2}{g}k_3(R^2-r^2)$$

$$f_\theta=\frac{\rho\omega^2}{g}[k_3R^2-k_4r^2]$$

Again the hoop stress is also maximum at the centre *i.e.*, at $r=0$

So $$p_{r\,max}=f_{\theta\,max}=\frac{\rho\omega^2}{g}\cdot k_3R^2$$

Hollow Cylinder

Stresses are $$p_r=\frac{A}{2}+\frac{B}{r^2}-k_3\frac{\rho\omega^2r^2}{g}$$

$$f_\theta=\frac{A}{2}-\frac{B}{r^2}-k_4\frac{\rho\omega^2r^2}{g}$$

Radial stress is zero at inner and outer radii of the cylinder

Say R_1=Inner radius

R_2=Outer radius

$\therefore$ $$0=\frac{A}{2}+\frac{B}{R_1^2}-k_3\frac{\rho\omega^2R_1^2}{g}$$

$$0=\frac{A}{2}+\frac{B}{R_2^2}-k_3\frac{\rho\omega^2R_2^2}{g}$$

From these boundary conditions

$$B\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)-k_3\frac{\rho\omega^2}{g}(R_1^2-R_2^2)=0$$

$$B\frac{(R_2^2-R_1^2)}{R_1^2R_2^2}=-k_3\frac{\rho\omega^2}{g}(R_2^2-R_1^2)$$

$$B=-k_3\frac{\rho\omega^2}{g}\times R_1^2R_2^2$$

Again $$\frac{A}{2}=-\frac{B}{R_1^2}+k_3\frac{\rho\omega^2R_1^2}{g}$$

$$=+k_3\frac{\rho\omega^2}{g}\times R_2^2+k_3\frac{\rho\omega^2R_1^2}{g}$$

$$=k_3\frac{\rho\omega^2}{g}(R_1^2+R_2^2)$$

The expressions for stresses will now be

Radial stress, $$p_r=k_3\frac{\rho\omega^2}{g}\left[(R_1^2+R_2^2)-\frac{R_1^2R_2^2}{r^2}-r^2\right]$$

To obtain maximum value of p_r

$$\frac{dp_r}{dr}=0=+\frac{2R_1^2R_2^2}{r^3}-2r$$

or
$$r^4 = R_1^2 R_2^2$$
$$r = \sqrt{R_1 R_2}$$

Maximum radial stress occurs at the radius of $\sqrt{R_1 R_2}$

$$p_{r\ max} = \frac{k_3 \rho\omega^2}{g}\left[(R_1^2+R_2^2) - \frac{R_1^2 R_2^2}{R_1 R_2} - R_1 R_2\right]$$
$$= \frac{k_3 \rho\omega^2}{g}[R_1^2+R_2^2-2R_1R_2]$$
$$= \frac{k_3 \rho\omega^2}{g}(R_2-R_1)^2$$

The expression for the circumferential or hoop stress will be

$$f_\theta = k_3 \frac{\rho\omega^2}{g}(R_1^2+R_2^2) + k_3 \frac{\rho\omega^2}{g}\frac{R_1^2R_2^2}{r^2} - k_4 \frac{\rho\omega^2}{g} r^2$$

Obviously this will be maximum when r is minimum *i.e.*, at inner radius, R_1

$$f_{\theta\ max} = k_3 \frac{\rho\omega^2}{g}(R_1^2+R_2^2) + k_4 \frac{\rho\omega^2}{g} R_2^2 - k_4 \frac{\rho\omega^2}{g} \times R_1^2$$
$$= \frac{\rho\omega^2}{g}[k_3(2R_2^2+R_1^2) - k_4 R_1^2]$$

Example 18·4-1. A long cylinder of diameter 60 cm is rotating at 3000 r.p.m. Calculate the maximum stress in the cylinder. Draw the variation of radial and circumferential stresses along the radius.

Weight density $=0{\cdot}0078$ kg/cm³

$g=980$ cm/sec², Poisson's ratio$=0{\cdot}3$

Solution. $R=30$ cm

Angular speed, $\omega = \dfrac{2\times\pi\times 3000}{60} = 314{\cdot}16$ rad/sec.

$$k_3 = \frac{3m-2}{8(m-1)} = \frac{3-0{\cdot}6}{8(1-0{\cdot}3)} = 0{\cdot}4286$$

$$k_4 = \frac{m+2}{8(m-1)} = \frac{1+0{\cdot}6}{8(1-0{\cdot}3)} = 0{\cdot}2857$$

Maximum radial and circumferential stresses occur at the centre

$$p_{r\ max} = f_{\theta\ max} = k_3 \frac{\rho\omega^2}{g} \times R^2$$
$$= \frac{0{\cdot}4286\times 0{\cdot}0078\times(314{\cdot}16)^2}{980}\times 30^2 = 303{\cdot}0 \text{ kg/cm}^2$$

Radial stress

$$p_r = k_3 \frac{\rho\omega^2}{g}[R^2-r^2]$$
$$= \frac{0{\cdot}4286\times 0{\cdot}0078\times(314{\cdot}16)^2}{980}(900-r^2)$$

$=0{\cdot}3367\,(900-r^2) \quad =303{\cdot}0$ kg/cm² at $r=0$
$=0{\cdot}3367\,(900-6^2) \quad =290{\cdot}90$ kg/cm² at $r=6$ cm
$=0{\cdot}3367\,(900-12^2) \quad =254{\cdot}54$ kg/cm² at $r=12$ cm
$=0{\cdot}3367\,(900-18^2) \quad =193{\cdot}94$ kg/cm² at $r=18$ cm
$=0{\cdot}3367\,(900-24^2) \quad =109{\cdot}09$ kg/cm² at $r=24$ cm
$=0{\cdot}3367\,(900-900) \quad =0$ at $r=30$ cm

Hoop stress

$$f_0=k_3\,\frac{\rho\omega^2}{g}\,R^2-k_4\,\frac{\rho\omega^2}{g}\,r^2=\frac{\rho\omega^2}{g}\,[k_3R^2-k_4r]$$

$$=\frac{0{\cdot}0078\times(314{\cdot}16)^2}{980}\,[0{\cdot}4286\times900-0{\cdot}2857\times r^2]$$

$=0{\cdot}785\,(385{\cdot}74-0{\cdot}2857\,r^2)=303{\cdot}0$ at $r=0$
$=0{\cdot}785(385{\cdot}74-0{\cdot}2857\times6^2)=294{\cdot}73$ kg/cm² at $r=6$ cm
$=0{\cdot}785(385{\cdot}74-0{\cdot}2857\times12^2)=270{\cdot}51$ kg/cm² at $r=12$ cm
$=0{\cdot}785\,(385{\cdot}74-0{\cdot}2857\times18^2)=230{\cdot}14$ kg/cm² at $r=18$ cm
$=0{\cdot}785\,(385{\cdot}74-0{\cdot}2857\times24^2)=173{\cdot}62$ kg/cm² at $r=24$ cm
$=0{\cdot}785\,(385{\cdot}74-0{\cdot}2857\times30^2)=100{\cdot}96$ kg/cm² at $r=30$ cm

Fig. 18·7 shows the distribution of hoop and radial stresses along the radius of the long cylinder.

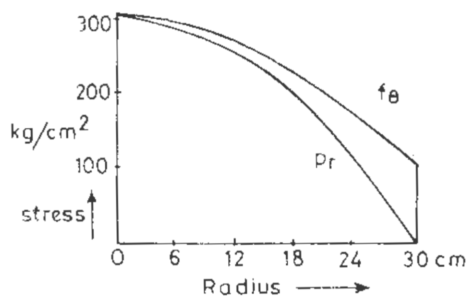


Fig. 18·7

Example 18·4-2. Calculate the maximum stress in a long cylinder 5 cm inside diameter and 25 cm outside diameter rotating at 5000 r.p.m.

Given :

Poisson's ratio $=0{\cdot}3$

Weight density $=0{\cdot}07644$ N/cm³

$g=980$ cm/sec²

Solution. Inner Radius, $R_1=2{\cdot}5$ cm, Outer radius$=12{\cdot}5$ cm

Maximum stress occurs at the inner radius of the cylinder

$$f_{0\,max}=\frac{\rho\omega^2}{g}\,[k_3\,(2R_2^2+R_1^3)-k_4\,R_1^2]$$

$$k_3=\frac{3m-2}{8(m-1)}=\frac{3-0{\cdot}6}{8(1-0{\cdot}3)}=0{\cdot}4286$$

$$k_4=\frac{m+2}{8(m-1)}=\frac{1+0{\cdot}6}{8(1-0{\cdot}3)}=0{\cdot}2857$$

$$\omega=\frac{2\pi\times5000}{60}=523{\cdot}6 \text{ radians/sec}$$

$$\frac{\rho\omega^2}{g}=\frac{0{\cdot}07644\times(523{\cdot}6)^2}{980}=21{\cdot}384 \text{ N/cm}^4$$

$$f_{\theta\,max}=21{\cdot}384\,[0{\cdot}4286\,(2\times 12{\cdot}5^2+2{\cdot}5^2)-0{\cdot}285{\cdot}7\times 2{\cdot}5^2]$$
$$=21{\cdot}34\,[0{\cdot}4286\,(318{\cdot}75)-0{\cdot}2857\times 6{\cdot}25]$$
$$=21{\cdot}34\,[136{\cdot}616-1{\cdot}785]=2877{\cdot}3\text{ N/cm}^2=28{\cdot}77\text{ N/mm}^2$$

Radial stress

$$p_r=k_3\,\frac{\rho\omega^2}{g}\left[(R_1^2+R_2^2)-\frac{R_1^2R_2^2}{r^2}-r^2\right]$$
$$=\frac{0{\cdot}4286\times 0{\cdot}07466\times(523{\cdot}6)^2}{980}\left[(12{\cdot}5^2+2{\cdot}5)^2-\frac{12{\cdot}5^2\times 2{\cdot}5^2}{r^2}-r^2\right]$$
$$=8{\cdot}952\left[162{\cdot}5-\frac{976{\cdot}56}{r^2}-r^2\right]$$
$$=8{\cdot}952\left[162{\cdot}5-\frac{976{\cdot}56}{2{\cdot}5^2}-2{\cdot}5^2\right]=0\text{ at } r=2{\cdot}5\text{ cm}$$
$$=8{\cdot}952\left[162{\cdot}5-\frac{976{\cdot}56}{5^2}-5^2\right]=881{\cdot}2\text{ N/cm}^2\text{ at } r=5\text{ cm}$$
$$=8{\cdot}952\left[162{\cdot}5-\frac{976{\cdot}56}{7{\cdot}5^2}-7{\cdot}5^2\right]=795{\cdot}73\text{ N/cm}^2\text{ at } r=7{\cdot}5\text{ cm}$$
$$=8{\cdot}952\left[162{\cdot}5-\frac{976{\cdot}56}{10^2}-10^2\right]=472{\cdot}08\text{ N/cm}^2\text{ at } r=10\text{ cm}$$
$$=8{\cdot}952\left[162{\cdot}5-\frac{976{\cdot}56}{12{\cdot}5^2}-12{\cdot}5^2\right]=0\text{ at } r=12{\cdot}5\text{ cm}$$

pr_{max} occurs at $r^2=R_2R_2=2{\cdot}5\times 12{\cdot}5=31{\cdot}25$

$$pr_{max}=8{\cdot}952\left[162{\cdot}5-\frac{976{\cdot}56}{31{\cdot}25}-31{\cdot}25\right]=895{\cdot}2\text{ N/cm}^2\text{ at } r=5{\cdot}59\text{ cm}$$

Hoop Stress

$$f_\theta=\frac{\rho\omega^2}{g}\left[k_3(R_1^2+R_2^2)+k_3\;.\;\frac{R_1^2R_2^2}{r^2}-k_4\;.\;r^2\right]$$
$$=\frac{0{\cdot}07466\times(523{\cdot}6)^2}{980}\left[0{\cdot}4286(12{\cdot}5^2+2{\cdot}5^2)+0{\cdot}4286\times\frac{12{\cdot}5^2\times 2{\cdot}5^2}{r^2}-0{\cdot}2857\times r^2\right]$$
$$=20{\cdot}88\left[69{\cdot}6475+\frac{418{\cdot}55}{r^2}-0{\cdot}2857\,r^2\right]$$
$$=20{\cdot}88\left[69{\cdot}6475+\frac{418{\cdot}55}{2{\cdot}5^2}-0{\cdot}2857\times 2{\cdot}5^2\right]$$
$$=2877{\cdot}3\text{ N/cm}^2\quad\text{at}\quad r=2{\cdot}5\text{ cm}$$
$$=20{\cdot}88\left[69{\cdot}6475+\frac{418{\cdot}55}{5^2}-0{\cdot}2857\times 5^2\right]=20{\cdot}88[69{\cdot}6475+16{\cdot}742-7{\cdot}142]$$
$$=1654{\cdot}69\text{ N/cm}^2\quad\text{at}\quad r=5\text{ cm}$$
$$=20{\cdot}88\left[69{\cdot}6475+\frac{418{\cdot}55}{7{\cdot}5^2}-0{\cdot}2857\times 5^2\right]$$
$$=20{\cdot}88[69{\cdot}6475+7{\cdot}44-16{\cdot}07]=1274{\cdot}05\text{ N/cm}^2\quad\text{at}\quad r=7{\cdot}5\text{ cm}$$

$$=20{\cdot}88\left[69\ 6475+\frac{418{\cdot}55}{10^2}-0{\cdot}2857\times10^2\right]$$

$=945{\cdot}09$ N/cm² at $r=10$ cm

$$=20{\cdot}88\left[69{\cdot}6475+\frac{418{\cdot}55}{12{\cdot}5^2}-0{\cdot}2857\times12{\cdot}5^2\right]$$

$=20{\cdot}88[69{\cdot}6475+2{\cdot}6787-44{\cdot}640]=578{\cdot}0$ N/cm² at $r=12{\cdot}5$ cm

Fig. 18·8 shows the distribution of circumferential stress f_θ and radial stress p_r along the radius of the long cylinder.

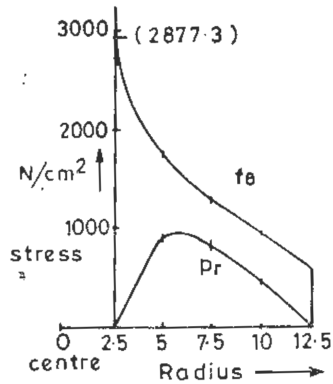


Fig. 18·8

Exercise 18·4-1. A long cylinder of steel of diameter 40 cm is rotating about its axis at an angular speed of 300 radians/sec. Draw the radial stress and circumferential stress distribution along its radius. Determine the maximum and minimum values of radial and circumferential stresses.

Given $\rho=0{\cdot}07644$ N/cm³

$1/m=0{\cdot}3$.

[**Ans.** 12·035 N/mm², 0 ; 12·035 N/mm², 4·012 N/mm²]

Exercise 18·4-2. A long cylinder of steel of outer diameter 75 cm and inner diameter 25 cm is rotating about its axis at 4000 r p.m. Draw the radial stress and circumferentia stress distribution along the radius. Determine the maximum and minimum values of circum-l ferential stress.

What is the maximum radial stress and where is occurs.

$\rho=0{\cdot}078$ kg/cm³, $1/m=0{\cdot}3$, $g=980$ cm/sec²

[**Ans.** 1715·2 kg/cm² and 467·84 kg/cm² ; 374·13 kg/cm² at $r=21{\cdot}65$ cm]

18·5. TEMPERATURE STRESSES IN THIN DISCS

Consider a thin disc rotating at a high speed and subjected to temperature variation at the same time. Say the stresses developed in the disc are f_θ, circumferential stress and p_r, radial stress, α is the coefficient of linear expansion of the disc and T is the temperature change. At angular speed ω, there is change in the radius of the disc. Refer to Fig. 18·2 and 18·3, considering an element *abcd* at radius r, of radial thickness δr and subtending an angle $\delta\theta$ at the centre. At high speed ω say

r changes to $r+u$

dr changes to $dr+du$

Circumferential strain, $$\epsilon_\theta=\frac{f_\theta}{E}-\frac{p_r}{mE}+\alpha T=\frac{u}{r} \quad ...(1)$$

Radial strain, $$\epsilon_r=\frac{p_r}{E}-\frac{f_\theta}{mE}+\alpha T=\frac{du}{dr} \quad ...(2)$$

Equation of equilibrium

$$f_\theta - p_r = r\,\frac{dp_r}{dr} + \frac{\rho\omega^2 r^2}{g} \qquad \text{...(3)}$$

From equation (1)

$$\frac{du}{dr} = \left(\frac{f_\theta}{E} - \frac{p_r}{mE} + \alpha T\right) + \frac{r}{E}\left(\frac{df_\theta}{dr} - \frac{dp_r}{mdr} + E\alpha\,\frac{dT}{dr}\right)$$

Equating this with equation (2),

$$\frac{1}{E}\left(p_r - \frac{f_\theta}{m} + E\alpha T\right) = \frac{1}{E}\left(f_\theta - \frac{p_r}{m} + E\alpha T\right) + \frac{r}{E}\left(\frac{df_\theta}{dr} - \frac{dp_r}{mdr} + E\alpha\,\frac{dT}{dr}\right)$$

$$p_r\left(1 + \frac{1}{m}\right) - f_\theta\left(1 + \frac{1}{m}\right) = r\,\frac{df_\theta}{dr} - \frac{r}{m}\,\frac{dp_r}{dr} + rE\alpha\,\frac{dT}{dr}$$

$$(f_\theta - p_r)\left(1 + \frac{1}{m}\right) = -r\frac{df_\theta}{dr} + \frac{r}{m}\,\frac{dp_r}{dr} - rE\alpha\,\frac{dT}{dr}$$

From equation (3) substituting the value of $f_\theta - p_r$

$$\left(1 + \frac{1}{m}\right) r\,\frac{dp_r}{dr} + \left(1 + \frac{1}{m}\right)\frac{\rho\omega^2 r^2}{g} = -r\frac{df_\theta}{dr} + \frac{r}{m}\,\frac{dp_r}{dr} - rE\alpha\,\frac{dT}{dr}$$

$$r\,\frac{dp_r}{dr} + r\,\frac{df_\theta}{dr} = -\left(1 + \frac{1}{m}\right)\frac{\rho\omega^2 r^2}{g} - rE\alpha\,\frac{dT}{dr}$$

or

$$\frac{d}{dr}\,(f_\theta + p_r) = -\left(1 + \frac{1}{m}\right)\frac{\rho\omega^2 r}{g} - E\alpha\,\frac{dT}{dr} \qquad \text{...(4)}$$

Integrating equation (1),

$$f_\theta + p_r = -\left(1 + \frac{1}{m}\right)\frac{\rho\omega^2 r^2}{2g} - E\alpha T + A \qquad \text{...(5)}$$

where A is the constant of integration.

Substracting equation (1) from equation (5),

$$2p_r = -\left(1 + \frac{1}{m}\right)\frac{\rho\omega^2 r^2}{2g} - \frac{\rho\omega^2 r^2}{g} - r\,\frac{dp^2}{dr} - E\alpha T + A.$$

$$2p_r + r\,\frac{dp_r}{dr} = -\left(1 + \frac{1}{m}\right)\frac{\rho\omega^2 r^2}{2g} - \frac{\rho\omega^2 r^2}{g} - E\,\alpha\,T + A$$

Multiplying throughout by r

$$2rp_r + r^2\,\frac{dp_r}{dr} = -\left(\frac{3m+1}{2m}\right)\frac{\rho\omega^2 r^3}{g} - E\,\alpha\,Tr + Ar \qquad \text{...(6)}$$

Integrating equation (6),

$$r^2 p_r = -\left(\frac{3m+1}{8m}\right)\frac{\rho\omega^2 r^4}{g} - E\,\alpha\int Tr\,dr + \frac{Ar^2}{2} + B$$

where B is the constant of integration

or

$$p_r = \frac{A}{2} + \frac{B}{r^2} - \left(\frac{3m+1}{8m}\right)\frac{\rho\omega^2 r^2}{g} - \frac{E\alpha}{r^2}\int Tr\,dr \qquad \text{...(7)}$$

From equation (5)

$$f_\theta = -\left(1+\frac{1}{m}\right)\frac{\rho\omega^2 r^2}{2g} - E\,\alpha\,T + A - \frac{A}{2} - \frac{B}{r^2}$$

$$+\left(\frac{3m+1}{8m}\right)\frac{\rho\omega^2 r^2}{g} + \frac{E\alpha}{r^2}\int Tr\,dr$$

$$= \frac{A}{2} - \frac{B}{r^2} - \frac{m+3}{8m}\cdot\frac{\rho\omega^2 r^2}{g} - E\,\alpha\,T + \frac{E\alpha}{r^2}\int Tr\,dr \qquad \ldots(8)$$

Constants A and B can be determined by using the boundary conditions for radial stress.

Fot a solid disc, $B=0$ because stresses cannot be infinite at the centre.

Example 18·5-1. A thin uniform steel disc of diameter 50 cm is rotating about its axis at 5000 r.p.m. Determine the stresses developed at the centre of the disc and at its periphery if the disc has a linear variation of temperature of 50°C between the centre and its outer edge.

Given : $\alpha = 11\times10^{-6}/°C$, $\rho = 0{\cdot}0078$ kg/cm³, $g = 910$ cm/sec²
$E = 2{\cdot}1\times10^6$ kg/cm², $1/m = 0{\cdot}3$

Solution. For a solid disc

$$p_r = \frac{A}{2} - \frac{3m+1}{8m}\cdot\frac{\rho\omega^2 r^2}{g} - \frac{E\alpha}{r^2}\int Tr\,dr$$

The variation of the temperature with radius can be written as

$$T = \frac{50\,r}{25}\ °C, \text{ where } r \text{ is in cm}$$

$$=0 \quad \text{at} \quad r=0$$

$$=50°C \quad \text{at} \quad r=25 \text{ cm}$$

or $T = 2r$ °C

$$\therefore \quad p_r = \frac{A}{2} - \frac{3m+1}{8m}\cdot\frac{\rho\omega^2 r^2}{g} - \frac{E\alpha}{r^2}\int 2r^2 dr$$

Angular speed, $\omega = \dfrac{2\times\pi\times5000}{60} = 523{\cdot}6$ rad/sec

$p_r = 0$ at $r = 25$ cm, outer radius

$$0 = \frac{A}{2} - \left(\frac{3m+1}{8m}\right)\frac{\rho\omega^2\times25^2}{g} - \frac{2{\cdot}1\times10^6\times11\times10^{-6}}{25^2}\int 2r^2\,dr$$

$$= \frac{A}{2} - \left(\frac{3m+1}{8m}\right)625\,\frac{\rho\omega^2}{g} - 0{\cdot}03696\left|\frac{2r^3}{3}\right|_0^{25}$$

$$= \frac{A}{2} - 625\left(\frac{3+0{\cdot}3}{8}\right)\frac{\rho\omega^2}{g} - 0{\cdot}03696\times2\times\frac{25^3}{3}$$

$$\frac{A}{2} = \frac{625\times3{\cdot}3}{8}\times0{\cdot}0078\times\frac{(523{\cdot}6)^2}{980} + 385{\cdot}0$$

$$= 562{\cdot}56 + 385{\cdot}0 = 947{\cdot}56 \text{ kg/cm}^2$$

Maximum radial stress occurs at the centre of the disc

$$p_{r\ max}=\frac{A}{2}-0-0=947{\cdot}56\ \text{kg/cm}^2$$

Hoop stress

$$f_\theta=\frac{A}{2}-\frac{m+3}{8m}\cdot\frac{\rho\omega^2r^2}{g}-E\alpha T+\frac{E\alpha}{r^2}\int Tr\,dr$$

where $T=2r$

$$=\frac{A}{2}-\left(\frac{m+3}{8m}\right)\frac{\rho\omega^2r^2}{g}-2\ E\alpha r+\frac{E\alpha}{r^2}\int 2r^2\,dr$$

At the centre of the disc

$$(f_\theta)=\frac{A}{2}=947{\cdot}56\ \text{kg/cm}^2$$

f_θ at the periphery *i.e.* at $r=25$ cm

$$f_\theta'=\frac{A}{2}-\left(\frac{m+3}{8m}\right)\frac{\rho\omega^2}{g}\times25^2-2\times2{\cdot}1\times10^6\times11\times10^{-6}\times25$$

$$+\frac{E\alpha}{25^2}\int_0^{25}2\,r^2\,dr$$

$$=947{\cdot}56-\frac{1{\cdot}9}{8}\times0{\cdot}0078\times\frac{(523{\cdot}6)^2\times625}{980}-1155$$

$$+\frac{2{\cdot}1\times10^{+6}\times11\times10^{-6}}{625}\left|\frac{2r^3}{3}\right|_0^{25}$$

$$=947{\cdot}56-545{\cdot}56-1155+\frac{2{\cdot}1\times11}{625}\times\frac{2}{3}\times25^3$$

$$=-753+385==-365\ \text{kg/cm}^2$$

Example 18·5-2. A thin disc of outer radius 30 cm and inner radius 10 cm is rotating about its axis at 3500 r.p.m. The disc has a linear variation of temperature of 60°C between the inner and outer (hotter edge). Calculate the maximum stress.

$$E=208\times10^3\ \text{N/mm}^2\qquad \rho=0{\cdot}07644\ \text{N/cm}^3$$

$$\frac{1}{m}=0{\cdot}3,\ g=980\ \text{cm/sec}^2,\ \alpha=11\times10^{-6}\text{ per °C}$$

Solution. $E=208\times10^5\ \text{N/cm}^2$

Angular speed, $\omega=\dfrac{2\pi\times3500}{60}=366{\cdot}52$ radians/second

Radial stress at any radius

$$p_r=\frac{A}{2}+\frac{B}{r^2}-\frac{3m+1}{8m}\ \frac{\rho\omega^2r^2}{g}-\frac{E\alpha}{r^2}\int Tr\,dr$$

Temperature variation can be written as

$T=3\ (r-10)$ *i.e.* at $r=10$ cm, $T=0$; at $r=30$ cm, $T=60°$

Moreover $p_r=0$ at $r=R_1=10$ cm and also at $r=R_2=30$ cm

$$0=\frac{A}{2}+\frac{B}{100}-\frac{3m+1}{8m}\times\frac{\rho\omega^2\,10^2}{g}$$
$$-\frac{208\times10^5\times11\times10^{-6}}{100}\int 3(r^2-10\,r)\,dr$$

$$0=\frac{A}{2}+\frac{B}{900}-\frac{3m+1}{8m}\times\frac{\rho\omega^2\times900}{g}$$
$$-\frac{208\times10^5\times11\times10^{-6}}{900}\int_{10}^{30}(3r^2-30\,r)\,dr$$

or
$$\frac{A}{2}+\frac{B}{100}=\frac{3\cdot3}{8}\times\frac{0\cdot07644\times(366\cdot52)^2100}{980}-0=432\cdot22$$

$$\frac{A}{2}+\frac{B}{900}=\frac{3\cdot3}{8}\times\frac{0\cdot07644\times(366\cdot52)^2\times900}{980}$$
$$+\frac{208\times10^5\times11\times10^{-6}}{900}\left[\frac{r^3}{3}-15\,r^2\right]_{10}^{30}$$
$$=3889\cdot8+0\cdot254\;(13500+500)=7448\cdot9$$

$$\frac{B}{900}-\frac{B}{100}=7448\cdot9-432\cdot22=7016\cdot68$$

$$B=-\frac{7016\cdot68\times9\times10^4}{800}=-78\cdot93\times10^4$$

$$\frac{A}{2}=432\cdot22-\frac{B}{100}=432\cdot22+7893=8325\cdot22$$

Maximum stress occurs at the inner radius, $r=10$ cm

$$f_{\theta\;max}=\frac{A}{2}-\frac{B}{100}-\frac{m+3}{8m}\cdot\frac{\rho\omega^2\times100}{g}$$
$$-E\,\alpha\,T+\frac{E\,\alpha}{100}\int(3r^2-30\,r)\,dr$$
$$=\frac{A}{2}-\frac{B}{100}-\frac{1\cdot9}{8}\times\frac{0\cdot07644\times(366\cdot52)^2\times100}{980}$$
$$=8325\cdot22+7893-256\cdot86=15961\cdot36\text{ N/cm}^2$$
$$=159\cdot6\text{ N/mm}^2.$$

Exercise 18·5-1. A thin uniform steel disc of diameter 40 cm is rotating at 2500 r.p.m about its axis. Determine the maximum stress if the disc has a linear variation of temperature of 40°C between the centre and the outer edge.

Given : $\alpha=11\times10^{-6}$ per °C, $\rho=0\cdot0078$ kg/cm³, $g=980$ cm/sec²
$E=2\cdot1\times10^6$ kg/cm², $1/m=0\cdot3$

Determine the hoop stress at the centre and at the periphery of the disc.

[**Ans.** 398·0 kg/cm², −269·82 kg/cm²]

Exercise 18·5-2. A thin disc of outer radius 35 cm and inner radius 15 cm is rotating about its axis at 4000 r.p.m. The disc has a linear variation of temperature of 40°C between the inner and outer (hotter) edges. Calculate the maximum stress in the disc.

$E=208\times10^3$ N/mm², $g=980$ cm/sec²

$\rho=0·07644$ N/cm³ $\alpha=11\times10^{-6}$/°C

$1/m=0·3$ **[Ans.** 99·6 N/mm²]

Problem 18·1. A composite ring is made by fitting a steel ring over a copper ring. The diameter of the ring at the common surface is 1·60 metres. The radial thickness of both the rings is 20 mm and their axial width is 30 mm. Determine the stresses set up in the steel and copper rings if the composite ring is rotating at 2000 r.p.m.

For steel $E=210\times10^3$ N/mm², $\rho_s=0·0078$ kg/cm³

For copper $E=105\times10^3$ N/mm² $\rho_c=0·0090$ kg/cm³

$g=9·81$ m/sec²

Solution. Fig. 18·9 shows the composite ring made of steel and copper rings.

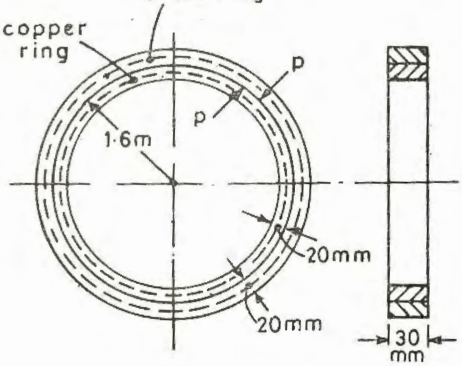


Fig. 18·9

Radius at the common surfac
$=80$ cm$=800$ mm

Mean radius of steel ring
$R_s=800+10=810$ mm

Mean radius of copper ring
$R_c=800-10=790$ mm

Density of steel$=0·0078\times9·8$ N/cm²
$=76·44\times10^{-6}$ N/mm³

Density of copper$=0·009\times9·8$ N/cm³
$=88·2\times10^{-6}$ N/mm³

$$\text{Angular speed}=\omega=\frac{2\pi N}{60}=\frac{2\times\pi\times2000}{60}=209·44 \text{ rad/sec}$$

Stresses due to rotation

$$\text{Hoop stress in steel,} \quad f_s=\frac{\rho\omega^2R_s^2}{g}=\frac{76·44\times10^{-6}\times(209·44)^2\times(810)^2}{9·81\times1000}$$

$$=224·25 \text{ N/mm}^2 \quad \text{(tensile)}$$

$$\text{Hoop stress in copper,} \ f_c=\frac{f\omega^2R_c^2}{g}=\frac{88·2\times10^{-6}\times(209·44)^2\times(790)^2}{9·81\times1000}$$

$$=246·13 \text{ N/mm}^2 \quad \text{(tensile)}$$

As $f_c>f_s$, showing thereby that centrifugal force developed in copper ring is more than the centrifugal force on the steel ring *i.e.*, copper ring exerts radial pressure on the steel ring. Say the radial pressure is p N/mm².

Stresses due to radial pressure

Due to the radial pressure p at the common surface there will be compressive hoop stress in copper ring and tensile hoop stress in steel ring.

Tensile hoop stress in steel ring,

$$f_s'=\frac{pD_s}{2t}=\frac{pR_s}{t}=\frac{p\times 810}{t}$$

Compressive hoop stress in copper ring,

$$f_c'=\frac{p\times 790}{t}$$

Resultant hoop stress in steel ring,

$$=224{\cdot}25+\frac{810\,p}{t}=224{\cdot}25+40{\cdot}5\,p \text{ since } t=20 \text{ mm}$$

Resultant hoop stress in copper ring,

$$=246{\cdot}13-\frac{790\,p}{t}=246{\cdot}13-39{\cdot}5\,p$$

Strain compatibility

Since both the rings are rotating together, circumferential strain in steel ring will be the same as the circumferential strain in copper ring.

So $\frac{1}{E_s}(224{\cdot}25+40{\cdot}5\,p)=\frac{1}{E_c}(246{\cdot}13-39{\cdot}5\,p)$

but $E_s=2\,E_c$

So $224{\cdot}5+40{\cdot}5\,p=2\times 246{\cdot}13-79\,p$

$119{\cdot}5\,p=268{\cdot}01$

$p=2{\cdot}24 \text{ N/mm}^2$

So,

Resultant hoop stress in steel ring $=224{\cdot}25+40{\cdot}5\times p$
$=314{\cdot}97 \text{ N/mm}^2$

Resultant hoop stress in copper ring $=246{\cdot}13-39{\cdot}5\times p$
$=157{\cdot}65 \text{ N/mm}^2$

Problem 18·2. A circular saw 3 mm thick × 60 cm diameter is secured upon a shaft of 8 cm diameter. The material of the saw has a density of 0·0078 kg/cm³ and Poisson's ratio = 0·3. Determine the permissible speed if the allowable hoop stress is 2400 kg/cm² and find the maximum radial stress. Acceleration due to gravity = 981 cm/sec².

Solution. Density, $\rho=0{\cdot}0078 \text{ kg/cm}^3$

Inner radius $R_1=4$ cm

Outer radius, $R_2=30$ cm

Constants $k_1=\frac{3m+1}{8m}=\frac{3{\cdot}3}{8},\quad k_2=\frac{m+3}{8m}=\frac{1{\cdot}9}{8}$

Say the permissible speed $=\omega$ radian/sec

Maximum allowable hoop stress

$=2400 \text{ kg/cm}^2$

Maximum stress occurs at the inner radius

$$f_{\theta max}=\frac{k_1\rho\omega^2}{g}(R_1^2+R_2^2)+\frac{k_1\rho\omega^2}{g}\,\frac{R_1^2R_2^2}{R_1^2}-k_2\,.\,\frac{\rho\omega^2}{g}\times R_1^2$$

$$=\frac{\rho\omega^2}{g}\,[2k_1R_2^2-k_2R_1^2]$$

$$2400=\frac{0{\cdot}0078\times\omega^2}{981}\left[2\times\frac{3{\cdot}3}{8}\times900-\frac{1{\cdot}9}{8}\times16\right]$$

$$\omega^2=\frac{2400\times981}{0{\cdot}0078\times738{\cdot}7}=40{\cdot}86\times10^4$$

$$\omega=6{\cdot}39\times10^2=639\text{ rad/sec}$$

$$N=\frac{639\times60}{2\pi}=6102\text{ revolution per minute.}$$

Maximum radial stress

Occurs at $\quad r=\sqrt{R_1R_2}=\sqrt{4\times30}=10{\cdot}954$ cm

$$p_{r_{max}}=\frac{k_1\rho\omega^2}{g}\,(R_1^2+R_2^2)-\frac{k_1\rho\omega^2}{g}\times\frac{R_1^2R_2^2}{120}-k_1\frac{\rho\omega^2}{g}\times120$$

$$=\frac{k_1\rho\omega^2}{g}\left[16+900-\frac{16\times900}{120}-120\right]=\frac{k_1\rho\omega^2}{g}\,(676)$$

$$=\frac{3{\cdot}3}{8}\times\frac{0{\cdot}0078}{981}\times40{\cdot}86\times10^4\times676=905{\cdot}93\text{ kg/cm}^2$$

Maximum radial stress $=905{\cdot}93$ kg/cm² at the radius of $10{\cdot}954$ cm.

Problem 18·3. Determine the stresses due to the centrifugal force in a rotor with an outer radius 50 cm and radius of the hole 10 cm. The outer portion of the rotor is cut by slots 20 cm deep for windings, (as shown in the Fig. 18·10). The rotor is of steel and rotates at 3000 r.p.m. The weight of the windings in the slots is the same as that of the material removed.

ρ for steel $\quad=0{\cdot}076$ N/cm³

Poisson's ratio $\quad=0{\cdot}3$ and $g=9{\cdot}81$ m/sec²

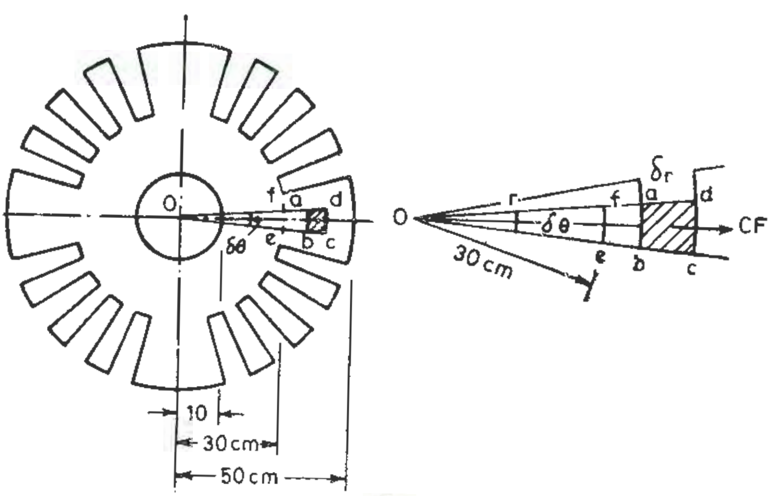


Fig. 18·10

Solution. Because the slots are cut in the outer portion of the rotor, the part of the rotor between the radii 30 to 50 cm can support no tensile hoop stress. The centrifugal force due to this rotating ring of 30 to 50 cm radius produces a tensile radial stress across the surface of the disc of 30 cm radius.

Say the radial stress at radius 30 cm $=p_0$

Say thickness of the rotor $=t$ cm.

Consider a small element *abcd* subtending an angle $\delta\theta$ at the centre, at a radius r with radial thickness δr,

Centrifugal force on the small element

$$=\frac{\rho\omega^2}{g}\,r\,(rd\theta\,.\,t\,dr)$$

Area of the section at radius 30 cm

$$=(30d\theta\times t)$$

Stress
$$p_0=\int_{30}^{50}\frac{\rho\omega^2r^2d\theta\,.\,t\,dr}{g\;30\;d\theta\,.\,t}=\int_{30}^{50}\frac{\rho\omega^2r^2dr}{30\,g}$$

$$p_0=\frac{\rho\omega^2}{30g}\left|\frac{r^3}{3}\right|_{30}^{50}=\frac{\rho\omega^2}{90g}\times\frac{98\times10^3}{}$$

$$\omega=2\times\pi\times\frac{2000}{60}=209{\cdot}44\text{ rad/scc}$$

So,
$$p_0=\frac{0{\cdot}07644\times(209{\cdot}44)^2}{90\times981}\times98\times10^3=3721{\cdot}8\text{ N/cm}^2.$$

Now on a disc of outer radius 30 cm and inner radius 10 cm, a radial pressure p_0 tensile is acting. Due to this p_0, the hoop stress (or the circumferential stress) developed at radius 30 cm is

$$f_0=p_0\times\frac{2\times30^2}{30^2-10^2}=3721{\cdot}8\times\frac{1800}{800}=8374{\cdot}08\text{ N/cm}^2$$

(Refer to the formula on thick cylinders)

Considering whole of the rotor as a hollow disc, let us find the circumferential stress at the radius 30 cm,

$$R_1=10\text{ cm}\qquad R_2=50\text{ cm}\qquad r=30\text{ cm}$$

$$f_\theta=k_1\frac{\rho\omega^2}{g}(R_1^2+R_2^2)+\frac{k_1\rho\omega^2}{g}\times\frac{R_1^2R_2^2}{r^2}-k_2\frac{\rho\omega^2}{g}\,.\,r^2$$

where
$$k_1=\frac{3m+1}{8m}=\frac{3{\cdot}3}{8}\text{ and }k_2=\frac{m+3}{8m}=\frac{1{\cdot}9}{8}$$

$$\frac{\rho\omega^2}{g}=\frac{0{\cdot}07644\times209{\cdot}44^2}{981}=3{\cdot}418\text{ N/cm}^4$$

f_θ at $r=30$ cm

$$=\frac{3{\cdot}3}{8}\times3{\cdot}418(50^2+10^2)+\frac{3{\cdot}3}{8}\times3{\cdot}418\times\frac{50^2\times10^2}{30^2}-\frac{1{\cdot}9}{8}\times3{\cdot}418\times30^2$$

$$=3665{\cdot}80+391{\cdot}6458-730{\cdot}5975$$

$$=3326{\cdot}85\text{ N/cm}^2$$

Total maximum circumferential stress at the inner edge of the slot

$$=8374{\cdot}08+3326{\cdot}85$$
$$=11700{\cdot}93 \text{ N/cm}^2=117{\cdot}0 \text{ N/mm}^2.$$

Problem 18·4. A thin circular disc of external radius R_2 is forced on to a rigid shaft of radius R_1. Prove that when the angular speed is ω, the pressure between the disc and the shaft will be reduced by

$$\frac{\rho\omega^2}{g}\times\frac{(R_2^2-R_1^2)\{(3m+1)R_2^2+(m-1)R_1^2\}}{4\{(m+1)R_2^2+(m-1)R_1^2\}}$$

Solution. The thin circular disc is forced onto the solid shaft, say the initial pressure between the shaft and the disc at the common surface is p.

Due to the radial pressure p, there will be initial hoop stress in the disc and at the inner radius, R_1

Hoop stress, $f_\theta'=p\left(\frac{R_2^2+R_1^2}{R_2^2-R_1^2}\right)$ ...refer to the chapter on thick cylinders

Circumferential strain in the disc at the inner radius

$$\epsilon_\theta'=\frac{f_\theta'}{E}+\frac{p}{mE} \quad \text{Since } p \text{ is compressive and } f_\theta' \text{ is tensile.}$$

$$=\frac{p}{E}\left\{\frac{R_2^2+R_1^2}{R_2^2-R_1^2}+\frac{1}{m}\right\} \qquad ...(1)$$

When the disc and the shaft are rotating at ω radians/sec, say the radial pressure between the disc and shaft is p'.

At the inner radius of the disc

Circumferential stress due to p',

$$f_\theta''=p'\left(\frac{R_2^2+R_1^2}{R_2^2-R_1^2}\right) \quad \text{tensile}$$

Circumferential stress due to rotation

$$f_\theta'''=k_1\frac{\rho\omega^2}{g}(R_1^2+R_2^2)+k_1\frac{\rho\omega^2}{g}R_2^2-k_2\frac{\rho\omega^2}{g}R_1^2$$

where $k_1=\frac{3m+1}{8m}$ and $k_2=\frac{m+3}{8m}$.

$$f_\theta'''=\frac{\rho\omega^2}{g}\left[\left(\frac{3m+1}{8m}\right)(R_2^2+R_1^2)+\frac{3m+1}{8m}R_2^2-\frac{m+3}{8m}+R_1^2\right]$$

$$=\frac{\rho\omega^2}{g}\left[\left(\frac{3m+1}{8m}\right)(2R_2^2+R_1^2)-\frac{m+3}{8m}.\,R_1^2\right]$$

Resultant hoop stress, $f_\theta=f_\theta''+f_\theta'''$

$$=p'\frac{(R_2^2+R_1^2)}{R_2^2-R_1^2}+\frac{\rho\omega^2}{g}\left[\left(\frac{3m+1}{8m}\right)(2R_2^2+R_1^2)-\frac{m+3}{8m}R_1^2\right]$$

Hoop strain, $\epsilon_\theta=\frac{f_\theta}{E}+\frac{p'}{mE}$ since f_θ is tensile and p' is compressive

$$\epsilon = \frac{p'(R_2^2+R_1^2)}{E(R_2^2-R_1^2)} + \frac{\rho\omega^2}{gE}\left[\left(\frac{3m+1}{8m}\right)(2R_2^2+R_1^2) - \frac{m+3}{8m} \cdot R_1^2\right] + \frac{p'}{mE} \qquad ...(2)$$

Equating the srains $\epsilon_{\theta'} = \epsilon_\theta$ we get

$$\frac{p}{E}\left[\frac{R_2^2+R_1^2}{R_2^2-R_1^2} + \frac{1}{m}\right] = \frac{p'}{E}\left\{\frac{R_2^2+R_1^2}{R_2^2+R_1^2} + \frac{1}{m}\right\} + \frac{\rho\omega^2}{gE}\left[\left(\frac{3m+1}{8m}\right)(2R_2^2+R_1^2) - \frac{m+3}{8m}R_1^2\right]$$

or $$(p-p')\left[\frac{R_2^2+R_1^2}{R_2^2-R_1^2} + \frac{1}{m}\right] = \frac{\rho\omega^2}{g}\left[\left(\frac{3m+1}{8m}\right)(R_2^2+R_1^2) + \frac{3m+1}{8m} \cdot R_2^2 - \frac{m+3}{8m}R_1^2\right]$$

or $$(p-p')\left[\frac{(mR_2^2+mR_1^2)+(R_2^2-R_1^2)}{m(R_2^2-R_1^2)}\right]$$

$$= \frac{\rho\omega^2}{g}\left[\frac{(3m+1)R_2^2+(3m+1)R_1^2+(3m+1)R_2^2-(m+3)R_1^2}{8m}\right]$$

$$(p-p')\left[\frac{m(R_2^2+R_1^2)+(R_2^2-R_1^2)}{(R_2^2-R_1^2)}\right] = \frac{\rho\omega^2}{8g}[(6m+2)R_2^2+(2m-2)R_1^2]$$

$$(p-p')\left[\frac{(m+1)R_2^2+(m-1)R_1^2}{R_2^2-R_1^2}\right] = \frac{\rho\omega^2}{4g}[(3m+1)R_2^2+(m-1)R_1^2]$$

or $$(p-p') = \frac{\rho\omega^2}{4g} \times \frac{[(3m+1)R_2^2+(m-1)R_1^2](R_2^2-R_1^2)}{(m+1)R_2^2+(m-1)R_1^2}$$

or Reduction in radial pressure

$$(p-p') = \frac{\rho\omega^2}{4g}(R_2^2-R_1^2)\left[\frac{(3m+1)R_2^2+(m-1)R_1^2}{(m+1)R_2^2+(m-1)R_1^2}\right]$$

Problem 18·5. If a disc of inside and outside radii R_1 and R_2 is made up in two parts which are shrunk together, the common radius being R_3, show that the hoop stresses at R_1 and R_2 will be equal to a rotational speed given by

$$\omega^2 = \frac{4pgR_3^2}{\rho(1+\nu)(R_3^2-R_1^2)(R_2^2-R_3^2)}.$$

Solution. A disc made up in two parts inner disc (with radii R_1 and R_3) and outer disc (with radii R_3 and R_2) is shown in the figure 18·11, p is the junction pressure. Due to the junction pressure, there will be compressive hoop stress developed in inner disc and tensile hoop stress developed in the outer disc.

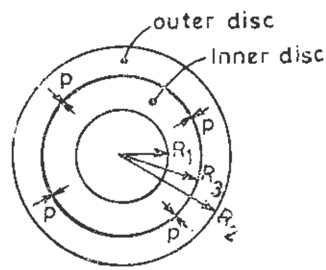


Fig. 18·11

Stresses due to junction pressure

Inner disc at radius R_1, $f_\theta' = -2p \cdot \dfrac{R_3^2}{R_3^2-R_1^2}$ compressive

Outer disc at radius R_2, $f_\theta'' = +2p \cdot \dfrac{R_3^2}{R_2^2-R_3^2}$ tensile

(Refer to the chapter on thick cylinders)

Stresses due to rotation

The expression for the hoop stress is

$$f_\theta = k_1 \frac{\rho\omega^2}{g}(R_1^2+R_2^2)+k_1\frac{\rho\omega^2}{g}\frac{R_1^2 R_2^2}{r^2}-k_2\frac{\rho\omega^2 r^2}{g}$$

where $$k_1=\frac{3m+1}{8m}\;;\quad k_2=\frac{m+3}{8m}$$

and $$k_1+k_2=\frac{4m+4}{8m}=\frac{m+1}{2m}$$

At the inner radius R_1

$$f_{\theta1}=k_1\frac{\rho\omega^2}{g}(R_1^2+R_2^2)+k_1\frac{\rho\omega^2}{g}\times R_2^2-k_2\frac{\rho\omega^2}{g}R_1^2$$

At the outer radius R_2

$$f_{\theta2}=k_1\frac{\rho\omega^2}{g}(R_1^2+R_2^2)+k_1\frac{\rho\omega^2}{g}R_1^2-k_2\frac{\rho\omega^2}{g}R_2^2$$

Resultant stresses

At the inner radius, $f_1=f_{\theta1}+f_\theta'$

$$k_1\frac{\rho\omega^2}{g}(R_1^2+R_2^2)+k_1\frac{\rho\omega^2}{g}R_2^2-k_2\frac{\rho\omega^2 R_1^2}{g}-2p\frac{R_3^2}{R_3^2-R_1^2}$$

At the outer radius R_2, $f_2=f_{\theta2}+f_\theta''$

$$=\frac{k_1\rho\omega^2}{g}(R_1^2+R_2^2)+k_1\frac{\rho\omega^2}{g}R_1^2-k_2\frac{\rho\omega^2 R_2^2}{g}+\frac{2pR_3^2}{R_2^2-R_3^2}$$

But as per the condition given

$$f_1=f_2$$

or
$$\frac{k_1\rho\omega^2(R_1^2+R_2^2)}{g}+k_1\frac{\rho\omega^2R_2^2}{g}-k_2\frac{\rho\omega^2R_1^2}{g}-2p.\frac{R_3^2}{R_3^2-R_1^2}$$
$$=\frac{k_1\rho\omega^2(R_1^2+R_2^2)}{g}+k_1\frac{\rho\omega^2R_1^2}{g}-k_2\rho\omega^2R_2^2+2p\frac{R_3^2}{R_2^2-R_3^2}$$

or
$$\frac{k_1\rho\omega^2}{g}(R_2^2-R_1^2)+k_2\rho\omega^2(R_2^2-R_1^2)=\frac{2p\,R_3^2}{R_2^2-R_3^2}+\frac{2p\,R_3^2}{R_3^2-R_1^2}$$

$$\frac{\rho\omega^2}{g}(R_2^2-R_1^2)(k_1+k_2)=2pR_3^2\left(\frac{1}{R_2^2-R_3^2}+\frac{1}{R_3^2-R_1^2}\right)$$

$$\frac{\rho\omega^2}{g}(R_2^2-R_1^2)\left(\frac{m+1}{2m}\right)=2p\,R_3^2\left(\frac{R_3^2-R_1^2+R_2^2-R_3^2}{(R_2^2-R_3^2)(R_3^2-R_1^2)}\right)$$

$$\frac{\rho\omega^2}{g}(R_2^2-R_1^2)\frac{(1+\nu)}{2}=2p\,R_3^2\,\frac{(R_2^2-R_1^2)}{(R_2^2-R_3^2)(R_3^2-R_1^2)}$$

$$\omega^2=\frac{4\,pg\,R_3^2}{\rho(1+\nu)(R_3^2-R_1^2)(R_2^2-R_3^2)}$$

Problem 18·6. A thin hollow disc of outer radius R_2 is shrunk over another solid disc of the same thickness but radius R_1, such that the junction pressure between the two is p.

Show that in order that the outer disc may not be loosened over the inner disc at the angular velocity ω, the minimum value of p should be

$$\frac{3+\nu}{8g} \cdot \rho\omega^2 \, (R_2^2-R_1^2) \qquad \text{where} \qquad \nu=\text{Poisson's ratio}$$

$$\text{and} \qquad \rho=\text{Weight density}$$

Solution. When the outer disc is loosened over the inner disc, the radial pressure between the two becomes zero.

Initial hoop strain at R_1

Due to junction pressure p,

Hoop stress in outer disc $=p \times \dfrac{R_2^2+R_1^2}{R_2^2-R_1^2}$ tensile

Hoop strain in outer disc $=\dfrac{p}{E} \times \dfrac{R_2^2+R_1^2}{R_2^2-R_1^2}+\dfrac{p}{mE}$ tensile

Hoop stress in the inner disc $=-p$ compressive

Hoop strain in inner disc $=-\dfrac{p}{E}+\dfrac{p}{mE}$ compressive

Total hoop strain
$$\epsilon_\theta=\frac{p}{E} \times \frac{R_2^2+R_1^2}{R_2^2-R_1^2}+\frac{p}{mE}+\frac{p}{E}-\frac{p}{mE}$$
$$=\frac{p}{E}\left(\frac{2R_2^2}{R_2^2-R_1^2}\right) \qquad \ldots(1)$$

Hoop strain due to rotation at R_1

Hoop stress in inner disc at R_1

$$f_\theta=\frac{\rho\omega^2}{g}\left[\frac{3m+1}{8m} \times R_1^2-\frac{m+3}{8m} R_1^2\right]$$
$$=\frac{\rho\omega^2 R_1^2}{g}\left[\frac{3m+1-m-3}{8m}\right]=\frac{\rho\omega^2 R_1^2}{g}\left(\frac{m-1}{4m}\right)$$

Hoop stress in outer disc at R_1

$$f_\theta'=\frac{\rho\omega^2}{g}\left[\frac{3m+1}{8m}(R_1^2+R_2^2)+\frac{3m+1}{8m} \times R_2^2-\frac{m+3}{8m} \times R_1^2\right]$$
$$=\frac{\rho\omega^2}{g}\left[\frac{3m+1}{8m}(2R_2^2+R_1^2)-\frac{m+3}{8m} R_1^2\right]$$

Net hoop strain
$$=\frac{f_\theta'-f_\theta}{E}$$
$$=\frac{\rho\omega^2}{gE}\left[\frac{3m+1}{8m} \times 2R_2^2+\frac{3m+1}{8m} \cdot R_1^2 -\frac{m+3}{8m} \cdot R_1^2-\frac{m-1}{4m} \cdot R_1^2\right]$$
$$=\frac{\rho\omega^2}{gE}\left[\frac{3m+1}{4m} \quad R_2^2+R_1^2 \quad \left(\frac{3m+1-m-3-2m+2}{8m}\right)\right]$$

$$=\frac{\rho\omega^2}{gE}\left\{\left(\frac{3m+1}{4m}\right)R_2^2+R_1^2(0)\right\}$$

$$=\frac{\rho\omega^2}{gE}\times\frac{3m+1}{4m}\cdot R_2^2 \qquad \ldots(2)$$

From equations (1) and (2) for the outer disc to loosen over the inner disc

$$\frac{p}{E}\times\frac{2R_2^2}{R_2^2-R_1^2}=\frac{\rho\omega^2}{gE}\times\frac{m+3}{4m}R_2^2$$

$$p=\frac{\rho\omega^2}{g}\times\left(\frac{3m+1}{8m}\right)(R_2^2-R_1^2)$$

where $\frac{1}{m}=\nu$, Poisson's ratio

So, $$p=\frac{\rho\omega^2}{g}\left(\frac{3+\nu}{8}\right)(R_2^2-R_1^2)$$

Problem 18·7. A thin steel disc of 80 cm radius is shrunk over a steel shaft of 15 cm diameter, such that the shrinkage pressure at the common surface is 150 N/mm².

At what speed will the disc be loosened on the shaft ? Neglect the change in the dimensions of the shaft.

Solution.

ρ, density of steel $=0·07644$ N/cm³

$E=2\times10^7$ N/cm² ; $\frac{1}{m}=0·3$

$g=980$ cm/sec²

Junction pressure, $p=150$ N/mm² $=15000$ N/cm²

For the disc,

inner radius $R_1=7·5$ cm

outer radius $R_2=40$ cm

Hoop stress due to shrinkage at the common surface in the disc

$$=p\times\frac{R_2^2+R_1^2}{R_2^2-R_1^2}=15000\times\frac{40^2+7·5^2}{40^2-7·5^2}$$

$$=15000\times\frac{1656·25}{1543·75}=16093·12\text{ N/cm}^2$$

Shrinkage strain $$=\frac{16093·12}{E}-\frac{15000}{mE} \quad \text{(neglecting strain in shaft)}$$

$$=\frac{15643·12}{E} \quad \text{since } \frac{1}{m}=0·3 \qquad \ldots(1)$$

When the assembly in rotating say at ω radians/second, the dlsc is loosened and the radial stress between the two becomes zero.

Hoop stress in the disc at inner radius

$$f_\theta=\frac{\rho\omega^2}{g}\left[\frac{3\cdot3}{8}\ (R_1^2+R_2^2)+\frac{3\cdot3}{8}\ R_2^2-\frac{1\cdot9}{8}\times R_1^2\right]$$

$$\text{as}\quad \frac{3m+1}{8m}=\frac{3\cdot3}{8}\ \text{and}\ \frac{1+3m}{8m}=\frac{1\cdot9}{8}$$

$$=\frac{\rho\omega^2}{g}\left[\frac{3\cdot3}{3}\ (1600+56\cdot25)\ +\frac{3\cdot3}{8}\times1600-\frac{1\cdot9}{8}\times56\cdot25\right]$$

$$=\frac{\rho\omega^2}{g}\ [1343\cdot20-13\cdot36]=\frac{1329\cdot84}{g}\times\rho\omega^2$$

Hoop strain $\quad =\dfrac{f_\theta}{E}=\dfrac{1329\cdot84+\rho\omega^2}{gE}$ ...(2)

Due to rotation, when the disc is loosened on the shaft, the shrinkage hoop strain becomes zero, therefore

$$1329\cdot84\ \frac{\rho\omega^2}{E}=\frac{15643\cdot12}{E}$$

or
$$\omega^2=\frac{15643\cdot12\times g}{1329\cdot84\times\rho}=\frac{15643\cdot12\times980}{1329\cdot84\times0\cdot07644}$$

Angular speed, $\quad \omega=388\cdot84$ radian/sec

$=3708$ revolutions per minute

Problem 18 8. A steel ring is shrunk on a cast iron hollow disc. Find the change in the shrink fit pressure produced by the inertia forces at 3000 r.p.m. If $R_1=4$ cm, $R_2=10$ cm and $R_3=20$ cm

Given $\quad E_{steel}=2100$ tonnes/cm²

$E_{C.I.}=1100$ tonnes/cm²

$1/m$ per steel and C.I. $\quad =0\cdot3$

$\rho_{steel}=0\cdot0079$ kg/cm³

$\rho_{C.I.}=0\cdot0072$ kg/cm³

$g=980$ cm/sec².

If the initial junction pressure is 100 kg/cm², calculate the speed at which the outer ring of steel will start slipping over the cast iron disc.

Solution. Say the initial shrink fit pressure$=p$ kg/cm²

Steel ring, Inner radius, $\quad R_2=10$ cm

Outer radius, $\quad R_3=20$ cm

Hoop stress at R_2 in steel ring

$$=p\cdot\frac{R_3^2+R_2^2}{R_3^2-R_2^2}=p\times\frac{20^2+10^2}{20^2-10^2}=+1\cdot66\ p\ \text{(tensile)}$$

Cast iron disc, Inner radius, $\quad R_1=4$ cm

Outer radius, $\quad R_2=10$ cm

Hoop stress at R_2 in cast iron disc

$$=-p\cdot\frac{R_2^2+R_1^2}{R_2^2-R_1^2}=-p\times\frac{100+16}{100-16}$$
$$=-1\cdot38\ p\ \text{(compressive)}$$

To obtain the hoop stresses, we can refer to the chapter on thick cylinders. Moreover when the disc is stationery, *i.e.*, $\omega=0$, the equation for p_r and f_θ will be

$$p_r=\frac{A}{2}+\frac{B}{r^2}$$

$$f_\theta=\frac{A}{2}-\frac{B}{r^2}$$

These equations are the same Lame's equations derived in the chapter 6 on thick cylinders.

Hoop strain in steel ring at $R_2=\dfrac{1{\cdot}66p}{Es}+\dfrac{p}{mEs}$

$$\epsilon_{\theta_s}=\frac{1{\cdot}66\,p}{2{\cdot}1\times10^6}+\frac{0{\cdot}3\,p}{2{\cdot}1\times10^6}=0{\cdot}933\times10^{-6}\,p$$

Hoop strain in cast iron disc at R_2

$$=\frac{-1{\cdot}38\,p}{E_c}+\frac{0{\cdot}3\,p}{E_c}=-\frac{1{\cdot}08\,p}{1{\cdot}1\times10^6}$$

$$\epsilon_{\theta_c}=-0{\cdot}98\times10^{-6}\,p$$

Total hoop strain at $R_2=\epsilon_{\theta_s}-\epsilon_{\theta_c}=(0{\cdot}933+0{\cdot}98)\times10^{-6}\,p$

$$\epsilon=1{\cdot}913\times10^{-6}\,p \qquad ...(1)$$

When the assembly is rotating at 3000 r.p.m., say the junction pressure is p'.

Hoop stresses due to junction pressure p′

At R_2 in steel ring $=p'\times1{\cdot}66$ (tensile)
At R_2 in cast iron disc $=-p'\times1{\cdot}38$ (compressive)

Hoop stresses due to rotation at R_2

$$\omega=\frac{2\times\pi\times3000}{60}=314{\cdot}16\text{ rad/sec.}$$

In steel ring, $f_{\theta s}=k_1\dfrac{\rho\omega^2}{g}(R_2^2+R_3^2)+k_1\dfrac{\rho\omega^2}{g}\times\dfrac{R_3^2}{}-k_2\dfrac{\rho\omega^2}{g}\times R_2^2$

where $k_1=\dfrac{3m+1}{8m}=\dfrac{3{\cdot}3}{8}$; $k_2=\dfrac{m+3}{8m}=\dfrac{1{\cdot}9}{8}$

$$=\frac{\rho\omega^2}{g}\left[\frac{3{\cdot}3}{8}(10^2+20^2)+\frac{3{\cdot}3}{8}\times20^2-\frac{1{\cdot}9}{8}\times10^2\right]$$

$$=\frac{\rho\omega^2}{g}[206{\cdot}25+165-23{\cdot}75]=\frac{\rho\omega^2}{g}\times347{\cdot}5$$

$$=\frac{0{\cdot}0079\times(314{\cdot}16)^2\times347{\cdot}5}{980}=276{\cdot}48\text{ kg/cm}^2$$

Total hoop stress in steel ring

$$=(276{\cdot}48+1{\cdot}66p')$$

Hoop strain in steel ring$=\left(\dfrac{276.48+1.66\,p'}{E_s}\right)+\dfrac{p'}{mE_s}$

$$\epsilon_{\theta s}'=\frac{276.48+1.96\,p'}{E_s}.$$

In cast iron disc

$$f_{\theta c}=k_1\frac{\rho\omega^2}{g}(R_1^2+R_2^2)+k_1\frac{\rho\omega^2}{g}\times R_1^2-k_2\frac{\rho\omega^2}{g}\times R_2^2$$

$$=\frac{\rho\omega^2}{g}\left[\frac{3.3}{8}(100+16)+\frac{3.3}{8}\times 16-\frac{1.9}{8}\times 100\right]$$

$$=\frac{\rho\omega^2}{g}[54.45-23.75]=\frac{0.0072\times(314.16)^2}{980}\times 30.7$$

$$=22.26 \text{ kg/cm}^2$$

Total hoop stress$=(22.26-1.38\,p')$

Hoop strain in cast iron disc

$$=\left(\frac{22.26-1.38\,p'}{E_c}\right)+\frac{p'}{mE_c}$$

$$\epsilon_{\theta c}=\frac{22.26-1.08\,p'}{E_c}$$

ϵ'=Total hoop strain

$$=\epsilon_{\theta s}'-\epsilon_{\theta c}=\frac{276.48+1.96\,p'}{E_s}-\frac{22.26-1.08\,p'}{E_c} \qquad \ldots(2)$$

Now equating the hoop strains $\epsilon=\epsilon'$

$$1.913\times 10^{-6}p'=\frac{276.48+1.96\,p'}{2.1\times 10^6}-\frac{22.26-1.08\,p'}{1.1\times 10^6}$$

$$=131.657\times 10^{-6}+0.933p'\times 10^{-6}-20.23\times 10^{-6}+0.98\times p'\times 10^{-6}$$

or $\quad 1.913\times 10^{-6}(p-p')=(111.427)\times 10^{-6}$

Reduction in shrink fit pressure

or $$p-p'=\frac{111.427}{1.913}=58.25 \text{ kg/cm}^2$$

Now the initial junction pressure$=p=100$ kg/cm².

If the outer steel ring starts slipping over the cast iron disc, the junction pressure will become zero *i.e.*, $p'=0$ at speed ω' (say)

Hoop stress due to rotation in steel ring at radius 10 cm

$$f_{\theta s}'=\frac{\rho\omega'^2}{g}(347.5)=\frac{0.0079\times 347.5}{980}\times\omega'^2$$

Hoop strain $$=\frac{f_{\theta s}'}{E_s}=\frac{0.0079\times 347.5\,\omega'^2}{980\times 2.1\times 10^6}=0.0133\times 10^{-7}\,\omega'^2$$

Hoop stress due to rotation in cast iron disc at radius 10 cm

$$f_{\theta c}'=\frac{\rho\omega'^2}{g}(30.7)=\frac{(0.0072\times\omega'^2\times 30.7}{980}$$

Hoop strain $$=\frac{f_{\theta c}'}{E_c}=\frac{0.0072\times\omega'^2\times 30.7}{980\times 1.1\times 10^6}=0.0020\times 10^{-7}\omega'^2$$

Total hoop strain $=(0.0133-0.0020)\times 10^{-7}$
$=0.0113\times 10^{-7}\omega'^2$ ...(3)

From equations (1) and (3)

$$1.913\times 10^{-6}\times p=0.0113\times 10^{-7}\omega^2$$

or

$$\omega'^2=\frac{1.913\times 10^{-6}\times 100}{0.0113\times 10^{-7}}=169.29\times 1000$$

$$\omega'=411.45 \text{ radians/sec}$$

$$=3929 \text{ revolutions per minute.}$$

Problem 18·9. A thin steel disc 80 cm diameter is shrunk on a steel shaft of 20 cm diameter, the shrinkage allowance is 1/1800 of the radius at the common surface.

(*i*) At what speed the disc will be loosened on the shaft

(*ii*) What are the maximum stresses in the shaft and the disc when stationery

(*iii*) What will be the maximum stress in the disc at half the rotational speed calculated in part (1).

Given $\rho_{steel}=0.0078$ kg/cm³

$1/m=0.3$

Acceleration due to gravity $=980$ cm/sec²

$E=2\times 10^7$ N/cm².

Solution.

Common radius, $R_1=10$ cm

Radius, $R_2=40$ cm

Shrinkage allowance $=\frac{1}{1800}\times R_1$

Shrinkage strain $=\frac{1}{1800}\times\frac{R_1}{R_1}=\frac{1}{1800}$

Say the shrinkage pressure at the common radius $=p$.

Then at R_1, hoop stress in disc

$$=p\cdot\frac{R_2^2+R_1^2}{R_2^2-R_1^2}=p\times\frac{1600+100}{1600-100}=p\times\frac{17}{15} \text{ (tensile)}$$

at R_1, hoop stress in shaft $=-p$ (compressive)

Hoop strain in disc $=\frac{17}{15}\frac{p}{E}-\frac{p}{mE}=\frac{0.833\,p}{E}$ (Taking $1/m=0.3$)

Strain in shaft $=-\frac{p}{E}+\frac{p}{mE}=-\frac{0.7\,p}{E}$

Total hoop strain $=\frac{0.833\,p}{E}+\frac{0.7\,p}{E}=\frac{1.533\,p}{E}=\frac{1}{1800}$

$$p=\frac{2\times 10^7}{1800\times 1.533}=7248 \text{ N/cm}^2.$$

(*ii*) Maximum stress in disc

$$=7248\times\frac{17}{15}=8214.4 \text{ N/mm}^2=82.144 \text{ N/mm}^2$$

Maximum stress in shaft $=-p=-7248$ N/cm²
(when stationery) $=-72.48$ N/mm²,

Say at speed ω radians, the disc will be loosened on the shaft. *i.e.*, at this speed total hoop strain of disc and shaft at the common radius will be equal to the shrinkage strain provided. As a result the radial stress between shaft and disc will become zero.

Rotational stresses at R_1

In the disc, $$f_\theta=\frac{\rho\omega^2}{g}\left[(R_1^2+R_2^2)k_1+R_2^2k_1-k_2R_1^2\right]$$

where $$k_1=\frac{3m+1}{8m}=\frac{3\cdot3}{8} \text{ and } k_2=\frac{1+3m}{8m}=\frac{1\cdot9}{8}$$

$$=\frac{\rho\omega^2}{g}\left[\frac{3\cdot3}{8}(2\times40^2+10^2)-\frac{1\cdot9}{8}\times10^2\right]$$

$$=\frac{\rho\omega^2}{g}\left[1361\cdot25-23\cdot75\right]=\frac{1337\cdot5\,\rho\omega^2}{g}$$

Strain $$\epsilon_\theta'=\frac{1337\cdot5\rho\omega^2}{gE}. \quad \text{where } \rho=0\cdot0078\times9\cdot8 \text{ N/cm}^3$$

In the shaft, $$f_\theta=\frac{\rho\omega^2}{e}\left[k_1R_1^2-k_2R_2^2\right]=\frac{\rho\omega^2}{g}\left[\frac{3\cdot3}{8}\times100-\frac{1\cdot9}{8}\times100\right]$$

$$=\frac{17\cdot5\,\rho\omega^2}{g}$$

Strain $$\epsilon_\theta''=\frac{17\cdot5\,\rho\omega^2}{gE}$$

Net hoop strain $$=\frac{1337\cdot5\,\rho\omega^2-17\cdot5\,\rho\omega^2}{gE}=\frac{1320\,\rho\omega^2}{gE}$$

But the strain provided by shrinkage $=\frac{1}{1800}$

Therefore $$\frac{1320\,\rho\omega^2}{gE}=\frac{1}{1800}$$

$$\omega^2=\frac{gE}{1800\times1320\times\rho}=\frac{980\times2\times10^7}{1800\times1320\times0\cdot07644}$$

$$\omega=328\cdot5 \text{ radians/sec}$$

$$=3137 \text{ Revolutions per minute.}$$

(*iii*) $\omega'=\frac{1}{2}\omega$, the rotational speed is reduced to half $f_\theta \propto \omega^2$, so the hoop stress in the disc due to rotation will be $\frac{1}{4}\times$ maximum hoop stress at ω

$$f_\theta'=\frac{\rho\omega^2}{g}\times\frac{1337\cdot5}{4}=\frac{0\cdot07644\times(328\cdot5)^2\times1337\cdot5}{980\times4}$$

$$=2814\cdot4 \text{ N/mm}^2.$$

Secondly the shrinkage allowance will be reduced only by $\frac{1}{4}$ th of $\frac{1}{1800}$. Remaining shrinkage allowance is $\frac{3}{4}\times\frac{1}{1800}$.

Therefore junction pressure $p'=3/4\times p=7248\times 3/4$
(where disc is rotating at half the speed of ω) $=5436\ \text{N/cm}^2$
Hoop stress in the disc at R_1 due to p' $=5436\times 17/15=6160{\cdot}8\ \text{N/cm}^2$
Total hoop stress in the disc $=2814{\cdot}4+6160{\cdot}8$
$=8975{\cdot}2\ \text{N/cm}^2=89{\cdot}75\ \text{N/mm}^2$.

Problem 18.10. A steel disc of a turbine is to be designed so that the radial and circumferential stresses are constant throughout and each equal to 120 N/mm^2, between the radius of 300 mm and 500 mm, when running at 5000 r.p.m. If the axial thickness at the outer radius of this zone is 20 mm, what should be the thickness at the inner radius.

ρ for steel$=0{\cdot}0079\ \text{kg/cm}^3$

g, acceleration due to gravity$=980\ \text{cm/sec}^2$

Solution. Density, $\rho=0{\cdot}0079\times 9{\cdot}8\ \text{N/cm}^3=0{\cdot}07742\ \text{N/cm}^3$

Uniform stress $f=120\ \text{N/mm}^2=120{,}00\ \text{N/cm}^2$

$g=980\ \text{cm/sec}^2$

Angular velocity, $\omega=\dfrac{2\times\pi\times 5000}{60}=523{\cdot}6$ rad/see.

Thickness at radius 500 mm or 50 cm, $t=2$ cm

$$\frac{\rho\omega^2r^2}{2fg}=\frac{0{\cdot}07742\times(523{\cdot}6)^2\times 50^2}{2\times 120{,}00\times 980}=2{\cdot}256$$

$$e^{-\frac{\rho\omega^2r^2}{2fg}}=e^{-2{\cdot}266}=\frac{1}{9{\cdot}541}=0{\cdot}1084$$

$$t=t_0\,e^{-2{\cdot}256}$$

Thickness at centre, $t_0=\dfrac{t}{0{\cdot}1048}=\dfrac{2}{0{\cdot}1048}=19{\cdot}084$ cm.

Thickness at the radius of 300 mm or 30 cm

$$\frac{\rho\omega^2r'^2}{2fg}=\frac{0{\cdot}007742\times(523{\cdot}6)^2\times 30^2}{2\times 12000\times 980}=0{\cdot}812$$

$$t'=t_0\,e^{-0{\cdot}812}=t_0.\times 0{\cdot}444$$

$$=19{\cdot}084\times{\cdot}444=8{\cdot}473\text{ cm}$$

Thickness at the inner radius$=8{\cdot}473$ cm.

Problem 18'11. A rotor disc of a steam turbine has inside diameter 15 cm and outside diameter 75 cm and axial width of 4 cm on its periphery blades are fixed at an angular pitch of 3°. The weight of each blade is 0'32 kg with effective radius of 40 cm. Determine the maximum rotational speed as per the maximum principal stress theory of failure.

Yield strength$=280$ *MPa*.

$$\rho=0{\cdot}07644\ \text{N/cm}^3$$

$$g=980\ \text{cm/sec}^2$$

$$\frac{1}{m}=0{\cdot}3.$$

Solution. Angular pitch of blades$=3°$

Number of blades $=\dfrac{360}{3}=120$

Weight of each blade	$=0{\cdot}32$ kg
Total weight	$=0{\cdot}32\times120$
Effective radius	$=40$ cm

Say the rotational speed $=\omega$ radians/second

Centrifugal force on the periphery due to blades

$$=\frac{(120\times0{\cdot}32)}{980}\omega^2\times40=1{\cdot}567\,\omega^2\text{ kg}$$

$$=1{\cdot}567\times9{\cdot}8\,\omega^2=15{\cdot}36\,\omega^2\text{ N}$$

Resisting area $=\pi\times75\times4=942{\cdot}48$ cm^2

Radial stress at the periphery of the disc

$$=\frac{15{\cdot}36\times\omega^2}{942{\cdot}48}=\text{N}=1{\cdot}63\times10^{-2}\,\omega^2\text{ N/cm}^2$$

Radial stress at the inner radius, $7{\cdot}5$ cm $=0$

Now radial stress $\quad p_r=\frac{A}{2}+\frac{B}{r^2}-\frac{3m+1}{8m}\times\frac{\rho\omega^2r^2}{g}$

Using the boundary conditions

$$0=\frac{A}{2}+\frac{B}{7{\cdot}5^2}-\frac{3{\cdot}3}{8}\times\frac{0{\cdot}07644\,\omega^2\times7{\cdot}5^2}{980}$$

$$1{\cdot}63\times10^{-2}\omega^2=\frac{A}{2}+\frac{B}{37{\cdot}5^2}-\frac{3{\cdot}3}{8}\times\frac{0{\cdot}07644\,\omega^2\times37{\cdot}5^2}{980}$$

or
$$\frac{A}{2}+\frac{B}{56{\cdot}25}=0{\cdot}181\times10^{-2}\,\omega^2$$

$$\frac{A}{2}+\frac{B}{1406{\cdot}25}=4{\cdot}525\times10^{-2}\,\omega^2+1{\cdot}63\times10^{-2}\,\omega^2=6{\cdot}155\times10^{-2}\,\omega^2$$

From these equations

$$\frac{B}{1406{\cdot}25}-\frac{B}{56{\cdot}25}=5{\cdot}974\times10^{-2}\times\omega^2$$

$$B=-\frac{5{\cdot}974\times10^{-2}\,\omega^2\times1406{\cdot}25\times56{\cdot}25}{1350}$$

$$=-3{\cdot}5\,\omega^2$$

$$\frac{A}{2}=0{\cdot}181\times10^{-2}\,\omega^2-\frac{B}{56{\cdot}25}$$

$$=0{\cdot}181\times10^{-2}\,\omega^2=6{\cdot}22\times10^{-2}\,\omega^2=6{\cdot}401\times10^{-2}\,\omega^2$$

Maximum stress occurs at the inner radius, $r=7{\cdot}5$ cm

$$f_{\theta max}=6{\cdot}401\times10^{-2}\,\omega^2=\frac{B}{7{\cdot}5^2}-\frac{m+3}{8m}\cdot\frac{\rho\omega^2\times7{\cdot}5^2}{9}$$

$$=6{\cdot}401\times10^{-2}\,\omega^2+6{\cdot}22\times10^{-2}\,\omega^2-\frac{1{\cdot}9}{8}\times\frac{0{\cdot}07644\times\omega^2\times7{\cdot}5^2}{980}$$

$$=12{\cdot}621\times10^{-2}\,\omega^2-0{\cdot}1\times10^{-2}\,\omega^2=12{\cdot}521\times10^{-2}\,\omega^2$$

$=280$ *MPa* (as per the maximum principal stress theory)

$=280\times10^6\times1$ N/m^2

$=280{,}00$ N/cm^2

So $\quad 12{\cdot}521\times10^{-2}\,\omega^2=280{,}00$

$$\omega^2=\frac{280}{12{\cdot}521}\times10^4=22{\cdot}362\times10^4$$

$$\omega=472{\cdot}89 \text{ radians/second or } N=4515 \text{ rpm.}$$

Problem 18·12. A steel disc of uniform thickness t having a central hole of radius R_1 and outer radius R_2 is shrunk on a shaft producing a radial pressure p at the common surface, when the assembly is stationery. Now the assembly is rotated at an angular speed ω radians/second. The coefficient of friction between the shaft and the disc is μ. Show that maximum power is transmitted when $\omega=\frac{\omega_0}{\sqrt{3}}$, where ω_0 is the angular velocity at which the radial pressure at the common surface becomes zero and show that maximum power transmitted is $2{\cdot}4184\ \mu p\omega_0 R_1^2 t$ Watt.

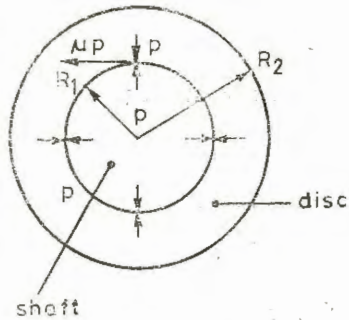


Fig. 18·12

Solution. Fig. 18·12 shows a disc of outer radius R_2 fitted over a shaft of radius R_1. When the shaft and the disc are stationery, the radial pressure at the common surface is p.

Hoop strain at the common surface when assembly is stationery

Hoop stress in disc at $\quad R_1=p\times\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$ (tensile)

Hoop strain in disc at $\quad R_1=\frac{p}{E}\ \frac{R_2^2+R_1^2}{R_2^2-R_1^2}+\frac{p}{mE}$ (because p is compressive)

Hoop strain in shaft at $\quad R_1=-\frac{p}{E}+\frac{p}{mE}$ (compressive)

(because the hoop stress in shaft is also p compressive)

Refer to the formulae of thick cylinders.

$$\epsilon_\theta\text{, total hoop strain}\quad =\frac{p}{E}\ \frac{R_2^2+R_1^2}{R_2^2-R_1^2}+\frac{p}{mE}+\frac{p}{E}-\frac{p}{mE}$$

$$=\frac{p}{E}\left[\frac{R_2^2+R_1^2+R_2^2-R_1^2}{R_2^2-R_1^2}\right]=\frac{2p\ R_2^2}{R_2^2-R_1^2} \qquad \ldots(1)$$

Hoop strain at the common surface when assembly is rotating

Say angular speed $=\omega$

Junction pressure $=p'$

Due to p', hoop stress in disc at $R_1=p'\ \frac{R_2^2+R_1^2}{R_2^2-R_1^2}$ (tensile)

hoop stress in shaft at $R_1=-p'$ (compressive)

Due to ω, hoop stress in disc at R_1

$$=\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{g}(R_1^2+R_2^2)+\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{g}\times R_2^2-\frac{m+3}{8m}\times\frac{\rho\omega^2}{g}\times R_1^2$$

$$=\frac{3m+1}{8m}\frac{\rho\omega^2}{g}\left[2R_2^2+R_1^2-\frac{m+3}{3m+1}\cdot R_1^2\right]$$

Total hoop stress in disc at R_1

$$=\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{g}\left[2R_2^2+R_1^2-\frac{m+3}{3m+1}\cdot R_1^2\right]+p'\frac{R_2^2+R_1^2}{R_2^2-R_1^2}$$

Hoop strain in disc at R_1

$$=\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{gE}\left[2R_2^2+R_1^2-\frac{m+3}{3m+1}\cdot R_1^2\right]+\frac{p'}{E}\frac{R_2^2+R_1^2}{R_2^2-R_1^2}+\frac{p'}{mE}$$

Due to ω, hoop stress in shaft at R_1

$$=\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{g}R_1^2-\frac{m+3}{8m}\cdot\frac{\rho\omega^2}{g}\cdot R_1^2$$

$$=\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{g}\left[R_1^2-\frac{m+3}{3m+1}\cdot R_1^2\right]$$

Total hoop stress in shaft at R_1

$$=\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{g}\left[R_1^2-\frac{m+3}{3m+1}\cdot R_1^2\right]-p'$$

Hoop strain in shaft at R_1

$$=\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{gE}\left[R_1^2-\frac{m+3}{3m+1}R_1^2\right]-\frac{p'}{E}+\frac{p'}{mE}$$

Since p' is compressive

Total hoop strain at R_1

$$\epsilon_\theta'=\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{gE}\left[2R_2^2+R_1^2-\frac{m+3}{3m+1}\cdot R_1^2\right]+\frac{p'}{E}\cdot\frac{R_2^2+R_1^2}{R_2^2-R_1^2}+\frac{p'}{mE}$$

$$-\frac{3m+1}{8m}\frac{\rho\omega^2}{gE}\left[R_1^2-\frac{m+3}{3m+1}\cdot R_1^2\right]+\frac{p'}{E}-\frac{p'}{mE}$$

$$=\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{gE}\left[2R_2^2+R_1^2-\frac{m+3}{3m+1}\cdot R_1^2-R_1^2+\frac{m+3}{3m+1}R_1^2\right]+\frac{p'}{E}\times\frac{2R_2^2}{R_2^2-R_1^2}$$

$$=\frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{gE}\times 2R_2^2+\frac{p'}{E}\cdot\frac{2R_2^2}{R_2^2-R_1^2} \qquad \ldots(2)$$

Equating $\epsilon_\theta = \epsilon_\theta'$ we get

$$\frac{p}{E} \times \frac{2R_2^2}{(R_2^2-R_1^2)} = \frac{p'}{E} \times \frac{2R_2^2}{(R_2^2-R_1^2)} + \frac{3m+1}{8m} \cdot \frac{\rho\omega^2}{gE} \times 2R_2^2$$

$$(p-p') = \frac{3m+1}{8m} \cdot \frac{\rho\omega^2}{g} (R_2^2-R_1^2)$$

Now when $p'=0$, speed is ω_0

So $$p = \frac{3m+1}{8m} \cdot \frac{\rho\omega_0^2}{g} (R_2^2-R_1^2)$$

or $$p' = \frac{3m+1}{8m} \cdot \frac{\rho}{g} (R_2^2-R_1^2)(\omega_0^2-\omega^2) = K(\omega_0^2-\omega^2)$$

Coefficient of friction between the shaft and disc $=\mu$

Power transmitted per revolution

$$=(\mu p' \times 2\pi R_1 \times t) R_1 \times \omega = \mu \times 2\pi R_1^2 t)(\omega p')$$

$$P = (\mu \times 2\pi \times R_1^2 t)(K)(\omega_0^2\omega - \omega^3)$$

For maximum power

$$\frac{dP}{d\omega} = 0 = (2\pi\mu R_1^2 t)(K)(\omega_0^2 - 3\omega^2)$$

or $$3\omega^2 = \omega_0^2$$

$$\omega = \frac{\omega_0}{\sqrt{3}} \text{ for maximum power transmission}$$

$$P_{max} = 2\mu R_1^2.\pi t K\left(\omega_0^2 \times \frac{\omega_0}{\sqrt{3}} - \frac{\omega_0}{3\sqrt{3}}\right)$$

$$= 2\mu R_1^2 \pi t K \left(\frac{2\omega_0^3}{3\sqrt{3}}\right)$$

$$= \frac{4}{3\sqrt{3}} \mu\pi R_1^2 t K(\omega_0^2)\omega_0$$

But $$\omega_0 = \frac{8m}{3m+1} \cdot \frac{g}{\rho(R_2^2-R_1^2)} \quad p = \frac{p}{K}$$

$$P_{max} = \frac{4}{3\sqrt{3}} \times \mu\pi R_1^2 t \cdot \frac{K}{K} p \cdot \omega_0$$

$$= \frac{4}{3\sqrt{3}} \times \mu\pi R_1^2 t p \omega_0$$

$= 2{\cdot}4184 \mu R_1^2 t p \omega_0$ Nm if ρ is in N/m^3

$= 2{\cdot}4184 \mu.R_1^2.t p \omega_0$ Watt

SUMMARY

1. For a thin ring rotating at angular speed ω, mean radius R,

Circumferential stress, $f_c = \frac{\rho V^2}{g} = \frac{\rho\omega^2 R^2}{g}$

ρ = weight density, V = linear velocity $=\omega R$

2. For a hollow thin disc rotating at angular speed ω, the radial and circumferential stresses are

$$p_r = \frac{A}{2} + \frac{B}{r^2} - \frac{3m+1}{8m} \cdot \frac{\rho\omega^2 r^2}{g}$$

$$f_\theta = \frac{A}{2} - \frac{B}{r^2} - \frac{m+3}{8m} \cdot \frac{\rho\omega^2 r^2}{g} \quad \text{where} \quad \frac{1}{m} = \text{Poisson's ratio}$$

If R_1=inner radius and R_2=outer radius

$p_r=0$ at $r=R_1$ and R_2, constants A and B are determined

$$\frac{A}{2} = \frac{3m+1}{8m} \cdot \frac{\rho\omega^2}{g} (R_1^2+R_2^2) \qquad B = -\frac{3m+1}{8m} \cdot \frac{\rho\omega^2}{g} R_1^2 . R_2^2$$

$$f_{\theta\,max} = \frac{\rho\omega^2}{g}\left[\frac{3m+1}{8m}(2R_2^2+R_1^2) - \frac{m+3}{8m} . R_1^2\right] \text{at inner radius}$$

$$p_{r\,max} = \frac{\rho\omega^2}{g} \frac{3m+1}{8m} (R_2-R_1)^2 \quad \text{at} \quad r=\sqrt{R_1R_2}$$

For a solid disc, $R_1=0$, $R_2=R$ expressions will be

$$p_r = \frac{A}{2} - \frac{3m+1}{8m} \cdot \frac{\rho\omega^2 r^2}{g}$$

$$f_\theta = \frac{A}{2} - \frac{m+3}{8m} \cdot \frac{\rho\omega^2 r^2}{g}$$

$$p_r=0 \quad \text{at} \quad r=R$$

$$\frac{A}{2} = \frac{3m+1}{8m} \cdot \frac{8\omega^2 R^2}{g}$$

$$f_{\theta\,max} = p_{r\,max} = \frac{3m+1}{8m} \cdot \frac{\rho\omega^2 R^2}{g} \quad \text{t the centre}$$

3. **Disc of uniform strength**

Thickness at any radius, $t = t_0\, e^{-\frac{\rho\omega^2 r^2}{2fg}}$

where t_0=thickness at the centre

f=constant strength throughout the disc

4. For rotating long hollow cylinders, radial and hoop stresses are

$$p_r = \frac{A}{2} + \frac{B}{r^2} - \frac{3m-2}{8(m-1)} \cdot \frac{\rho\omega^2 r^2}{g}$$

$$f_\theta = \frac{A}{2} - \frac{B}{r^2} - \frac{(m+2)}{8(m-1)} \frac{\rho\omega^2 r^2}{g}$$

$p_r=0$ at $r=R_1$ and R_2 and constants A and B are determined.

$$p_{r\,max} = \frac{3m-2}{8(m-1)} \frac{\rho\omega^2}{g} (R_2-R_1)^2 \text{ at } r=\sqrt{R_1R_2}$$

$$f_{\theta\,max} = \frac{3m-2}{8(m-1)} \frac{\rho\omega^2}{g} (2R_2^2+R_1^2) - \frac{m+2}{8(m-1)} \frac{\rho\omega^2}{g} R_1^2 \text{ at } R_1$$

5. Temperature stresses in thin discs

Radial stress, $$p_r = \frac{A}{2} + \frac{B}{r^2} - \frac{3m+1}{8m} \cdot \frac{\rho\omega^2 r^2}{g} - \frac{E\alpha}{r^2}\int Tr\,dr$$

where T=temperature

E=Young's modulus of elasticity

α=coefficient of linear expansion

Circumferential stress, $$f_\theta = \frac{A}{2} - \frac{B}{r^2} - \frac{m+3}{8m} \cdot \frac{\rho\omega^2 r^2}{g} - E\,\alpha\,T + \frac{E\alpha}{r^2}\int Tr\,dr$$

MULTIPLE CHOICE QUESTIONS

1. A thin rim is rotating about its axis. The linear velocity at the periphery is 10 m/sec. If the density of the material is 0·098 N/cm², the maximum stress developed in the rim
   (*a*) 0·1 N/mm² (*b*) 0·2 N/mm²
   (*c*) 0·5 N/mm² (*d*) 1 N/mm²

2. A thin hollow disc of inner radius 2 cm and outer radius 8 cm is rotating at a high speed, ω radians/second. The maximum radial stress occurs at the radius
   (*a*) 2 cm (*b*) 4 cm
   (*c*) 6 cm (*d*) 8 cm

3. A thin solid disc of diameter 40 cm is rotating at N r.p.m. If the maximum radial stress developed is 200 N/mm², then the maximum hoop developed will be
   (*a*) 200 N/mm² (*b*) 250 N/mm²
   (*c*) 300 N/mm² (*d*) 500 N/mm²

4. A long cylinder of inner radius R_1 and outer radius R_2 is rotating at angular speed ω radians/second. If ρ is the weight density, g=acceleration due to gravity, $1/m$=Poisson's ratio, then expression for the maximum radial stress is
   (*a*) $\frac{3m-2}{8(m-1)} \cdot \frac{\rho\omega^2}{g} \cdot (R_2^2 - R_1^2)$ (*b*) $\frac{3m-2}{8(m-1)} \frac{\rho\omega^2}{g} (R_2 - R_1)^2$
   (*c*) $\frac{3m-2}{8(m-1)} \frac{\rho\omega^2}{g} (R_2^2 + R_1^2)$ (*d*) None of the above

5. A solid thin disc is rotating about its axis at angular speed ω radians/second. If the expression $\rho\omega^2/g$=0·8 kg/cm⁴, and radius of the disc is 50 cm, the radial stress at outer periphery is
   (*a*) 2000 kg/cm² (*b*) 1000 kg/cm²
   (*c*) 500 kg/cm² (*d*) None of the above

6. A thin disc of inner radius 5 cm and outer radius 25 cm is shrunk on a solid shaft of diameter 5 cm. If the junction pressure between the disc and the shaft is 50 N/mm², then hoop stress developed in the shaft at radius 5 cm.
   (*a*) 100 N/mm² (*b*) 75 N/mm²
   (*c*) 50 N/mm² (*d*) 25 N/mm²

7. A thin disc of inner radius 5 cm and outer radius 25 cm is shrunk on a solid shaft of diameter 5 cm. If the junction pressure between the disc and shaft is 60 N/mm², then hoop stress developed at the periphery of the disc
   (*a*) 5 N/mm² (*b*) 6 N/mm²
   (*c*) 7·5 N/mm² (*d*) 10 N/mm².

8. A thin disc of inner radius R_1 and outer radius R_2 is rotating at 1000 r.p.m. The maximum stress developed is 70 N/mm². If the yield strength of the material is 280 N/mm². The speed at which the disc will fail according to the maximum principal stress theory of failure is

(*a*) 1000 rpm (*b*) 2000 rpm

(*c*) 3000 rpm (*d*) 4000 rpm

ANSWERS

1. (*d*)	2. (*b*)	3. (*a*)	4. (*b*)
5. (*d*)	6. (*c*)	7. (*a*)	8. (*b*)

EXERCISES

18·1. A composite ring is made of steel and brass rings. The diameter of the ring at the common surface is 120 cm. The radial thickness of both the rings is 10 mm, and axial width is 20 mm each. Determine the stresses set up in steel and brass rings if composite ring is rotating at 2500 r.p.m.

$E_{steel}=2080$ tonnes/cm² $E_{brass}=1040$ tonnes/cm²

$\rho_{steel}=0{\cdot}0078$ kg/cm³ $\rho_{brass}=0{\cdot}00883$ kg/cm³

$g=9{\cdot}8$ m/sec²

[**Ans.** 2797·5 kg/cm² (in steel ring), 1398·75 kg/cm² (in brass)]

18·2. A circular saw 2 mm thick × 36 cm diameter is secured upon a 4 cm diameter shaft. The material of the saw has density 0·0077 kg/cm³, and Poisson's ratio = 0·285. Determine the permissible speed if the allowable hoop stress is 200 N/mm² and find the maximum value of the radial stress. $g=981$ cm/sec².

[**Ans.** 9425 RPM ; 78·8 N/mm²]

18·3. Determine the stresses due to the centrifugal force in a rotor with an outer radius 65 cm and radius of the hole 10 cm. The outer portion of the rotor is cut by slots 25 cm deep for windings. The rotor is of steel and rotates at 1800 r.p.m. The weight of the windings in the slots is the same as that of the material removed.

Given : $\rho=0{\cdot}0078$ kg/cm³, $g=981$ cm/sec²

$1/m=0{\cdot}3$ [**Ans.** 1485·3 kg/cm²]

18·4. A thin circular disc of external radius 300 mm is forced on to a rigid shaft of radius 100 mm such that the radial pressure at the junction of the two is 50 N/mm². The assembly rotates at 300 radians/second. What is the final junction pressure between the two.

[**Ans.** 15·62 N/mm²]

18·5. A disc of inside and outside diameters 20 cm and 50 cm, is made up in two parts, which are shrunk together. the common diameter being 35 cm. The junction pressure at the common surface is 40 N/mm². A what speed, the hoop stress at the inner and outer radii of the disc will be equal ?

$g=981$ cm/sec², $\rho=0{\cdot}0078$ kg/cm³

Poisson's ratio = 0·29 [**Ans.** 8223 RPM]

18·6. A thin hollow steel disc of outer radius 50 cm is shrunk over another solid disc of same thickness and radius 10 cm, such that the shrinkage pressure at the common surface is 100 N/mm². At what speed will the disc be loosened on the shaft ? Neglect the strain in the shaft.

$\rho=0{\cdot}07644$ N/cm³ $E=2\times10^7$, N/cm²

$g=980$ cm/sec² $1/m=0{\cdot}3$ [**Ans.** 2018 RPM]

18·7. A thin hollow steel disc of outer radius 50 cm is shrunk on another solid disc of the same thickness but radius 10 cm, such that the junction pressure between the two is p. What should be the minimum value of junction pressure p so that outer disc may not be loosened over the inner disc at an angular speed of 2500 r.p.m.

$\rho=0{\cdot}0078$ kg/cm³, $1/m=0{\cdot}3$, $g=980$ cm/sec²

[**Ans.** 540·06 kg/cm²]

18·8. A steel ring is shrunk on a cast iron hollow disc. Find the change in shrink fit pressure produced by the inertia forces at 2500 r.p.m. If $R_1=5$ cm, $R_2=15$ cm and $R_3=25$ cm

$E_{steel}=210$ kN/mm² $\quad E_{CI}=110$ kN/mm²

$\rho_{steel}=0{\cdot}0019$ kg/cm³ $\quad \rho_{CI}=0{\cdot}0072$ kg/cm³

$1/m$ for steel and C.I. = 0·3

[**Ans.** 5·42 N/mm²

18·9. A thin steel disc 50 cm diameter is shrunk over a solid shaft of diameter 10 cm. The shrinkage allowance is 1/1000.

(*a*) What are the maximum stresses in the disc and the shaft when stationery ?

(*b*) At what speed the disc will be loosened on the shaft ?

(*c*) What will be maximum stress in the disc at half the rotational speed calculated in part (*b*) ?

[**Ans.** (*a*) 104 N/mm², −96 N/mm² ; 7013 RPM ; 128·43 N/mm²]

18·10. A steel disc of a turbine is to be designed so that the radial and circumferential stresses are to be constant and each equal to 800 kg/cm², between the radii of 36 cm and 20 cm when running at 3500 r.p.m. If the axial thickness at the outer radius of this zone is 15 mm, what should be the thickness at the inner radius ?

$\rho=0{\cdot}0079$ kg/cm³ $\quad g=980$ cm/sec²

[**Ans.** 27·5 mm]

18·11. A steel rotor disc of a steam turbine has a uniform thickness of 5 cm. The outer diameter of the disc is 60 cm and inner diameter 10 cm. There are 100 blades each of weight 0·3 kg fixed evenly around the periphery of the disc at an effective radius of 33 cm.

Yield strength of the material = 300 *MPa*

$\rho=0{\cdot}0078$ kg/cm³. Determine the maximum rotational speed as per the maximum shear stress theory of failure. [**Ans.** 5835 RPM]

19

Bending of Curved Bars

In chapter 8 on Theory of simple bending we assumed the beam to be initially straight before the application of a bending moment and derived the relationship $M/I=E/R=f/y$, and studied about the stresses developed and deflections in beams. But in this chapter we will study the effect of bending moment on bars of large initial curvature.

19·1. STRESSES IN A CURVED BAR

Fig. 19·1 shows a portion of a curved bar of initial radius of curvature R, subtending an angle θ at the centre of curvature O. This curved bar is subjected to a bending moment M

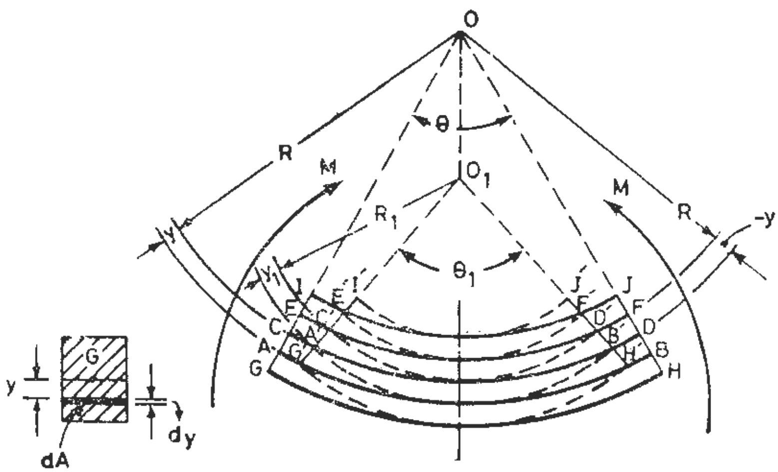


Fig. 19·1

tending to increase the curvature of the bar. To find out the stresses developed in the bar, let us derive a relationship between bending moment M, radius of curvature R and dimensions of the section of the bar, for which following assumptions are taken

1. The transverse sections of the bar which are plane before the application of a bending moment remain plane after bending.
2. The material obeys Hooke's law and stress is directly proportional to strain.

Consider a small portion $IJHG$ of the curved bar in its initial unstrained position, where AB is a layer at a radial distance of y from the centroidal layer CD *i.e.*, a layer passing through the centroidal axis of the sections. At layer AB, stresses due to the bending moment M are to be determined.

After the application of the bending moment, say $I'J'H'G'$ is the final shape of the bar. The centroidal layer is now $C'D'$ and the layer AB takes the new position $A'B'$. Say the final centre of curvature is O_1 and final radius of curvature is R_1 and θ_1 is the angle subtended by the length $C'D'$ at the centre.

Say f is the stress in the strained layer $A'B'$ under the bending moment M tending to increase the curvature (or tending to reduce the radius of curvature), and ϵ is the strain in the same layer.

Strain $$\epsilon=\frac{A'B'-AB}{AB}=\frac{(R_1+y_1)\theta_1-(R+y)\theta}{(R+y)\theta}$$

where y_1 = distance between centroidal layer $C'D'$ and layer $A'B'$, in the final position

or $$\epsilon=\frac{R_1+y_1}{R+y}\cdot\frac{\theta_1}{\theta}-1$$

Moreover, ϵ_0 = strain the centroidal layer *i.e.*, when $y=0$

$$=\frac{R_1}{R}\;\frac{\theta_1}{\theta}-1$$

or $$1+\epsilon=\frac{R_1+y_1}{R+y}\cdot\frac{\theta_1}{\theta} \qquad \ldots(1)$$

and $$1+\theta_0=\frac{R_1}{R}\times\frac{\theta_1}{\theta} \qquad \ldots(2)$$

Dividing equation (1) by equation (2)

$$\frac{1+\epsilon}{1+\epsilon_0}=\frac{R_1+y_1}{R+y}\cdot\frac{R}{R_1}=\frac{1+\dfrac{y_1}{R_1}}{1+\dfrac{y}{R}}$$

or $$\epsilon=(1+\epsilon_0)\frac{\left(1+\dfrac{y_1}{R_1}\right)}{1+\dfrac{y}{R}}-1$$

or $$\epsilon=\frac{\epsilon_0\dfrac{y_1}{R_1}+\dfrac{y_1}{R_1}+\epsilon_0-\dfrac{y}{R}}{1+\dfrac{y}{R}}$$

Now $y_1 \simeq y$ — considering that change in thickness is negligible.

So strain, $$\epsilon=\frac{\epsilon_0\dfrac{y}{R_1}+\dfrac{y}{R_1}+\epsilon_0-\dfrac{y}{R}+\epsilon_0\dfrac{y}{R}-\epsilon_0\dfrac{y}{R}}{1+\dfrac{y}{R}}$$

$\left(\text{Here, we have added and substracted the term } \frac{\epsilon_0 y}{R}\right)$

or
$$\epsilon = \epsilon_0 + \frac{(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right) y}{1+\frac{y}{R}} \quad ...(3)$$

The stress in the layer AB (which is tensile as is obvious from the diagram, *i.e.*, layers below the centroidal layer are in tension and layers above the centroidal layer are in compression for the bending moment shown.

Stress,
$$f = E\epsilon = E\left[\epsilon_0 + \frac{(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right) y}{1+\frac{y}{R}}\right]$$

where E=Young's modulus of the material

Total force on the section, $F = \int f \cdot dA$

Considering a small strip of elementary area dA, at a distance of y from the centroidal layer CD.

or
$$F = E\int \epsilon_0 dA + E\int \frac{(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right) y \cdot dA}{1+\frac{y}{R}}$$

$$= E\int \epsilon_0 dA + E(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\int \frac{y}{1+\frac{y}{R}}\, dA$$

$$= E\epsilon_0 A + E(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\int \frac{y}{1+\frac{y}{R}} \cdot dA \quad ...(4)$$

where A is the area of cross section of the bar.

Now the total resisting moment will be given by

$$M = \int f \cdot y \cdot dA$$

$$= E\int \epsilon_0 \cdot y\, dA + E\int \frac{(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)}{1+\frac{y}{R}} \cdot y^2\, dA$$

$$= E\epsilon_0 \times 0 + E\,(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\int \frac{y^2}{1+y/R}\, dA$$

Because $\int y dA = 0$, *i.e.*, first moment of any area about its centroidal layer is zero.

So
$$M = E\,(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\int \frac{y^2}{1+y/R} \cdot dA$$

Let us assume $\int \frac{y^2}{1+y/R}\, dA = Ah^2$, a quantity which depends upon the disposition of section and the radius of curvature.

Therefore $$M = E\,(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right) Ah^2 \qquad ...(5)$$

From equation (4)

$$F = E\,\varepsilon_0\,A + E\,(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\int \frac{y}{1+y/R}\cdot dA$$

$=0$, because the bar is in equilibrium and the net force on the section is zero.

Now $$\int \frac{y}{1+y/R}\, dA = \int \frac{Ry - y^2 + y^2}{R+y}\, dA = \int y\,dA - \frac{1}{R}\int \frac{Ry^2}{R+y}\, dA$$

$$= 0 - \frac{1}{R}\int \frac{y^2}{1+y/R}\, dA = -\frac{Ah^2}{R}$$

Considering the equation (4) again

$$E\,\varepsilon_0\,A = E\,(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\times\frac{Ah^2}{R} \quad \text{(since } F=0)$$

or $$\frac{\varepsilon_0\,R}{h^2} = (1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right) \qquad ...(6)$$

Substituting this value of $(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)$ in equation (5)

$$M = E\left(\frac{\varepsilon_0\,R}{h^2}\right) Ah^2 = R\,\varepsilon_0\,EA$$

or $$\varepsilon_0 = \frac{M}{EAR} \qquad ...(7)$$

Substituting the value of ε_0 in the equation for stress

$$f = E\,.\,\frac{M}{EAR} + E\,.\,\frac{\varepsilon_0\,.\,R}{h^2}\times\frac{y}{1+y/R}$$

$$f = \frac{M}{AR} + E\times\frac{y}{1+y/R}\times\frac{R}{h^2}\times\varepsilon_0$$

Putting the value of ε_0 again

$$f = \frac{M}{AR} + E\times\frac{y}{1+y/R}\times\frac{R}{h^2}\times\frac{M}{EAR}$$

$$= \frac{M}{AR} + \frac{M}{AR}\times\frac{Ry}{1+y/R}\times\frac{1}{h^2}$$

$$= \frac{M}{AR}\left[1 + \frac{R^2}{h^2}\times\frac{y}{R+y}\right] \text{...(tensile)} \qquad ...(8)$$

On the other side of the centroidal layer y will be negative as for the layer EF shown in the Fig.

f' = stress when y is negative

$$= \frac{M}{AR}\left[\frac{y}{R-y} \times \frac{R^2}{h^2} - 1\right] \text{compressive} \qquad \ldots(9)$$

The expression given in equations (8) and (9) are for the stresses due to the bending moment which tends to increase the curvature. If the bending moment tends to straighten the bar or tends to decrease the curvatue, then $\theta_1 < \theta$ and $R_1 > R$ and the stresses will be reversed

Bending moment tending to decrease the curvature

For y to be positive

$$f = \frac{M}{AR}\left[1 + \frac{R^2}{h^2} \times \frac{y}{R+y}\right] \text{ compressive} \qquad \ldots(10)$$

On the other side of the centroidal layer, where y is negative

$$f' = \frac{M}{AR}\left[\frac{y}{R-y} \times \frac{R^2}{h^2} - 1\right] \text{ (tensile)} \qquad \ldots(11)$$

19·2. Ah² FOR RECTANGULAR SECTION

Fig. 19·2 shows the rectangular cross section of breadth B and depth D of a curved bar with radius of curvature R, *i.e.* the radius from the centre of curvature C to the centroid G of the section. Consider a strip of thickness dy at a distance y from the centroidal layer. Area of the strip, $dA = B\,dy$

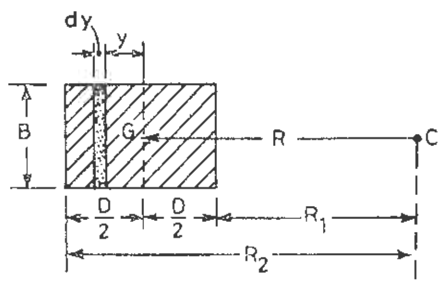


Fig. 19 2

$$Ah^2 = \int \frac{Ry^2}{R+y}\, dA = \int_{-D/2}^{D/2} \frac{RBy^2}{R+y}\, dy$$

$$= \int_{-D/2}^{+D/2} ByR\,dy - \int_{-D/2}^{+D/2} BR^2 dy + \int_{-D/2}^{+D/2} \frac{BR^3}{R+y}\, dy$$

$$Ah^2 = 0 - R^2BD + BR^3 \; l_n \; \frac{R+D/2}{R-D/2}$$

where $A = BD$

$$h^2 = -R^2 \times \frac{BD}{BD} + \frac{BR^3}{BD} \; l_n \; \frac{2R+D}{2R-D}$$

$$\frac{h^2}{R^2} = \frac{R}{D} \; l_n \; \frac{2R+D}{2R-D} - 1$$

$$= \frac{R}{D} \; l_n \; \frac{R_2}{R_1} - 1 = \frac{BR}{A} \; l_n \; \frac{R_2}{R_1} - 1 = \frac{R}{A}\left[B \; l_n \; \frac{R_2}{R_1}\right] - 1$$

where R = Radius upto the centroidal layer

R_1 = Radius upto the inner surface of the curved bar

R_2 = Radius upto the outer surface of the curved bar.

Example 19·2-1. A circular ring of rectangular section, with a slit is loaded as shown in the Fig. 19·3. Determine the magnitude of the force P if the maximum resultant stress along the section $a-b$ is not to exceed 150 N/mm². Draw the stress distribution along ab.

Solution. Mean radius of curvature,

$$R = 11 \text{ cm}$$

Radius of curvature of inner surface,

$$R_1 = 8 \text{ cm}$$

Radius of curvature of outer surface,

$$R_2 = 14 \text{ cm}$$

Breadth $B = 4$ cm

Depth $D = 6$ cm

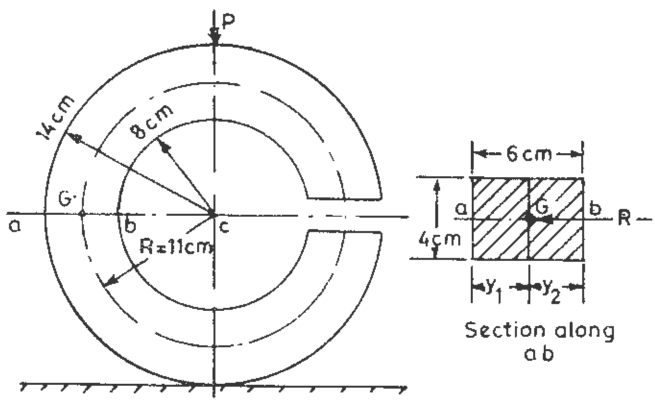


Fig. 19·3

$$\frac{h^2}{R^2} = \frac{R}{D} \; l_n \; \frac{R_2}{R_1} - 1$$

$$= \frac{11}{6} \; l_n \; \frac{14}{8} - 1 = \frac{11}{6} \times 0{\cdot}5596 - 1 = 0{\cdot}02593$$

or $$\frac{R^2}{h^2} = 38{\cdot}56$$

Maximum resultant stress will occur at the inner radius *i.e.*, at the point b.

Bending moment, $M = P \times R = 11 \; P$ N-cm

Direct stress, $$f_d = \frac{P}{4 \times 6} = \frac{P}{24} \text{ N/cm}^2 \text{ (compressive)}$$

Resultant stress at the point

$$b = \frac{M}{AR}\left(\frac{R^2}{h^2} \times \frac{y_2}{R - y_2} - 1\right) + \frac{P}{A}$$

$$y_2 = 3 \text{ cm}$$

So $$f_{max}=\frac{PR}{AR}\left(\frac{R^2}{h^2}\times\frac{y_2}{R-y_2}\right)-\frac{P}{A}+\frac{P}{A}$$

$$150\times 100=\frac{P}{A}\left(\frac{R^2}{h^2}\times\frac{y_2}{R-y_2}\right)=\frac{P}{24}\left(38{\cdot}56\times\frac{3}{11-3}\right)$$

$$P=\frac{24\times 15000\times 8}{3\times 38{\cdot}56}=24896{\cdot}26\text{ N}=24{\cdot}9\text{ kN}$$

Stress Distribution (along Gb)

$$f=\frac{P}{A}\cdot\frac{R^2}{h^2}\times\frac{y}{R-y}\text{......}y\text{ varies from 0 to 3}$$

$$=0\quad\text{at}\quad y=0\text{ cm}$$

$$=\frac{24{\cdot}9}{24}\times 38{\cdot}56\times\frac{1}{11-1}=4{\cdot}0\text{ kN/cm}^2\quad\text{at}\quad y=1\text{ cm}$$

$$=\frac{24{\cdot}9}{24}\times 38{\cdot}56\times\frac{2}{11-2}=8{\cdot}89\text{ kN/cm}^2\quad\text{at}\quad y=2\text{ cm}$$

$$=\frac{24{\cdot}9}{24}\times 38{\cdot}56\times\frac{3}{11-3}=15{\cdot}00\text{ kN/cm}^2\quad\text{at}\quad y=3\text{ cm}$$

Along Ga y varies from 0 to 3

Resultant stress, $f=\frac{M}{AR}\left(1+\frac{R^2}{h^2}\times\frac{y}{R+y}\right)-\frac{P}{A}$. Since the direct stress is compressive

$$=\frac{P}{A}\times\frac{R^2}{h^2}\times\frac{y}{R+y}$$

$$=0\quad\text{at}\quad y=0\text{ cm}$$

$$=\frac{24{\cdot}9}{24}\times 38{\cdot}56\times\frac{1}{11+1}=3{\cdot}333\text{ kN/cm}^2\quad\text{at}\quad y=1\text{ cm}$$

$$=\frac{24{\cdot}9}{24}\times 38{\cdot}56\times\frac{2}{11+2}=6{\cdot}154\text{ kN/cm}^2\quad\text{at}\quad y=2\text{ cm}$$

$$=\frac{24{\cdot}9}{24}\times 38{\cdot}56\times\frac{3}{11+3}=8{\cdot}571\text{ kN/cm}^2\quad\text{at}\quad y=3\text{ cm}$$

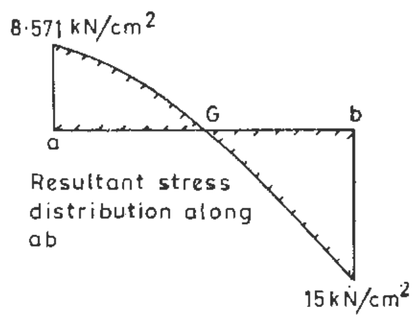


Fig. 19·4

Fig. 19·4 shows the stress distribution along the radial thickness *ab* of the section *ab* which has maximum bending moment *PR*. In this case the resultant stress at centroid *G* is zero.

Exercise 19·2-1. A curved bar of rectangular cross section 4 cm × 6 cm is subjected to a bending moment of 2 kNm, its centre line is curved to a radius of 20 cm Determine the maximum tensile and compressive stress in beam, if the bending moment tends to increase the curvature. What is the stress at the CG of the section.

Plot the stress distribution diagram to a suitable scale along any section.

[**Ans.** $+113$ N/mm², -143 N/mm², $+4·16$ N/mm² (at C.G.)]

19·3. VALUE OF h² FOR SECTIONS MADE UP OF RECTANGULAR STRIPS

Sections such as *T*, *I*, channel section and bar section are made of rectangular strips. The value of h^2 for each section can be determined by considering each strip separately. In the ease of single rectangular strip section

$$\frac{h^2}{R^2}=\frac{R}{A}\left[B\, l_n \frac{R_2}{R_1}\right]-1$$

where B is the breadth, R_2 is the radius of outer fibres and R_1 is the radius of the inner fibres of the section. Using this expression, let us determine h^2/R^2 for various sections.

(*i*) **T-section.** Fig. 19·5 shows a T section with following dimension,

Breadth of the flange $=B$

Breadth of the web $=b$

$R=$ Radius of curvature upto centroid G of the section

$R_1=$ Radius upto extreme edge of web

$R_2=$ Radius upto inner edge of flange

$R_3=$ Radius upto outer edge of flange

$A=$ Area of cross section of T section

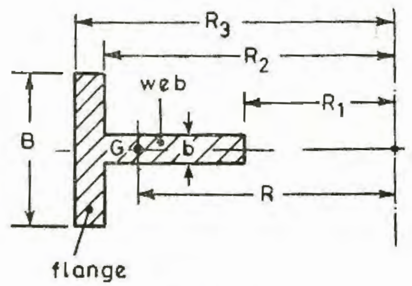


Fig. 19·5

$$=B\,(R_3-R_2)+b\,(R_2-R_1)$$

$$\frac{h^2}{R^2}=\frac{R}{A}\left[B\, l_n \frac{R_3}{R_2}+b\, l_n \frac{R_2}{R_1}\right]-1$$

(*ii*) **I-section.** Fig. 19·6 shows an I section with flange and web of breadths B and b respectively.

$R=$ Radius of curvature upto centroid G of the section

$R_1=$ Radius upto outer edge of inner flange

$R_2=$ Radius upto inner edge of inner flange

$R_3=$ Radius upto inner edge of outer flange

$R_4=$ Radius upto outer edge of inner flange

$A=$ area of cross section $=B(R_4-R_3)+b(R_3-R_2)+B(R_2-R_1)$

$$\frac{h^2}{R^2}=\frac{R}{A}\left[B\, l_n \frac{R_4}{R_3}+b\, l_n \frac{R_3}{R_2}+B\, l_n \frac{R_2}{R_1}\right]-1.$$

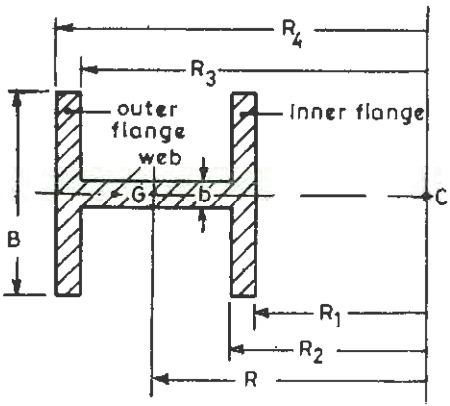


Fig. 19·6

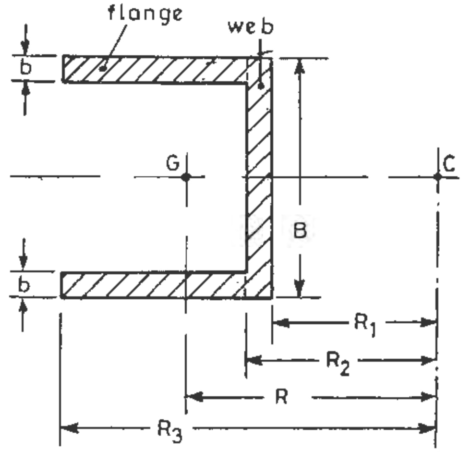


Fig. 19·7

***(iii)* Channel section.** Fig. 19·7 shows a channel section with

B=breadth of web

b=breadth of flanges

R=Radius of curvature upto centroid G of the section

R_1=Radius upto inner surface

R_2=Radius upto outer edge of web

R_3=Radius upto the outer edge of flange

A=Area of cross section$=B(R_2-R_1)+2b(R_3-R_2)$

$$\frac{h^2}{R^2}=\frac{R}{A}\left[2h\ l_n\ \frac{R_3}{R_2}+B\ l_n\ \frac{R_2}{R_1}\right]-1$$

(iv) **Box section.** Fig. 19·8 shows a box section with

B=Breadth

b=Thickness as shown

R=Radius of curvature upto the centroid G of the section

R_1=Radius upto inner surface

R_2=Radius upto inside edge as shown

R_3=Radius upto other inside edge

R_4=Radius upto outer surface of the section

A=Area of cross section
$=B\ (R_2-R_1)+2b(R_3-R_2)$
$+B(R_4-R_3)$

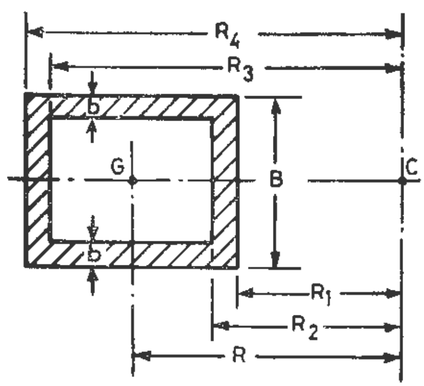


Fig. 19·8

$$\frac{h^2}{R^2}=\frac{R}{A}\left[B\ l_n\ \frac{R_2}{R_1}+2b\ l_n\ \frac{R_3}{R_2}+Bl_n\ \frac{R_4}{R_3}\right]-1.$$

Example 19 3-1. A curved beam whose centroidal line is a circular arc of 12 cm radius. The cross section of the beam is of T-shape with dimensions as shown in the Fig. 19·9. Determine the maximum tensile and compressive stresses set up by a bending moment of 6 tonne-cms ; tending to decrease the curvature.

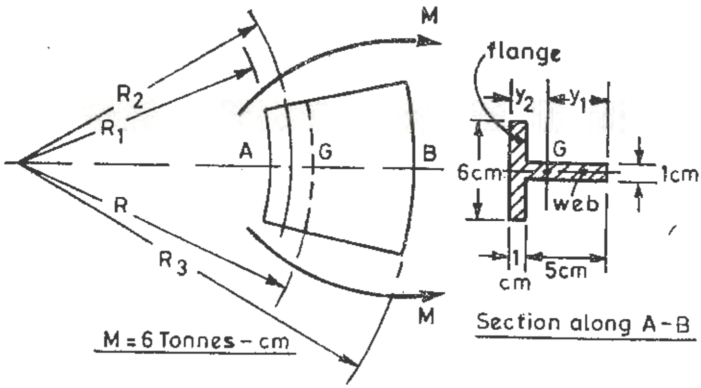


Fig. 19·9

Solution. The Figure shows the curved bar with T-section subjected to a bending moment M tending to decrease the curvature ; therefore there will be tensile stresses between to G, and compressive stresses between G to B.

Let us first calculate the distance of centroid from the outer edge of web

$$y_1 = \frac{5\times1\times2{\cdot}5+6\times1\times5{\cdot}5}{5+6} = \frac{12{\cdot}5+33}{11} = 4{\cdot}136 \text{ cm}$$

$$y_2 = 6-4{\cdot}136 = 1{\cdot}884 \text{ cm}$$

Radius of curvature, $R=12$ cm (given)

Radius upto inner surface, $R_1 = 12-1{\cdot}884 = 10{\cdot}136$ cm

Radius upto outer edge of flange, $R_2 = 11{\cdot}136$ cm.

Radius upto outer edge of web, $R_3 = R_1+6 = 10{\cdot}136+6 = 16{\cdot}136$ cm

$$B=6 \text{ cm},\ b=1 \text{ cm},\ \text{Area, } A=6\times1+1\times5=11 \text{ cm}^2$$

$$\frac{h^2}{R^2} = \frac{R}{A}\left[B\, l_n \frac{R_2}{R_1} + b\, l_n \frac{R_3}{R_2}\right]-1$$

$$\frac{h^2}{R_2} = \frac{12}{11}\left[6\times l_n \frac{11{\cdot}136}{10{\cdot}136} + 1\, l_n \frac{16{\cdot}136}{11{\cdot}136}\right]-1$$

$$= \frac{12}{11}\left[6\times0{\cdot}935+0{\cdot}3712\right]-1 = 1{\cdot}01694-1 = 0{\cdot}01694$$

$$\frac{R^2}{h^2} = 59{\cdot}03$$

Maximum compressive stress at point B

$$= \frac{M}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y_1}{R+y_1}\right] = \frac{6\times1}{11\times12}\left[1+\frac{59{\cdot}03\times4{\cdot}136}{16{\cdot}136}\right]$$

$$= 0{\cdot}733 \text{ tonne/cm}^2$$

Maximum tensile stress at point A

$$=\frac{M}{AR}\left[\frac{y_2}{R-y_2}\times\frac{R^2}{h^2}-1\right]=\frac{6}{11\times 12}\left[\frac{1\cdot884}{12-1\cdot884}\times\frac{59\cdot03}{1}-1\right]$$

$=0\cdot453$ tonne/cm^2

Example 19·3-2. Fig. 19·10 shows a press applying 200 kN force on a job. Determine the stresses at the points a and b. The section is hollow as shown.

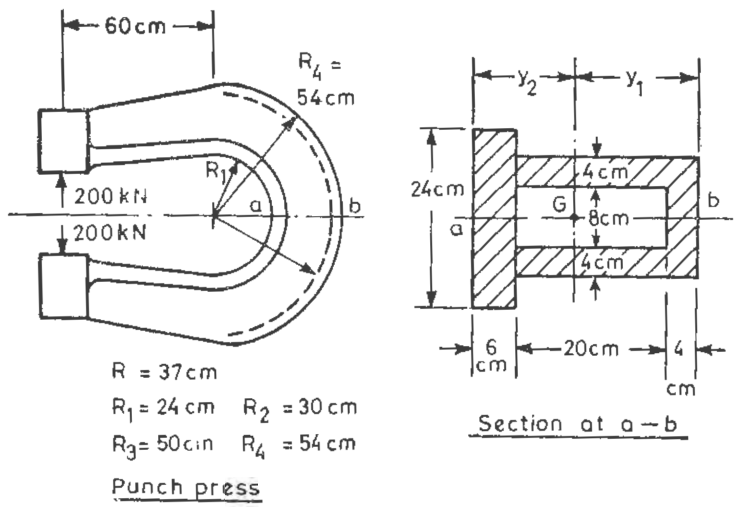


Fig. 19·10

Solution. Let us first determine the position of the centroid

$$y_1=\frac{16\times 4\times 2+2\times 20\times 14\times 4+24\times 6\times 27}{16\times 4+2\times 20\times 4+24\times 6}$$

$$=\frac{128+2240+3888}{368}=17\text{ cm}$$

$$y_2=30-17=13\text{ cm}.$$

Radius of curvature, $R=24+13=37$ cm

Area of cross section, $A=24\times 6+2\times 4\times 20+4\times 16=368$ cm^2

$$\frac{h^2}{R^2}=\frac{R}{A}\left[24\ l_n\frac{30}{24}+2\times 4\times l_n\frac{50}{30}+16\times l_n\frac{54}{50}\right]-1$$

$$=\frac{37}{368}\ [24\times 0\cdot 2232+2\times 4\times 0\cdot 5105+16\times 0\cdot 0769]-1$$

$$=\frac{37}{368}\ [5\cdot 3568+4\cdot 084+1\cdot 2304]-1$$

$$=\frac{37}{368}\times 10\cdot 6712-1=1\cdot 07292-1=0\cdot 07292$$

$$\frac{R^2}{h^2}=13\cdot 714$$

Bending moment, $M = \text{Force} \times (60 + R) = 200 \times 97$ kN cm

Direct tensile stress, $f_d = \dfrac{200}{368} = 0{\cdot}543$ kN/cm²

Bending stress due to M at a

$$= \frac{M}{AR}\left[\frac{y_2}{R - y_2} \times \frac{R^2}{h^2} - 1\right]$$

$$= \frac{200 \times 97}{368 \times 37}\left[\frac{13}{37 - 13} \times 13{\cdot}714 - 1\right]$$

$= 9{\cdot}159$ kN/cm² (tensile)

Bending stress due to M at b

$$= \frac{M}{AR}\left[1 + \frac{y_1}{R + y_1} \times \frac{R^2}{h^2}\right]$$

$$= \frac{200 \times 97}{368 \times 37}\left[1 + \frac{17}{37 + 17} \times 13{\cdot}714\right]$$

$= 7{\cdot}576$ kN/cm² (compressive)

Resultant stress at the point a

$= 9{\cdot}159 + 0{\cdot}543 = 9{\cdot}702$ kN/cm²

$= 97{\cdot}02$ N/mm² (tensile)

Resultant stress at the point b

$= 7{\cdot}576 - 0{\cdot}543 = 7{\cdot}033$ kN/cm²

$= 70{\cdot}33$ N/mm².

Exercise 19·3-1. An open ring of channel section is subjected to a compressive force of 50 kN as shown in Fig. 19·11. Determine the maximum tensile and compressive stresses along the section *ab*.

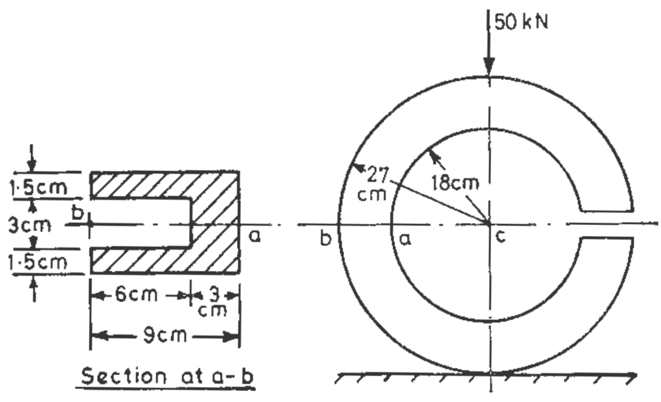


Fig. 19·11

[**Ans.** $f_a = 246{\cdot}69$ N/mm² (compressive), $f_b = 230{\cdot}24$ N/mm² (tensile)]

Exercise 19·3-2. A load $P=15$ kN is applied on a C-clamp as shown in the Fig. 19·12. Determine the stresses at the points a and b.

[**Ans.** $f_a=257·55$ N/mm² (tensile), 112·65 N/mm² (compressive)]

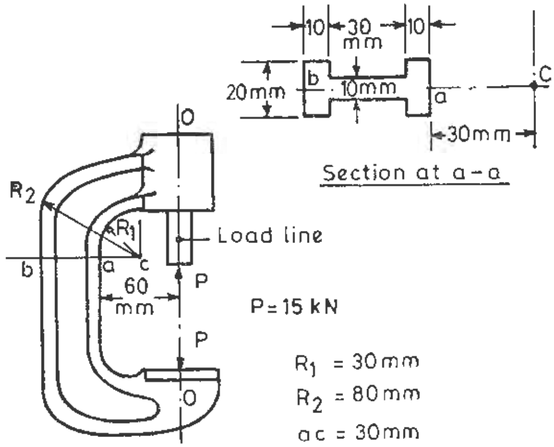


Fig. 19·12

19·4. Ah² FOR A TRAPEZOIDAL SECTION

Fig. 19·13 shows a trepezoidal section of a curved bar with breadths B_1 and B_2 and depth D and radius of curvature R. Say C is the centre of curvature, and G is the centroid of the section. Then

$$y_1=\frac{B_1+2B_2}{B_1+B_2}\times\frac{D}{3}$$

$$y_2=\frac{B_2+2B_1}{B_1+B_2}\times\frac{D}{3}$$

A=area of cross section

$$=\frac{B_1+B_2}{2}\times D$$

Consider a strip of depth dy at a distance of y from the centroidal layer.

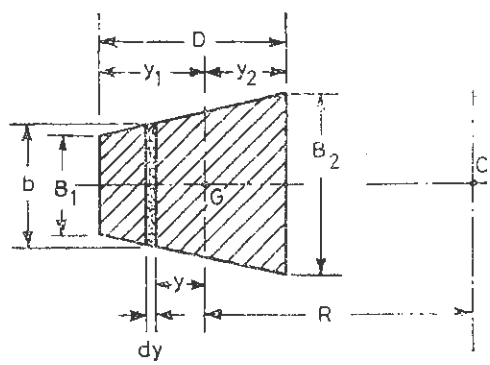


Fig. 19·13

If b=breadth of the strip

Area of the strip, $\quad dA=bdy=\left[B_1+\frac{B_2-B_1}{D}(y_1-y)\right]dy$

Now $\quad Ah^2=\int\frac{Ry^2}{R+y}dA=-AR^2+\int_{-y_2}^{y_1}\frac{R^3\,dA}{R+y}$

$$=-AR^2+R^3\int_{-y_2}^{y_1}\left[B_1+\frac{B_2-B_1}{D}(y_1-y)\right]\frac{1}{R+y}\,dy$$

$$= -AR^2 + R^3 \int_{-y_2}^{y_1} \frac{B_1}{R+y} \cdot dy + R^3 \int_{-y_2}^{y_1} \frac{B_2-B_1}{D} \cdot \frac{y_1}{R+y}\, dy$$

$$- R^3 \int_{-y_2}^{+y_1} \left(\frac{B_2-B_1}{D}\right) \frac{y}{R+y}\, dy$$

$$= -AR^2 + R^3 B_1\, ln\, \frac{R+y_1}{R-y_2} + R^3 \left(\frac{B_2-B_1}{D}\right) y_1\, ln \frac{R+y_1}{R-y_2}$$

$$- R^3 \int_{-y_2}^{y_1} \left(\frac{B_2-B_1}{D}\right) dy + R^3 \int_{-y_2}^{y_1} \left(\frac{B_2-B_1}{D}\right) \frac{R}{R+y} \cdot dy$$

$$= -AR^2 + R^3 B_1\, ln\, \frac{R+y_1}{R-y_2} + R^3 \left(\frac{B_2-B_1}{D}\right) y_1\, ln\, \frac{R+y_1}{R-y_2}$$

$$- R^3 \left(\frac{B_2-B_1}{D}\right)(y_1+y_2) + \left(\frac{B_2-B_1}{D}\right) R^4\, ln\, \frac{R+y_1}{R-y_2}$$

$$\frac{h^2}{R^2} = -1 + \frac{R}{A} B_1\, ln\, \frac{R+y_1}{R-y_2} + \frac{R}{A}\left(\frac{B_2-B_1}{D}\right) y_1\, ln \frac{R+y_1}{R-y_2}$$

$$- \frac{R}{A}(B_2-B_1) + \frac{R^2}{A}\left(\frac{B_2-B_1}{D}\right) ln\, \frac{R+y_1}{R-y_2}$$

$$\frac{h^2}{R^2} = \frac{R}{A}\left[\left\{ B_1 + \frac{B_2-B_1}{D}(y_1+R)\right\} ln \frac{R+y_1}{R-y_2} - (B_2-B_1)\right] - 1$$

Example 19·4-1. Determine the maximum compressive and tensile stresses in the critical section of a crane hook lifting a load of 5 tonnes. The dimensions of the hook are shown in the Fig. 19·14. The line of application of the load is at a distance of 8 cm from the inner fibre. (Rounding off of the corners of the cross section are not to be taken into account).

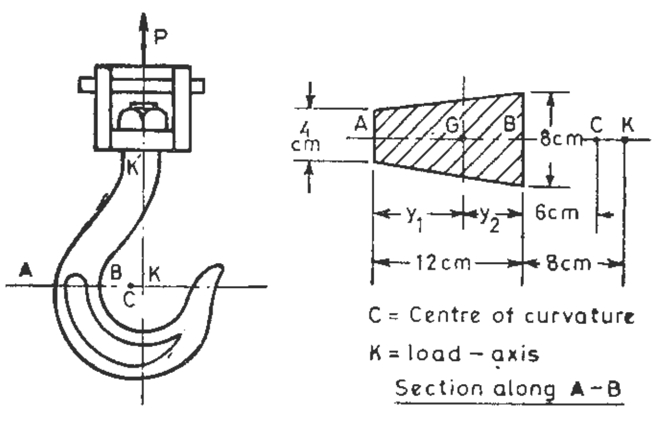


Fig. 19·14

Solution. Fig. 19·14 shows, a crane hook and the trapezoidal section. The load line KK' is away from the centre of the curvature C.

Position of CG of the section

$$y_1=\frac{B_1+2B_2}{B_1+B_2}\times\frac{D}{3} \quad \text{where} \quad B_1=4 \text{ cm} ; B_2=8 \text{ cm} ; D=12 \text{ cm}$$

$$=\frac{4+16}{4+8}\times\frac{12}{3}=\frac{20}{3} \text{ cm}$$

So $$y_2=\frac{16}{3} \text{ cm}$$

Radius of curvature, $R=6+\frac{16}{3}=\frac{34}{3}$ cm

Area of cross section, $A=\frac{B_1+B_2}{2}\times D=\frac{4+8}{2}\times 12=72 \text{ cm}^2$

Now $$\frac{h^2}{R^2}=-1+\frac{R}{A}\left[\left\{B_1+\frac{B_2-B_1}{D}(y_1+R)\right\} ln \frac{R+y_1}{R-y_2}-(B_2-B_1)\right]$$

Substituting the values

$$\frac{h^2}{R^2}=-1+\frac{34}{3\times 72}\left[\left\{4+\frac{8-4}{12}\left(\frac{34}{3}+\frac{20}{3}\right)\right\} ln \frac{\frac{34}{3}+\frac{20}{3}}{\frac{34}{3}-\frac{16}{3}}-(8-4)\right]$$

$$=-1+\frac{34}{216}[10 \ ln\ 3-4]=-1+\frac{34}{216}[10\times 1{\cdot}09876-4]$$

$$=-1+1{\cdot}0999=+0{\cdot}0999$$

$$\frac{R^2}{h^2}=10{\cdot}01$$

Distance $KG=y_2+8=\frac{16}{3}+8=\frac{40}{3}$ cm

Bending moment, $M=5\times\frac{40}{3}=\frac{200}{3}$ tonne-cm

This bending moment tends to reduce the curvature so the portion GA will be in compression and portion GB will be in tension.

Direct stress, $f_d=\frac{5}{72}=0{\cdot}0694 \text{ tonne/cm}^2$ (tensile)

Maximum compressive stress at A,

$$f_A=\frac{M}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y_1}{R+y_1}\right]-f_d$$

$$=\frac{200\times 3}{3\times 72\times 34}\left[1+10{\cdot}01\times\frac{20}{3}\times\frac{1}{18}\right]-0{\cdot}0624$$

as $R+y_1=18$ cm

$$=0{\cdot}0817\ [4{\cdot}7074]-0{\cdot}0694=0{\cdot}3152 \text{ tonne/cm}^2$$

Maximum tensile stress at B

$$f_B = \frac{M}{AR}\left[\frac{y_2}{R-y_2}\times\frac{R^2}{h^2}-1\right]+f_d$$

$$= \frac{200\times 3}{3\times 72\times 34}\left[\frac{16}{3}\times\frac{10{\cdot}01}{6}-1\right]+0{\cdot}0694$$

$$= 0{\cdot}0817\ [7{\cdot}898]+0{\cdot}0694 = 0{\cdot}7146 \text{ tonne/cm}^2.$$

Exercise 19·4-1. The section of a crane hook is a trapezium. At the critical section, the inner and outer sides are 40 mm and 25 mm respectively and depth is 75 mm. The centre of curvature of the section is at a distance of 60 mm from the inner fibres and the load line is 50 mm from the inner fibres. Determine the maximum load the hook can carry if the maximum stress is not to exceed 120 N/mm². [**Ans.** 30·56 kN]

19·5. Ah^2 FOR A CIRCULAR SECTION

Fig. 19·15 shows the circular section of diameter d of a curved bar of radius of curvature R, from the centre of curvature C upto the centroid G of the section.

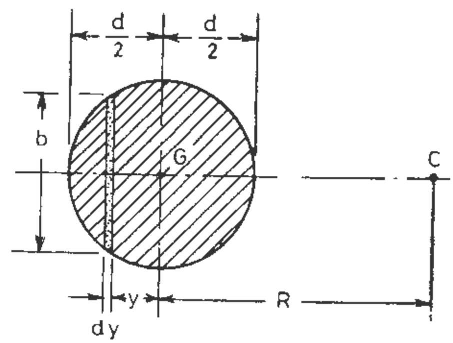


Fig. 19·15

Area of cross section, $A = \frac{\pi}{4} d^2$

Consider a strip of depth dy at a distance of y from the centroidal layer as shown.

Breadth of the layer,

$$b = \sqrt{\left(\frac{d}{2}\right)^2 - y^2}$$

Area of the strip,

$$dA = b\,dy = \sqrt{\left(\frac{d}{2}\right)^2 - y^2}\ .\ dy$$

Now
$$Ah^2 = \int \frac{Ry^2}{y+R}\, dA$$

or
$$h^2 = \frac{R}{A}\int \frac{y^2}{R+y}\, dA = \frac{4R}{\pi d^2}\int_{-d/2}^{+d/2} \frac{y^2}{(R+y)} \times \sqrt{\left(\frac{d}{2}\right)^2 - y^2}\ .\ dy$$

$$= \frac{d^2}{16}\left[1+\frac{1}{2}\left(\frac{d}{2R}\right)^2+\frac{15}{16}\left(\frac{d}{2R}\right)^4+\dots\right]$$

Example 19·5-1. A curved bar is formed of a tube of 8 cm outside diameter and thickness 0·5 cm. The centre line of this beam is a circular arc of radius 15 cm. Determine the greatest tensile and compressive stresses set up by a bending moment of 2 kNm tending to increase its curvature.

Solution. The Fig. 19·16 shows the cross section of a curved bar of radius of curvature $R=15$ cm.

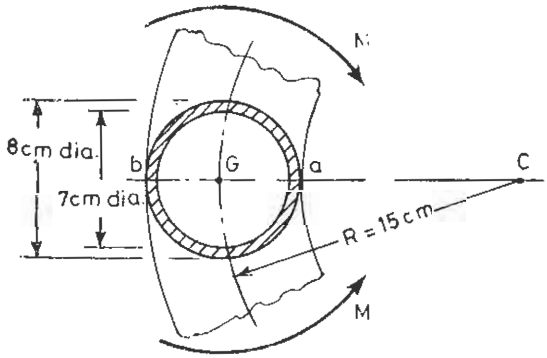


Fig. 19·16

Area of cross section,

$$A=\frac{\pi}{4}(8^2-7^2)=11{\cdot}781 \text{ cm}^2$$

Area of inner circle,

$$A_1=\frac{\pi}{4}(7^2)=38{\cdot}485 \text{ cm}^2$$

Area of outer circle,

$$A_2=\frac{\pi}{4}(8^2)=50{\cdot}266 \text{ cm}^2$$

Bending moment, $M=2$ kNm $=2\times10^5$ N cm

Now $$Ah^2=\int\frac{Ry^2dA}{R+y}$$

For a circular section

$$Ah^2=A\frac{d^2}{16}\left[1+\frac{1}{2}\left(\frac{d}{2R}\right)^2+\frac{5}{16}\left(\frac{d}{2R}\right)^4+\ldots\right]$$

For inner circle $$h_1^2=\frac{d_1^2}{16}\left[1+\frac{1}{2}\left(\frac{7}{30}\right)^2+\frac{5}{16}\left(\frac{7}{30}\right)^4+\ldots\right]$$

$$=\frac{49}{16}[1+0{\cdot}0272+0{\cdot}0009]=3{\cdot}14856$$

For outer circle $$h_2^2=\frac{d_2^2}{16}\left[1+\frac{1}{2}\left(\frac{8}{30}\right)^2+\frac{5}{16}\left(\frac{8}{30}\right)^4+\ldots\right]$$

$$=\frac{8^2}{16}[1+0{\cdot}03555+0{\cdot}00158]=4{\cdot}14852$$

$$Ah^2=A_2h_2^2-A_1h_1^2=50{\cdot}266\times4{\cdot}14852-38{\cdot}485\times3{\cdot}14856$$
$$=208{\cdot}53-121{\cdot}17=87{\cdot}36$$

$$h^2=\frac{87{\cdot}36}{11{\cdot}781}=7{\cdot}415$$

$$\frac{R^2}{h^2}=\frac{15\times15}{7{\cdot}415}=30{\cdot}344$$

Maximum tensile stress at b,

$$f_b=\frac{M}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y_1}{R+y_1}\right] \text{ where } y_1=4 \text{ cm}$$

$$=\frac{2\times10^5}{11{\cdot}781\times15}\left[1+\frac{30{\cdot}344\times4}{(15+4)}\right]$$

$$=1131{\cdot}766\,(1+6{\cdot}388)=8361{\cdot}5 \text{ N/cm}^2$$

Maximum compressive stress at a,

$$f_a = \frac{M}{AR}\left[\frac{R^2}{h^2} \times \frac{y_2}{R-y_2} - 1\right]$$

$$= \frac{2\times 10^5}{11{\cdot}781\times 15}\left[\frac{30{\cdot}344\times 4}{(15-4)} - 1\right] = 11356{\cdot}36 \text{ N/cm}^2$$

Exercise 19·5-1. A bar of circular cross section is bent in the shape of a horse shoe. The diameter of the section is 8 cm and the mean radius R is 8 cm as shown in the Fig. 19·17. Two equal and opposite forces of 15 kN each are applied so as to straighten the bar. Determine the greatest tensile and compressive stresses and plot a diagram showing the variation of the normal stresses along the central section.

[**Ans.** 39·22 N/mm² (maximum compressive stress)
99·83 N/mm² (maximum tensile stress)]

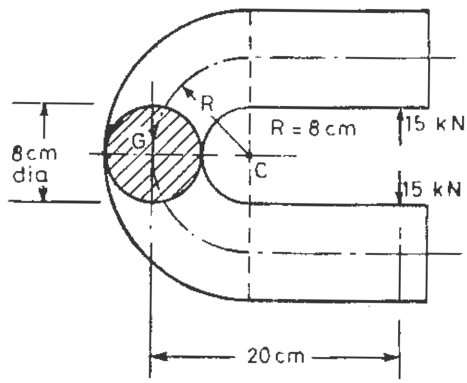


Fig. 19·17

19·6. RING SUBJECTED TO A DIAMETRAL LOAD

Fig. 19·18 shows a circular ring of mean radius R subjected to a diametral pull P. Consider a section CD at an angle θ from the line of application of the load *i.e.*, Y_1Y_2, and determine the bending moment and stresses in this section. Due to symmetry, the ring can be divided into four equal quadrants. Say M_1 is the bending moment on the section AB along the line of symmetry X_1X_2.

Taking moments about CD,

Bending moment at the section CD,

$$M = M_1 + \frac{P}{2}(R - R\sin\theta)$$

From equation (5) of article 19·1.

$$M = E(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)Ah^2$$

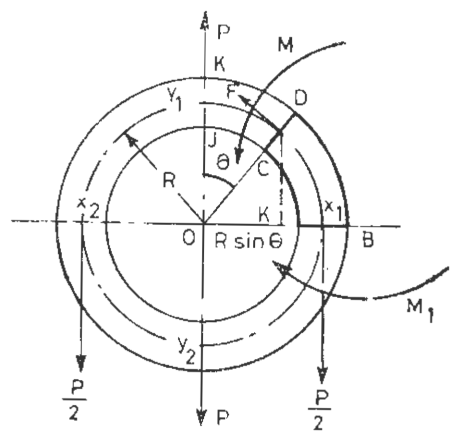


Fig. 19·18

where E=Young's modulus

ϵ_0=Strain in centroidal layer

R=Initial radius of curvature

R_1=Radius of curvature after bending

$$Ah^2 = \int \frac{Ry^2}{R+y}\,dA$$

So $E(1+\epsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)Ah^2=M_1+\frac{PR}{2}-\frac{PR}{2}\sin 2\theta$

Multiplying this equation throughout by $Rd\theta$ and integrating for one quadrant

i.e., $\theta=0$ to $\frac{\pi}{2}$

$$\int_0^{\pi/2} E\,(1+\epsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right) Ah^2\, Rd\theta$$

$$=\int_0^{\pi/2} M_1 Rd\theta+\int_0^{\pi/2}\frac{PR^2}{2}\,d\theta-\int_0^{\pi/2}\frac{PR^2}{2}\sin\theta\, d\theta$$

$$\therefore\quad \int_0^{\pi/2}\frac{E(1+\epsilon_0)}{R_1}\,Ah^2\,Rd\theta-E(1+\epsilon_0)\,Ah^2\,\frac{\pi}{2}$$

$$=M_1R\,\frac{\pi}{2}+\frac{PR^2\pi}{4}-\frac{PR^2}{2} \qquad \ldots(1)$$

Now $(1+\epsilon_0)=\frac{R_1}{R}\cdot\frac{\theta_1}{\theta}$ (equation (2) of article 19·1)

Now for one quadrant initial angle $\theta=90^\circ=\pi/2$ and final angle $\theta_1=90^\circ=\pi/2$ due to symmetry, *i.e.*, angle $\angle Y_1OX_1$ remains 90° even after the application of diametral load

So $(1+\epsilon_0)\,\frac{R}{R_1}=1$

Substituting this in equation (1) above, we get

$$\int_0^{\pi/2} EAh^2\,.\,d\theta-E(1+\epsilon_0)Ah^2\,\frac{\pi}{2}=M_1R\frac{\pi}{2}+\frac{PR^2}{2}\left(\frac{\pi}{2}-1\right)$$

or

$$-E\epsilon_0Ah^2\frac{\pi}{2}=M_1R\,\frac{\pi}{2}+\frac{PR^2}{2}\left(\frac{\pi}{2}-1\right) \qquad \ldots(2)$$

Again by equation (4) of article (19·1)

Normal force on the section,

$$F=E\epsilon_0A+E(1+\epsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\int\frac{y}{1+\frac{y}{R}}\,dA$$

Now $\displaystyle\int\frac{y}{1+\frac{y}{R}}\,dA=\int\frac{Ry}{R+y}\,dA=\int\frac{Ry+y^2-y^2}{R+y}\,dA=\int ydA-\int\frac{y^2}{R+y}\,dA$

$$=0-\frac{1}{R}\int\frac{Ry^2}{R+y}\,dA=-\frac{Ah^2}{R} \qquad \ldots(3)$$

Therefore normal force, $F=E\epsilon_0A-E(1+\epsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{Ah^2}{R}$

Normal force on the section CD,

$$F=\frac{P}{2}\sin\theta$$

$$\therefore \quad \frac{P}{2}\sin\theta=EA\epsilon_0-EA(1+\epsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{h^2}{R}$$

$$=EA\epsilon_0-\frac{M}{R} \text{ (refer to equation 5 again)}$$

$$=EA\epsilon_0-\frac{M_1}{R}-\frac{P}{2R}(R-R\sin\theta)$$

$$=EA\epsilon_0-\frac{M_1}{R}-\frac{P}{2}+\frac{P}{2}\sin\theta$$

So, $$\epsilon_0=\frac{M_1}{EAR}+\frac{P}{2EA}$$

Substituting the value of ϵ_0 in equation (2) above

$$-EAh^2\times\frac{\pi}{2}\left[\frac{M_1}{EAR}+\frac{P}{2EA}\right]=\frac{M_1R\pi}{2}+\frac{PR^2}{2}\left(\frac{\pi}{2}-1\right)$$

$$-\frac{M_1}{R}\times h^2\times\frac{\pi}{2}-\frac{P}{4}\times h^2\pi=\frac{M_1R\pi}{2}+\frac{PR^2}{2}\left(\frac{\pi}{2}-1\right)$$

$$-\frac{P}{4}\cdot h^2\pi-\frac{PR^2}{4}\pi+\frac{PR^2}{2}=M_1R\frac{\pi}{2}+\frac{M_1}{R}h^2\frac{\pi}{2}$$

$$\frac{PR^2}{\pi}-\frac{PR^2}{2}-\frac{Ph^2}{2}=\frac{M_1}{R}\left[R^2+h^2\right]$$

$$M_1=\frac{PR^3}{\pi(R^2+h^2)}-\frac{PR}{2}=\frac{PR}{2}\left[\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-1\right] \quad \ldots(4)$$

Now $$M=M_1+\frac{PR}{2}(1-\sin\theta)$$

$$=\frac{PR}{2}\times\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-\frac{PR}{2}+\frac{PR}{2}-\frac{PR}{2}\sin\theta$$

$$=\frac{PR}{2}\left[\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-\sin\theta\right]$$

M will be maximum when $\theta=0°$

$$M_{max}=\frac{PR}{\pi(R^2+h^2)}$$

M will be zero when $\theta=\sin^{-1}\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}$.

So there will be 4 sections, one in each quadrant where the bending moment M will be zero and consequently the stress due to bending will be zero.

Now substituting the value of M_1 in equation

$$\epsilon_0=\frac{M_1}{EAR}+\frac{P}{2EA}=\frac{P}{EA}\times\frac{R^2}{\pi(R^2+h^2)} \qquad \ldots(5)$$

and $$E(1+\epsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)=\frac{M}{Ah^2}$$

Stress, $$f=E\epsilon_0+E(1+\epsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{y}{1+\frac{y}{R}}$$

[Refer to equation (3) of article 19·1, $f=\epsilon E$]

$$=\frac{P}{A}\times\frac{R^2}{\pi(R^2+h^2)}+\frac{M}{Ah^2}\times\frac{Ry}{R+y}$$

$$=\frac{P}{A}\times\frac{R^2}{\pi(R^2+h^2)}+\frac{Ry}{R+y}\times\frac{1}{Ah^2}\left[\frac{PR}{2}\times\frac{R^2}{(R^2+h^2)}\times\frac{2}{\pi}-\frac{PR}{2}\sin\theta\right]$$

$$=\frac{P}{A}\left[\frac{R^2}{\pi(R^2+h^2)}+\frac{R^2}{2h^2}\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-\sin\theta\right)\frac{y}{R+y}\right]$$

Direct stress at any section, $f_d=\frac{P}{2}\ \frac{\sin\theta}{A}$

Resultant stress at any point on the section

$$f_R=\frac{P}{A}\left[\frac{R^2}{\pi(R^2+h^2)}+\frac{R^2}{2h^2}\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-\sin\theta\right)\frac{y}{R+y}\right]+\frac{P}{2A}\sin\theta$$

Stress along Y_1Y_2 axis, where $\theta=0°$

$$f_R=\frac{P}{A}\left[\frac{R^2}{\pi(R^2+h^2)}+\frac{R^2}{2h^2}\times\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}\right)\frac{y}{R+y}\right]$$

$$=\frac{PR^2}{A\pi(R^2+h^2)}\left[1+\frac{R^2}{h^2}\times\frac{y}{R+y}\right]$$

at the point K, $y=d/2$...where d=diameter of the rod of the ring

$$f_k=\frac{PR^2}{A\pi(R^2+h^2)}\left[1+\frac{R^2}{h^2}\times\frac{d}{2R+d}\right]\text{ tensile}$$

at the point J, $y=-d/2$

$$f_J=\frac{PR^2}{\pi A(R^2+h^2)}\left[\frac{R^2}{h^2}\times\frac{d}{2R-d}-1\right]\text{ compressive.}$$

It can be observed that maximum stress occurs at the point J, where the diametral load is applied.

Stresses along X_1X_2 axis, where $\theta=90°$.

$$f_R'=\frac{P}{A}\left[\frac{R^2}{\pi(R^2+h^2)}+\frac{R^2}{2h^2}\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-1\right)\frac{y}{R+y}\right]+\frac{P}{2A}$$

At the point B, $y=+d/2$

$$f_B'=\frac{P}{A}\left[\frac{R^2}{\pi(R^2+h^2)}+\frac{R^2}{2h^2}\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-1\right)\frac{d}{2R+d}\right]+\frac{P}{2A}$$

(compressive)

At the point A, $y=-d/2$

$$f_{A}'=\frac{P}{A}\left[\frac{R^2}{\pi(R^2+h^2)}-\frac{R^2}{2h^2}\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-1\right)\frac{d}{2R-d}\right]+\frac{P}{2A}\text{(tensile)}.$$

Example 19·6-1. A ring is made of round steel 2 bar cm diameter and the mean diameter of the ring is 12 cm. Determine the greatest intensities of tensile and compressive stresses along a diameter XX if the ring is subjected to a pull of 10 kN along diameter YY.

Solution. Fig. 19·19 shows a ring of mean diameter 12 cm, bar diameter 2 cm, subjected to a diametral pull P.

Radius of curvature, $R=6$ cm
Bar diameter, $d=2$ cm
Pull, $P=10$ kN

Area of cross section, $A=\frac{\pi}{4}(2)^2=3·1416\text{ cm}^2$

$$h^2=\frac{d^2}{16}\left[1+\frac{1}{2}\left(\frac{d}{2R}\right)^2+\frac{5}{16}\left(\frac{d}{2R}\right)^4+\ldots\right]$$

$$=\frac{2^2}{16}\left[1+\frac{1}{2}\left(\frac{2}{12}\right)^2+\frac{5}{16}\left(\frac{2}{12}\right)^4+\ldots\right]$$

$$=\frac{1}{4}\left[1+\frac{1}{72}+\frac{5}{16}\times\frac{1}{36}\times\frac{1}{36}+\ldots\right]$$

$$=\frac{1}{4}[1+0·01388+0·00024]=0·25353$$

or $$\frac{R^2}{h^2}=\frac{6\times6}{0·25353}=142.$$

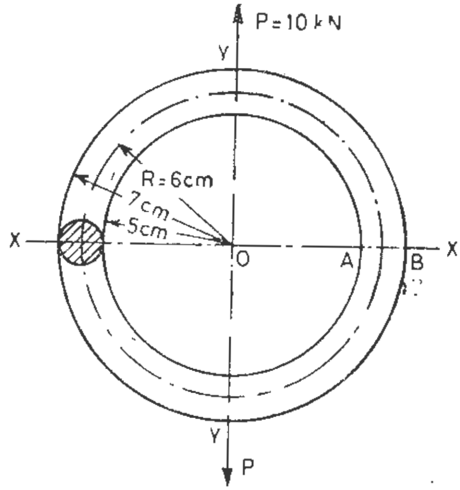


Fig. 19·19

Stresses

$$f_A=\frac{P}{A}\left[\frac{R^2}{\pi(R^2+h^2)}-\frac{R^2}{2h^2}\times\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-1\right)\frac{d}{2R-d}\right]+\frac{P}{2A}\quad\text{tensile}$$

$$=\frac{10,000}{3·1416}\left[\frac{36}{\pi(36+0·25353)}-\frac{142}{2}\left(\frac{36}{(36+0·25353)}\times\frac{\pi}{2}-1\right)\frac{2}{12-2}\right]+\frac{10,000}{2\times3·1416}$$

$$=3183·09[0·316-14·2(0·632-1)]+1591·54$$

$$=17639·41+1591·54=19230·95\text{ N/cm}^2=192·30\text{ N/mm}^2$$

$$f_B=\frac{P}{A}\left[\frac{R^2}{\pi(R^2+h^2)}+\frac{R^2}{2h^2}\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-1\right)\frac{d}{2R+d}\right]+\frac{P}{2A}\quad\text{compressive,}$$

$$=3183·09\left[0·316+\frac{142}{2}(0·632-1)\times\frac{2}{12+2}\right]+1591·54$$

$$=3183·09[0·316-3·7326]+1591·54=-9283·71\text{ N/cm}^2$$

$$=-92·83\text{ N/mm}^2\text{ (compressive stress)}$$

Exercise 19·6-1. A ring is made of round steel rod of diameter 2·4 cm. The mean diameter of the ring is 24 cm. The ring is pulled by a force of 100 kg. Determine the greatest intensities of tensile and compressive stresses along the diameter of loading.

[**Ans.** 260·95 kg/cm² (tensile), 303·33 kg/cm² (compressive)]

19·7. CHAIN LINK SUBJECTED TO A TENSILE LOAD

Figure 19·20 shows a chain link of mean radius R, length of the straight portion l, subjected to pull P. Consider a section CD at an angle θ from the line of application Y_1Y_2 of the pull P. Let us determine the bending moment and stresses in this section. Due to symmetry, the ring can be divided into four equal parts as shown. Say M_1 is the bending moment on the section AB along the line OX_1.

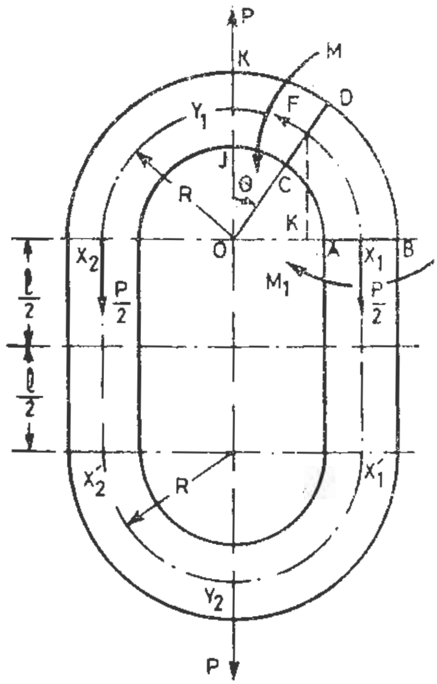


Fig. 19·20

Taking moments at the section CD,

$$M = M_1 + \frac{P}{2}(R - R\sin\theta)$$

From equation (5) of article 19·1

$$M = E(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)Ah^2$$

where

E = Young's modulus

ϵ_0 = Strain in the centroidal layer

R = Initial radius of curvature

R_1 = Final radius of curvature

$$Ah^2 = \int \frac{Ry^2}{R+y}\,dA$$

Therefore,

$$E(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)Ah^2 = M_1 + \frac{P}{2}(R - R\sin\theta)$$

Multiplying throughout by $Rd\theta$ and integrating from 0 to $\pi/2$

$$\int_0^{\pi/2} E(1+\epsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)Ah^2\,Rd\theta$$

$$= \int_0^{\pi/2} M_1 \,.\, Rd\theta + \int_0^{\pi/2} \frac{PR^2}{2}\,d\theta - \int_0^{\pi/2} \frac{PR^2}{2}\sin\theta\,d\theta$$

$$\therefore \quad \int_0^{\pi/2} \frac{E(1+\epsilon_0)}{R_1} \times R\,Ah^2\,d\theta - E(1+\epsilon_0)\,Ah^2\,\frac{\pi}{2}$$

$$= M_1R\,\frac{\pi}{2} + \frac{PR^2}{4}\times\pi - \frac{PR^2}{2} \qquad \ldots(1)$$

Now, $(1+\varepsilon_0)=\dfrac{R_1}{R}\cdot\dfrac{\theta_1}{\theta}$ (equation (2) of article 19·1)

In this case initial angle $\theta=\angle X_1OY_1=90°$

But final angle θ_1 will not be 90°, but there will be slight change from 90°.

Slope at $X_1=\dfrac{M_1\,l}{EI\times 2}$

where I is the moment of inertia of the section

$$\therefore\qquad \int_0^{\pi/2}\frac{R(1+\varepsilon_0)}{R_1}\,d\theta=\frac{\pi}{2}-\frac{M_1\,l}{2\,EI}$$

Substituting in equation (1)

$$EAh^2\left(\frac{\pi}{2}-\frac{M_1\,l}{2\,EI}\right)-E\,(1+\epsilon_0)\,Ah^2\,\frac{\pi}{2}=M_1\,R\,\frac{\pi}{2}+\frac{PR^2}{2}\left(\frac{\pi}{2}-1\right)$$

$$-EAh^2\times\frac{M_1\,l}{2EI}-E\;\epsilon_0\,Ah^2\;\frac{\pi}{2}=M_1R\;\frac{\pi}{2}+\frac{PR^2}{2}\left(\frac{\pi}{2}-1\right)\qquad ...(2)$$

Again by equation (4) of article 19·1

Normal force on the section CD,

$$F=E\,\varepsilon_0\,A+E\,(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\int\frac{Ry}{R+y}\,dA$$

Now $$\int\frac{Ry}{R+y}\,dA=\int\frac{Ry+y^2-y^2}{R+y}\,dA=\int y\,dA-\int\frac{y^2}{R+y}\,dA$$

$$=0-\frac{1}{R}\int\frac{Ry^2}{R+y}\,dA=-\frac{Ah^2}{R}\qquad ...(3)$$

Therefore normal force,

$$F=E\,\varepsilon_0\,A-E\,(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{Ah^2}{R}$$

$$=\frac{P}{2}\sin\theta$$

$$\therefore\qquad \frac{P}{2}\sin\theta=EA\,\varepsilon_0-EAh^2\,(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\times\frac{1}{R}$$

$$=EA\,\varepsilon_0-\frac{M}{R}\text{ (by equation (5) of article 19·1)}$$

$$=\varepsilon_0\,EA-\frac{1}{R}\left[M_1+\frac{PR}{2}\,(1-\sin\theta)\right]$$

$$=\epsilon_0\,EA-\frac{M_1}{R}-\frac{PR}{2R}+\frac{P}{2}\,\sin\theta$$

So $$\epsilon_0=\frac{M_1}{EAR}+\frac{P}{2\,EA}\qquad ...(4)$$

Substituting the value of ϵ_0 in equation (2) above

$$-EAh^2\times\frac{M_1\,l}{2\,EI}-EAh^2\,\frac{\pi}{2}\left(\frac{M_1}{EAR}+\frac{P}{2EA}\right)=\frac{M_1\,R\,\pi}{2}+\frac{PR^2}{2}\left(\frac{\pi}{2}-1\right)$$

$$-Ah^2\,\frac{M_1\,l}{2\,I}-M_1\,h^2\,\frac{\pi}{2}-\frac{Ph^2\pi}{4}=\frac{M_1R\pi}{2}+\frac{PR^2\pi}{4}-\frac{PR^2}{2}$$

$$M_1\left(\frac{\pi R}{2}+\frac{Ah^2\,l}{2\,I}+\frac{h^2\pi}{2\,R}\right)=\frac{PR^2}{2}\left(1-\frac{\pi}{2}\right)-\frac{Ph^2\pi}{4}$$

where $I=Ak^2$; $k=$radius of gyration of the section

$$M_1\left(\frac{\pi R}{2}+\frac{h^2l}{2k^2}+\frac{\pi}{2}\times\frac{h^2}{R}\right)=\frac{PR^2}{2}\left(1-\frac{\pi}{2}\right)-Ph^2\,\frac{\pi}{4}$$

Dividing throughout by $\pi/2$

$$M_1\left(R+\frac{h^2l}{\pi k^2}+\frac{h^2}{R}\right)=\frac{PR^2}{2}\left(\frac{2}{\pi}-1\right)-\frac{Ph^2}{2}$$

$$M_1=\frac{P\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}\right)}{R+\frac{h^2l}{\pi k^2}+\frac{h^2}{R}} \quad ...(5)$$

But $$M=M_1+\frac{R}{2}\,(R-R\sin\theta)$$

$$=\frac{P\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}\right)}{R+\frac{h^2l}{\pi k^2}+\frac{h^2}{R}}+\frac{PR}{2}\,(1-\sin\theta) \quad ...(6)$$

Substituting the value of M_1 in equation (4)

$$\epsilon_0=\frac{1}{EAR}\left[\frac{P\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}\right)}{R+\frac{h^2l}{\pi k^2}+\frac{h^2}{R}}\right]+\frac{P}{2EA}$$

Stress at any layer at a distance of y from the neutral layer is

$$f=E\,\epsilon_0+E\,(1+\epsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{y}{1+\frac{y}{R}}$$

Moreover $E\,(1+\epsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)=\frac{M}{Ah^2}$

$$f=E\,\epsilon_0+\frac{M}{Ah^2}\times\frac{y}{1+\frac{y}{R}}$$

$$=E\,\epsilon_0+\frac{M}{Ah^2}\times\frac{Ry}{R+y}$$

Putting the values of M and ϵ_0, stress due to bending moment

$$f_b=\frac{1}{AR}\left[\frac{P\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}\right)}{R+\frac{h^2 l}{\pi k^2}+\frac{h^2}{R}}\right]+\frac{P}{2A}+\frac{Ry}{(R+y)}\times\frac{1}{Ah^2}$$

$$\times\left[\frac{P\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}\right)}{R+\frac{h^2 l}{\pi k^2}+\frac{h^2}{R}}\right]+\frac{Ry}{R+y}\times\frac{1}{Ah^2}\times\frac{PR}{2}\ (1-\sin\theta)$$

Direct stress due to F, $f_d=\frac{P}{2A}\sin\theta$

Resultant stress, $\quad f_R=f_b+f_d$

$$=\frac{P}{AR}\left[\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{h^2 l}{\pi k^2}+\frac{h^2}{R}}\right]+\frac{PR}{Ah^2}\times\frac{y}{R+y}\left[\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{h^2 l}{\pi k^2}+\frac{h^2}{R}}\right]$$

$$+\frac{Ry}{R+y}\cdot\frac{1}{Ah^2}\times\frac{PR}{2}\ (1-\sin\theta)+\frac{P}{2A}+\frac{P}{2A}\sin\theta$$

$$=\frac{P}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y}{R+y}\right]\left[\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{h^2 l}{\pi k^2}+\frac{h^2}{R}}\right]$$

$$+\frac{PR^2}{2Ah^2}\ (1-\sin\theta)\ \frac{y}{R+y}+\frac{P}{2A}\ (1+\sin\theta) \qquad \ldots(7)$$

This is the equation for resultant stress in any section along the curved portions $X_2\,Y_1\,X_1$ and $X_2'\,Y_2\,X_1'$ of the chain link.

The bending moment M_1 on the straight portion $X_1\,X_1'$ and $X_2\,X_2'$ will remain constant and for the straight portion bending stress will be found with the help of general flexural formula. To obtain the resultant stress in this straight portion, direct tensile stress $P/2A$ will be added to the bending stress.

On the inner surface of the ring which is also called *intrados* stress can be obtained by putting $y=-d/2$ in equation (5) where d is the diameter of the bar of the chain link. Similarly for the outer surface which is also known as *extrados* the resultant stress is obtained by replacing y by $+d/2$ in equation (6) above.

Maximum stress along $Y_1\,O\,Y_2$ axis

$\theta=0°$

$$f_R=\frac{P}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y}{R+y}\right]\left[\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{h^2 l}{\pi k^2}+\frac{h^2}{2}}\right]$$

$$+\frac{PR^2}{2h^2A}\times\frac{y}{R+y}+\frac{P}{2A}$$

At the intrados $\qquad y=-\dfrac{d}{2}$

$$f_{RI}=\frac{P}{AR}\left[1-\frac{R^2}{h^2}\times\frac{d}{2R-d}\right]\left[\frac{\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{h^2}{2}}{R+\dfrac{h^2 l}{\pi^2 k}+\dfrac{h^2}{R}}\right]-\frac{PR^2}{2Ah^2}\times\frac{d}{2R-d}+\frac{P}{2A}$$

At the extrados $\qquad y=+\dfrac{d}{2}$

$$f_{RE}=\frac{P}{AR}\left[1+\frac{R^2}{h^2}\times\frac{d}{2R+d}\right]\left[\frac{\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{h^2}{2}}{R+\dfrac{h^2 l}{\pi^2 k}+\dfrac{h^2}{R}}\right]+\frac{PR^2}{2Ah^2}\times\frac{d}{2R+d}+\frac{P}{2A}$$

Maximum stresses along X_1 O X_2 axis

$$\theta=90^\circ$$

$$f_R=\frac{P}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y}{R+y}\right]\left[\frac{\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{h^2}{2}}{R+\dfrac{h^2 l}{\pi k^2}+\dfrac{h^2}{R}}\right]+\frac{P}{A}$$

At the intrados $\qquad y=-\dfrac{d}{2}$

$$f_{RI}'=\frac{P}{AR}\left[1-\frac{R^2}{h^2}\times\frac{d}{2R-d}\right]\left[\frac{\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{h^2}{2}}{R+\dfrac{h^2 l}{\pi k^2}+\dfrac{h^2}{R}}\right]+\frac{P}{A}$$

At the extrados $\qquad y=+\dfrac{d}{2}$

$$f_{RE}'=\frac{P}{AR}\left[1+\frac{R^2}{h^2}\times\frac{d}{2R+d}\right]\left[\frac{\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{h^2}{2}}{R+\dfrac{h^2 l}{\pi k^2}+\dfrac{h^2}{R}}\right]+\frac{P}{A}$$

Maximum stress in straight portion

$X_1\,X_1'$ or $X_2\,X_2'$

Bending moment $\qquad M_1=\dfrac{P\left(\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{h^2}{2}\right)}{R+\dfrac{h^2 l}{\pi k^2}+\dfrac{h^2}{R}}$

Stress due to bending, $f_b = \frac{32\, M_1}{\pi d^3}$

Direct stress, $f_d = \frac{P}{2A} = \frac{2P}{\pi d^2}$

f_R, resultant stress at introdos and extrados

$$= \frac{32\, M_1}{\pi d^3} \pm \frac{2P}{\pi d^2}.$$

Example 19·7-1. A chain link is made of round steel rod of 1 cm diameter. If $R=3$ cm and $l=5$ cm, determine the maximum stress along the section where tensile load is applied. If $P=1$ kN.

Solution. $R=3$ cm, $d=1$ cm, $l=5$ cm, and $P=1$ kN

$$h^2 = \frac{d^2}{16}\left[1+\frac{1}{2}\left(\frac{d}{2R}\right)^2+\frac{5}{16}\left(\frac{d}{2R}\right)^4+\ldots\right]$$

$$= \frac{1}{16}\left[1+\frac{1}{2}\left(\frac{1}{6}\right)^2+\frac{5}{16}\left(\frac{1}{6}\right)^4\right]$$

$$= \frac{1}{16}\,[1+0{\cdot}01390+0{\cdot}00024] = 0{\cdot}06338$$

$$\frac{R^2}{h^2} = \frac{3\times 3}{0{\cdot}06338} = 142$$

Area, $A = \frac{\pi}{4}\,(1)^2 = 0{\cdot}7854$ cm^2,

Radius of gyration, $k = \frac{d}{4} = \frac{1}{4}$ cm $\quad k^2 = \frac{1}{16}$

$$\frac{R^2}{\pi} - \frac{R^2}{2} - \frac{h^2}{2} = \frac{9}{\pi} - \frac{9}{2} - \frac{0{\cdot}06338}{2}$$

$$= 2{\cdot}8648 - 4{\cdot}500 - 0{\cdot}03169 = -1{\cdot}66689$$

$$R + \frac{h^2 l}{\pi k^2} + \frac{h^2}{R} = 3 + \frac{0{\cdot}06338\times 5\times 4\times 4}{\pi\times 1} + \frac{0{\cdot}06338}{3}$$

$$= 3 + 1{\cdot}6139 + 0{\cdot}0211 = 4{\cdot}635$$

$$\left(\frac{R^2}{\pi} - \frac{R^2}{2} - \frac{h^2}{2}\right) \Big/ \left(R + \frac{h^2 l}{\pi k^2} + \frac{h^2}{R}\right) = -\frac{1{\cdot}66689}{4{\cdot}635} = -0{\cdot}3596$$

$\theta = 0°$, Therefore

Maximum stress at intrades,

$$f_{Ri} = \frac{P}{AR}\left[1 - \frac{R^2}{h^2}\times\frac{d}{2R-d}\right]\left[\frac{\frac{R^2}{\pi} - \frac{R^2}{2} - \frac{h^2}{2}}{R + \frac{h^2 l}{\pi k^2} + \frac{h^2}{R}}\right]$$

$$- \frac{PR^2}{2Ah^2}\times\frac{d}{2R-d} + \frac{P}{2A}$$

$$=\frac{1000}{0{\cdot}7854\times3}\left[1-142\times\frac{1}{5}\right][-0{\cdot}3596]$$

$$-\frac{1000\times142}{2\times0{\cdot}7854}\times\frac{1}{5}+\frac{1000}{0{\cdot}7859\times2}$$

$$=424{\cdot}4[9{\cdot}853]-18079{\cdot}96+636{\cdot}62$$

$$=4181{\cdot}61-18079{\cdot}96+636{\cdot}62=-13261{\cdot}73 \text{ N/cm}^2$$

$$=-132{\cdot}61 \text{ N/mm}^2 \quad \text{(compressive)}$$

Maximum stress at extrados,

$$f_{RE}=\frac{P}{AR}\left[1+\frac{R^2}{h^2}\times\frac{d}{2R+d}\right]\left[\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{h^2l}{\pi k^2}+\frac{h^2}{R}}\right]$$

$$+\frac{P}{2A}\times\frac{R^2}{h^2}\times\frac{d}{2R+d}+\frac{P}{2A}$$

$$=\frac{1000}{0{\cdot}7854\times3}\left[1+1\,42\times\frac{1}{7}\right][-0{\cdot}3596]$$

$$+\frac{1000}{2\times0{\cdot}7854}\times142\times\frac{1}{7}+\frac{1000}{0{\cdot}7854\times2}$$

$$=-3248{\cdot}50+12914{\cdot}26+636{\cdot}62=+10302{\cdot}38 \text{ N/cm}^2$$

$$=+103{\cdot}02 \text{ N/mm}^2 \quad \text{(tensile)}$$

Maximum stress occurs at the intrados, *i.e.* where the load is applied.

Exercise 19·7-1. A chain link is made of round steel rod of 1 cm diameter. If $R=3$ cm and $l=5$ cm, determine the maximum stresses along the section at the end of the straight portion. A load of 1000 N (tensile) is applied on the chain link.

[**Ans.** 54·54 N/mm² (tensile), 19·75 N/mm² (compressive)]

19·8. DEFLECTION OF CURVED BARS

In order to estimate the stiffness of a curved beam subjected to bending moment it is necessary to determine the deflection of the curved beam and in such cases the influence of the initial curvature of the beam on its deflection is considerable. Fig. 19·21 shows centre line

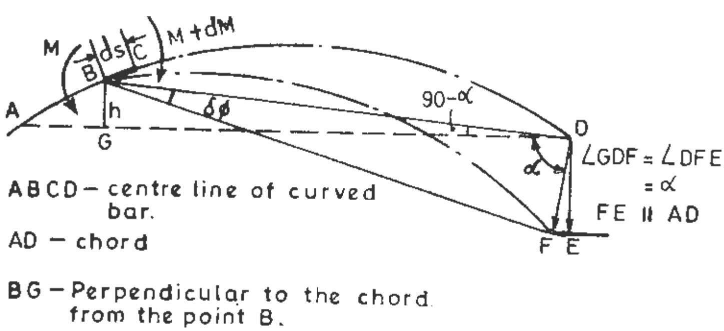


Fig. 19·21

ABCD of a curved bar subjected to variable bending moment. Consider a small portion *BC* of length *ds* along the centre line. Say the bending moment at *B* is M and $M+\delta M$ at *C*. Due to the bending moment say the centre line of the curved beam takes new position *ABF* and the element *BC* rotates by an angle $d\phi=\angle DBF$ at the point *B*. The angular rotation is small and the displacement of the point *D* is also small.

Displacement $DF \simeq BD\ d\phi$

$\angle BDF \simeq 90°$ for very small displacement DF

Components of the displacement are *DE* perpendicular to the chord *AD* and *EF* parallel to the chord *AD* *i.e.* the line joining the ends of the centre line of the curved beam considered.

FE shows negative displacement towards the point *A*.

Deflection of the point *D* with respect to *A* is δ_{DA} and considering the small length *ds* only say the deflection is

$$\Delta\delta_{DA}=-EF=-DF\cos\alpha \ldots\ldots \text{where } \angle DEF=\alpha$$

$$=-(BD\ d\phi)\cos\alpha$$

$$\angle ADF=\alpha$$

Therefore $\angle BDG=90°-\alpha$ or $\angle DBG=\alpha$

and $BD\cos\alpha=BG$

Therefore $\Delta\ \delta_{DA}=-(BD\cos\alpha)\ d\phi=-BG\ d\phi=-h\ d\phi$

where h is the perpendicular distance of the point *B* from the chord *AD*.

Moreover $d\phi=\dfrac{M\ ds}{EI}$

So $\Delta\delta_{DA}=-\dfrac{M\ h\ ds}{EI}$

Total deflection of *D* with respect to *A*

$$\delta_{DA}=-\int_D^A \frac{h\ M\ ds}{EI}$$

(*i*) Deflection of a closed ring

Fig. 19·22 shows the quadrant of a ring of mean radius of curvature R subjected to diametral pull P along OY_1. We have to determine the deflection along the load line or along the chord Y_2Y_1.

OY_1 is half the chord Y_2Y_1. Consider a small length *ds* at *C* at an angular displacement θ.

CC_1=perpendicular distance on chord from the point *C*

$=h=R\sin\theta$

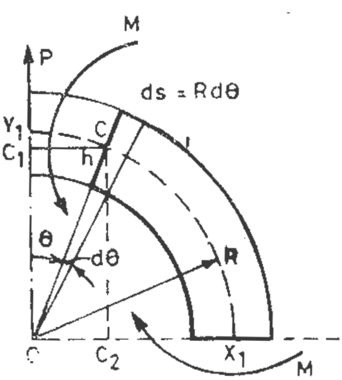


Fig. 19·22

Bending moment at the section C,

$$M=M_1+\frac{PR}{2}(1-\sin\theta)$$

$$=\frac{PR^3}{\pi(R^2+h^2)}-\frac{PR}{2}+\frac{PR}{2}(1-\sin\theta)$$

$$=\frac{PR}{2}\left[\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-\sin\theta\right] \text{ (see equation 4 of article 19·6)}$$

$\delta_{Y_1\,O}$=deflection along the load

$$=-\int_0^{Y_1}\frac{(R\sin\theta)}{EI}\left[\frac{PR}{2}\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-\sin\theta\right)\right]Rd\theta$$

$$\delta_{Y_1Y_2}=-\frac{PR^3}{2EI}\times 2\int_0^{\pi/2}\left[\frac{2}{\pi}\times\frac{R^2}{R^2+h^2}\times\sin\theta-\sin^2\theta\right]d\theta$$

Note that we have considered only one quadrant, when we consider the complete ring, the deflection along the load will be $\delta_{Y_1Y_2}$.

$$=-\frac{PR^3}{EI}\left\{\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)\Big|-\cos\theta\Big|_0^{\pi/2}\right\}+\frac{PR^3}{EI}\left|\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right|_0^{\pi/2}$$

$$=-\frac{PR^3}{EI}\left(\frac{R^2}{R^2+h^2}\right)\times\frac{2}{\pi}+\frac{PR^3}{EI}\times\frac{\pi}{2}\times\frac{1}{2}$$

$$=-\frac{PR^3}{EI}\times\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)+\frac{PR^3}{EI}\times\frac{\pi}{4}$$

$$=\frac{PR^3}{EI}\left[\frac{\pi}{4}-\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)\right]$$

(*ii*) **Deflection perpendicular to load line**

Refer to the Fig. 19·22 again, now the chord is OX_1 and perpendicular distance $CC_2=h$ on the chord OX_1 from C ; $h=R\cos\theta$

$\delta_{X_1\,O}$=deflection perpendicular to load line

$$=-\int_0^{X_1}\left(\frac{R\cos\theta}{EI}\right)\left[\frac{PR}{2}\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi}-\sin\theta\right)\right]Rd\theta$$

Total deflection, $$\delta_{X_1X_2}=-2\int_0^{\pi/2}\frac{PR^3}{2EI}\left[\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)\cos\theta-\sin\theta\cos\theta\right]d\theta$$

$$=-\frac{PR^3}{EI}\left|\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)\sin\theta+\frac{\cos 2\theta}{4}\right|_0^{\pi/2}$$

$$=-\frac{PR^3}{EI}\times\left[\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)\times 1+\frac{1}{4}(-1-1)\right]$$

$$=\frac{PR^3}{EI}\left[\frac{1}{2}-\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)\right]$$

Example 19·8-1. A ring with a mean diameter of 120 mm and a circular cross section of 40 mm in diameter is subjected to a diametral compressive load of 10 kN. Calculate the deflection of the ring along the load line. $E=200$ GN/m².

Solution. Since the diametral load is compressive, there will be reduction in diameter along the load line and increase in diameter perpendicular to the load line.

$$R=60 \text{ mm}=6 \text{ cm} \; ; \; d=40 \text{ mm}=4 \text{ cm} \; ; \; P=10\times 10^3 \text{ N}$$

$$E=200 \text{ GN/m}^2=200\times 10^5 \text{ N/cm}^2$$

$$h^2=\frac{d^2}{16}\left[1+\frac{1}{2}\left(\frac{d}{2R}\right)^2+\frac{5}{16}\left(\frac{d}{2R}\right)^4+\ldots\right]$$

$$=\frac{4^2}{16}\left[1+\frac{1}{2}\left(\frac{4}{12}\right)^2+\frac{5}{16}\left(\frac{4}{12}\right)^4+\ldots\right]-1$$

$$=1[1+0{\cdot}0555+0{\cdot}0038]=1{\cdot}0593=1{\cdot}0593$$

$$\frac{R^2}{R^2+h^2}=\frac{36}{37{\cdot}0593}=0{\cdot}97$$

$$I=\frac{\pi d^4}{64}=\frac{\pi}{64}(4)^4=4\pi=12{\cdot}5664 \text{ cm}^4$$

Deflection along the load line

$$=-\frac{PR^3}{EI}\left[\frac{\pi}{4}-\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)\right]$$

$$=-\frac{10\times 10^3}{200\times 10^5}\times\frac{6^3}{12{\cdot}5664}\left[\frac{\pi}{4}-\frac{2}{4}\times 0{\cdot}97\right]$$

$$=-0{\cdot}0398\times 10^{-3}\,[0{\cdot}7854-{\cdot}6167]\times 216 \text{ cm}$$

$$=-0{\cdot}0067\times 10^{-3}\times 216 \text{ cm}$$

$$=-1{\cdot}443\times 10^{-3} \text{ cm} \qquad \text{(reduction in diameter).}$$

Exercise 19·8-1. A ring with a mean diameter of 150 mm and a circular cross section of 30 mm is subjected to a diametral tensile load of 4 kN. Calculate the deflection of the ring along the direction perpendicular to load line. $E=200$ kN/mm² [**Ans.** $2{\cdot}786\times 10^{-2}$ mm]

19·9. DEFLECTION OF A CHAIN LINK

Fig. 19·23 shows the quarter of a chain link subjected to axial tensile load P. The radius of curvature of the link is R and length of the straight portion is l. Consider a section at an angle θ from the axis OY_1. We have to determine the deflection along the load P or along the chord Y_2Y_1.

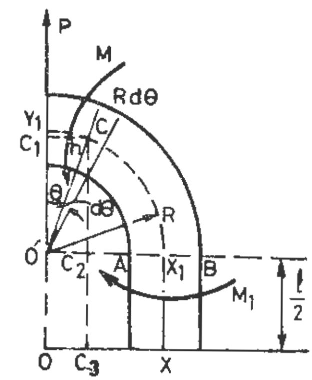


Fig. 19·23

$CC_1=h=$perpendicular from C to the chord

$=R \sin\theta$

Bending moment at the section

$$M=\frac{P\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}\right)}{R+\frac{lh^2}{\pi k^2}+\frac{h^2}{R}}+\frac{PR}{2}(1-\sin\theta)$$

(Refer to equation 6 of article 19·7)

$$=PK'+\frac{PR}{2}(1-\sin\theta)\quad \text{where } K'=\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{lh^2}{\pi k^2}+\frac{h^2}{R}}$$

Over the length $l/2$, the bending moment is constant and is equal to

$M_1=PK'$ where the value of K' is as above.

Deflection along the line of loading

$$\delta_{Y_1Y_2}'=-2\int_0^{\pi/2}\frac{R\sin\theta}{EI}\left\{PK'+\frac{PR}{2}(1-\sin\theta)\right\}Rd\theta-\frac{2R}{EI}\times PK'\times\frac{l}{2}$$

$$=-\frac{2R^2}{EI}\int_0^{\pi/2}\left\{PK'\sin\theta+\frac{PR}{2}\sin\theta-\frac{PR}{2}\sin^2\theta\right\}d\theta-\frac{RPK'l}{EI}$$

$$=-\frac{2R^2}{EI}\Big|-PK'\cos\theta\Big|_0^{\pi/2}-\frac{2R^2}{EI}\Big|-\frac{PR}{2}\cos\theta\Big|_0^{\pi/2}$$

$$+\frac{2R^2}{EI}\left|\frac{PR}{2}\left(\frac{\theta}{2}-\frac{\sin 2\theta}{2}\right)\right|_0^{\pi/2}-\frac{RPK'l}{EI}$$

$$=\frac{2R^2}{EI}\times(-PK')-\frac{2R^2}{EI}\left(\frac{PR}{2}\times 1\right)+\frac{2R^2}{EI}\left[\frac{PR\pi}{8}\right]-\frac{RPK'l}{EI}$$

$$=-\frac{2R^2}{EI}\left[PK'+\frac{PR}{2}-\frac{PR\pi}{8}\right]-\frac{RPK'l}{EI}$$

$$=\frac{PR^2}{EI}\left[\frac{\pi R}{4}-2K'-R\right]-\frac{RPK'l}{EI}.$$

Deflection due to direct load $P/2$ is

$$\delta'=\frac{Pl}{2AE}$$

Total deflection $\quad \delta_{Y_1Y_2}=\delta_{Y_1Y_2}'+\delta'=\frac{PR^2}{EI}\left[\frac{\pi R}{4}-2K'-R\right]-\frac{RPK'l}{EI}+\frac{Pl}{2AE}$

$$\text{where}\quad K'=\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{lh^2}{\pi k^2}+\frac{h^2}{R}}$$

(*ii*) **Deflection perpendicular to load line**

In this case chord is XX and $CC_2=h=R\cos\theta+l/2$

$$\delta_{xx}=-\int_0^{\pi/2}\frac{R\cos\theta}{EI}\left\{PK'+\frac{PR}{2}(1-\sin\theta)\right\}Rd\theta-\frac{PK'lR}{EI}$$

From theory of simple bending deflection in straight portion

$=\left(\frac{M_1}{EI}\times l\right)R$ where M_1 is bending moment on straight portion and l is the length of the straight portion.

$$\delta_{xx}=-\frac{2R^2}{EI}\left[\int_0^{\pi/2}PK'\cos\theta\,d\theta+\int_0^{\pi/2}\frac{PR}{2}\cos\theta\,d\theta-\int_0^{\pi/2}\frac{PR}{4}\sin 2\theta\,d\theta\right]-\frac{PK'lR}{EI}$$

$$=-\frac{2PR^2}{EI}\left[K'\Big|\sin\theta\Big|_0^{\pi/2}+\frac{R}{2}\sin\theta\Big|_0^{\pi/2}+\frac{R}{8}\Big|\cos 2\theta\Big|_0^{\pi/2}\right]-\frac{PK'lR}{EI}$$

$$=-\frac{2PR^2}{EI}\left[K'+\frac{R}{2}+\frac{R}{8}(-2)\right]-\frac{PK'lR}{EI}$$

$$=-\frac{2PR^2}{EI}\left(K'+\frac{R}{4}\right)-\frac{PK'lR}{EI}$$

where $$K'=\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{lh^2}{\pi k^2}+\frac{h^2}{R}}$$ where k=radius of gyration.

Example 19·9-1. A chain link is made of a steel rod of 12 mm diameter. The straight portion is 60 mm in length and the ends are 60 mm in radius. Determine the deflection of the link along the load line when subjected to a load of 10 kN. $E=200\times10^3$ N/mm².

Solution. Rod diameter, $d=1{\cdot}2$ cm.

Area of cross section, $A=\frac{\pi d^2}{4}=1{\cdot}131$ cm²

Length of straight portion $l=6$ cm

Radius of curvature, $R=6$ cm

Load, $P=10$ kN

$E=200\times10^5$ N/cm²

Radius of gyration, $k=d/4=0{\cdot}3$ cm

$$h^2=\frac{d^2}{16}\left[1+\frac{1}{2}\left(\frac{d}{2R}\right)^2+\frac{5}{16}\left(\frac{d}{2R}\right)^4+\ldots\right]$$

$$=\frac{1{\cdot}2^2}{16}\left[1+\frac{1}{2}\left(\frac{1{\cdot}2}{12}\right)^2+\frac{5}{16}\left(\frac{1{\cdot}2}{12}\right)^4+\ldots\right]$$

$$=0{\cdot}09[1+0{\cdot}005+0{\cdot}00003]=0{\cdot}09045$$

$$K'=\frac{\frac{R^3}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{lh^2}{\pi k^2}+\frac{h^2}{R}}=\frac{\frac{36}{\pi}-\frac{36}{2}-\frac{0.09045}{2}}{6+\frac{6\times0.09045}{\pi\times0.09}+\frac{0.09045}{6}}$$

$$=\frac{11.459-18-0.0452}{6+1.919+0.0150}=\frac{-6.5862}{7.934}=-0.83$$

$$I=\frac{\pi d^4}{64}=\frac{\pi}{64}\,(1.2)^4=0.1018\text{ cm}^4$$

Deflection along the load line

$$\delta_{Y_1Y_2}=\frac{PR^2}{EI}\left[\frac{\pi R}{4}-2K'-R\right]-\frac{PRK'l}{EI}+\frac{Pl}{2AE}$$

$$=\frac{10\times10^3\times6^3}{200\times10^5\times0.1018}\left[\frac{\pi\times6}{4}+2\times0.83-6\right]$$

$$+\frac{10\times10^3\times6\times0.83\times6}{200\times10^5\times0.1078}+\frac{10\times10^3\times6}{2\times1.131\times200\times10^5}$$

$$=1060.9\times10^{-3}[0.3724]+146.76\times10^{-3}+1.326\times10^{-3}\text{ cm}$$

$$=10^{-3}[395.08+146.76+1.326]=543.166\times10^{-3}\text{ cm}$$

$$=5.43\text{ mm.}$$

Exercise 19·9-1. A chain link is made of a steel rod of 12 mm diameter. The straight portion is 60 mm in length and the ends are 60 mm in radius. Determine the deflection of the link along the direction perpendicular to the load line if the chain link is subjected to a load of 100 kg.

$E=2000\times10^3$ kg/cm². [**Ans.** 0·3837 mm]

Problem 19·1. A curved bar of rectangular section with breadth B and depth 6 cm is bent to a radius of curvature 6 cm. It is subjected to a bending moment of 4000 kg cm tending to reduce the curvature. Determine the breadth of the section if the maximum stress is not to exceed 500 kg/cm².

Solution. Fig. 19·24 shows a curved bar of rectangular section $B\times6$ cm subjected to a bending moment tending to reduce the curvature.

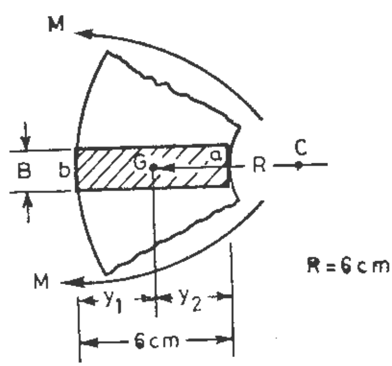


Fig. 19·24

$M=4000$ kg-cm

$y_1=y_2=3$ cm

$R=6$ cm

$A=6\times B$ cm²

$$\frac{h^2}{R^2}=\frac{R}{D}\ ln\ \frac{2R+D}{2R-D}-1$$

$$=\frac{6}{6}\times\ ln\ \frac{12+6}{12-6}-1=ln\ 3-1$$

$$=1.09876-1=0.09876$$

$$\frac{R^2}{h^2}=10.126$$

In this case maximum tensile stress will occur at the inner fiber at point *a*

$$\text{Stress, } f_a=\frac{M}{AR}\left(\frac{R^2}{h^2}\times\frac{y_2}{R-y_2}-1\right)=\frac{4000}{6\times B\times 6}\left(10{\cdot}126\times\frac{3}{6-3}-1\right)$$

$$=\frac{1014}{B}=500 \text{ kg/cm}^2$$

or Breadth, $B=\frac{1014}{500}=2{\cdot}028$ cm.

Problem 19·2. For the frame of a punching machine shown in Fig. 19 25 determine the circumferential stresses at A and B on a section inclined at an angle $\theta=45°$ to the vertical Force $P=100$ kN.

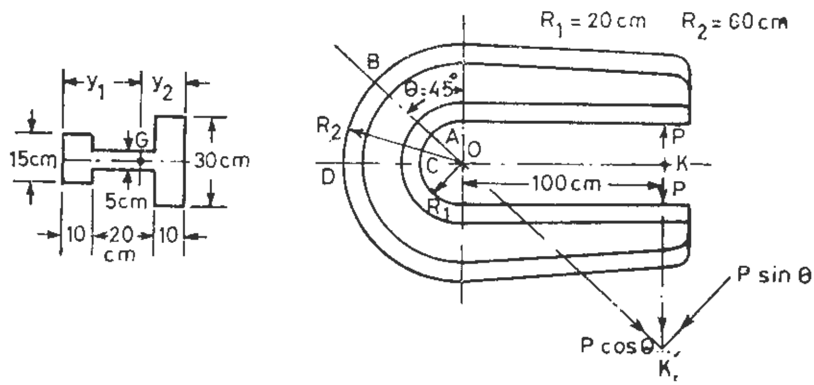


Fig. 19·25

Solution. Force $P=100$ kN

Perpendicular force on the section $AB=P \sin 45°=\frac{P}{\sqrt{2}}$

Tangential force on the section $AB=P \cos 45°=\frac{P}{\sqrt{2}}$

Area of cross section, $A=30\times 10+5\times 20+15\times 10=550$ cm²

Location of G

$$y_2=\frac{30\times 10\times 5+20\times 5\times 20+15\times 10\times 35}{550}$$

$$=\frac{1500+2000+5250}{550}=15{\cdot}9 \text{ cm}$$

$$y_1=40-15{\cdot}9=24{\cdot}1 \text{ cm}$$

Radius of curvature, $R=R_1+y_2=20+15{\cdot}9=35{\cdot}9$ cm

Bending moment on the section,

$$M=\frac{P}{\sqrt{2}}\times(OK'+OA+y_2)$$

$$=\frac{100}{\sqrt{2}}\times(100\times\sqrt{2}+35{\cdot}9)$$

$$=12538{\cdot}9 \text{ kN cm}$$

Direct force on the section,

$$P'=\frac{P}{\sqrt{2}}=\frac{100}{\sqrt{2}}=70{\cdot}7 \text{ kN}$$

$$\frac{h^2}{R^2}=\frac{R}{A}\left[30\ ln\frac{30}{20}+5\ ln\ \frac{50}{30}+15\ ln\ \frac{60}{50}\right]-1$$

$$\frac{h^2}{R^2}=\frac{35{\cdot}9}{550}\ [30\times0{\cdot}4056+5\times0{\cdot}5106+15\times0{\cdot}1824]-1$$

$$=0{\cdot}0652[12{\cdot}168+2{\cdot}553+2{\cdot}736]-1=1{\cdot}1382-1=0{\cdot}1382$$

$$\frac{R^2}{h^2}=7{\cdot}236$$

Tensile stress at point A

$$=\frac{M}{AR}\left[\frac{y_2}{R-y_2}\times\frac{R^2}{h^2}-1\right]+\frac{P'}{A}$$

$$=\frac{12538{\cdot}9\times1000}{550\times35{\cdot}9}\left[\frac{15{\cdot}9}{35{\cdot}9-15{\cdot}9}\times7{\cdot}236-1\right]+\frac{70{\cdot}7\times1000}{550}$$

$$=635{\cdot}04\ [4{\cdot}752]+128{\cdot}54$$

$$=3146{\cdot}25 \text{ N/cm}^2=31{\cdot}46 \text{ N/mm}^2$$

Compressive stress at point B

$$=\frac{M}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y_1}{R+y_1}\right]-\frac{P'}{A}$$

$$=\frac{12538{\cdot}9\times1000}{550\times35{\cdot}9}\left[1+\frac{7{\cdot}236\times24{\cdot}1}{35{\cdot}9+24{\cdot}1}\right]-\frac{70{\cdot}7\times1000}{550}$$

$$=635{\cdot}04\ [3{\cdot}906]-128{\cdot}54$$

$$=2351{\cdot}9 \text{ N/cm}^2=23{\cdot}52 \text{ N/mm}^2.$$

Problem 19·3. The radius of the inner fibres of a curved bar of trapezoidal section is equal to the depth of the cross section. The base of the trapezium on the concave side is four times the base on the convex side. Determine the ratio of the stresses in the extreme fibres of the curved bar to the stresses in the same fibres of a straight bar subjected to the same bending moment.

Solution.

Now $R_2=D$

$\therefore$ $R_1=2R_2=2D$

$B_2=4B_1$

$$y_1=\frac{B_1+2B_2}{B_1+B_2}\times\frac{D}{3}$$

$$=\frac{B_1+8B_1}{B_1+4B_1}\times\frac{D}{3}=0{\cdot}6\ D$$

So $y_2=0{\cdot}4\ D$

Area, $A=(B_1+B_2)\dfrac{D}{2}$

$$=\frac{5B_1D}{2}=2{\cdot}5\ B_1D$$

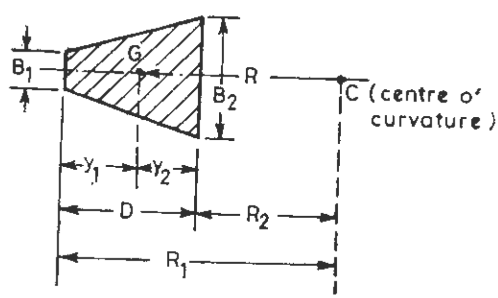


Fig. 19·26

$$R = R_2 + y_2 = 1{\cdot}4\ D$$

$$\frac{h^2}{R^2} = \frac{R}{A}\left[\left\{ B_1 + \frac{B_2 - B_2}{D}\ (y_1 + R)\right\} l_n \left(\frac{R+y_1}{R-y_2}\right) - (B_2 - B_1)\ \right] - 1$$

$$= \frac{R}{A}\left[\left\{ B_1 + \frac{3B_1}{D} \times R_1\ \right\} l_n \left(\frac{1{\cdot}4D + 0{\cdot}6D}{1{\cdot}4D - 0{\cdot}4D}\right) - (3B_1)\right] - 1$$

$$= \frac{1{\cdot}4D}{A}\ [7B_1\ l_n\ 2 - 3B_1] - 1$$

$$= \frac{1{\cdot}4D}{2{\cdot}5\ B_1 D}\ [7B_1\ l_n\ 2 - 3B_1] - 1$$

$$= 0{\cdot}56[7 \times 0{\cdot}693 - 3] - 1 = 0{\cdot}03656$$

$$\frac{R^2}{h^2} = 27{\cdot}35.$$

Let us consider that this curved bar is subjected to a bending moment, M tending to reduce the curvature.

Maximum tensile stress,
(when $y = -y_2$)

$$f_2 = \frac{M}{AR}\left[\frac{y_2}{R - y_2} \cdot \frac{R^2}{h^2} - 1\right]$$

$$= \frac{M}{2{\cdot}5B_1D \times 1{\cdot}4D}\left[\frac{0{\cdot}4D}{1{\cdot}4D - 0{\cdot}4D} \times 27{\cdot}35 - 1\right]$$

$$= \frac{M}{3{\cdot}5B_1D^2}[10{\cdot}94 - 1]$$

$$= \frac{2{\cdot}84\ M}{B_1D^2} \qquad \ldots(1)$$

Maximum compressive stress,
(when $y = +y_1$)

$$f_1 = \frac{M}{AR}\left[\frac{y_1}{R + y_1} \cdot \frac{R^2}{h^2} + 1\right]$$

$$= \frac{M}{3{\cdot}5B_1D^2}\left[\frac{0.6D}{1{\cdot}4D + 0{\cdot}6D} \times 27{\cdot}35 + 1\right]$$

$$= \frac{2{\cdot}627M}{B_1D^2} \qquad \ldots(2)$$

Stresses in the straight bar

Let us divide the section in two triangles as shown, as to calculate the moment of inertia I_{YY}.

Area of triangle I, $\quad A_1 = \dfrac{B_1D}{2}$

Area of triangle II, $\quad A_2 = \dfrac{B_2D}{2} = 2B_1D$

I_{YY} of triangle I, about its $C.G.$

$$= \frac{B_1D^3}{36}$$

Distance, $h_1=(0{\cdot}6-0{\cdot}333)\ D=0{\cdot}2667\ D$

I_{YY} of triangle *II*, about its *C.G.*

$$=\frac{B_2D^3}{36}=\frac{4B_1D^3}{36}$$

Distance, $h_2=(0{\cdot}4-0{\cdot}333)\ D=0{\cdot}06667\ D$

$$I_{YY} \text{ of the whole section}=\frac{B_1D^3}{36}+\frac{B_1D}{2}(0{\cdot}2667\ D)^2+\frac{B_1D^3}{9}+2B_1D\ (0{\cdot}0667\ D)^2$$

$$=\frac{5}{36}B_1D^3+\frac{B_1D}{2}\times\frac{64}{900}\ D^2+\frac{2B_1D\times D^2\times 4}{900}$$

$$=\frac{11}{60}\ B_1D^3$$

Maximum tensile stress,

$$f_2'=\frac{M}{I_{YY}}\times y_2 \qquad (y=-y_2)$$

$$=\frac{60\ M}{11\ B_1D^3}\times 0{\cdot}4\ D=\frac{2{\cdot}18\ M}{B_1D^2}$$

Maximum compressing stress,

$(y=+y_1)$ $$f_1'=\frac{60\ M}{11\ B_1D^3}\times 0{\cdot}6\ \ D=\frac{3{\cdot}27\ M}{B_1\ D^3}$$

So the ratios $$\frac{f_2}{f_2'}=\frac{2{\cdot}84}{2{\cdot}18}=1{\cdot}303$$

and $$\frac{f_1}{f_1'}=\frac{2{\cdot}627}{3{\cdot}27}=0{\cdot}804.$$

Problem 19·4. The cross section of a triangular hook has a base of 5 cm and altitude 7·5 cm and a radius of curvature of 5 cm at the inner face of the shank. If the allowable stress in tension is 100 N/mm² and that in compression is 80 N/mm², what load may be applied along a line 7·5 cm from the inner face of the shank.

Solution. Fig. 19·27 shows a triangular section with base 5 cm and altitude 7·5 cm. Radius of curvature R_1 is equal to 5 cm. C.G. of the section lies at a distance of $\frac{7{\cdot}5}{3}$ cm from the base.

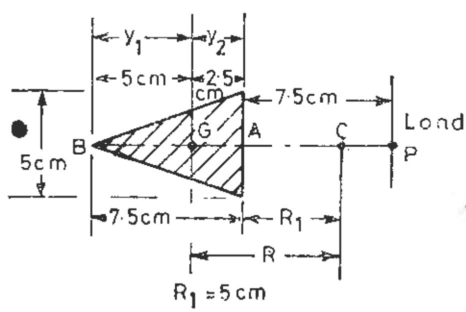


Fig. 19·27

So $y_2=2{\cdot}5$ cm, $y_1=5$ cm

Radius of curvature,

$R=5+2{\cdot}5=7{\cdot}5$

Breadth, $B_1=0$ and $B_2=5$ cm and depth $D=7{\cdot}5$ cm

Area of cross section,

$$A=\frac{5\times 7{\cdot}5}{2}=18{\cdot}75\ \text{cm}^2$$

$$\frac{h^2}{R^2}=\frac{R}{A}\left[\left\{B_1+\frac{B_2-B_1}{D}\ (y_1+R)\right]\ ln\ \frac{R+y_1}{R-y_2}-(B_2-B_1)\right]-1$$

$$=\frac{7\cdot5}{18\cdot75}\left[\left\{0+\frac{5}{7\cdot5}\times(5+7\cdot5)\right\}\ ln\ \frac{7\cdot5+5}{7\cdot5-2\cdot5}-5\right]-1$$

$$=\frac{7\cdot5}{18\cdot75}\ [8\cdot33\times0\cdot916-5]-1=1\cdot052-1=0\cdot052$$

$$\frac{R^2}{h^2}=19\cdot23$$

Say load applied $=P$ Newtons ;

Bending moment $M=(7\cdot5+2\cdot5)\ P=10\ P$ N cm

Maximum tensile stress at A,

$$=\frac{M}{AR}\left[\frac{y_2}{R-y_2}\times\frac{R^2}{h^2}-1\right]+\frac{P}{A}$$

$$=\frac{10\ P}{18\cdot75\times7\cdot5}\left[\frac{2\cdot5}{7\cdot5-2\cdot5}\times19\cdot23-1\right]+\frac{P}{18\cdot75}=10000\ \text{N/cm}^2$$

or $P[0\cdot6126+0\cdot0533]=10000$

or $P=15017\cdot2$ N$=15\cdot017$ kN

Maximum compressive stress at B,

$$=\frac{M}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y_1}{R+y_1}\right]-\frac{P}{A}$$

$$=\frac{10\ P}{18\cdot75\times7\cdot5}\left[1+19\cdot23\times\frac{5}{7\cdot5+5}\right]-\frac{P}{18\ 75}$$

$$=P[0\cdot6180-0\cdot0533]=8000\ \text{N/cm}^2$$

$$P=14166\cdot8\ \text{N}=14\cdot166\ \text{kN}$$

So the maximum permissible load is $14\cdot166$ kN.

Problem 19·5. A chain link is made of round steel rod of diameter 12 mm. If $R=40$ mm and $l=60$ mm, draw the stress distribution diagram along the intrados if the link is subjected to a tensile load of $1\cdot5$ kN.

Solution.

Bar diameter, $d=12$ mm

Radius of curvature, $R=40$ mm

Length of straight portion,

$l=60$ mm

Load, $P=1500$ N

$$h^2=\frac{d^2}{16}\left[1+\frac{1}{2}\left(\frac{d}{2R}\right)^2+\frac{5}{16}\left(\frac{d}{2R}\right)^4+\ldots\ldots\right]$$

$$=\frac{(12)^2}{16}\left[1+\frac{1}{2}\left(\frac{12}{80}\right)^2+\frac{5}{16}\left(\frac{12}{80}\right)^4+\ldots\ldots\right]$$

$$=\frac{144}{16}\ [1+0\cdot01125+0\cdot00016]=9\cdot10269$$

$$\frac{R^2}{h^2}=\frac{40^2}{9\cdot10269}=175\cdot77$$

Area of cross section, $A=\frac{\pi}{4}(12)^2=113\cdot098$ mm^2

$$\frac{P}{A}=\frac{1500}{113\cdot098}=13\cdot26 \text{ N/mm}^2$$

Radius of gyration $\quad =k^2=\frac{d^2}{16}=\frac{(12)^2}{16}=9$ mm

Moreover

$$\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}=\frac{1600}{\pi}-\frac{1600}{2}-\frac{9\cdot10269}{2}=509\cdot29-800-4\cdot55$$
$$=-295\cdot26$$

$$R+\frac{h^2l}{\pi k^2}+\frac{h^2}{R}=40+\frac{9\cdot10269\times60}{\pi\times9}+\frac{9\cdot10269}{40}=40+19\cdot316+0\cdot227$$
$$=59\cdot543$$

$$\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{h^2l}{\pi k^2}+\frac{h^2}{R}}=-\frac{295\cdot26}{59\cdot543}=-4\cdot959$$

Along the intrados $y=-d/2$, the equation for the resultant stress is

$$f_{RI}=\frac{P}{AR}\left[1-\frac{R^2}{h^2}\times\frac{d}{2R-d}\right]\left[\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}}{R+\frac{h^2l}{\pi k^2}+\frac{h^2}{R}}\right]$$
$$-\frac{PR^2}{2Ah^2}(1-\sin\theta)\frac{d}{2R-d}+\frac{P}{2A}(1+\sin\theta)$$

Substituting the values

$$f_{RI}=\frac{1500}{113\cdot098\times40}\left[1-175\cdot77\times\frac{12}{80-12}\right][-4\cdot959]$$
$$-\frac{1500}{2\times113\cdot098}\times175\cdot77(1-\sin\theta)\frac{12}{80-12}+\frac{1500}{2\times113\cdot098}(1+\sin\theta)$$
$$=49\cdot357-205\cdot695(1-\sin\theta)+6\cdot631(1+\sin\theta)$$
$$=49\cdot357-205\cdot695+6\cdot631=-149\ 707 \text{ N/mm}^2 \text{ at } \theta=0°$$
$$=49\cdot357-205\cdot695(1-0\cdot2588)+6\cdot631(1+0\cdot2588)$$
$$=49\cdot357-152\cdot461+8\cdot347=-94\cdot757 \text{ N/mm}^2 \text{ at } \theta=15°$$
$$=49\cdot357-205\cdot695(1-0\cdot5)+6\cdot631(1+0\cdot5)$$
$$=49\cdot357-102\cdot847+9\cdot946=-43\cdot544 \text{ N/mm}^2 \text{ at } \theta=30°$$
$$=49\cdot357-205\cdot695(1-0\cdot707)+6\cdot631(1+0\cdot707)$$
$$=49\cdot357-60\cdot268+11\cdot319=+0\cdot408 \text{ N/mm}^2 \text{ at } \theta=45°$$
$$=49\cdot357-205\cdot695(1-0\cdot866)+6\cdot631(1+0\cdot866)$$
$$=49\cdot357-27\cdot565+12\cdot373=+34\cdot167 \text{ N/mm}^2 \text{ at } \theta=60°$$
$$=49\cdot357-205\cdot695(1-0\cdot9659)+6\cdot631(1+0\cdot9659)$$

$=49{\cdot}357-7{\cdot}014+13{\cdot}036=55{\cdot}379$ N/mm² at $\theta=75°$

$=49{\cdot}357-205{\cdot}695\,(1-1)+6{\cdot}631\,(1+1)$

$=49{\cdot}557-0+13{\cdot}262=62{\cdot}619$ N/mm² at $\theta=90°$

Fig. 19·28 shows the variation of resultant stress along the intrados, showing that maximum compressive stress occurs at the inner fibre where load is applied and maximum tensile stress occurs at the section where the straight line portion in the link commences.

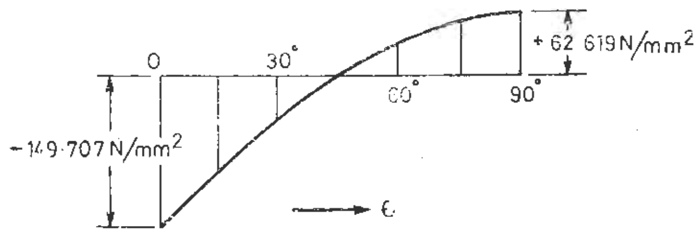


Fig. 19·28

SUMMARY

1. If a curved bar of radius of curvature R and area of cross section A is subjeeted to a (*i*) bending moment tending to increase the curvature

Stress in any layer. $f=\frac{M}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y}{R+y}\right]$ tensile for positive y

$=\frac{M}{AR}\left[\frac{R^2}{h^2}\times\frac{y}{R-y}-1\right]$ compressive for negative y

(*ii*) Bending moment tending to decrease the curvature

Stress in any layer, $f=\frac{M}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y}{R+y}\right]$ compressive for positive y

$=\frac{M}{AR}\left[\frac{R^2}{h^2}\times\frac{y}{R-y}-1\right]$ tensile for negative y.

2. For a rectangular section

$$\frac{h^2}{R^2}=\frac{R}{D}\,l_n\,\frac{R_2}{R_1}-1$$

where D=depth of the section

R_2=radius of outer surface of curved bar

R_1=radius of inner surface of curved bar.

3. For a circular section of diameter d

$$h^2=\frac{d^2}{16}\left[1+\frac{1}{2}\left(\frac{d}{2R}\right)^2+\frac{5}{16}\left(\frac{d}{2R}\right)^4+\ldots\ldots\right].$$

4. For a trapezoidal section

$$\frac{h^2}{R^2}=\frac{R}{A}\left[\left\{B_1+\frac{B_2-B_1}{D}(R_2)\right\}l_n\,\frac{R_1}{R_2}-(B_2-B_1)\right]-1$$

where B_1, B_2=width of the outer and inner surfaces respectively

R_1, R_2=radius of outer and inner surfaces respectively

D=depth of the section

A=area of the section.

5. A circular ring of mean radius R is subjected to a diametral pull P, then resultant stress on any section at an angle θ from the load line is

$$f_R = \frac{P}{A}\left[\frac{R^2}{\pi(R^2+h^2)} + \frac{R^2}{2h^2}\left(\frac{R^2}{R^2+h^2}\times\frac{2}{\pi} - \sin\theta\right)\frac{y}{R+y}\right] + \frac{P}{2A}\sin\theta$$

where $\qquad y$ = distance of the layer under consideration from the centroidal layer.

6. A chain link of radius at end R, length of the straight portion l, is subjected to an axial tensile load P.

Resultant stress on any section inclined at an angle θ from the load line

$$f_R = \frac{P}{AR}\left[1+\frac{R^2}{h^2}\times\frac{y}{R+y}\right]K' + \frac{PR^2}{2Ah^2}(1-\sin\theta)\frac{y}{R+y} + \frac{P}{2A}(1+\sin\theta)$$

where
$$K' = \frac{\frac{R^2}{\pi} - \frac{R^2}{2} - \frac{h^2}{2}}{R + \frac{h^2 l}{\pi k^2} + \frac{h^2}{R}}$$

y = distance of the layer under consideration from the centroidal layer

k = radius of gyration of the section.

7. Deflection in a curved bar along a chord

$$\delta_{chord} = -\int h\,\frac{M\,ds}{EI}$$

where $\qquad M$ = bending moment on a section

h = perpendicular distance from the point on centre line of the section to the chord

E = Young's modulus of elasticity

I = moment of inertia of the section.

8. A circular ring of mean radius R subjected to a diametral load P

Deflection along the load $= \dfrac{PR^3}{EI}\left[\dfrac{\pi}{4} - \dfrac{2}{\pi}\left(\dfrac{R^2}{R^2+h^2}\right)\right]$

Deflection perpendicular to the load line

$$= \frac{PR^3}{EI}\left[\frac{1}{2} - \frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)\right].$$

9. A chain link of mean radius at ends R and length of the straight portion l, subjected to axial load P

Deflection along the load line

$$= \frac{PR^2}{EI}\left[\frac{\pi R}{4} - 2K' - R\right] - \frac{PRK'l}{EI} + \frac{Pl}{2AE}$$

the value of K' is as above

Deflection perpendicular to the load line

$$= -\frac{2PR^2}{EI}\left[K' + \frac{R}{4}\right] - \frac{PRK'l}{EI}.$$

MULTIPLE CHOICE QUESTIONS

1. A bar of square section 4 cm × 4 cm is curved to a mean a radius of 8 cm. A bending moment M is applied on the bar. The moment M tries to straighten the bar. If the stress at the innermost fibres is 80 N/mm² tensile, then the stress at the outermost fibres is

   (a) 80 N/mm² (compressive) (b) 80 N/mm² (tensile)

   (c) More than 80 N/mm² (compressive) (d) Less than 80 N/mm² (compressive).

2. The most suitable section of a crane hook is

   (a) Square (b) Round

   (c) Hollow round (d) Trapezoidal.

3. If bar of square section 4 cm × 4 cm is curved to a mean radius of 80 metres. A bending moment M, tending to increase the curvature is applied on the bar. If the stress at the outermost fibres is 1000 kg/cm² tensile, then the stress at the innermost fibres will be appproximately equal to

   (a) 1500 kg/cm² (compressive) (b) 1250 kg/cm² (compressive)

   (c) 1125 kg/cm² (compressive) (d) 1000 kg/cm² (compressive)

4. A ring is subjected to a diametral tensile load. The variation of the stress at the intrados surface from the point of loading up to the section of symmetry is

   (a) Maximum tensile stress to maximum compressive stress

   (b) Throughout tensile stress

   (c) Maximum compressive stress to maximum tensile stress

   (d) Throughout compressive stress.

5. The distribution of stress along a section of a curved bar subjected to a bending moment tending to increase its curvature is

   (a) Linear (b) Uniform

   (c) Parabolic (d) Hyperbolic.

ANSWERS

1. (d) 2. (d) 3. (d) 4. (c) 5. (d)

EXERCISES

19·1. Prove that the ratio of the extreme tensile and compressive stresses in the case of a curved bar subjected to pure bending is approximately 1·76 if the bar is of rectangular section whose depth, $D=8$ cm and the radius of curvature, $R=10$ cm.

19·2. A curved bar of rectangular section with breadth B and depth $D=2B$ is bent to a radius of curvature equal to 1·2 D. It is subjected to a bending moment of 1 k Nm tending to increase its curvature. Determine the size of the section if the maximum stress is not to exceed 80 N/mm². [**Ans.** $B=24·35$ mm. $D=48·7$ mm]

19·3. For the frame of a punching machine shown in Fig. 19·25. Determine the circumferential stress at the points A and B on a section inclined at an angle $\theta=60°$ to the vertical. Take force $P=150$ kN. [**Ans.** 50·08 N/mm² (tensile), 36·86 N/mm² (compressive)]

19·4. A curved beam with a circular centre line has a trapezoidal cross section shown in Fig. 19·29 and is subjected to pure bending in its plane of symmetry. The face b_1 is of the concave side of the beam. If h=10 cm, and a=10 cm, find the ratio of b_1/b_2 of base widths so that the extreme fibre stresses in tension and compression will be numerically equal. [**Ans.** 1·87]

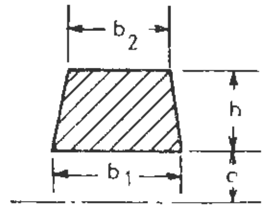


Fig. 19·29

19·5. The cross section of a triangular hook has a base of 4 cm and altitude 6 cm and a radius of curvature of 5 cm at the inner face of the shank. If a load of 500 kg is applied along a line 8 cm from the inner face of the shank, determine the maximum tensile and compressive stresses developed in the critical section of the shank.

[**Ans.** 579·02 kg/cm² (tensile), 560·48 kg/cm² (compressive)]

19·6. A chain link is made of round steel rod of diameter 12 mm. If R=40 mm and l=60 mm, draw the stress distribution diagram along the extrados for an angle of 90° starting from the outermost edge (along the direction of loading) of the curved bar, if the bar is subjected to a tensile load of 10 kN.

[**Ans.** 119·32 N/mm² (tensile) to −27·08 N/mm² (compressive) stress]

20

Unsymmetrical Bending and Shear Centre

In chapter 8 on theory of simple bending, an assumption is taken that section of the beam is symmetrical about the plane of bending. This condition is satisfied if the plane of the loads contains the axis of symmetry of all the sections of the beam. Beam sections like circular, rectangular, square and I sections are symmetrical about the plane of bending and about the neutral axis, while T-section and channel section with web horizontal are symmetrical about the plane of bending but unsymmetrical about the neutral axis. Then a beam section such as angle-section, is not symmetrical about both the centroidal axis. If the cross section of the beam has an axis of symmetry then this axis of symmetry is always a principal axis of inertia, and if beam section has two axes of symmetry, then these are two principal axes.

If the load line on a beam does not coincide with one of the principal axes of the section, the bending takes place in a plane different from the plane of principal axes. This type of bending is known as unsymmetrical bending. There are two reasons of unsymmetrical bending as follows :

I. The section is symmetrical like I section, rectangular section, circular section but the load-line is inclined to both the principal axes.

II. The section itself is unsymmetrical like angle section or a channel section with vertical-web and load line is along any centroidal axis.

Fig. 20·1 (*a*) shows a beam with I section with load-line coinciding with *Y-Y* principal axis. I-section has two axes of symmetry and both these axes are principal axis. Section is symmetrical about *Y-Y* plane, *i.e.*, the plane of bending. This type of bending is known as symmetrical bending.

Fig. 20·1 (*b*) shows a cantilever with rectangular section, which has two axes of symmetry which are principal axes but the load line is inclined at an angle α with *Y-Y* axis. This is first type of unsymmetrical bending. Then Fig. 20·1 (*c*) shows a cantilever with angle-section which does not have any axes of symmetry, *i.e.*, *X-X* and *Y-Y* are not the axes of symmetry. Load line is coinciding with *Y-Y* axis. This is the second type of unsymmetrical bending.

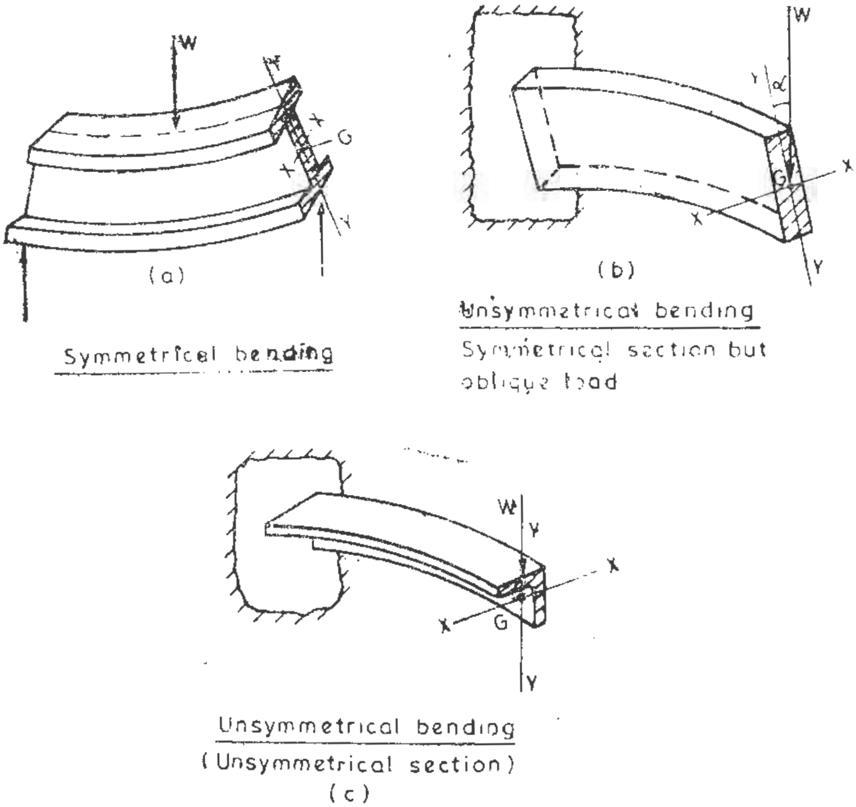


Fig. 20·1

Before we proceed further let us study about the principal axes of a section.

20·1. PRINCIPAL AXES

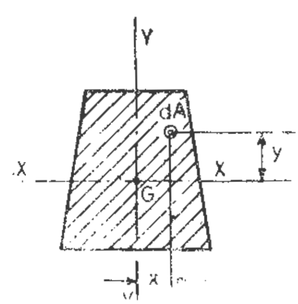


Fig. 20·2

Fig. 20·2 shows a beam section which is symmetrical about the plane of bending *Y-Y*, a requirement of the theory of simple bending or symmetrical bending. *G* is the centroid of the section. *XX* and *YY* are the two perpendicular axes passing through the centroid. Say the bending moment on the section (in the plane *YY* of the beam) is *M*, about the axis *XX*. Consider a small element of area *dA* with (x, y) co-ordinates

Stress on the element, $f = \frac{M}{I_{xx}} \cdot y$ ...(1)

Force on the element, $dF = \frac{MydA}{I_{xx}}$

Bending moment about YY axis,

$$dM=\frac{My\,xdA}{I_{xx}}$$

Total moment, $$M'=\int\frac{MyxdA}{I_{xx}} \qquad ...(2)$$

If no bending is to take place about YY axis, then

$$M'=0$$

or $$\int\frac{MyxdA}{I_{xx}}=0$$

or $$\frac{M}{I_{xx}}\int xy\,.\,dA=0$$

or $$\int xydA=0 \qquad ...(3)$$

The expression $\int xydA$ is called a **product of inertia,** of the area about X-X and YY axis, represented by I_{xy}. If the product of inertia is zero about the two co-ordinate axes passing through the centroid, then the bending is symmetrical or pure bending. Such axes (about which product of inertia is zero) are called **Principal axes** of the section and moment of inertia about the principal axes are called **Principal moments of inertia.**

The product of inertia may be positive, negative or zero depending upon the section and co-ordinate axis. The product of inertia of a section with respect to two perpendicular axes is zero if either one of the axes is an axis of symmetry.

Example 20·1-1. Show that product of inertia of a T-section about a centroidal axis is zero.

Solution. Fig. 20·3 shows a T-section with flange $B\times h_1$ and web $b\times h_2$. The section is symmetrical about YY axis. Say G is the centroid the section on the axis YY, and X-X and YY are the centroidal axes

$$I_{xy}=I_{xy}' \text{ for flange}+I_{xy}'' \text{ for web}$$

For flange x varies from

$$-\frac{B}{2}\text{ to }+\frac{B}{2}$$

For web x varies from

$$-\frac{b}{2}\text{ to }+\frac{b}{2}$$

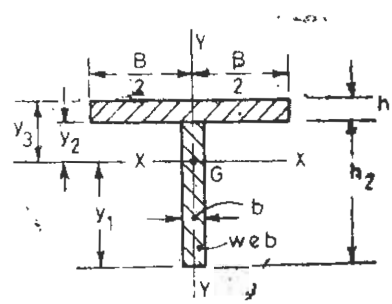


Fig. 20·3

Now, $$I_{xy}=\int_{y_2}^{y_3}\int_{-B/2}^{+B/2} xy\,dx\,dy+\int_{-y_1}^{y_2}\int_{-b/2}^{+b/2} xy\,dy\,dy$$

$$= \int_{y_2}^{y_3} \left| \frac{x^2}{2} \right|_{-B/2}^{+B/2} y\,dy + \int_{-y_1}^{y_2} \left| \frac{x^2}{2} \right|_{-b/2}^{+b/2} y\,dy$$

$$= O \times \underset{\text{(for flange)}}{\int_{y_2}^{y_3}} y\,dy + O \times \underset{\text{(for web)}}{\int_{-y_1}^{y_2}} y\,dy = 0$$

Example 20·1-2. Determine the product of inertia about axes X and Y for a triangular section shown in the Fig. 20·4.

Solution. Consider a small element of area dA at co-ordinates x, y.

Product of inertia about XY axis,

$$I_{xy} = \int_0^{20} \int_0^{2y} xy\,dx\,dy$$

Note that limiting value of $x = 40$ mm $= 2\times$ limiting value of y

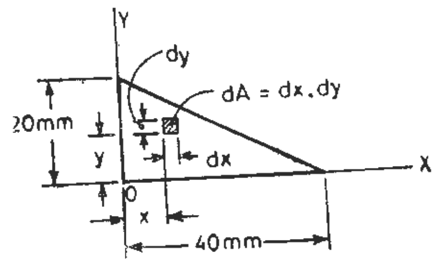


Fig. 20·4

$$= \int_0^{20} \left[\int_0^{2y} x\,dx \right] y\,dy \text{ and also } = \int_0^{40} \left[\int_0^{x/2} y\,dy \right] x\,dx$$

$$= \int_0^{20} \left| \frac{x^2}{2} \right|_0^{2y} y\,dy = \int_0^{20} 2y^2.y\,dy = \int_0^{20} 2y^3\,dy$$

$$= \left| \frac{y^4}{2} \right|_0^{20} = \frac{20^4}{2} = 80000 \text{ mm}^4.$$

Exercise 20·1-1. Consider an I section with flanges $B \times t_1$ and web $H \times t_2$ and show that product of inertia about its centroidal axes is zero.

Exercise 20·1-2. Fig. 20·5 shows a rectangular section with breadth b and height h. Determine the product of inertia of the section about X-Y axis.

$$\left[\textbf{Ans. } \frac{b^2h^2}{4} \right]$$

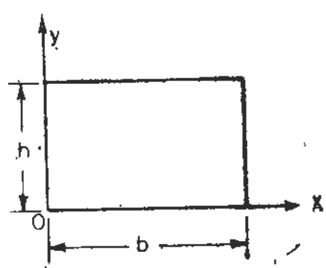


Fig. 20·5

20·2. PARALLEL AXES THEOREM FOR PRODUCT OF INERTIA

Fig. 20·6 shows a section with its centroid at G and GX', GY' are the two rectangular co-ordinates passing through G. Say the product of inertia about $X'Y'$ is $I_{\bar{x}\bar{y}}$, let us determine the product of inertia about the axes OX and OY i.e. I_{xy}.

Say distance of G from OX axis $=\bar{y}$
and distance of G from OY axis $=\bar{x}$

Consider a small element of area

$$dx \,.\, dy$$

Say co-ordinates of the element about the centroidal axes GX', GY' are x', y'.

Then co-ordinates of the element about X-Y axes are

$$x=\bar{x}+x' \qquad \text{and} \qquad y=\bar{y}+y'$$

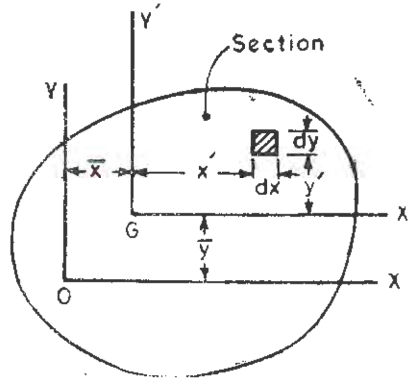


Fig. 20·6

So product of inertia,
$$I_{xy}=\int xy\, dA=\int (\bar{x}+x')(\bar{y}+y')\, dxdy$$
$$=\int x'y'\, dA+\bar{x}\bar{y}\int dA+\bar{y}\int x'\, dA+\bar{x}\int y'dA$$
$$=I_{\bar{x}\bar{y}}+\bar{x}\,\bar{y}\,A+0+0$$

$$\left(\text{because} \int x'\, dA=\int y'\, dA=0 \text{ about centroidal axes}\right)$$

$$I_{xy}=I_{\bar{x}\bar{y}}+A\,\bar{x}\,\bar{y}$$

i.e. the product of inertia of any section with respect to any set of co-ordinate axes in its plane is equal to the product of inertia of the section with respect to the centroidal axes parallel to the co-ordinate axes plus the product of the area and the co-ordinates of the centroid of the section with respect to the given set of co-ordinate axes.

Example 20·2-1. Fig. 20·7 shows an unequal channel section, determine its product of inertia I_{xy} and $I_{\bar{x}\bar{y}}$.

Solution. Let us break up the section into 3 rectangular strips I, II and III as shown and write the co-ordinates of their centroids with respect to the given set of axes YOX.

Strip	*Area*	$\bar{x}$	$\bar{y}$	$A\,\bar{x}\,\bar{y}$
I	20 cm^2	5 cm	1 cm	100 cm^4
II	8 cm^2	0·5 cm	6 cm	24 cm^4
III	12 cm^2	3 cm	11 cm	396 cm^4

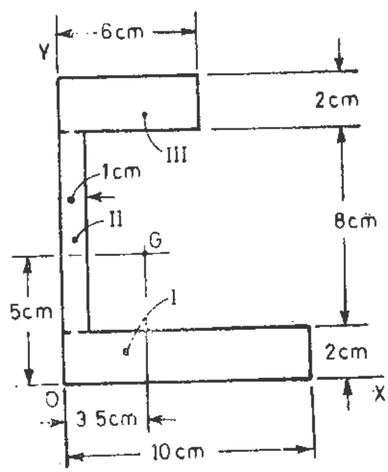


Fig. 20·7

Remember that the product of inertia of these rectangular strips about their principal axes passing through the respective centroids is zero, because rectangular strips have two axes of symmetry.

$$(I_{xy})_I = 0+100 \text{ cm}^4$$
$$(I_{xy})_{II} = 0+24 \text{ cm}^4$$
$$(I_{xy})_{III} = 0+396 \text{ cm}^4$$
$$I_{xy} = 520 \text{ cm}^4$$

(using the parallel axis theorem for product of inertia)

To determine $I_{\bar{x}\bar{y}}$, let us first determine the position of the centroid of the section.

$$\bar{x} = \frac{20\times5+8\times0{\cdot}5+12\times3}{20+8+12} = 3{\cdot}5 \text{ cm}$$

$$\bar{y} = \frac{20\times1+8\times6+12\times11}{20+8+12} = 5 \text{ cm}$$

Area of cross section, $A = 20+8+12 = 40 \text{ cm}^2$

$$I_{\bar{x}\bar{y}} = I_{xy} - A\,\bar{x}\,\bar{y} = 520 - 40\times3{\cdot}5\times5 = -180 \text{ cm}^4$$

Exercise 20·2·1. Fig. 20·8 shows an an unequal angle section, determine its product of inertia I_{xy} and $I_{\bar{x}\bar{y}}$ (through the centroidal axes)

[**Ans.** $24{\cdot}75\times10^4$ mm^4, $-32{\cdot}30\times10^4$ mm^4]

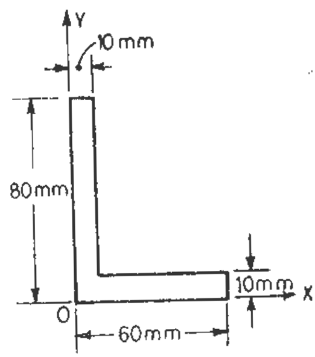


Fig. 20·8

20·3. DETERMINATION OF PRINCIPAL AXES

In article 20·1 we have learnt that principal axes pass through the centroid of a section and product of inertia of the section about principal axes is zero. Fig. 20·9 shows a section with centroid G. XX and YY are two co-ordinate axes passing through G. Say UU and VV is another set of axes passing through the centroid G and inclined at an angle θ to the X-Y co-ordinate. Consider an element of area dA at point P having co-ordinates (x, y). Say u, v are the co-ordinates of the point P in U-V co-ordinate axes.

So $\quad u = GA' = GD + DA' = GD + AE$ (as shown in the enlarged figure)

where $\quad GD = GA \cos\theta = x\cos\theta$

$AE = DA' = y\sin\theta$

or $\quad u = x\cos\theta + y\sin\theta$

$v = GB' = PA' = PE - A'E$

$= PE - AD \quad$ since $A'E = AD$

$= PA\cos\theta - x\sin\theta = y\cos\theta - x\sin\theta$

Similarly x, y co-ordinates can be written in terms of u, v co-ordinates.

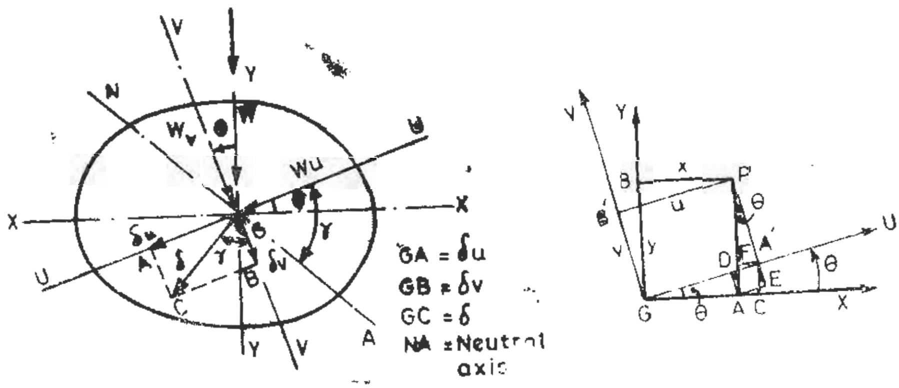


Fig. 20·9

$$x = GC - AC = GC - A'F = u\cos\theta - v\sin\theta$$

(as $PA' = v$ and $GA' = u$)

$$y = GB = PA = AF + FP = A'C + FP = u\sin\theta + v\cos\theta$$

Second moment of area about U-U

$$I_{uu} = \int v^2 dA = \int (y\cos\theta - x\sin\theta)^2\, dA$$

$$= \int y^2\cos^2\theta\, dA + \int x^2\sin^2\theta\, dA - \int 2xy\sin\theta\cos\theta\, dA$$

$$= I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - \int \sin 2\theta\,.\, xy\, dA$$

$$= I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - I_{xy}\sin 2\theta$$

$$= \tfrac{1}{2}(I_{xx} + I_{yy}) + \tfrac{1}{2}(I_{xx} - I_{yy})\cos 2\theta - I_{xy}\sin 2\theta \qquad \ldots(1)$$

Second moment of area about V-V

$$I_{vv} = \int u^2\, dA = \int (x\cos\theta + y\sin\theta)^2\, dA$$

$$= \int x^2\cos^2\theta\, dA + \int y^2\sin^2\theta\, dA + \int 2xy\sin\theta\cos\theta\,.\, dA$$

$$= I_{yy}\cos^2\theta + I_{xx}\sin^2\theta + I_{xy}\sin 2\theta$$

$$= \tfrac{1}{2}(I_{xx} + I_{yy}) + \tfrac{1}{2}(I_{yy} - I_{xx})\cos 2\theta + I_{xy}\sin 2\theta \qquad \ldots(2)$$

From equations (1) and (2),

$$I_{uu} + I_{vv} = I_{xx}(\sin^2\theta + \cos^2\theta) + I_{yy}(\sin^2\theta + \cos^2\theta) = I_{xx} + I_{yy} \qquad \ldots(3)$$

Product of inertia about UV axes

$$I_{uv} = \int uv\, dA = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\, dA$$

$$=\int xy\,(\cos^2\theta-\sin^2\theta)\,dA+\int y^2\sin\theta\cos\theta\,dA-\int x^2\sin\theta\cos\theta\,dA$$

$$I_{uv}=I_{xy}\ \cos 2\theta+I_{xx}\ \frac{\sin 2\theta}{2}-I_{yy}\ \frac{\sin 2\theta}{2}$$

But as per the condition of pure bending or symmetrical bending $I_{uv}=0$, then U and V will be the principal axes

or
$$2\ I_{xy}\cos 2\theta+(I_{xx}-I_{yy})\sin 2\theta=0$$

or
$$\tan 2\theta=\frac{2\ I_{xy}}{I_{yy}-I_{xx}}=\frac{I_{xy}}{(I_{yy}-I_{xx})/2} \qquad \ldots(4)$$

Say θ_1 and θ_2 are two values of θ given by equation (4)

$$\theta_2=\theta_1+90^\circ$$

$$\sin 2\theta_1=\frac{I_{xy}}{\sqrt{\left(\frac{I_{yy}-I_{xx}}{2}\right)^2+I_{xy}^2}}\quad\text{and}\quad\cos 2\theta_1=\frac{(I_{yy}-I_{xx})/2}{\sqrt{\left(\frac{I_{yy}-I_{xx}}{2}\right)^2+I_{xy}^2}}$$

Substituting these values of $\sin 2\theta_1$ and $\cos 2\theta_1$

$$(I_{uu})_{\theta_1}=\frac{1}{2}(I_{xx}+I_{yy})+\frac{\frac{1}{2}(I_{xx}-I_{yy})\,\frac{1}{2}(I_{yy}-I_{xx})}{\sqrt{\left[\frac{1}{2}(I_{yy}-I_{xx})\right]^2+I_{xy}^2}}$$

$$-\frac{I_{xy}.I_{xy}}{\sqrt{\left[\frac{1}{2}(I_{yy}-I_{xx})\right]^2+I_{xy}^2}}$$

$$(I_{uu})_{\theta_1}=\frac{1}{2}(I_{xx}+I_{yy})-\sqrt{\left[\frac{1}{2}(I_{yy}-I_{xx})\right]^2+I_{xy}^2} \qquad \ldots(5)$$

Similarly
$$(I_{vv})_{\theta_1}=\frac{1}{2}(I_{xx}+I_{yy})+\sqrt{\left[\frac{1}{2}(I_{yy}-I_{xx})\right]^2+I_{xy}^2} \qquad \ldots(6)$$

Now for
$$\theta_2=\theta_1+\pi/2$$

$$\sin 2\theta_2=\sin(2\theta_1+\pi)=-\sin 2\theta_1$$

$$\cos 2\theta_2=\cos(2\theta_1+\pi)=-\cos 2\theta_1$$

Substituting these values in equations (1) and (2)

$$(I_{uu})_{\theta_2}=\frac{1}{2}(I_{xx}+I_{yy})+\sqrt{\left[\frac{1}{2}(I_{yy}-I_{xx})\right]^2+I_{xy}^2} \qquad \ldots(7)$$

$$(I_{vv})_{\theta_2}=\frac{1}{2}(I_{xx}+I_{yy})-\sqrt{\left[\frac{1}{2}(I_{yy}-I_{xx})\right]^2+I_{xy}^2} \qquad \ldots(8)$$

From equations 5, 6, 7, 8 we learn that

$$(I_{uu})_{\theta_1}=(I_{vv})_{\theta_2},\ \text{and}\ (I_{vv})_{\theta_1}=(I_{uu})_{\theta_2}$$

Maximum and minimum values of I_{uu} and I_{vv}

$$I_{uu}=\frac{1}{2}(I_{xx}+I_{yy})+\frac{1}{2}(I_{yy}-I_{xx})\cos 2\theta+I_{xy}\sin 2\theta$$

For maximum value of I_{uu},

$$\frac{dI_{uu}}{d\theta}=0$$

i.e. $$\frac{1}{2}(I_{yy}-I_{xx})(-2\sin 2\theta)+I_{xy}\times 2\cos 2\theta=0$$

or $$\tan 2\theta=\frac{I_{xy}}{(I_{yy}-I_{xx})/2}$$

This shows that the values of $(I_{uu})_{\theta_1}$ and $(I_{vv})_{\theta_1}$ are the maximum and minimum values of I_{uu} and I_{vv}. These values are called the principal values of moment of inertia as $I_{uv}=0$. The directions θ_1 and θ_2 are called the principal directions.

Moment of inertia about any axis

If the principal moments of inertia I_{uu} and I_{vv} are known then moment of inertia about any axis inclined at an angle θ to the principal axes can be determined. Say u, v are the co-ordinates of an element of area dA in the U-V principal axes system. X and Y are the co-ordinates axes inclined at an angle θ to the U-V axes.

x co-ordinate of element $=u\cos\theta-v\sin\theta$

y co-ordinate of element $=u\sin\theta+v\cos\theta$

Moment of inertia, $$I_{yy}=\int x^2\,dA=\int(u\cos\theta-v\sin\theta)^2\,dA$$

$$=\int u^2\cos^2\theta\,dA+\int v^2\sin^2\theta\,dA-\int 2\,uv\sin\cos\theta\,dA$$

$$=I_{vv}\cos^2\theta+I_{uu}\sin^2\theta-0\quad\text{since}\int uv\,dA=0$$

$$=I_{vv}\cos^2\theta+I_{uu}\sin^2\theta \qquad ...(9)$$

Similarly $$I_{xx}=I_{uu}\cos^2\theta+I_{vv}\sin^2\theta \qquad ...(10)$$

From equations (9) and (10)

$$I_{xx}+I_{yy}=I_{uu}+I_{vv}=J,$$

polar moment of inertia about an axis passing through G and normal to the section.

Example 20·3-1. Determine the principal moments of inertia for the equal angle shown in the Fig. 20·10.

Solution. Let us consider the angle section in two portions I and II as shown and determine the position of the centroid

$$\bar{x}=\bar{y}=\frac{10\times 0{\cdot}5+9\times 5{\cdot}5}{19}=2{\cdot}87\text{ cm}$$

(due to symmetry $\bar{x}=\bar{y}$)

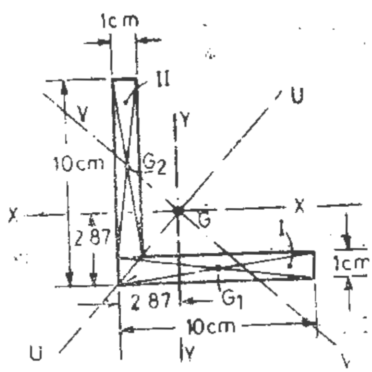


Fig 20·10

Moment of inertia, $I_{xx}=I_{yy}$

$$=\frac{10\times 1^3}{12}+10\,(2{\cdot}87-0{\cdot}5)^2+\frac{9\times 1^3}{12}+9\,(2{\cdot}87-0{\cdot}5)^2$$
$$=0{\cdot}833+56{\cdot}169+0{\cdot}750+50{\cdot}552$$
$$=108{\cdot}304 \text{ cm}^4$$

Co-ordinates of centroid of portion I

$$=[(5-2{\cdot}87),-(2{\cdot}87-0{\cdot}5)]=(2{\cdot}13,\ -2{\cdot}37)$$

Co-ordinates of centroid of portion II

$$=[-(2{\cdot}87-0{\cdot}5),\ (5{\cdot}5-2{\cdot}87)]=(-2{\cdot}37,\ 2{\cdot}63)$$

Product of inertia, $I_{xy}=10\,(2{\cdot}13)\,(-2{\cdot}37)+9\,(2{\cdot}63)\,(-2{\cdot}37)$

(as the product of inertia about their own centroidal axes is zero, since portions I and II are rectangles).

So $I_{xy}=-50{\cdot}481-56{\cdot}098=-106{\cdot}579 \text{ cm}^4$

If θ=angle of the principal axes UU with respect to X-axis

$$\tan 2\theta=\frac{I_{xy}}{(I_{yy}-I_{xx})/2}=-\frac{106{\cdot}579}{0{\cdot}0}=\infty$$

or $2\theta=90°$, or $\theta=45°$

Principal angles are $\theta_1=45°$, $\theta_2=90°+45°=135°$

Principal moments of inertia

$$I_{uu}=\frac{1}{2}\,(I_{xx}+I_{yy})+\frac{1}{2}\,(I_{xx}-I_{yy})\cos 2\theta_1-I_{xy}\sin 2\theta_1$$
$$=\frac{1}{2}\,(108{\cdot}304+108{\cdot}304)+\frac{1}{2}\times 0\times\cos 90°+106{\cdot}579\times\sin 90°$$
$$=108{\cdot}304+106{\cdot}579=214{\cdot}883 \text{ cm}^4$$
$$I_{vv}=I_{xx}+I_{yy}-I_{uu}=2\times 108{\cdot}304-214{\cdot}883$$
$$=1{\cdot}725 \text{ cm}^4.$$

Example 20 3-2. Fig. 20·11 shows ın I section 15 cm×20 cm. Axes $X'X'$ and $Y'Y'$ are inclined at an angle of 30° to the axes of symmetry. Determine the moment of inertia about these axes. Calculate also the product of inertia $I_{x'y'}$.

Solution. The I section shown has two axes of symmetry *i.e.*, UU and VV passing through the centroid G. Therefore, these are the principal axes and I_{uu} and I_{vv} are the principal moments of inertia. The angle of inclinations of UU and VV axes with respect to $X'X'$ and $Y'Y'$ axes is $\theta=30°$.

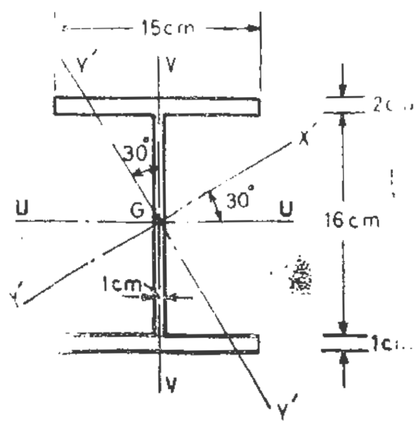


Fig. 20·11

$$\sin^2\theta=0{\cdot}25 \qquad \cos^2\theta=0{\cdot}75$$

$$I_{y'y'}=I_{VV}\cos^2\theta+I_{uu}\sin^2\theta$$

$$I_{x'x'}=I_{uu}\cos^2\theta+I_{VV}\sin^2\theta$$

Now $$I_{uu}=\frac{15\times20^3}{12}-\frac{14\times16^3}{12}=10{,}000-4778{\cdot}667=5221{\cdot}333\text{ cm}^4$$

$$I_{VV}=\frac{2\times15^3}{12}+\frac{16\times1^3}{12}+\frac{2\times15^3}{12}=562{\cdot}5+1{\cdot}333+562{\cdot}5$$

$$=1126{\cdot}333\text{ cm}^4$$

Therefore, $$I_{Y'Y'}=1126{\cdot}333\times0{\cdot}75+5221{\cdot}333\times0{\cdot}25$$

$$\doteq844{\cdot}749+1305{\cdot}333=2150{\cdot}082\text{ cm}^4$$

$$I_{X'X'}=5221{\cdot}333\times0{\cdot}75+1126{\cdot}33\times0{\cdot}25$$

$$=3915{\cdot}999+281{\cdot}583=4197{\cdot}582\text{ cm}^4$$

Now $$I_{UU}=\frac{1}{2}(I_{X'X'}+I_{Y'Y'})+\frac{1}{2}(I_{X'X'}-I_{Y'Y'})\cos2\theta-I_{X'Y'}\sin2\theta$$

$$5221{\cdot}333=\frac{1}{2}(4197{\cdot}582+2150{\cdot}082)$$

$$+\frac{1}{2}(4197{\cdot}582-2150{\cdot}082)\cos60^\circ-I_{X'Y'}\times0{\cdot}866$$

$$5221{\cdot}333=3173{\cdot}832+511{\cdot}875-I_{X'Y'}\times0{\cdot}866$$

$$I_{X'Y'}=-\frac{1515{\cdot}626}{0{\cdot}866}=-1750{\cdot}145\text{ cm}^4$$

Exercise 20·3-1. Determine the principal angles and principal moments of inertia for the section shown in the Fig. 20·12.

[**Ans.** 29° 31′, 119° 31′ ; $I_{UU}=360{\cdot}044$ cm^4, $I_{VV}=38{\cdot}29$ cm^4]

Exercise 20·3-2. Consider a rectangular section of 6 cm width and 12 cm depth. Determine I_{XX}, I_{YY} and I_{XY} about XX and YY axis inclined at an angle of 45° to the principal axes.

[**Ans.** 540·0 cm^4, 540·0 cm^4, −324·0 cm^4]

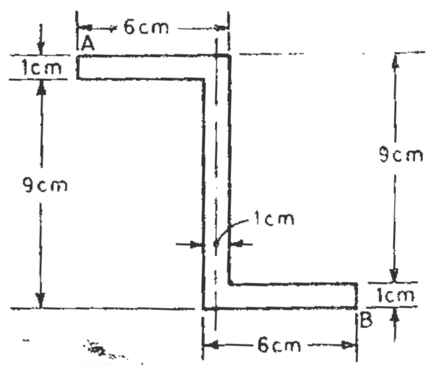


Fig. 20·12

20·4. STRESSES DUE TO UNSYMMETRICAL BENDING

When the load line on a beams does not coincide with one of the principle axes of th section, unsymmetrical bending takes place. Fig. 20·13 (*a*) shows a rectangular sectio symmetrical about XX and YY axis or with U-U and V-V principal axes. Load line is incline at an angle ϕ to the principal axes VV, and passing through G (centroid) or C (shear centre) the section.

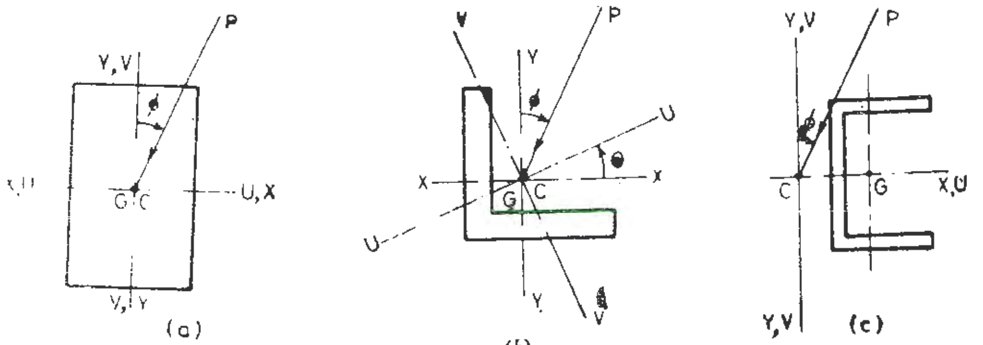


Fig. 20·13

Fig. 20·13 (*b*) shows an angle section which does not have any axis of symmetry. Principal axes UU and VV are inclined to axes XX and YY at an angle θ. Load line is inclined at an angle ϕ to the vertical or at an angle $(90-\phi-\theta)$ to the axis U-U. Load line is passing through G (centroid of the section) or C (shear centre).

Fig. 20·13 (*c*) shows a channel section which has one axis of symmetry *i.e.*, XX. Therefore, UU and VV are the principal axis. G is the centroid of the section while C is the shear centre. Load line is inclined at an angle ϕ to the vertical (or the axis VV) and passing through the shear centre of the section.

Shear centre for any transverse section of a beam is the point of inter section of the bending axis and the plane of transverse section. If a load passes through the shear centre there will be only bending of the beam and no twisting will occur. If a section has two axes of symmetry, then shear centre coincides with the centre of gravity or centroid of the section as in the case of a rectangular, circular or I section. *For sections having one axis of symmetry only, shear centre does not coincide with centroid but lies on the axis of symmetry,* as shown in the case of a channel section.

For a beam subjected to symmetrical bending only, following assumptions are made :

(*i*) The beam is initially straight and of uniform section throughout

(*ii*) Load or loads are assumed to act through the axis of bending

(*iii*) Load or loads act in a direction perpendicular to the bending axes and load line passes through the centre of transverse section.

Fig. 20·14 shows the cross section of a beam subjected to bending moment M, in the plane YY. G is the centroid of the section and XX and YY are the two co-ordinate axes passing through G. Moreover UU and VV are the principal axes inclined at an angle θ to the XX and YY axis respectively. Let us determine the stresses due to bending at the point P having the co-ordinates u, v corresponding to principal axes. Moment in the plane YY can be resolved into two components M_1 and M_2.

M_1, moment in the plane $UU = M \sin \theta$

M_2, moment in the plane $VV = M \cos \theta$

The components M_1 and M_2 have their axes along VV and UU respectively.

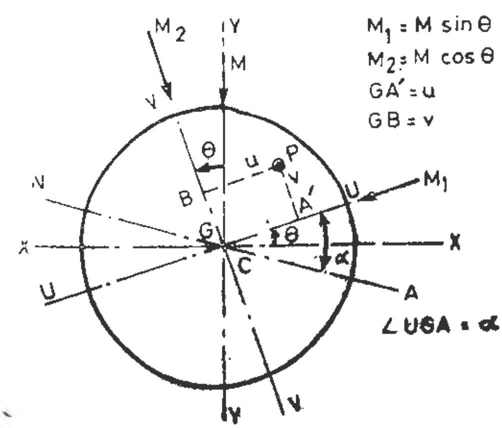


Fig. 20·14

Resultant bending stress at the point P

$$f_b=\frac{M_1 . u}{I_{vv}}+\frac{M_2 . v}{I_{uu}}=\frac{M \sin \theta . u}{I_{vv}}+\frac{M \cos \theta . v}{I_{uu}}$$

$$=M\left[\frac{v \cos \theta}{I_{uu}}+\frac{u \sin \theta}{I_{vv}}\right] \quad ...(1)$$

The exact nature (whether tensile or compressive) depends upon the quadrant in which the point P lies. In other words sign of co-ordinates u and v is to be taken into account while determining the resultant bending stress.

The equation of the neutral axis can be determined by considering the resultant bending stress. At the neutral axis bending stress is zero *i.e.*,

$$M\left[\frac{v \cos \theta}{I_{uu}}+\frac{u \sin \theta}{I_{vv}}\right]=0$$

or

$$v=-\frac{\sin \theta}{\cos \theta}\times\frac{I_{uu}}{I_{vv}} . u$$

$$=-\tan \alpha . u \quad ...(2)$$

where

$$\tan \alpha=\frac{\sin \theta}{\cos \theta} . \frac{I_{uu}}{I_{vv}}=\tan \theta\left(\frac{I_{uu}}{I_{vv}}\right)$$

This is the equation of a straight line passing through the centroid G of the section. All the points of the section on one side of the neutral axis have stresses of the same nature and all the points of the section on the other side of the neutral axis have stresses of opposite nature.

Example 20·4-1. A 40 mm×40 mm×5 mm angle section shown in the Fig. 20·15 is used as a simply supported beam over a span of 2·4 metres. It carries a 0·200 kN of load along the line YG, where G is the centroid of the section. Determine resultant bending stresses on point A, B and C *i.e.*, outer corners of the section, along the middle section of the beam.

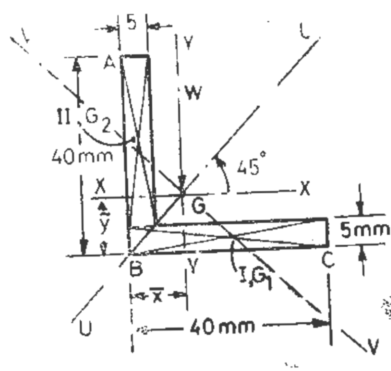


Fig. 20·15

Solution. Let us first determine the position of the centroid

$$\bar{x}=\bar{y}=\frac{40\times5\times2·5+35\times5\times22·5}{200+175}$$

$$=\frac{500+3937·5}{375}=11·83 \text{ mm}$$

Moments of inertia, $I_{xx}=\frac{5\times35^3}{12}+5\times35(22·5-11·83)^2+\frac{40\times5^3}{12}+40\times5(11·83-2·5)^2$

$$=17864·583+19923·557+416·667+17409·780$$

$$=55614·537 \text{ mm}^4=5·561\times10^4 \text{ mm}^4$$

$$=I_{yy} \text{ (because it is equal angle section)}$$

Co-ordinates of G_1 (centroid of portion I)

$$=+(20-11\cdot83),\ -(11\cdot83-2\cdot5)$$
$$=(8\cdot17,\ -9\cdot33)$$

Co-ordinates of centroid

$$G_2=-(11\cdot83-2\cdot5),\ +(22\cdot5-11\cdot83)=-9\cdot33,\ +10\cdot67$$

Product of inertia, $I_{xy}=40\times5(8\cdot17)(-9\cdot33)+35\times5(-9\cdot33)(10\cdot67)$

(Product of inertia about their centroidal axes is zero because portion I and II are rectangular strips)

$$I_{xy}=-15245\cdot22-17421\cdot44=-32666\cdot66\ \text{mm}^4$$
$$=-3\cdot266\times10^4\ \text{mm}^4.$$

Principal angle, θ

$$\tan 2\theta=\frac{I_{xy}}{\frac{1}{2}(I_{yy}-I_{xx})}=\frac{-3\cdot266\times10^4}{0}=\alpha$$
$$=\tan 90^\circ\quad \theta=45^\circ.$$

Principal Moment of Inertia

$$I_{uu}=\tfrac{1}{2}(I_{xx}+I_{yy})+\tfrac{1}{2}(I_{xx}-I_{yy})\cos 90^\circ-I_{xy}\sin 90^\circ$$
$$=\tfrac{1}{2}(5\cdot561+5\cdot50)\times10^4+\tfrac{1}{2}\times0\times\cos 90^\circ+3\cdot266\times10^4$$
$$=5\cdot561+3\cdot266)\times10^4=8\cdot827\times10^4\ \text{mm}^4$$
$$I_{vv}=I_{xx}+I_{yy}-I_{uu}=(5\cdot561+5\cdot561-8\cdot827)\times10^4$$
$$=2\cdot295\times10^4\ \text{mm}^4$$

Bending moment $\quad M=\frac{Wl}{4}=\frac{0\cdot200\times10^3\times2\cdot4\times10^3}{4}=0\cdot120\times10^6\ \text{N mm}$

Components of bending moment,

$$M_1=M\sin 45^\circ=0\cdot120\times0\cdot707\times10^6=84\cdot84\times10^3\ \text{N mm}$$
$$M_2=M\cos 45^\circ=0\cdot120\times0\cdot707\times10^6=84\cdot84\times10^3\ \text{N mm}$$

u-v co-ordinates of the points

Point A. $\quad x=-11\cdot83,\ y=40-11\cdot83=28\cdot17$

$$u=x\cos\theta+y\sin\theta=-11\cdot83\times0\cdot707+28\cdot17\times0\cdot707=11\cdot55\ \text{mm}$$
$$v=y\cos\theta-x\sin\theta=28\cdot17\times0\cdot707+11\cdot83\times0\cdot707=28\cdot28\ \text{mm}$$

Point B. $\quad x=-11\cdot83,\ \ y=-11\cdot83$

$$u=-11\cdot83\times0\cdot707-11\cdot83\times0\cdot707=-16\cdot727\ \text{mm},\ V=0$$

Point C. $\quad x=40-11\cdot83=28\cdot17.\ \ y=-11\cdot83$

$$u=28\cdot17\times\cos 45^\circ-11\cdot83\sin 45^\circ$$
$$=28\cdot17\times0\cdot707-11\cdot83\times0\cdot707=11\cdot55\ \text{mm}$$
$$v=-11\cdot83\times0\cdot707-28\cdot17\times0\cdot707=-28\cdot28\ \text{mm}$$

Resultant bending stresses at points A, B and C.

$$f_A=\frac{M_1u}{I_{vv}}+\frac{M_2v}{I_{vv}}=84\cdot84\times10^3\left[\frac{11\cdot55}{2\cdot295\times10^4}+\frac{28\cdot28}{8\cdot827\times10^4}\right]$$
$$=69\cdot88\ \text{N/mm}^2$$

$$f_B = 84 \cdot 84 \times 10^3 \left[\frac{-11 \cdot 627}{2 \cdot 295 \times 10^4} + \frac{0}{8 \cdot 827 \times 10^4} \right] = -42 \cdot 98 \text{ N/mm}^2$$

$$f_C = 84 \cdot 84 \times 10^3 \left[\frac{11 \cdot 55}{2 \cdot 295 \times 10^4} - \frac{28 \cdot 28}{8 \cdot 827 \times 10^4} \right] = +15 \cdot 51 \text{ N/mm}^2.$$

Example 20·4-2. Fig. 20·16 shows I section of a cantilever 1·2 metres long subjected to a load $W=40$ kg at free end along the direction $Y'G$ inclined at 15° to the vertical. Determine the resultant bending staess at corners A and B, at the fixed section of the cantilever.

Solution. I section is symmetrical about XX and YY axis, therefore XX and YY are the principal axes UU and VV.

Moment of Inertia

$$I_{uu} = I_{xx} = \frac{3 \times 5^3}{12} - \frac{2 \cdot 8 \times 4 \cdot 5^3}{12}$$

$$= 31 \cdot 25 - 21 \cdot 26$$

$$= 9 \cdot 99 \text{ cm}^4$$

$$I_{vv} = I_{yy} = \frac{0 \cdot 25 \times 2 \times 3^3}{12} + \frac{4 \cdot 5 \times (0 \cdot 2)^3}{12}$$

$$= 1 \cdot 125 + 0 \cdot 003 = 1 \cdot 128 \text{ cm}^4$$

Maxm. Bending moment

$$M = Wl = 40 \times 120 = 4800 \text{ kg-cm}$$

Components of bending moment

$$M_1 = M \sin 15° = 4800 \times 0 \cdot 2588$$

$$= 1242 \cdot 24 \text{ kg-cm}$$

$$M_2 = M \cos 15° = 4800 \times 0 \cdot 9659$$

$$= 4636 \cdot 32 \text{ kg-cm}$$

Due to bending moment M_1, there will be tensile stresses at points B and and compressive stresses at points D and A

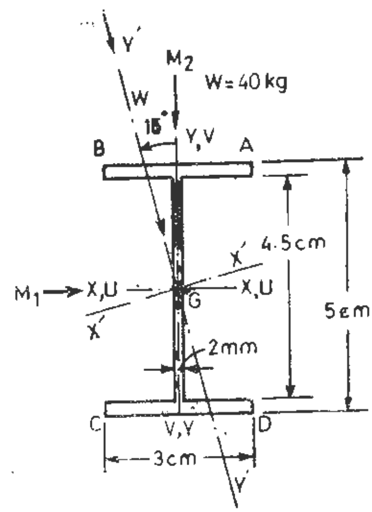


Fig. 20·16

Due to bending moment M_2 there will be tensile stress on points A and B and compressive stress on points C and D.

Resultant bending stress on A,

$$f_A = \frac{M_2 \times 2 \cdot 5}{I_{xx}} - \frac{M_1 \times 1 \cdot 5}{I_{yy}}$$

$$= \frac{4636 \cdot 32 \times 2 \cdot 5}{9 \cdot 99} - \frac{1242 \cdot 24 \times 1 \cdot 5}{1 \cdot 128} = 1160 \cdot 24 - 1651 \cdot 91$$

$$= -491 \cdot 67 \text{ kg/cm}^2$$

Resultant stress on B, $f_B = \dfrac{M_2 \times 2 \cdot 5}{I_{xx}} + \dfrac{M_1 \times 1 \cdot 5}{I_{yy}}$

$$= \frac{4636 \cdot 32 \times 2 \cdot 5}{9 \cdot 99} + \frac{1242 \cdot 24 \times 1 \cdot 5}{1 \cdot 128} = 1160 \cdot 24 + 1651 \cdot 91$$

$$= 2812 \cdot 15 \text{ kg/cm}^2.$$

Exercise 20·4-1. Fig. 20·12 shows Z-section of a beam simply supported over a span of 2 metres. A vertical load $W=2$ kN acts at the centre of the beam and passes through the centroid of the section. Determine the resultant bending stress at points A and B.

[**Ans.** $-17·05$ N/mm², $+17·05$ N/mm²]

Exercise 20·4-2. A cantilever of rectangular section of breadth=4 cm and depth 6 cm is subjected to an inclined load W at the free end. The length of the cantilever is 2 metres and the angle of inclination of the load to the vertical is 25°. What is the maximum value of W if the maximum stress due to bending is not to exceed 200 N/mm².

[**Ans.** 1558·24 N]

20·5. DEFLECTION OF BEAMS DUE TO UNSYMMETRICAL BENDING

Fig. 20·17 shows the transverse section of a beam with centroid G. X-X and Y-Y are two rectangular co-ordinate axes and U-U and V-V are the principal axes inclined at an angle θ to the XY set of co-ordinate axes. Say the beam is subjected to a load W along the line YG. This load can be resolved into two components *i.e.*,

$$W_u = W \sin \theta$$

(along UG direction)

$$W_v = W \cos \theta$$

(along VG direction)

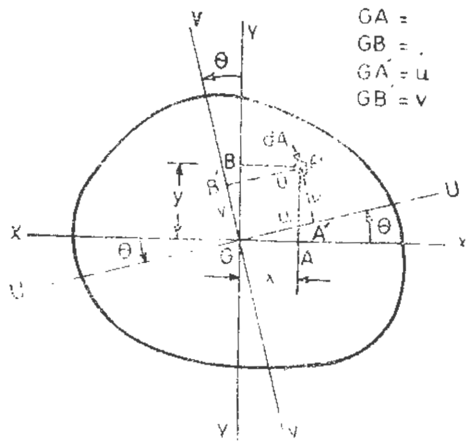


Fig. 20·17

Say deflection due to W_u is GA in the direction GU

i.e., $$GA = \delta_u = \frac{K.W_u.l^3}{E\,I_{vv}}$$

where K is a constant depending upon the end conditions of the beam and position of the load along the beam.

Deflection due to W_v is GB in the direction GV

i.e., $$GB = \delta_v = \frac{KW_v\,l^3}{EI_{uu}}$$

Total deflection, $$\delta = \sqrt{\delta_u^2 + \delta_v^2} = \frac{Kl^3}{E}\sqrt{\left[\frac{W_u^2}{I_{vv}^2} + \frac{W_v^2}{I_{uu}^2}\right]}$$

$$= \frac{KWl^3}{E}\sqrt{\frac{\sin^2\theta}{I_{vv}^2} + \frac{\cos^2\theta}{I_{uu}^2}}$$

Total deflection δ is along the direction GC, at angle γ to VV axis

$$\tan\gamma = \frac{CG}{GB} = \frac{GA}{GB} = \frac{W_u}{I_{vv}} \times \frac{I_{uu}}{W_v}$$

$$= \frac{W\sin\theta}{W\cos\theta} \times \frac{I_{uu}}{I_{vv}} = \tan\theta\;\frac{I_{uu}}{I_{vv}}$$

Comparing this with the equation (2) of article 20 (4)

$$\tan\alpha = \tan\theta \,.\, \frac{I_{uu}}{I_{vv}},$$

where α is the angle of inclination of the neutral axis with respect to UU axis

and $$\tan\gamma = \tan\theta \cdot \frac{I_{uu}}{I_{vv}}$$

where γ is the angle of inclinations of direction of δ with respect to VV axis

$\gamma=\alpha$, showing thereby that resultant deflection δ takes place in a direction perpendicular to the neutral axis.

Example 20·5-1. A simply supported beam of length 2 metres carries a central load 4 kN inclined at 30° to the vertical and passing through the centroid of the section. Determine (1) maximum tensile stress (2) maximum compressive stress and (3) deflection due to the load (4) direction of neutral axis. Given $E=200\times10^5$ N/cm².

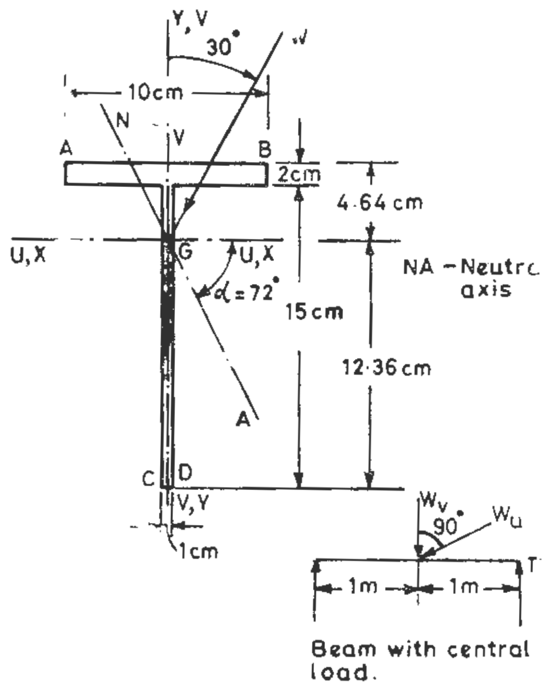


Fig. 20·18

Solution. Let us first determine the position of the centroid of the T section shown in the Fig. 20·18.

$$\bar{y}=\frac{15\times1\times7{\cdot}5+10\times2\times(15+1)}{15+20}=12{\cdot}36 \text{ cm}$$

The section is symmetrical about vertical axis, therefore the principal axes pass through the centroid G and are along U-U and V-V axes shown.

So $$I_{xx}=I_{uu}=\frac{10\times2^3}{12}+20\,(4{\cdot}64-1{\cdot}0)^2+\frac{1\times15^3}{12}+15\,(12{\cdot}36-7{\cdot}5)^2$$

$$=6{\cdot}667+264{\cdot}992+281{\cdot}250+354{\cdot}294$$
$$=907{\cdot}203 \text{ cm}^4$$

$$I_{yy}=I_{vv}=\frac{2\times10^3}{12}+\frac{15\times1^3}{12}=166{\cdot}667+1{\cdot}250=167{\cdot}917 \text{ cm}$$

Load, $W=4000$ N

Components of W, $W_v=4000\times\cos 30°=4000\times 0.866=3464$ N

$W_u=4000\times\sin 30°=4000\times 0.500=2000$ N

Bending moment, $M_v=\dfrac{W_v\times l}{4}=\dfrac{3464\times 200}{4}=173{,}200$ N cm

Bending moment, $M_u=\dfrac{W_u\times l}{4}=\dfrac{2000\times 200}{4}=100{,}000$ N cm

Due to M_v there will be maximum compressive stress on A and B and maximum tensile stress at C and D.

Due to M_u there will be maximum compressive stress at B and D and maximum tensile stress at A and D.

So maximum compressive stress at B,

$$f_B=\frac{M_v\times 4.64}{I_{uu}}+\frac{M_u\times 5}{I_{vv}}$$

$$=\frac{173200\times 4.64}{907.203}+\frac{100000\times 5}{167.917}=885.852+2977.661$$

$$=3863.5\ \text{N/cm}^2=38.63\ \text{N/mm}^2$$

Maximum tensile stress at C

$$f_C=\frac{M_v\times 12.36}{I_{uu}}+\frac{M_u\times 0.5}{I_{vv}}$$

$$=\frac{173200\times 12.36}{907.203}+\frac{100000\times 0.5}{167.917}$$

or $$=2359.727+297.766=2657.493\ \text{N/cm}^2$$

$$=26.57\ \text{N/mm}^2$$

Deflection $$\delta=\frac{KWl^3}{E}\sqrt{\frac{\sin^2\theta}{I_{vv}^2}+\frac{\cos^2\theta}{I_{uu}^2}}$$

where $K=1/48$ as the beam is simply supported and carries a concentrated load at its centre

So $$\delta=\frac{KWl^3}{EI_{uu}}\sqrt{\sin^2\theta\times\left(\frac{I_{uu}}{I_{vv}}\right)^2+\cos^2\theta}$$

Now $\sin\theta=0.5$; $\sin^2\theta=0.25$

$\cos\theta=0.866$, $\cos^2\theta=0.75$

$$\delta=\frac{1}{48}\times\frac{4000\times(200)^3}{200\times 10^5\times 907.203}\sqrt{0.25\times\left(\frac{907.203}{169.917}\right)^2+0.75}$$

$$=0.0367\sqrt{0.25\times 28.50+0.75}$$

$$=0.0367\times 2.8065=0.103\ \text{cm}=1.03\ \text{mm}$$

Position of the neutral axis

$$\tan\alpha=\tan\theta\,\frac{I_{uu}}{I_{vv}}=\tan 30°\times\frac{907.203}{169.917}=0.5774\times 5.339$$

$$=3.0828$$

$$\alpha\simeq 72°$$

Exercise 20·5-1. A cantilever 2·8 m long having T-section with flange 12 cm × 2 cm and web 13 cm × 2 cm carries a concentrated load W its free end but inclined at an angle of 45° to the vertical. Determine the maximum value of W if the deflection at the free end is not to exceed 2 mm. Given $E=200\times10^3$ N/mm². What is the direction of neutral axis with respect to the vertical axis. [**Ans.** 221·20 N, 15° 24′]

20·6. SHEAR CENTRE

In chapter 9 we studied about the distribution of shear stresses in the transverse section of a beam subjected to bending moment M and shear force F. Summation of shear stresses over the section of the beam gives a set of forces which must be in equilibrium with the applied shear force F. In the case of symmetrical sections such as rectangular and I sections, the applied shear force is balanced by the set of shear forces summed over the rectangular section or over the flanges and web of I section and the *shear centre* coincides with the centroid of the section. If the applied load is not placed at the shear centre, the section twists about this point and this point is also known as *centre of twist*. So the shear centre of a section can be defined as a point about which the applied shear force is balanced by the set of shear forces obtained by summing the shear stresses over the section.

For unsymmetrical sections such as angle section and channel section, summation of shear stresses in each leg gives a set of forces which should be in equilibrium with the applied shear force.

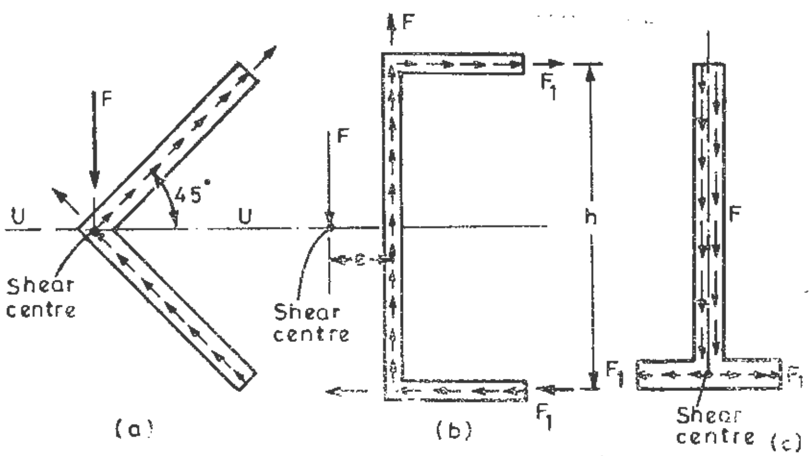


Fig. 20·19

Fig. 20·19 (*a*) shows an equal angle section with principal axes UU. We have learnt in previous examples that a principal axis of equal angle section passes through the centroid of the section and corner of the equal angle as shown in the Fig. Say this angle section is subjected to bending about a principal axis UU with shearing force F at right angles to this axis. The sum of the shear stresses along the legs, gives a shear force in the direction of each leg as shown. It is obvious that the resultant of these shear forces in legs passes through the corner of the angle and unless the applied force F is applied through this point, there will be twisting of the angle section in addition to bending. This point of the equal angle section is called its shear centre or centre of twist.

For a beam of channel section subjected to loads parallel to the web, as shown in Fig. 20·19 (*b*), the total shearing force carried by the the web must be equal to applied shear force F, then in flanges there are two equal and opposite forces say F_1 each. Then for equilibrium $F \times e$ is equal to $F_1.h$, and we can determine the position of the shear centre along the axis of symmetry *i.e.* $e = \dfrac{F_1 \times h}{F}$

Similarly Fig. 20·19 (*c*) shows a T-section and its shear centre. Vertical force in web F is equal to the applied shear force F and horizontal forces F_1 in two portions of the flange balance each other at shear centre.

Example 20·6-1. Fig. 20·20 shows a channel section, determine its shear centre.

Solution. Fig. 20·20 shows a channel section with flanges $b \times t_1$ and web $h \times t_2$. X-X is the horizontal symmetric axis of the section. Say F is the applied shear force, vertically downwards. Then shear force in the web will be F upwards. Say the shear force in the top flange $= F_1$.

Shear stress in the flange at a distance of x from right hand edge

$$= \frac{F a \bar{y}}{I_{xx}\, t}$$

where F = applied shear force

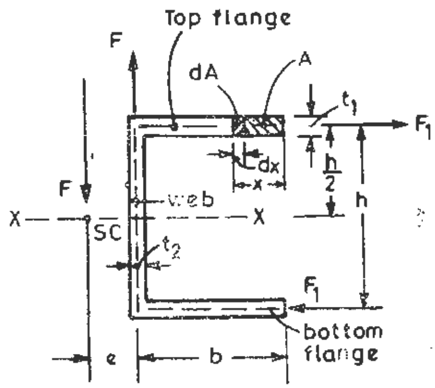


Fig. 20·20

$$a\bar{y} = (t_1 \,.\, x)\, \frac{h}{2}, \text{ first moment of area about axis } X\text{-}X$$

$$t = t_1 \qquad \text{(thickness of the flange)}$$

$$q = \frac{F.t_1 x}{I_{xx}.t_1}\, \frac{h}{2} = \frac{Fxh}{2\, I_{xx}}$$

Shear force in elementary area

$(t_1\, dx = dA)$ $\qquad = q.dA = q.t_1.dx$

Total shear force in top flange

$$= \int_0^b q.t_1.dx \qquad \text{where } b = \text{breadth of the flange}$$

(say) $$F_1 = \int_0^b \frac{Fxt_1 h}{2\, I_{xx}}\, dx = \frac{Ft_1 h}{I_{xx}}\, \frac{b^2}{4}$$

There will be equal and opposite shear force in the bottom flange.

Say shear centre is at a distance of e from web along the symmetric axis XX.

Then for equilibrium

$$F.e = F_1 h = \frac{F.t_1 h^2 b^2}{4\, I_{xx}} \quad \text{or} \quad e = \frac{t_1 b^2 h^2}{4\, I_{xx}}$$

$\therefore$ Moment of inertia, $$I_{xx}=\frac{t_2h^3}{12}+\frac{2\times b\times t_1^3}{12}+2\times b\times t_1\left(\frac{h}{2}\right)^2$$

in which, the expression $\frac{2bt_1^3}{12}$ is negligible in comparison to other terms

$$\therefore \qquad I_{xx}=\frac{t_2h^3}{12}+\frac{bt_1h^2}{2}=\frac{h^2}{12}\,(t_2h+6bt_1)$$

Substituting this in the expression for e

$$e=\frac{t_1b^2h^2}{4}\times\frac{12}{h^2(t_2h+6bt_1)}=\frac{3t_1b^2}{(t_2h+6bt_1)}$$

if we take bt_1=area of flange$=A_f$

ht_2=area of web$=A_w$

Then $$e=\frac{3b\,A_f}{A_w+6A_f}=\frac{3b}{6+\frac{A_w}{A_f}}$$

Exercise 20·6-1. A channel section has flanges 6 cm×1 cm and web 8 cm×0·5 cm. Determine the position of its shear centre. [Note that b=5·75 cm and h=9 cm.]

[**Ans.** e=2·543 cm]

Problem 20·1. Find the product of inertia of a quadrant of a circle about axes X and Y as shown in Fig. 20·21.

Solution. Fig. 20·21 shows the quadrant of a circle of radius R. Consider a small element at radius r, radial thickness dr and subtending an angle $d\theta$ at the centre.

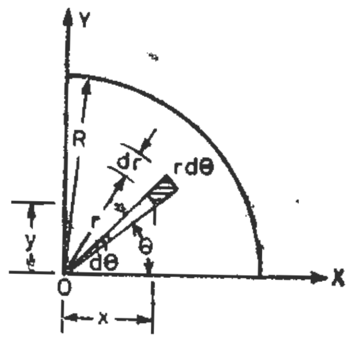


Fig. 20·21

Area of the element,

$$dA=r\,d\theta.dr$$

Co-ordinates of the element

$$x=r\cos\theta \quad\text{and}\quad y=r\sin\theta$$

Product of inertia, $$I_{r\theta}=\int_0^R\int_0^{\pi/2}(r\cos\theta.r\sin\theta)\,rd\theta\,dr$$

$$=\int_0^R\left\{r^3\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right\}dr$$

$$=\int_0^R r^3\left|-\frac{\cos 2\theta}{4}\right|_0^{\pi/2}dr=\int_0^R r^3\left(\frac{2}{4}\right)dr=\frac{R^4}{8}$$

Problem 20·2 A beam of angle section shown in Fig. 20·22 is simply supported over a span of 1·6 metres with 15 cm leg vertical. A uniformly distributed vertical load of 10 k N/m is applied throughout the span. Determine (*a*) maximum bending stress (*b*) direction of neutral axis (*c*) deflection at the centre. $E=210$ k N/mm².

Solution. Let us first determine the position of the centroid

$$\bar{y}=\frac{10\times1\times0{\cdot}5+14\times1\times8}{14+10}$$

$$=\frac{117}{24}=4{\cdot}875 \text{ cm}$$

$$\bar{x}=\frac{14\times0{\cdot}5+10\times5}{14+10}$$

$$=\frac{57}{24}=2{\cdot}375 \text{ cm}$$

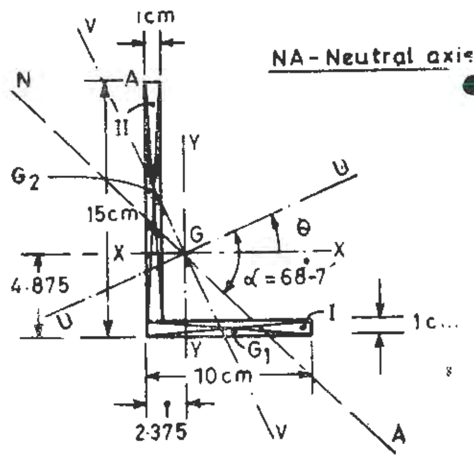


Fig. 20·22

Moment of inertia

$$I_{xx}=\frac{10\times1^3}{12}+10(4{\cdot}875-0{\cdot}5)^2+\frac{1\times14^3}{12}+14(7+1-4{\cdot}875)^2$$

$$=0{\cdot}833+191{\cdot}406+228{\cdot}667+136{\cdot}718$$

$$=557{\cdot}624 \text{ cm}^4$$

$$I_{yy}=\frac{1\times10^3}{12}+10(5-2{\cdot}375)^2+\frac{14\times1^3}{12}+14(2{\cdot}375-0{\cdot}5)^2$$

$$=83{\cdot}333+68{\cdot}906+1{\cdot}167+49{\cdot}218=202{\cdot}624 \text{ cm}^4$$

Co ordinates of G_2 and G_1 : $[-1{\cdot}875, (8-4{\cdot}875)]$ and $[(5-2{\cdot}375), -4{\cdot}375]$

$$I_{xy}=14(8-4{\cdot}875)(-1{\cdot}875)+10(5-2{\cdot}375)(-4{\cdot}375)$$

$$=-82{\cdot}031-114{\cdot}843=-196{\cdot}874 \text{ cm}^4$$

(Note that parallel axes theorem for product of inertia is used here and product of inertia of rectangular strips about their own centroidal axes is zero)

Directions of Principal axes

$$\tan 2\theta=-\frac{I_{xy}}{(I_{yy}-I_{xx})/2}=\frac{-196{\cdot}874}{(202{\cdot}624-557{\cdot}624)/2}$$

$$=\frac{196{\cdot}874}{177{\cdot}5}=1{\cdot}1091$$

$$2\theta=47^\circ\ 58' \quad \text{or} \quad \theta=23^\circ\ 59'$$

$$\cos 2\theta=0{\cdot}6695 \qquad \sin 2\theta=0{\cdot}7420.$$

Principal moments of Inertia

$$I_{uu}=\frac{1}{2}(I_{xx}+I_{yy})+\frac{1}{2}(I_{xx}-I_{yy})\cos\theta-I_{xy}\sin 2\theta$$

$$=\frac{1}{2}(557{\cdot}624+202{\cdot}624)+\frac{1}{2}(557{\cdot}624-202{\cdot}624)\times0{\cdot}6695+196{\cdot}874\times0{\cdot}742$$

$$=380{\cdot}124+177{\cdot}5\times0{\cdot}6695+146{\cdot}080=645{\cdot}040 \text{ cm}^4$$

$$I_{vv}=\frac{1}{2}(I_{xx}+I_{yy})+\frac{1}{2}(I_{yy}-I_{xx})\cos 2\theta+I_{xy}\sin 2\theta$$
$$=380{\cdot}124-177{\cdot}5\times0{\cdot}6695-196{\cdot}874\times0{\cdot}742$$
$$=380{\cdot}124-118{\cdot}836-146{\cdot}080=115{\cdot}208\text{ cm}^4$$

(*a*) **Maximum bending stress**

$$\omega=\text{rate of loading}=10\text{ kN/m}=10{,}000\text{ N/m}=100\text{ N/cm}$$

Components,

$$\omega_u=\omega\sin\theta=100\times0{\cdot}4065=40{\cdot}65\text{ N/cm}$$
$$\omega_v=\omega\cos\theta=100\times0{\cdot}9138=91{\cdot}38\text{ N/cm.}$$

The beam is simply supported and carries uniformly distributed load, the maximum bending moment occurs at the centre of the beam.

Bending moment $M_u=\dfrac{\omega_u l^2}{8}=\dfrac{40{\cdot}65\times160\times160}{8}=130080$ N cm

(above span length $l=160$ cm)

Bending moment $M_v=\dfrac{\omega_v l^2}{8}=\dfrac{91{\cdot}38\times160\times160}{8}=292416$ N cm

As is obvious, maximum bending stress occurs at the point A with co-ordinates

$$x=-2{\cdot}375,\quad y=15-4{\cdot}875=10{\cdot}125$$

Co-ordinates $u=x\cos\theta+y\sin\theta=-2{\cdot}375\times0{\cdot}9138+10{\cdot}125\times0{\cdot}4065$

$$=-2{\cdot}170+4{\cdot}116=+1{\cdot}946\text{ cm}$$
$$v=y\cos\theta-x\sin\theta=10{\cdot}125\times0{\cdot}9138+2{\cdot}375\times0{\cdot}4065$$
$$=9{\cdot}252+0{\cdot}965=10{\cdot}217\text{ cm}$$

Maximum Bending stress,

$$f_A=\frac{M_u.u}{I_{vv}}+\frac{M_v.v}{I_{uu}}$$
$$=\frac{130080\times1{\cdot}946}{115{\cdot}208}+\frac{29{\cdot}2416\times10{\cdot}217}{645{\cdot}040}=2197{\cdot}20+4631{\cdot}67$$
$$=6828{\cdot}87\text{ N/cm}^2=68{\cdot}28\text{ N/mm}^2$$

Direction of neutral axis

$$\tan\alpha=\tan\theta\,.\,\frac{I_{uu}}{I_{vv}}=0{\cdot}4448\times\frac{645{\cdot}040}{115{\cdot}208}=2{\cdot}4904$$
$$\alpha=68°\ 7'$$

Deflection at the centre $=\dfrac{kl^4\omega}{E}\sqrt{\dfrac{\sin^2\theta}{I_{vv}^2}+\dfrac{\cos^2\theta}{I_{uu}^2}}$

where ω = rate of loading

Constant, $k=\dfrac{5}{384}$

Span length, $l=160$ cm.

$E=210\times10^5$ N/cm²

Deflection, $$\delta=\frac{5}{384}\times\frac{160^4}{210\times10^5}\times100\times\frac{1}{I_{uu}}\int\sqrt{\sin^2 . \frac{I_{uu}^2}{I_{vv}^2}+\cos^2\theta}$$

$$=\frac{40\cdot635}{I_{uu}}\sqrt{(0\cdot4065)^2\times\left(\frac{645\cdot040}{115\cdot208}\right)^2+(0\cdot9138)^2}$$

$$=\frac{40\cdot635}{645\cdot040}\times\sqrt{5\ 180+0\cdot835}=0\cdot154\text{ cm}=1\cdot54\text{ mm}$$

(in the direction perpendicular to the neutral axis)

Problem 20·3. A cantilever of I section 3 m long carries a load of 2 kN at the free end and 3 kN at its middle. Line of load 2 kN is passing through the centroid of the section and inclined at an angle of 30° to the vertical and the line of application of load 3 kN is also passing through the centroid but inclined at 45° to the vertical on the other side of load 2 kN as shown in the Fig. 20·23. I section has two flanges 12 cm×2 cm and web 16 cm×1 cm. Determine the the resultant bending stress at the corners *A*, *B*, *C* and *D*.

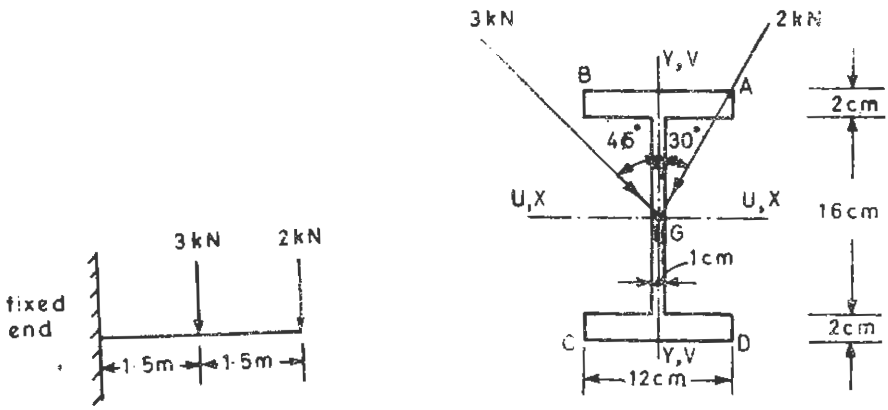


Fig. 20·23

Solution. Moment of Inertia,

$$I_{xx}=\frac{12\times20^3}{12}-\frac{11\times16^3}{12}=8000-3754\cdot667$$

$$=4245\cdot333\text{ cm}^4$$

$$I_{yy}=\frac{2\times2\times12^3}{12}+\frac{16\times1^3}{12}=576+1\cdot333$$

$$=577\cdot333\text{ cm}^4$$

I section shown is symmetrical about *XX* and *YY* axis, so principal axes *UU* and *VV* passing through the centroid of the section are along *XX* and *YY* axis.

Loads applied can be resolved into components along *U* and *V* directions.

Components of 2kN load

$$W_{u_1}=2000\times\sin 30°=100\text{ N}$$

$$W_{v_1}=2000\times\cos 30°=1732\text{ N}$$

Components of 3 kN load

$$W_{u_2}=3000\times\sin 45°=3000\times0\cdot707=2121\text{ N}$$

$$W_{v_2}=3000\times\cos 45°=3000\times0\cdot707=2121\text{ N}.$$

Bending moments at the fixed end

$$M_u = W_{u1}\times 3 + W_{u2}\times 1{\cdot}5$$
$$= -1000\times 3 + 2121\times 1{\cdot}5 \text{ Nm}$$
$$= 181{\cdot}5 \text{ Nm} = 0{\cdot}18\times 10^5 \text{ N cm}$$
$$M_v = W_{v1}\times 300 + W_{v2}\times 150$$
$$= 1732\times 300 + 2121\times 150$$
$$= 8{\cdot}38\times 10^5 \text{ N cm.}$$

Resultant bending stresses

$$f_A = -\frac{M_u\times 6}{I_{vv}} + \frac{M_v\times 10}{I_{uu}} = \frac{0{\cdot}18\times 10^5\times 6}{577{\cdot}333} + \frac{8{\cdot}38\times 10^5\times 10}{4245{\cdot}333}$$
$$= -1{\cdot}870 + 19{\cdot}739)\times 10^2 \text{ N/cm}^2 = 17{\cdot}869 \text{ N/mm}^2$$
$$f_B = +\frac{M_u\times 6}{I_{vv}} + \frac{M_v\times 10}{I_{uu}} = (+1{\cdot}870 + 19{\cdot}739)\times 10^2 \text{ N/mm}^2$$
$$= 21{\cdot}609 \text{ N/mm}^2.$$

Due to M_u there will be tensile stresses on points B and C and compressive stresses on points D and A.

Due to M_v there will tensile stress on points A and B, and compressive stress on points C and D.

Stress, $$f_C = +\frac{6M_u}{I_{vv}} - \frac{M_v\times 10}{I_{uu}} = (+1{\cdot}870 - 19{\cdot}739)\times 10^2 \text{ N/cm}^2$$
$$= -17{\cdot}869 \text{ N/mm}^2$$

Stress, $$f_D = -\frac{6M_u}{I_{vv}} - \frac{M_v\times 10}{I_{uu}} = (-1{\cdot}87 - 19{\cdot}739)\times 10^2 \text{ N/cm}^2$$
$$= -21{\cdot}609 \text{ N/mm}^2$$

Problem 20·4. Fig. 20·24 shows an unequal I section. Dctermine the position of its shear centre.

Solution. Fig. 20·24 shows an unequal I section, with flanges b_1+b_2 wide and t_1 thick. The web is h high and t_2 thick. The section is symmetric about X-X axis, therefore shear centre will lie on this axis. Fig. shows the direction of shear flow in flanges and in the web. Say the shear force in flange for width b_1 is F_1 and for width b_2 is F_2. The shear force in web is say F_3. The applied shear force is F acting at the shear centre.

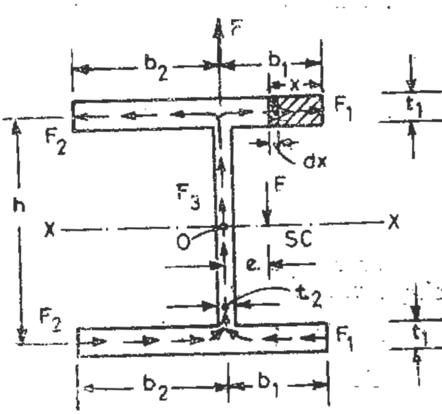


Fig. 20·24

For equilibrium

$$F_3 = F \qquad \ldots(1)$$

Shear stress in any layer,

$$q = \frac{Fay}{It}$$

where $$I = I_{xx} = 2(b_1+b_2)\,\frac{t_1^3}{12} + 2(b_1+b_2)t_1\times\frac{h^2}{4} + t_2\,.\,\frac{h^3}{12} \qquad \ldots(2)$$

and t = thickness of the section

Shear Force F_1

Considering an area $dA=t_1\, dx$.

$$a\bar{y}=x\,.\,t_1\,.\,\frac{h}{2}$$

$$F_1=\int_0^{b_1} q\,.\,dA=\frac{Fxt_1}{I_{xx}\,t_1}\quad\frac{h}{2}\times t_1\,dx$$

$$\int_0^{b_1}\frac{F}{2\;I_{xx}}\times h\,t_1\,.\,x\,dx$$

$$=\frac{Fht_1}{2I_{xx}}\left|\frac{x^2}{2}\right|_0^{b_1}=\frac{Fa\,t_1\,b_1^2}{4\,I_{xx}} \qquad \ldots(3)$$

Similarly the shear force in the other portion of the flange,

$$F_2=\frac{Fh\,t_1\,.\,b_2^2}{4I_{xx}}$$

Taking moments of the shear force about the centre of the web

$$F_2\times h=F_1\times h+F\,.\,e$$

$$(F_2-F_1)h=F\,.\,e$$

$$\frac{Fh^2\,t_1}{4I_{xx}}(b_2^2-b_1^2)=Fe$$

or distance of shear centre from the centre of the web

$$e=\frac{t_1h^2\,(b_2^2-b_1^2)}{4\,I_{xx}}.$$

Problem 20·5. Determine the position of the shear centre of the section of a beam shown in Fig. 20·25.

Solution. Fig. 20·25 shows the section for which the shear centre is to be determined. In the diagram direction of shear flow is given. Due to symmetry shear forces, $F_1=F_5$ shear forces, $F_2=F_4$. The section is symmetrical about the axis XX, therefore shear centre will lie on this axis.

Let us determine shear force F_1 or F_5

Shear stress in the vertical portion AB

$$q=\frac{FA\bar{y}}{I_{xx}t_1}$$

$$=\frac{F\,(b_1-y)\,t_1}{I_{xx}\,t_1}\times\left(\frac{h}{2}+y+\frac{b_1-y}{2}\right)$$

$$=\frac{F\,(b_1-y)}{I_{xx}}\left(\frac{h+b_1+y}{2}\right)$$

where F is the applied shear force on the section

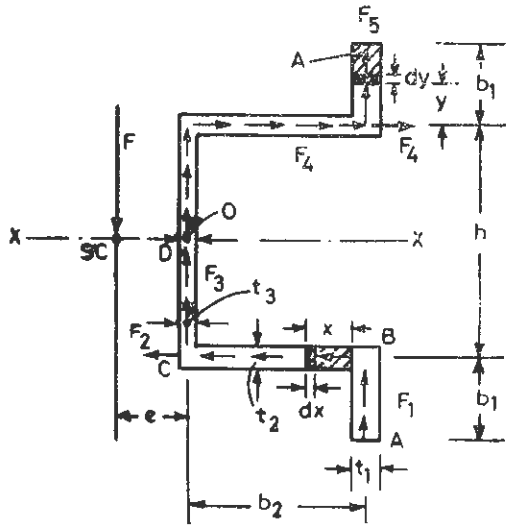


Fig. 20·25

Now, $dA=t_1\,dy$

Shear force
$$F_1=\int_0^{b_1}\frac{F\,(b_1-y)}{2\,I_{xx}}\,(h+b_1+y)\,t_1\,dy$$

$$=\frac{Ft_1}{2I_{xx}}\int_0^{b_1}(hb_1-hy+b_1^2-b_1y+b_1y-y^2)\,dy$$

$$=\frac{Ft_1}{2\,I_{xx}}\left|\,hb_1y-\frac{hy^2}{2}+b_1^2y-\frac{y^3}{3}\right|_0^{b_1}$$

$$=\frac{Ft_1}{2I_{xx}}\left[\,b_1^2h-\frac{h}{2}b_1^2+b_1^3-\frac{b_1^3}{3}\right]$$

$$=\frac{Ft_1}{2I_{xx}}\left[\frac{b_1^2h}{2}+\frac{2b_1^3}{3}\right]=\frac{Fb_1^2t_1}{12\,I_{xx}}\,[3h+4b_1]$$

Shear force in horizontal portion BC

$$F_2=\int_0^{b_2}\frac{F}{I_{xx}\,t_2}\times\left(b_1t_1\times\left(h+\frac{b_1}{2}\right)+t_2x\,.\,\frac{h}{2}\right)t_2dx$$

$$=\frac{F}{I_{xx}}\int_0^{b_2}\left(\frac{b_1t_1h}{2}+\frac{b_1^2t_1}{2}+\frac{t_2h}{2}\,x\right)dx$$

$$=\frac{F}{I_{xx}}\left|\frac{b_1t_1h}{2}\,x+\frac{b_1^2t_1}{2}\,x+\frac{t_2h}{2}\,.\,\frac{x^2}{2}\right|_0^{b_2}$$

$$=\frac{F}{I_{xx}}\left[\frac{b_1b_2t_1h}{2}+\frac{b_2b_1^2t_1}{2}+\frac{t_2b_2^2h}{4}\right]$$

$=F_4$ (due to symmetry)

Taking moments of the shear forces about the centre O of the vertical web

$$F.e+2F_1.b_2=F_2.h$$

$$\therefore\quad Fe=\frac{Fh}{I_{xx}}\left[\frac{b_1b_2t_1h}{2}+\frac{b_1^2b_2t_1}{2}+\frac{t_1b_2^2h}{4}\right]-\frac{Fb_2b_1^2t_1}{6\,I_{xx}}\,[3h+4b_1]$$

$$e=\frac{h^2}{I_{xx}}\left[\frac{b_1b_2t_1}{2}+\frac{t_2b_2^2}{4}\right]+\frac{hb_1^2b_2t_1}{2I_{xx}}-\frac{b_2b_1^2t_1h}{2\,I_{xx}}-\frac{2}{3}\times\frac{b_2b_1^3t_1}{I_{xx}}$$

$$=\frac{h^2}{I_{xx}}\times\frac{b_1b_2t_1}{2}+\frac{h^2t_2b_2^2}{4\,I_{xx}}-\frac{2}{3}\,\frac{b_2b_1^3t_1}{I_{xx}}$$

$$=\frac{b_1b_2t_1}{I_{xx}}\left[\frac{h^2}{2}-\frac{2}{3}b_1^2\right]+\frac{t_2h^2b_2^2}{4\,I_{xx}}$$

Moment of inertia I_{xx}

$$=\frac{2\times b_1^3\times t_1}{12}+2b_1t_1\left(\frac{h}{2}+\frac{b_1}{2}\right)^2$$

$$+\frac{2b_2t_2^3}{12}+2b_2t_2\left(\frac{h}{2}\right)^2+\frac{t_3h^3}{12}$$

Problem 20·6. Fig. 20·26 shows a section of a beam subjected to shear force. Determine the position of the shear centre of the beam.

Solution. The Fig. 20·26 shows a section with web $h\times t$, flanges $b_2\times t$ and projection $b_1\times t$. Say the applied force is F and shear force in different portions is F_1, F_2, F_3, F_4 and F_5 as shown.

Due to symmetry $F_1=F_5$ and $F_2=F_4$

Shear stress in any layer

$$q=\frac{Fa\bar{y}}{I_{xx}t}$$

Portion V. (*i.e.* vertical projection)

area $a=y.t$

$$\bar{y}=\left(\frac{h}{2}-b_1\right)+\frac{y}{2}$$

$$=\left(\frac{h-2b_1+y}{2}\right)$$

area, $dA=t.dy$

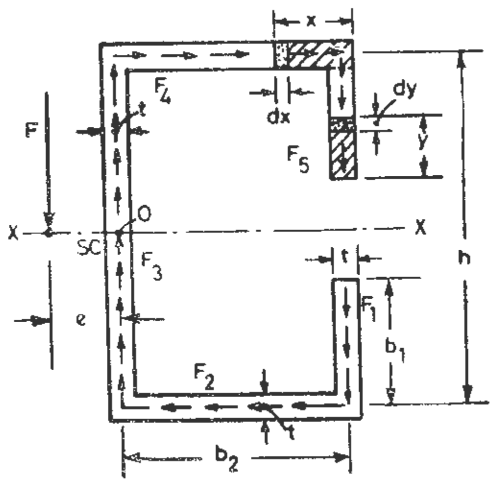


Fig. 20·26

Shear force $$F_5=\int_0^{b_1} q\,dA=\int_0^{b_1}\frac{Fyt}{I_{xx}t}\times\left(\frac{h-2b_1+y}{2}\right)t\,dy$$

$$=\frac{F\,t}{2\,I_{xx}}\int_0^{b_1}(hy-2b_1y+y^2)\,dy=\frac{Ft}{2\,I_{xx}}\left|\frac{hy^2}{2}-b_1y^2+\frac{y^3}{3}\right|_0^{b_1}$$

$$=\frac{F\,t}{2\,I_{xx}}\left[\frac{hb_1^2}{2}-b_1^3+\frac{b_1^3}{3}\right]=\frac{Ft}{2\,I_{xx}}\left[\frac{hb_1^2}{2}+\frac{2}{3}b_1^3\right]$$

$$=\frac{F\,t\,b_1^2}{12\,I_{xx}}\,[3h+4b_1] \qquad ...(1)$$

Portion IV-Flange

$$a\bar{y}=(t.x)\,\frac{h}{2}+b_1\times t\left(\frac{h}{2}-b_1+\frac{b_1}{2}\right)$$

$$=tx\,\frac{h}{2}+b_1t\left(\frac{h}{2}-\frac{b_1}{2}\right)$$

and $$dA=dx.t$$

Shear force $$F_4=\int_0^{b_2}\frac{F}{I_{xx}t}\times\left(\frac{txh}{2}+\frac{b_1t}{2}(h-b_1)\right)\times t\,dx$$

$$=\frac{F\,t}{2\,I_{xx}}\int_0^{b_2}(xh+b_1h-b_1^2)\,dx$$

$$=\frac{F\,t}{2\,I_{xx}}\left[\frac{b_2^2}{2}\times h+b_1b_1h-b_1^2b_2\right]$$

Taking moments of the shear forces about the centre O of the web

$$F.e=2\,F_5\times b_2+F_4\times h$$

$$e=\frac{t\,b_1^2\,b_2}{6\,I_{xx}}(3h+4b_1)+\frac{t\,h}{2\,I_{xx}}\left(\frac{b_2^2h}{2}+b_1b_2h-b_1^2b_2\right)$$

$$=\frac{t\,b_1^2b_2\,(3h+4b_1)}{6\,I_{xx}}+\frac{t\,h}{4\,I_{xx}}(b_2^2\,h+2b_1b_2h-2b_1^2b_2)$$

$$=\frac{1}{12\,I_{xx}}[6\,t\,h\,b_1^2b_2+8\,t\,b_1^3\,b_2+3t\,h^2b_2^2+6\,t\,b_1b_2h^2-6\,b_1^2b_2t\,h]$$

$$=\frac{t}{12\,I_{xx}}[8\,b_1^3b_2+3h^2b_2^2+6\,b_1b_2h^2]$$

where $$I_{xx}=\frac{t\times h^3}{12}+\frac{2\times b_2\times(t)^3}{12}+2b_2t\left(\frac{h}{2}\right)^2+\frac{2\times t\times b_1^3}{12}$$

$$+2b_1t\times\left(\frac{h}{2}-\frac{b_1}{2}\right)^2$$

$$=\frac{t\,h^3}{12}+\frac{b_2\,t^3}{6}+\frac{b_2\,t\,h^2}{2}+\frac{t\,b_1^3}{6}+\frac{b_1t}{t}(h-b_1)^2$$

Problem 20·7. For a section shown in the Fig. 20·27, determine the position of the shear centre. The thickness of the section is t throughout.

Solution. Due to symmetry

Shear force in portion AB, $F_1=F_4$, shear force in portion DC

Shear force in portion BO, $F_2=F_3$, shear force in portion CO

Shear force F_1

Shear stress, $q=\dfrac{F\,a\,\bar{y}}{I_{NA}\,t}$ where F=applied shear force

Shear force, $$F_1=\int_0^{a_1}q\,dA$$

where $a=z\,t$ (as shown)

So $dA=t\,dz$

Distance, $\bar{y}=(a_2\sin 45°-a_1\sin 45°)+\dfrac{z}{2}\sin 45°$

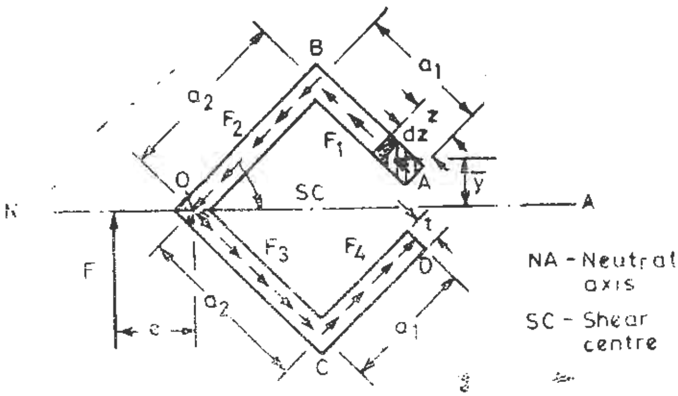


Fig. 20·27

$$=\left(a_2-a_1+\frac{z}{2}\right)\sin 45°=\frac{2a_2-2a_1+Z}{2\sqrt{2}}$$

Shear force, $$F_1=\int_0^{a_1}\frac{F\,z\,t}{I_{NA}\,t}\left(\frac{2a_2-2a_1+z}{2\sqrt{2}}\right)t\,dz$$

$$=\frac{F\,t}{2\sqrt{2}\,I_{NA}}\int_0^{a_1}2a_2\,z-2a_1z+z^2)\,dz$$

$$=\frac{F\,t}{2\sqrt{2}\,I_{NA}}\left[a_2a_1^2-a_1^3+\frac{a_1^3}{3}\right]=\frac{F\,t\,a_1^2\,(3a_2-2a_1)}{6\sqrt{2}\,I_{NA}}$$

Moment of I_{xx}

Moment of inertia of AB, about their principal axes

$$I_{uu}=\frac{a_1\,t^3}{12},\quad I_{vv}=\frac{t\,.\,a_1^3}{12}$$

$\theta=45°$ (as shown in Fig. 20·27)

$$I_{xx}=I_{uu}\cos^2\theta+I_{vv}\sin^2\theta=\frac{I_{uu}+I_{vv}}{2}$$

$$=\frac{a_1t}{24}\,(a_1^2+t^2)$$

$$I_{NA}'=I_{xx}+ta_1\left[\left(a_2-\frac{a_1}{2}\right)\sin 45°\right]^2$$

$$=I_{xx}+\frac{ta_1(2a_2-a_1)^2}{8}$$

$$=\frac{a_1t}{24}\,(a_1^2+t^2)+a_1t\left(\frac{a_2^2}{2}+\frac{a_1^2}{8}-\frac{a_1a_2}{2}\right)$$

$$=\frac{a_1 t}{24}\ (a_1^2+t^2+12a_1^2+3a_1^2-12a_1a_2)$$

$$=\frac{a_1 t}{24}\ (t^2+4a_1^2+12a_2^2-12a_1a_2)$$

Similarly moment of inertia of BO

$$I_{uu}'=\frac{a_2 t^3}{12},\quad I_{vv}'=\frac{t\,a_2^3}{12}$$

$$\theta=45^\circ$$

$$I_{xx}'=I_{uu}'\cos^2\theta+I_{vv}'\sin^2\theta=\frac{I_{uu}'+I_{vv}'}{2}$$

$$=\frac{a^2 t}{24}\ (a_2^2+t^2)$$

$$I_{NA}''=I_{xx}'+a_2 t\left(\frac{a_2}{2}\right)^2\sin^2 45^\circ$$

$$=\frac{a^2 t}{24}\ (a_2^2+t^2)+a_2 t\left(\frac{a_2^2}{8}\right)$$

$$=\frac{a_2 t}{24}\ (4a_2^2+t^2)$$

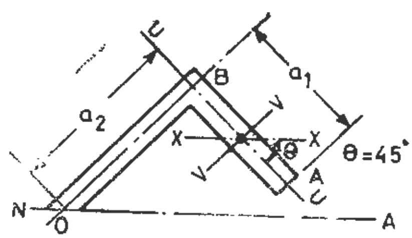


Fig. 20·28

Total moment of inertia of the section,

$$I_{NA}=2I_{NA}'+2I_{NA}''$$

$$I_{NA}=\frac{a_1 t}{12}\quad (t^2+4a_1^2+12\,a_2^2-12\,a_1a_2)+\frac{a_2 t}{12}\ (4a^2+t^2)$$

Taking moments of the shear forces about the point O

$$F\times e=F_1\times a_2+F_1\times a_2$$

$$=\frac{F\,t\,a_1^2\,a_2}{3\sqrt{2}\,I_{NA}}(3a_2-2a_1)$$

or

$$e=\frac{t\,a_1^2 a_2}{3\sqrt{2}\ I_{NA}}\ (3a_2-2a_1)$$

SUMMARY

1. Unsymmetrical bending occurs in a beam (*i*) if the section is symmetrical but load line is inclined to the principal axes (*ii*) if section itself is unsymmetrical.

2. Product of inertia, $I_{xy}=\int xydA$

Product of inertia of a section about its principal axes is zero.

3. For a symmetrical section, principal axes are along the axes of symmetry.

4. Parallel axes theorem for product of inertia

$$I_{xy}=I_{\bar{x}\bar{y}}+A\bar{x}\bar{y}$$

where I_{xy}=product of inertia about any co-ordinate axes X-Y

$I_{\bar{x}\bar{y}}$=Product of inertia about centroidal axes $\bar{X}$-$\bar{Y}$

$\bar{x}, \bar{y}$=Cordinates of the centroid of the section about XY co-ordinates.

5. If I_{xx}, I_{yy}, I_{xy} are moments of inertia about any co-ordinates axes X-Y passing through the centroid of the section, Inclination of Principal axes with respect to X-Y axes

$$\theta=\frac{1}{2}\tan^{-1}\frac{2I_{xy}}{I_{yy}-I_{xx}}$$

Principal moments of inertia

$$I_{uu},\ I_{vv}=\tfrac{1}{2}(I_{xx}+I_{yy})\pm\sqrt{[\tfrac{1}{2}(I_{yy}-I_{xx})]^2+I_{xy}^2}.$$

6. If principal moments of inertia of a section are I_{uu}, I_{vv} then moment of inertia about an axis X-X inclined at angle θ to U-V axis is

$$I_{xy}=I_{uu}\cos^2\theta+I_{vv}\sin^2\theta.$$

7. Stresses due to unsymmetrical bending, if u, v are the co-ordinates of a point and M is the bending moment applied on the section and θ is the angle of inclination of axis of M, with respect to the principal axes UU.

Resulant bending stress at the point

$$f_b=M\left[\frac{v\cos\theta}{I_{uu}}+\frac{u\sin\theta}{I_{vv}}\right].$$

8. Angle of inclination of neutral axis with respect to principal axis UU

$$\alpha=\tan^{-1}\left(\tan\theta\ .\ \frac{I_{uu}}{I_{vv}}\right).$$

9. Deflection of a beam under load W causing unsymetrical bending

$$\delta=\frac{KWl^3}{E}\sqrt{\frac{\sin^2\theta}{I_{vv}^2}+\frac{\cos^2\theta}{I_{uu}^2}}$$

where K=Constant depending upon end conditions of the beam and position of the load

θ=Angle of inclination of load W with respect to VV principal axes.

10. If the direction of the applied load on a beam passes through the shear centre of the section, no twisting takes place of the beam.

11. For a section symmetrical about two axes, shear centre lies at the centroid of the section.

12. For a section symmetrical about one axis only, shear centre lies along the axis of symmetry.

13. About the shear centre, the moment due to the applied shear force is balanced by the moment of the shear forces obtained by summing the shear stresses over the various portions of the section.

MULTIPLE CHOICE QUESTIONS

1. The product of inertia of a rectangular section of breadth 4 cm and depth 6 cm. about its centroid axes is

   (a) 72 cm^4 (b) 52 cm^4
   (c) 32 cm^4 (d) None of the above.

2. The product of inertia of a rectangular section of breadth 4 cm and depth 6 cm about the co-ordinate axes passing at one corner of the section and parallel to the sides is

   (a) 144 cm^4 (b) 72 cm^4
   (c) 52 cm^4 (d) 32 cm^4.

3. For an equal angle section, co ordinate axes XX and YY passing through centroid are parallel to its length. The principal axes are inclined to XY axes at an angle

   (a) 22·5° (b) 45·0°
   (c) 67·5° (d) None of the above.

4. For an equal angle section, moments of inertia I_{xx} and I_{yy} are both equal to 120 cm^4. If one principal moment of inertia is 210 cm^4, the magnitude of other principal moment of inertia is

   (a) 210 cm^4 (b) 120 cm^4
   (c) 60 cm^4 (d) 30 cm^4.

5. For a section, principal moments of inertia are I_{uu}=360 cm^4 and I_{vv}=160 cm^4. Moment of inertia of the section about an axis inclined at 30° to the U-U axis, is

   (a) 310 cm^4 (b) 260 cm^4
   (c) 210 cm^4 (d) 120 cm^4.

6. For an equal angle section $I_{xx}=I_{yy}$=32 cm^4 and I_{xy}=−20 cm^4. The magnitude of one principal moment of inertia is

   (a) 52 cm^4 (b) 42 cm^4
   (c) 32 cm^4 (d) 16 cm^4.

7. For a T-section, shear centre is located at

   (a) Centre of the vertical web (b) Centre of the horizontal flange
   (c) At the centroid of the section (d) None of the above.

8. For an I section (symmetrical about X-X and YY axis) shear centre lies at

   (a) Centroid of top flange (b) Centroid of bottom flange
   (c) Centroid of the web (d) None of the above.

9. For a channel section symmetrical about X-X axis, shear centre lies at

   (a) The centroid of the section (b) The centre of the vertical web
   (c) The centre of the top flange (d) None of the above.

10. If the applied load passes through the shear centre of the section of the beam, then there will be

(a) No bending in the beam (b) No twisting in the beam
(c) Bending and twisting in the beam (d) No deflection in the beam.

ANSWERS

1. (d)	2. (a)	3. (b)	4. (d)	5. (a)
6. (a)	7. (b)	8. (c)	9. (d)	10. (b).

EXERCISE

20·1. A section is a quadrant bounded by two concentric circles of radii 5 cm and 8 cm. Determine its product of inertia about axes OX and OY, passing through the centre of circles. [**Ans.** 433·875 cm^4]

20·2. A beam of angle section 12 cm×8 cm×2 cm is simply supported over a span of 2 metres with 12 cm leg vertical. A vertical load of 1 tonne is applied at the centre of the span. Determine (a) maximum bending stress (b) direction of netural axis (c) deflection at the centre of the beam. Given E=2100 tonnes/cm^2.

[**Ans.** (a) 1071·8 kg/cm^2, (b) 42° 55′ with respect to X-axis (c) 3·15 mm)

20·3. A beam 4·5 metres long is of a rectangular section 12 cm wide and 18 cm deep. The beam is simply supported at each end and carries a concentrated loads of 3 kN, 1·5 m apart from each support. The plane of the loads make an angle of 30° to the vertical, and passes through the centroid of the section. Find

(i) bending stress at the corner of the quadrant of the section, in which the load is applied

(ii) direction of neutral axis.

[**Ans.** (i) 11·22 N/mm^2, (ii) 52° 24′ with respect to horizontal axis]

20·4. An unequal I section is shown in Fig. 20·29. Shear centre lies along X-axis. Show that $\frac{e_1}{e_2}=\frac{t_2 b_2^3}{t_1 b_1^3}$. Determine the value of e_1, if b_1=6 cm, b_2=8 cm and $t_1=t_2=t_3$=1 cm and h=12 cm. [**Ans.** e_1=8·44 cm]

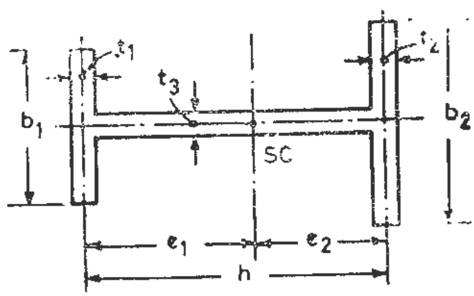


Fig. 20·29

20·5. Determine the position of the shear centre of the section of a beam shown in Fig. 20·25 if b_1=4 cm, b_2=6 cm, h=8 cm and $t_1=t_2=t_3$=1 cm. [**Ans.** [e=2·021 cm]

20·6. Determine the position of the shear centre of the section shown in Fig. 20·26, if b_1=3 cm, b_2=5 cm, h=10 cm, t=1 cm. [**Ans.** e=3·557 cm]

20·7. For a section shown in the Fig. 20·27, determine the position of the shear centre if a_1=4 cm a_2=6 cm, t=1 cm. [**Ans.** e=1·59 cm]

21

Mechanical Properties

Any machine member or a structure designed to sustain loads must have the necessary mechanical properties as strength, stiffness, toughness, hardness etc., before they can serve any other purpose in addition to sustaining the loads. In this chapter mechanical properties and how these are determined will be discussed. The behaviour of the materials under various types of loads and moments and how they fail will also be analysed briefly.

21·1. BEHAVIOUR OF MATERIALS UNDER STATIC TENSION

Members of engineering structures and devices are often subjected to steady axial tensile loads, and response of the material to other types of loading sometimes be explained or predicted on the basis of their behaviour in simple tension. In 1st chapter we have studied about the tensile test on the most commonly used structural material—mild steel and have acquainted ourselves with terms like stress, strain, yield point, elastic and plastic behaviour, ductility etc., etc.

When a solid bar is loaded in tension, it elongates as the load is increased. The mechanism by which elongation takes place in the solid material can be viewed as a simple separation of its atoms in the direction of loading. The atoms are displaced from their normal position of equilibrium and develop attractive forces between them which balance

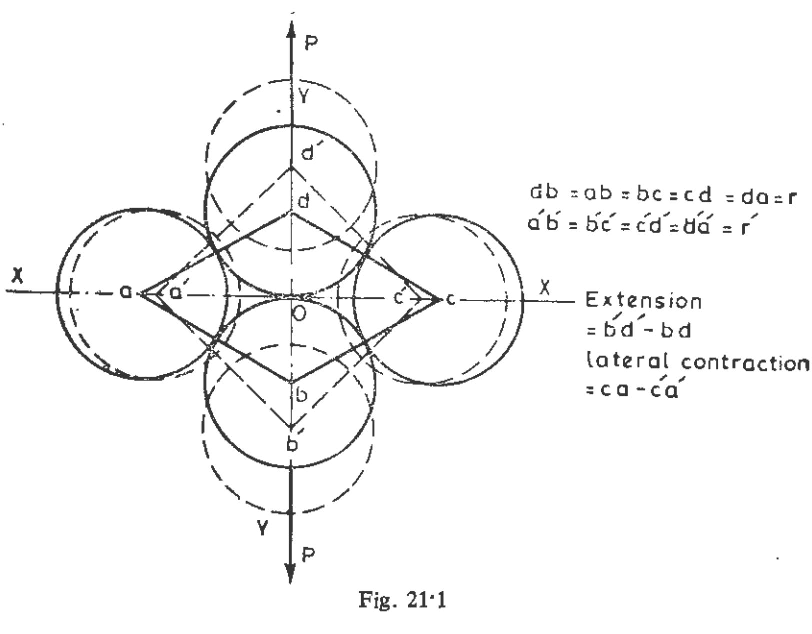


Fig. 21·1

the applied loads. In most crystalline materials, atoms are closely packed. Fig. 21·1 shows four atoms in a close packed structure, with r as the interatomic distance. When a tensile force P is applied in the direction db, the interatomic distance changes to r'. Distance db increases to $d'b'$, while the distance ac decreases to $a'c'$.

$$\text{Normal strain} = \frac{d'b' - db}{db}$$

$$\text{Lateral strain} = \frac{a'c' - ac}{ac}.$$

So long as the elongation involves only simple separation of atoms by very small amounts (not so large as shown in the figure) release of the applied force will allow the atoms to return to their normal equilibrium positions. The axially loaded bar will return to its original size and shape and the deformation is said to be *elastic*. Upto the elastic stage the deformation is reversible or recoverable.

Say extension along Y-axis

$$= \delta_y \quad \text{or} \quad dd' = \frac{\delta_y}{2}$$

$$\text{Contraction along } X\text{-axis} = \delta_x \quad \text{or} \quad cc' = \frac{\delta_x}{2}$$

$$\angle ocb = \angle ocd = 30° \text{ or } \angle bcd = 60° \text{ since } db = bc = cd$$

$$db = r \text{ and } ac = r\sqrt{3}$$

or $$ob = \frac{r}{2} \quad \text{and} \quad oc = \frac{r}{2}\sqrt{3}$$

Consider that while applying the load on atoms at b and d, position of atom b is fixed and atoms a, c and d are displaced, and so the centre o shifts to o'. Say the displacement is very small and distance between b and c remains r or $r' \simeq r$. Then displacement cc'' is perpendicular to line bc. Displacement of point c (*i.e.*, centre of atom at c) has two components

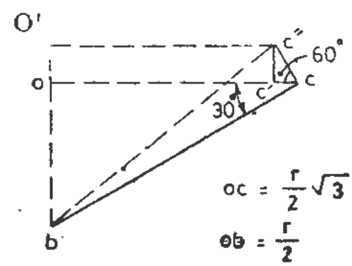


Fig. 21·2

$$c'c'' = \frac{\delta_y}{2}$$

and $$cc' = \frac{\delta_x}{2}$$

But $$\frac{\delta_x}{2} \times \frac{2}{\delta_y} = \tan\ 30° = \frac{1}{\sqrt{3}}$$

$$\frac{\delta_x}{\delta_y} = \frac{1}{\sqrt{3}}$$

$$\text{Now linear strain} = \frac{\delta_y}{bd} = \frac{\delta_y}{r}$$

$$\text{Lateral strain} = -\frac{\delta_x}{ac} = -\frac{\delta_x}{r\sqrt{3}}$$

$$\text{Poisson's ratio} \quad = \frac{\text{Lateral strain}}{\text{Linear strain}} = -\frac{\delta_x}{r\sqrt{3}} \times \frac{r}{\delta_y} = -\frac{\delta_x}{\sqrt{3}\,\delta_y} = -\frac{1}{3}$$

Most metals are found to have value of $1/m$ (Poisson's ratio) close to 1/3.

Now as the tension on a solid bar increases and atoms are pulled farther apart, a stage is reached where the elongation is no longer a simple separation of atoms and irrecoverable structural changes take place in the material and the material's behaviour is said to be in elastic. Some of the atoms or molecules of the material under the distortion produced by tensile force, slip to new equilibrium positions at which they form new bonds with other atoms, thus permitting an elongation in excess of that produced by the simple elastic separation of atoms. After the removal of the load, there is no tendency of the atoms to return to their original positions. Such deformation is also called the plastic deformation. This stage *i.e.*, onset of plastic deformation is said to be yielding of the material. The material which yields is said to be ductile.

The most common mechanism of yielding in crystalline materials is slip, in which two planes of atoms slip past each other causing one full section of the crystal to shift relative to the other. Slip occurs most easily on certain crystallographic planes depending upon the crystal structure. Generally, the planes of easy slip are those on which atoms are most closely spaced—those having the largest number of atoms per unit area.

The stress required to separate the two planes of atoms, breaking all the bonds simultaneously is much larger than the maximum elastic stress. Similarly the shearing stress necessary to shift one layer of atoms past another all at the same time is much larger than the actual shearing stress. The reason is that slip is progressive rather than simultaneous, it starts at one point in the slip plane where the presence of an imperfection in the crystal lattice makes it possible and moves through the crystal by a progressive shifting of atoms along the slip plane.

The imperfections usually responsible for slip are called *dislocations*. These are small groups of atoms in the crystal lattice that are displaced from their regular positions, distorting the lattice slightly. Dislocations are present in great numbers in all crystals. These are formed during crystal growth and by plastic deformation. (The reader is advised to refer to a book on Materials Science and to study the various types of dislocations such as edge and skew dislocations).

Most crystalline materials are aggregates of many crystals or grains. The directions of the planes of easy slip of individual crystals are oriented at random in all possible directions, throughout the material.

When a tensile stress is applied along the axis of the bar, the maximum shear stress occurs on planes at 45° to the axis of loading. This stress will coincide with the planes of easy slip in some crystals but not in the majority of the crystals. Hence there are weak and strong crystals and slip will generally start in weak crystals—*i.e.*, those which are most favourably oriented for slip. After slip has begun in certain crystals, its continued progress through the material involves slip in adjacent crystals and because of their different orientations, a greater stress is required—resulting in *strain hardening*.

Fig. 21·3 (*a*) shows the stress strain curve for a general ductile material. From O upto A is a straight line, beyond A the curve is not straight and the material has yielded. Stress at A is called the yield point stress. From A onwards, increasing stress upto the maximum load point, (where necking takes place) is required to continue the slip or the plastic deformation. The material is strain hardened from A to P_{max}. Fig. 21·3 (*b*) shows the stress strain curve for mild steel. O to A is a straight line, at B there is considerable extension with slight decrease in load (from B to C). This point B is called the *upper yield point*, C is the *lower yield point*. This type of yielding is called discontinuous yielding which is a typical characteristic of mild steel. At C, strain hardening in the material starts and ends at the maximum load point P_{max}, where necking takes place in the bar.

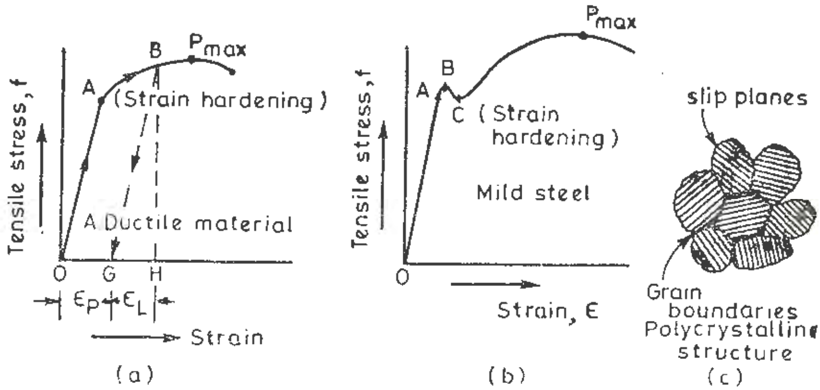


Fig. 21·3

The crystal boundaries offer more resistance to slip than the interiors of the crystals. Fig. 21·3(c) shows a polycrystalline material with random orientation of slip planes of individual crystals' The grain boundaries are harder than the interior of the crystal, with the net result that the slip through the boundaries becomes very difficult because dislocations are impeded both by atomic disorder at the grain boundary and by precipitation of impurity atoms along grain boundaries.

The range of mechanical behaviour in which yielding and strain hardening takes place is called *plastic stage*. In addition to the plastic strain there is a recoverable elastic strain. To provide the necessary internal stresses to balance the external loads, the atoms are always separated by a certain amount from their equilibrium spacings. As the yielding continues, the atoms are shifted to new equilibrium positions and at the same time interatomic spacing changes to develop the necessary inter atomic forces. Fig. 21·3 (a) shows unloading of the bar from the point B, where BG is the unloading stress-strain curve.

Elastic strain (recovered) $=GH=\epsilon_L$

Plastic strain (permanent)$=OG=\epsilon_P$.

In many materials, the rate of strain hardening decreases with increased strain and at the maximum load point *i.e.* P_{max}, strain hardening becomes zero *i.e.*, where the strain hardening no larger compensates for the increased stress caused by the reduction in area. At this stage constriction or neck begins to form in the specimen.

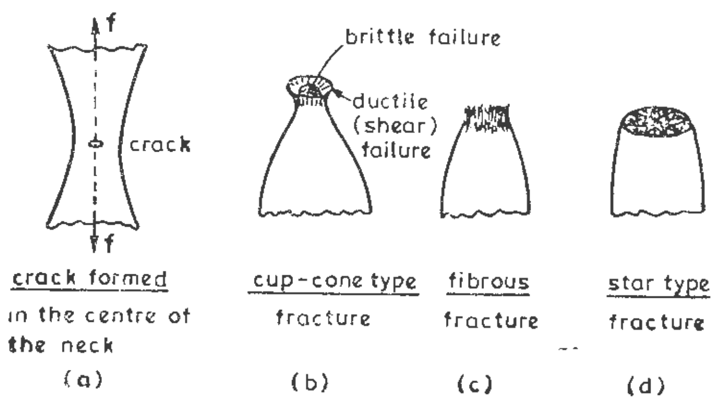


Fig. 21·4

The plastic range ends in the fracture of the bar, the break occurs at the smallest section of the n.ck. The very centre of the neck is in a state of triaxial tension, which encourages brittle type fracture. Fracture starts with a small crack in the centre of the neck, The crack spreads rapidly outwards. By the time the crack has spread nearly to the circumference, there is only a narrow ring of material like a tube supporting the load. This tube fails by shearing action all around resulting in a *cup and cone type fracture* for ductile materials.

In steels with increasing carbon content, the depth of the cup becomes shallower until for high carbon steel it may completely disappear. Fig. 21·4 (*a*) shows the formation of a fine crack in the centre of the neck. Fig. (*b*) shows the cup and cone type fracture for a ductile material as mild steel. Fig. (*c*) shows the fibrous fracture for very soft materials like wrought iron Fig. (*d*) shows the star type fracture for high carbon steel.

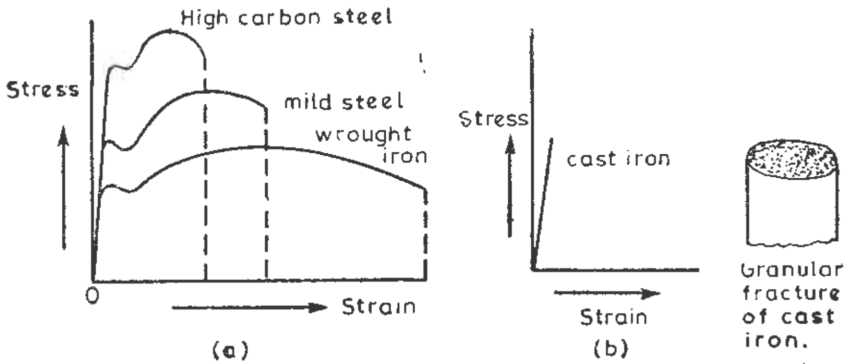


Fig. 21·5

Fig. 21·5 (*a*) shows a comparison between the stress strain curves for wrought iron, mild steel and high carbon steel. As the carbon percentage increases in steel, its ductility goes on decreasing but strength goes on increasing. If we compare the strain energy absorbed by the specimen uptil fracture, then it is observed that wrought iron absorbs maximum strain energy till breaking and high carbon steel absorbs the least amount of strain energy. In other words wrought iron is tougher than mild steel and mild steel is tougher than high carbon steel. The toughness of a material is defined as its ability to absorb energy and deform plastically before fracture. Toughness is proportional to the combined effect of strength and ductility.

When the carbon percentage in steel increases further and carbon comes out in the form of graphite flakes rendering the material weak, as in the case of cast iron, the material fails with very little extension with granular type of fracture showing separation of grains in the direction perpendicular to the axis of load. This type of fracture is called a brittle fracture.

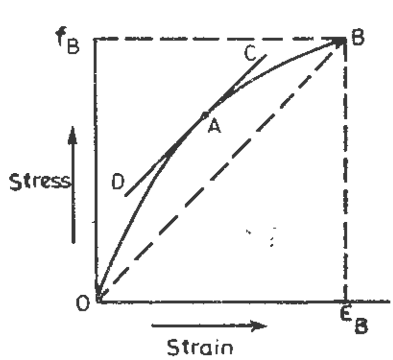
Non linear elastic properties. Some materials do not follow Hooke's law, therefore their stiffness does not remain constant but varies with stress. Sometimes average stiffness is taken at a given stress. This average stiffness is given by secant modulus.

$$E_{secant}=\left(\frac{f}{\epsilon}\right)=\frac{f_B}{\epsilon_B}$$

Secant modulus depends on the location of point B.

If the stiffness associated with a small increase in stress is designed, the instantaneous stiffness is determined from the slope of the tangent to the curve at that point. This slope is called the tangent modulus. As CD is the tangent to the curve OAB at the point A.

$$E_{tangent}=\left(\frac{df}{d\epsilon}\right)_{f=f_A}$$



Non-linear elastic curve

Fig. 21·6

The modulus of resilience (discussed in chapter 1) is not applicable in such cases because proportional limit does not exist in the non-linear behaviour of the material.

Though the material is elastic, the strain energy is not always fully recoverable. The Fig. 21·7 shows a typical non-linear stress-strain diagram for rubber. Curve 0 1 A is the loading curve and the area under this curve gives the strain energy absorbed per unit volume when the rubber is stretched.

Curve A 20 is the unloading curve and the area under this curve gives the strain energy recovered during unloading. And

Area between the loading and unloading curves = Strain energy lost in the form of internal friction between the molecules of rubber during one loading-unloading cycle.

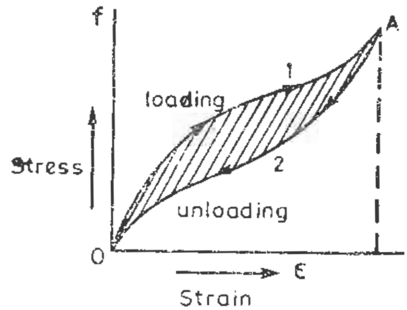


Fig. 21·7

This is a very good example of mechanical hysterisis and accounts for high damping capacity of rubber when used for vibration isolation supporting the vibrating machinery. Here the term resilience denotes the ratio between the recoverable strain energy and the energy absorbed by the material during deformation. A low resilience is desirable for good damping and a high resilience is desirable for low internal heat generation.

Repeated loading. In a tensile test on a ductile material, if after unloading, the member is loaded again, the atoms will simply be displaced to the position they occupied just before unloading, after which further yielding will take place Fig. 21·8 shows that load is applied beyond the yield point (loading curve $OABCD$ and then gradually released (unloading curve DEF), there will be permanent deformation in the material (shown by OF). On reloading, it will be observed that (*i*) material loses elasticity and it no longer obeys Hooke's law (*ii*) yield point is considerably raised (from point B to point H) and (*iii*) unloading and reloading curves form a mechanical hysterisis loop which represents the strain energy lost in friction.

The yield point is raised by a significant value, almost as high as the stress value at the end of the previous loading. The material is said to be strain-hardened or work-hardened as in the cases of processes like cold rolling and drawing. Repeated loading and unloading may raise the yield point near the ultimate stress point and the ductile material fails with only a very small elongation and with only a small reduction in area.

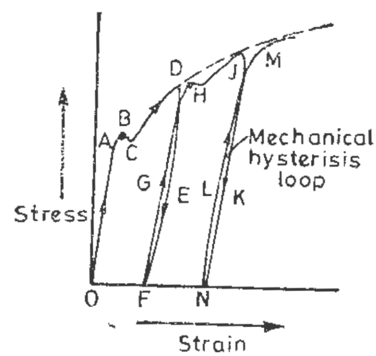


Fig. 21·8

21·2. BEHAVIOUR OF MATERIALS UNDER STATIC COMPRESSION

For a material, stress-strain diagrams for tension and compression generally differ. Similarly the ductility and mode of failure exhibited by a material under tensile and compressive loading also differ. It is in the plastic range for yielding that differences between the behaviour under tension and compression are the greatest. Behaviour in the elastic range is important for brittle material which do not exhibit yielding.

In crystalline materials, the elastic action in compression is exactly the same as the elastic action in tension but in the reverse direction. So the elastic stress-strain curve in compression is a linear extension of that in tension for many materials as shown in Fig. 21·9 (*a*).

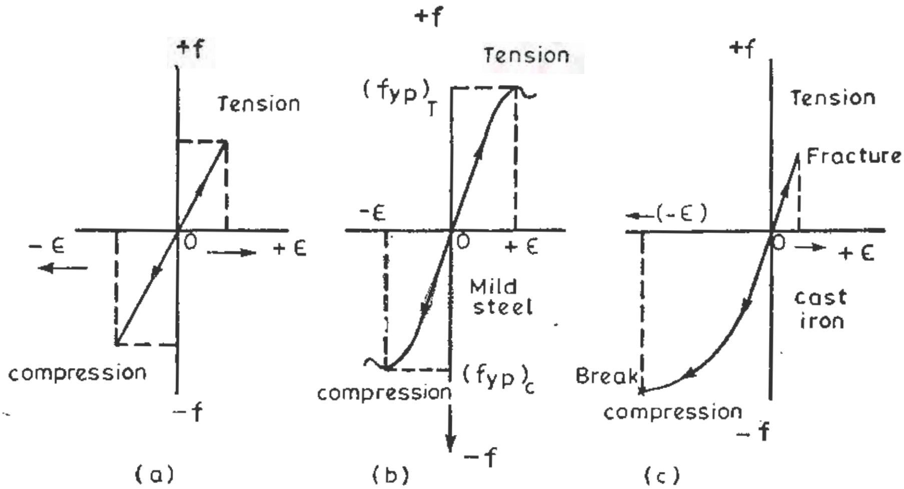


Fig. 21·9

In ductile materials subjected to compression slipping of atoms on crystallographic planes leads to yielding at a stress approximately the same as the yield stress in tension. This applies to discontinuous yielding also as in the case of mild steel, which has upper and lower yield points in compression that are usually the same as those for tension.

In brittle materials, slip leads to fracture along a single shear plane or a multitudes of small failures or shear (or slip) planes in all directions leading to fragmentation.

The axial compressive stress required to cause fracture in a brittle material is much greater than the required tensile stress. In tension, fracture is initiated by stress raisers in the form of cracks, holes and other imperfections even through the stress is well below that necessary to cause slip on the 45° shear plane. Since in compression, those imperfections cease to act as stress raisers. Instead if cracks or holes are present in the material, these tend to close up under compressive force and their effect vanishes. The stress can then reach the larger values needed to initiate slip. Imperfections oriented along the shear planes act as shear stress raisers. But these are far less effective than the stress raisers in tension. So the strength of a material is compression is often increased. Cast iron, concentre, soils are examples of this effect. The net result is that brittle materials are stronger in compression than in tension. Fig. 21·9 (*c*) shows the complete stress-strain diagram for a grey cast iron in tension and compression. The tensile strength of a typical grey cast iron is 150-160 N/mm^2 but its compressive strength is 750-800 N/mm^2.

The plastic range in compression extends from the end of the elastic stage to final fracture. Both the area of cross section and the strength of the material increase with compressive plastic strain, the former due to Poisson's effect and the latter due to strain hardening. Therefore the load which is the product of the area and the stress, always increases throughout the plastic range. The plastic range is potentially much larger in compression than in tension.

Testing. A universal testing machine fitted with compression plates is usually used to to apply the compressive loads. As it is impossible to make specimens having perfectly parallel ends it is desirable to provide some adjustment in compression plates so that they can be made to apply a uniformly distributed load over each end of the specimen. The simple adjustable compression plates have spherical seats as shown in Fig. 21·10.

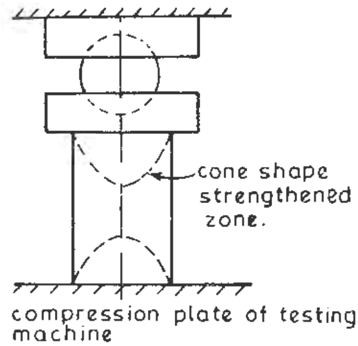


Fig. 21·10

Effects of eccentricity are more pronounced in compression than in tension because of the lateral instability involved while applying compressive loads. Therefore it is utmost necessary to avoid eccentricity in loading the specimen.

The majority of compression tests are made on other than ductile materials, the samples are tested upto fracture. In cast iron and concrete shear surfaces tend to run from one corner to the other corner of the specimen and not necessarily on 45° plane of the maximum shear. This is due to the end restraints which strengthen the material in a cone-shaped region at each end and leaves a weakness around the edge, as shown in Fig. 21·10.

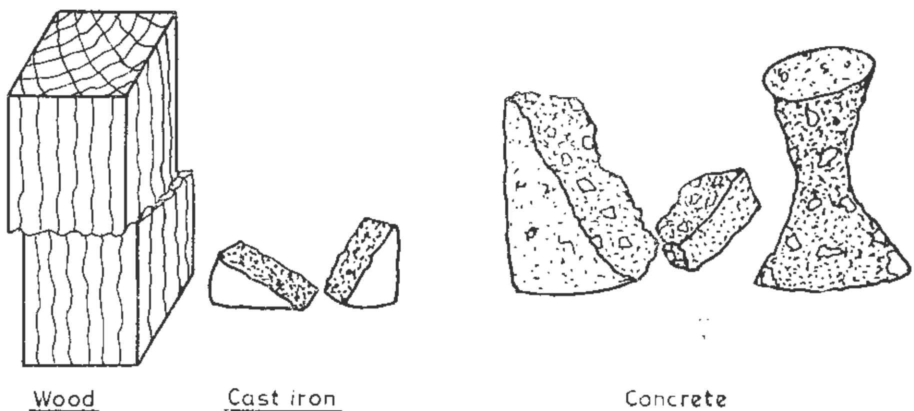


Fig. 21·11

Cylindrical specimens of concrete tend to fail along conical shear surfaces forming the typical hour-glass fracture of concrete.

Wood has fibrous structure and fibres are aligned in one direction and load is applied along the fibre direction. At the time of fracture each fibrous stick in wooden specimen breaks giving the type of fracture shown in Fig. 21·11.

Load reversal in compression. In crystalline materials loading in the plastic range in tension and then unloading results in permanent deformation with elastic recovery. When the material is loaded again in tension, yield strength is raised to a higher value. But instead of applying the tensile load on reloading, if the load is reversed and a compressive load is applied, an interesting effect is observed *i.e.* yield strength in compression is reduced.

Say a ductile material has equal strength in tension and in compression *i.e.* $(f_{yp})_T=(f_{yp})_C$ as shown by the points A and A' in the Fig. 21·12. A specimen made of such a ductile material is loaded in tension upto the point B and unloaded. (As shown by the loading curve OAB and unloading curve BC). Now the specimen is loaded again but in compression, it is observed that the compressive yield strength has been decreased *i.e.* from the stress at the point A', now it is reduced to the stress at the point D. This is the well known Bauschinger's effect.

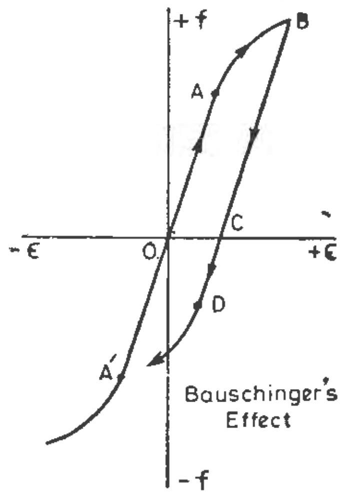


Fig. 21·12

One of the reasons of Bauschinger's effect is that yielding in a polycrystalline metal is non-uniform. The crystals are oriented at random and when the specimen is loaded in tension, there crystals yield by different amounts so that stress varies slightly from crystal to crystal. When the specimen or a machine member is uuloaded, it contracts until the average stress becomes zero. The crystals that yielded the least do not quite return to zero and remain in tension while the crystals that yielded the most go beyond zero and are under compression. Therefore there are microscopic residual stresses throughout the material, some in tension and some in compression. Now when the material is subjected to compressive load, the crystals that already have residual compressive stresses will yield at a lower than normal stress and therefore overall yield stress is lowered *i.e.* $f_D < f_{A'}$.

21·3. BEHAVIOUR OF THE MATERIALS UNDER BENDING

In pure bending, no shear stress is present and only the normal stresses are present across the section. Fig. 21·13 shows a beam $ABCD$ carrying loads W each, at distance 'a' from each support. The portion BC of beam is subjected to pure bending, as is obvious from the SF diagram, shear force is zero along the portion BC and the bending moment is constant and equal to Wa throughout its length of $(l-2a)$.

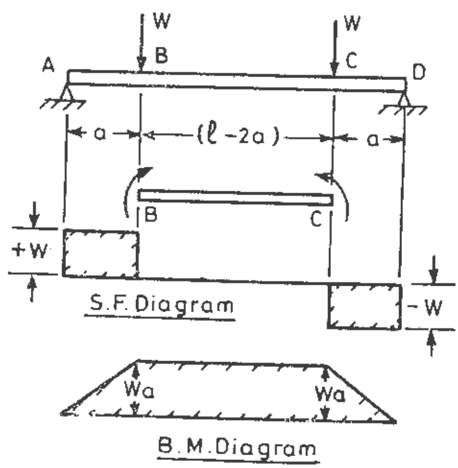


Fig. 21·13

For convenience the beam may be thought of as composed of longitudinal elements of infinitesimal cross section or fibres, each of which is in a state of simple tension or compression. In chapter 8 we have studied about the relation ship between bending moment, stress, section modulus and radius of curvature, and we have shown that variation of strain along the depth of the section is always linear even when the extreme fibres of the beam go to the plastic stage.

If the material were perfectly brittle, the flexure formula could be used all the way upto rupture $f_r = \frac{M_r \cdot y}{I_{xx}}$, is called the modulus of rupture where M_r is the bending moment causing rupture in the beam.

Since no material is actually perfectly brittle, stress f_r is never quite equal to the maximum stress in the beam at rupture. It is however a commonly used property for materials like ceramics, cast iron, concrete, wood and brittle plastics even though some of these have considerable plastic deformation before rupture.

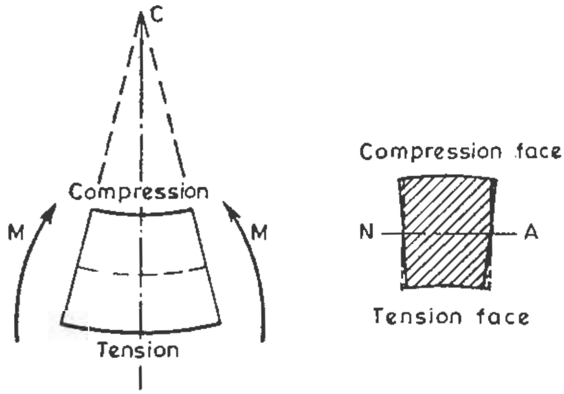


Fig. 21·14

Accompanying the change in length of the longitudinal fibres is a lateral strain, just as in simple tension and compression (due to the Poisson's effect) the fibres on the tension side of the beam contract laterally and those on the compression side expand laterally. Consequently the beam becomes wider on the compression side and narrower on the tension side. A transverse curvature is produced in the opposite direction from the longitudinal curvature, (as shown in Fig. 21·14 (*b*).

Yielding in pure Bending. The atomic mechanism of yielding in pure bending is the same as in simple tension ; slip along planes in the general direction of the maximum shearing stress at 45° with the axis of the beam. When the extreme fibres (those farthest from the neutral axis) reach the strain at which yielding begins in simple tension ; local yielding takes place. As bending continues, yielding progresses gradually inward towards the neutral surface. The stress in each fibre follows the stress-strain relationship for simple tension.

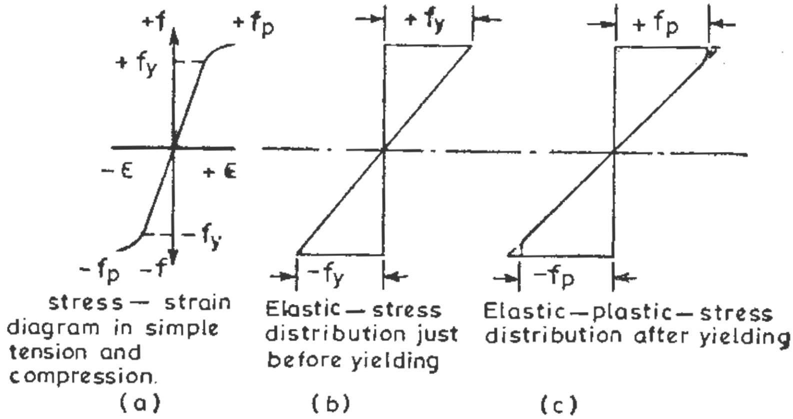


Fig. 21·15

Fig. 20·15 (*a*) shows the stress-strain diagram for the material of the beam, in simple tension and compression. Fig. 21·15 (*b*) shows the stress distribution across the section of the beam just before yielding and Fig. 21·15 (*c*) shows the stress-distribution diagram across the section, after yielding has started in extreme fibres of the beam.

Because of the concentration of the maximum stress in the extreme fibres and the support given by the inner fibres, the beam usually does not begin to yield until some what higher stresses are reached than are ordinarily observed in tension. When yielding does begin at some point, owing to imperfection it forms a small slip band starting at the extreme surface and progressing inward towards the neutral surface in the form of a wedge. This wedge acts like a notch having stress concentration at its tip and inner fibres therefore yield at stresses lower than the stress at extreme fibres.

Final failure in beams made of ductile materials usually involves either excessive deformation or lateral buckling of some kind.

Mild Steel. Beams of mild steel are of particular interest because of their wide use as structural members and because of the discontinuous behaviour in yielding of mild steel. After the yielding has progressed some distance from the outer surfaces, the stress distribution has the appearance shown in Fig. 21·16 (*a*) the maximum stress f_{yl} (stress at the lower yield

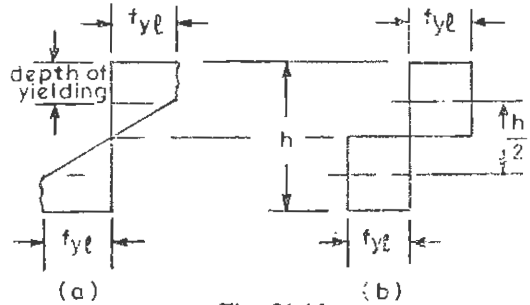


Fig. 21·16

point) is approximately constant over the depth of yielding. At the limit, as the yielded region approaches the centre of the beam, the stress distribution can be represented by two rectangles as shown in Fig. 21·16 (*b*).

This distribution is referred to as the fully plastic condition and the corresponding moment can easily be calculated. So long as the strain hardening does not occur, the bending moment cannot increase beyond this value, which is therefore called the ultimate moment, M_u

$$M_u=\left(f_{yl}\,.\,\frac{bh}{2}\right)\frac{h}{2}=f_{yl}\,.\,\frac{bh^2}{4}$$

where f_{yl}=lower yield point stress

b=breadth of the cross section

h=depth.

The bending moment at which yielding begins,

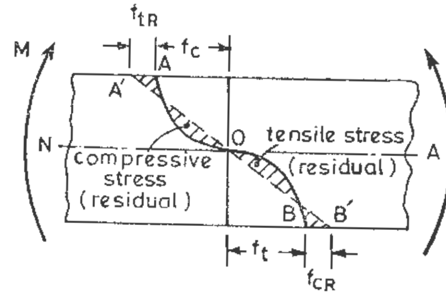
$$M_y=f_{yl}\,.\,\frac{I_{xx}}{h/2}$$

$$M_y=f_{yl}\,.\,\frac{bh^2}{6}\,.$$

Therefore $\frac{M_u}{M_y}=1\cdot5$ or the ultimate moment is 50 per cent more than the yield moment.

Ratio M_u/M_y depends upon the shape of the cross section of the beam, therefore it is called a *shape factor*. For circular sections the value of shape factor is approximately 1·8.

Residual Stresses. After a beam has been bent into the plastic range, removal of the load leaves the beam with internal residual stresses, because the stress-strain diagram for unloading is different than for loading. Fig. 21·17 shows the distribution of residual stresses in the beam after unloading. *AOB* is the stress distribution after the beam has been loaded producing stresses in the plastic range *i.e.*, beyond the yield point. When the beam is unloaded, the stress distribution for unloading is *A′OB′* and is linear as the strain distribution is always linear acrcsss the depth of the section during loading of the beam and also during unloading of the beam so as to satisfy the assumption that plane sections remain plane in pure bending.



Residual stresses after complete unloading.

Fig. 21·17

When the load is completely removed, the moment of stress distribution must be zero, so as to maintain equilibrium. Consequently the stress all across the cross section is educed further (*i.e.*, beyond zero) such that the stress in the outer fibres changes sing and produces an opposite moment to balance that of the remaining stress in the inner fibres. Line *A′OB′* represents the necessary superimposed linear stress distribution. The rostult is that in most parts of the unloaded beam, the residual stress is not zero. On the tenion side the re, are residual compressive stresses (f_{CR}) in the outer fibres and tensile residual stresses in interior. On the compression side, there are residual tensile stresses (f_{CR}) in the outer fibres and compressive residual stress in the interior, the net moment of the distribution is zero.

Experimental Methods. Fig. 21·18 (*a*) shows the experimental set up for pure bending on beam *ABCD*. The portion *BC* of the beam is subjected to pure bending or constant bending moment and no shear force. Following conditions must be satisfied during testing (1) Loading has to be in the plane of symmetry so as to avoid unsymmetrical bending (2) Freedom from longitudinal restraint (3) Constant bending moment with zero shear in the portion of the beam under consideration.

Experimental observations are made on load and either deflection or strain. Deflection in the centre can be measured with a dial gauge and strain on top and bottom surfaces can be measured with the help of electrical resistance strain gages.

Fig. 21·18 (*b*) shows the set up for 3-point loading of beam or bending with shear. The bending moment is not constant. Transverse shear transforms the stress in a beam from uniaxial to a biaxial state. The longitudinal fibres are no longer under simple tensile or compressive stresses, and the state of stress changes from point to point due to variation in bending moment.

In the elastic analysis of beams, it is assumed that the effects of transverse shear and those of the normal and bending stresses can be considered separately. But in the plastic range, this cannot be done without introducing a certain degree of approximation.

Bending tests with shear are also used to analyse the performance of full-sized members, as the bending test provides a direct means of evaluating the effects of such factors as shape factor on the structural stability of the members.

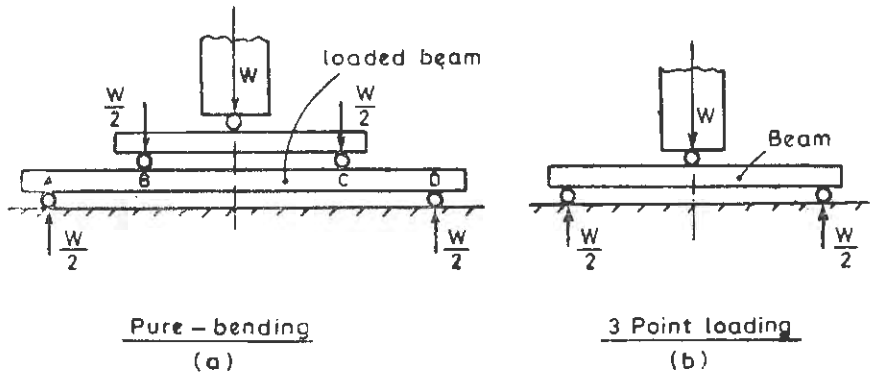


Fig. 21·18

21·4. BEHAVIOUR OF MATERIALS UNDER TORSION

Torsion tests are performed on materials to determine properties such as Modulus of rigidity, yield strength and modulus of rupture. Parts such as shafts, axles and drills are subjected to torsional loading in service and torsion test is performed on such full sized members otherwise a test specimen is made on which the test is performed. The specimen generally has a circular cross section and in the elastic range, shear stress varies linearly from zero at the centre to the maximum at the surface. In the case of a thin walled tube, shear stress is nearly uniform over the cross section of the specimen and it is preferable to use thin walled tube specimens for the determination of yield strength and modulus of rupture.

The torsion-test specimen shown in the figure 21·19(*a*) is gripped in the chucks of a torsion-testing machine. Twisting moment is gradually applied on the twisting head gripping

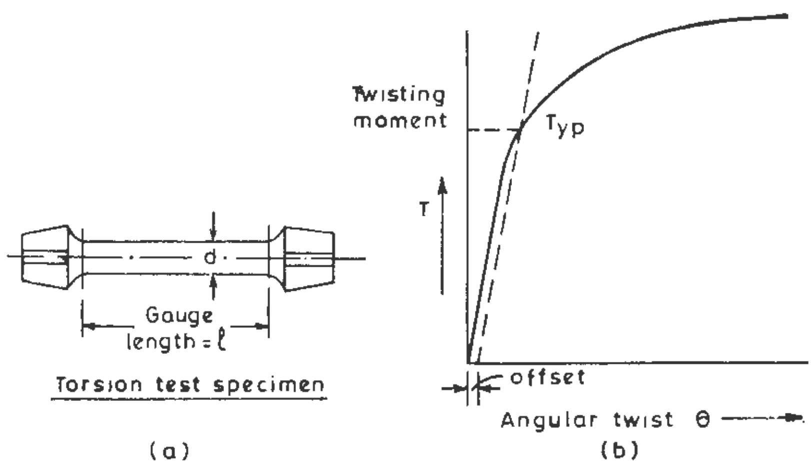


Fig. 21·19

one end of the specimen and torque T is measured on the weighing head connected to the other end of the specimen. Angular twist θ is measured with the help of a troptometer near one end of the test section with respect to the test section of the specimen at the other end. A torque Vs. θ (angular twist) diagram usually obtained for a ductile material is shown in the Fig 21·19 (*b*).

The elastic properties in torsion may be obtained by using the torque at the proportional limit or the torque at some offset angle of twist, generally 0·04 radian/metre of gauge length, and calculating the shear stress at the twisting moment T_{yp}, using the torsion formula.

Because of stress gradient across the radius of the solid shaft, the surface fibres are restrained from yielding by the less highly stressed inner fibres. Therefore, the first onset of yielding is not readily apparent. The use of a thin-walled tubular specimen minimises this effect because the shear stress is nearly uniform in the section of tube. However, an ultimate torsional shear strength or modulus of rupture is frequently determined by using T_{max} in the torsion formula.

Modulus of rigidity, $G = \frac{Tl}{J\theta}$

where θ=angular twist within the elastic limit corresponding to torque T.

Polar moment of inertia, $J = \frac{\pi d^4}{32}$

Modulus of rupture, $q_r = \frac{T_{max}}{J} \times \frac{d}{2}$

where d=diameter of solid circular section

l=Gauge length of the specimen.

TORSION FAILURE

Fig. 21·20 shows the state of stress at a point on the surface of the circular specimen tested under torsion. The maximum shear stress occurs on two mutually perpendicular planes, parallel and perpendicular to the longitudinal axis XX of the specimen. The principal stresses p_1 and p_2 make an angle of 45° with the longitudinal axis and are equal in magnitude to the

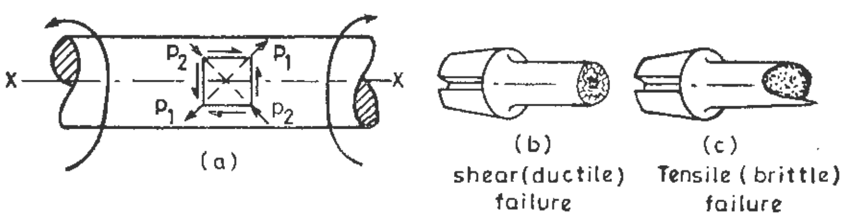


Fig. 21·20

shear stresses, p_1 is a tensile stress and p_2 is an equal compressive stress.

Torsion failures are different from tensile failures. Ductile materials fail in tension after considerable elongation and reduction in area, and showing cup and cone type fracture while in torsion a ductile material fails by shear along one of the planes of maximum shear stress. Generally the plane of fracture is normal to the longitudinal axis as shown in Fig. 21·20(b). A briltle material fails in torsion along a plane perpendicular to the direction of the maximum tensile stress. This plane bisects the angle between the two planes of maximum shear stress and makes an angle of 45° with the longitudinal axis, resulting in a helical fracture (as shown in Fig. 21·20(c)).

21·5. BEHAVIOUR OF MATERIALS UNDER IMPACT

The Impact tests are used in studying the **toughness** of the materials i.e. the ability of the material to absorb strain energy during plastic deformation. In static tensile test, the area under the load extension curve gives the strain energy absorbed by the specimen uptil breaking. In order to have high toughness, the material should possess high strength and large ductility. Brittle materials have low toughness since they exhibit very small deformation before fracture. The use of such materials in structures or machines is dangerous since fracture may occur suddenly withont any noticeble deformation.

In the case of polycrystalline materials there are two types of fractures (*i*) *brittle fracture* as in the case of cast iron (*ii*) shear fracture (or the ductile fracture) as in the case of mild steel and aluminium. The strength of the meterial can be described by two characteristics *i.e.*, (*i*) resistance of the material to separation and (*ii*) resistance of the material to sliding. If the resistance to sliding is greater than the resistance to separation, the material is brittle and if the resistance to separation is greater than the resistance to sliding, the material is ductile.

Three basic factors contribute to a brittle type of fracture *i.e.*, (*i*) a triaxial state of stress (*ii*) a low temperature and (*iii*) a high strain rate or rapid rate of loading. All the three factors need not be present at the same time to produce a brittle fracture.

(*i*) **Triaxiality of stresses**. Fig. 21·21. shows a round bar with a groove (or notch) subjected to axial tensile force P. Due to the presence of the groove or the notch, the stress at the root of the notch is very high due to the effect of the stress concentration. Maximum stress at the root of the notch depends upon the root-radius. The material in the centre of the

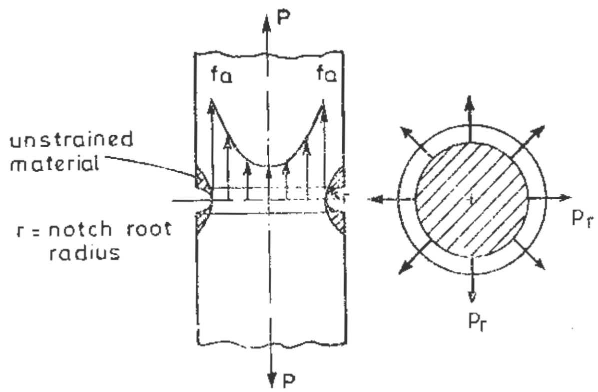


Fig. 21·21

bar, carrying the tensile load tries to contract laterally (*i.e.*, along the radius) because of Poisson's effect, but it is hindered by the resistance of the unstrained material. The result is that there are tensile stresses acting radially outward on the inner portion of the material, which produce a state of *triaxial tension*. These triaxial stresses f_a, p_r, p_r leads to the brittle failure of the material along the notch. Therefore, the impact test on ductile materials is generally performed on bars with a notch, so as to have the effect of triaxiality of stresses.

(*ii*) **Effect of Temperature**. Steels are used for building purposes and the notch impact strength of steel depends on temperature. The energy required for a given notched bar impact test falls rapidly and irregularly once the temperature drops below a critical temperature and usually a ductile steel breaks in a brittle manner.

In general, at high temperatures, fractures in steel occur with large deformation and high values of impact energy are obtained. The fracture is fibrous in character. As the temperature drops, the impact energy values fall more or less rapidly within a critical temperature range, and brittle fractures occur (*i.e.* fracture with a very small deformation). The fracture is granular having crystalline appearance. At transitional temperatures mixed fractures occur with an alternating sequence of the deformation and instantaneous fractures.

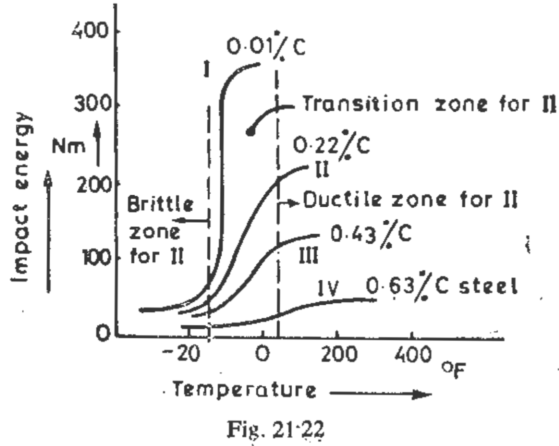


Fig. 21·22

It can be easily seen that the transition curves flatten out as the carbon content is increased in steel and also the maximum impact energy at which only ductile fracture occurs, falls as the carbon content is increased. In the Fig. 21·22, three zones for 0·22% carbon steel are shown, *i.e.*, brittle zone, transition zone and ductile zone. If the temperature of 0·22% *C* steel is less than $-140°$ F, brittle fracture occurs and if the temperature is more than 40°F, ductile fracture occurs.

(*iii*) **Effect of Straining Rate**. The plastic stress–strain curve of a ductile metal is raised by increasing the strain rate. In other words, if a tensile load is applied on a metallic specimen with a very high strain rate, its yield point is increased in comparison to the yield point obtained in static tensile test. This effect is also temperature dependent and is more pronounced near the melting point of the metal. The effect is fairly small at room temperature. For example increasing the strain rate by a factor of 100 increases the yield stress of copper by only 10 to 15 per cent at room temperature. But if at the temperature near the melting point if the strain rate is increased from 10^{-6} to 10^{+3} per second, the yield stress is almost doubled.

Especially in mild steel, the yield point is subjected to striking variations with strain rate, which is closely associated with the causes of discontinuous yielding. With high strain rates, the stress can reach much higher values before general yielding begins in mild steel.

The importance of increased yield strength at higher strain rates lies in its effect on ductility. The result is a decreased ductility and a greater tendency to brittle fracture, so increasing the rate of loading has the same general effect on ductility as increasing the triaxiality of stress.

Ductile and Brittle States. We have learnt that the following three factors control the ductile or the brittle type of fracture or yielding and fracture stress.

(*a*) Triaxiality (*b*) Temperature (*c*) Rate of loading.

Fig. 21·23 illustrates the effect of these 3 factors. Consider a material loaded at point *A* representing the given triaxiality, strain rate and temperature. As the stress is increased along the line AA', the material is yielded at Y_A as $Y_A < F_A$. Therefore, material will be in a ductile state and the final fracture will be ductile fracture. Now consider loading at point *B*, high triaxiality and strain rate and a lower temperature will produce brittle fracture. Obviously the point Y_B will never be reached and the material is in a brittle state. The transition value of triaxiality, strain rate or temperature is represented by the point *C*. To the right of this transition point *C*, the material is in a ductile state and to the left it is in a brittle state.

In the neighbourhood of the point *C*, there will be usually some yielding followed by fracture. This transition phenomenon occurs over a wide range of values.

Impact loading of unnotched samples provides evidence of transition strain rate above which ductile materials behave in a brittle manner. Low temperature testing of unnotched specimens shows the existence of a transition temperature. Similarly, the ductile materials tend to become brittle if triaxiality increases.

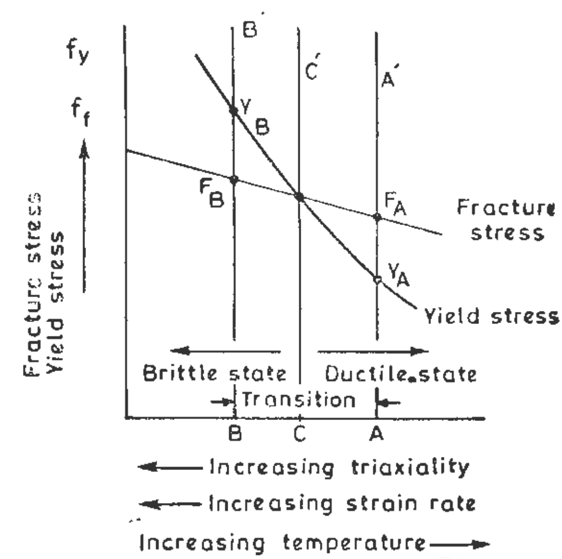


Fig. 21·23

Notch Effect. The strress concentration at the root of the notch provides large stress necessary to raise the yield stress and a high local strain rate at the root.

Say K_t=theoretical elastic stress concentration factor

f_{av}=average stress at the section containing notch

$$f_{max}=K_t \cdot f_{av}$$

or

$$\frac{f_{max}}{E}=K_t \cdot \frac{f_{av}}{E}$$

$$\epsilon_{max}=K_t\ \epsilon_{av}$$

Differentiating both the sides with respect to time

$$\mathring{\epsilon}_{max}=K_t\ \mathring{\epsilon}_{av}$$

This shows that the local strain rate at the root of the notch is multiplied by the same concentration factor as the stress.

If the combined effect of triaxiality, high strain rate, low temperature and stress concentration raises the yield stress above the fracture stress, a crack will form near the root of the notch, which is locally yielded. The immediate effect of crack formation is a sharp local increase in strain rate which further increases the yield stress and brittle fracture continues and the crack rapidly runs through the material.

Notch Sensitivity. The tendency of a ductile material to behave in a brittle manner in the presence of a notch is called *notch sensitivity.* This property also depends on strain rate, triaxiality and temperature. The effect of notch sensitivity is obtained by plotting a curve between impact energy and temperature for a notched bar impact test, keeping traxiality constant by using a standard notch for all specimens and keeping strain rate constant at some high value by standard impact loading. The high overall strain rate multiplied by the stress concentration factor of the notch produces local strain rates as high as 10^3 cm/cm/second.

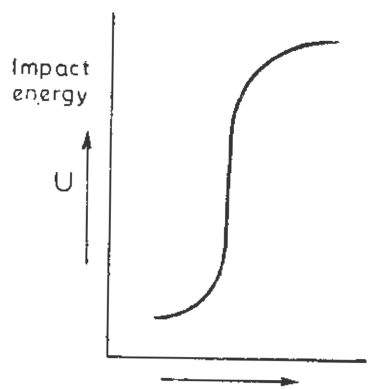


Fig. 21·24

Notch sensitivity is measured partly by the sharpness of the transition in the fracture energy or impact energy versus temperature curve shown in Fig. 21·24 for a low carbon steel. The sharper the transition the more notch sensitive is the material. In the case of low carbon steel, the transition is so abrupt that a single temperature T defines it.

Notched Bar Impact Test. This is a standard test on notch sensitivity combining all the three factors *i.e.*, triaxiality (notch), high strain rate (pendulum) and temperature. High temperatures upto 2000 °F are obtained in ovens/furnaces. Low temperature are obtained by (*i*) forced air circulation over dry ice (−109 °F) (*ii*) liquid nitrogen (−319°F) and (*iii*) liquid hydrogen (−423 °F). The pendulum of the impact testing machine must be carefully constructed with the striking edge at its centre of percussion to minimise vibrations.

In the case of Charpy impact test, standard specimen with a notch in the centre is supported like a beam loaded at the centre as shown in the Fig. 21·25 (*a*). The notch is on the tension side. While in the case of Izod impact test, the specimen is fixed as a cantilever loaded at the end.

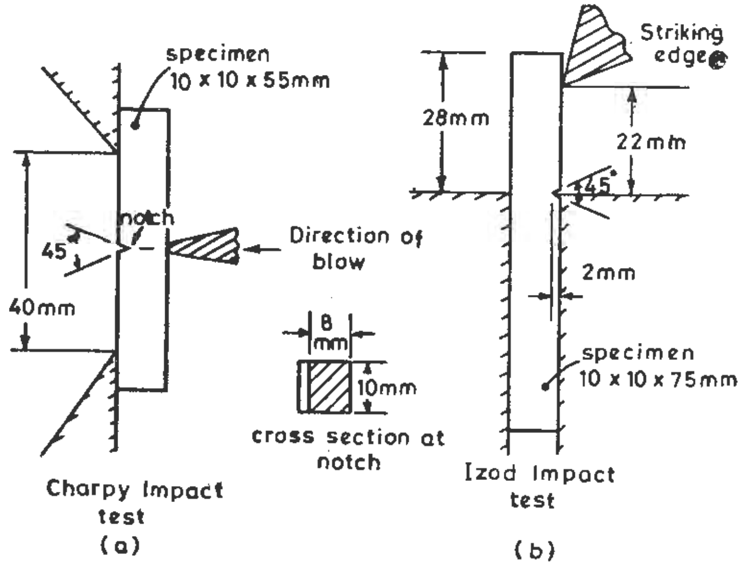


Fig. 21·25

The Charpy test has two advantages over the Izod test *i.e.* (*i*) It is easier to place the specimen in the machine, an important consideration in low temperature tests when the test must be performed within a few seconds after removing the specimen from a low temperature bath. (*ii*) It is also free from compressive stresses around the notch, which are produced in the Izod specimen by the vice, when we consider the complexity of the stress distribution introduced by the notch itself.

When the notched bar impact test is used to compare the notch sensitivities of materials, the significant information is simply a tabulation of comparative impact energy values.

21·6. HARDNESS

Hardness is the property of a material by virtue of which it resists penetration, indentation, scratch, wear and tear, abrasion and cutting. An appropriate definition of hardness is the resistance of the material to permanent deformation of its surface.

The relationship between hardness and atomic structure was first developed by a German mineralogist Mr. Mohs, who determined hardness by surface scratching of one material by another material. He assigned Mohs hardness number from 1 to 10. Mohs hardness number 10 was given to diamond, 9·7 was given to Tungsten carbide. But the measurement of hardness by scratching is difficult to standardise and to interpret. Therefore for most engineering applications, Mohs' scale does not define the hardness number in a clear quantitative manner.

Hardness Measurement. One way in which surface may be deformed permanently is by *indentation*. An indentor having a diamond point or a hardened steel ball is pressed on the surface of the material and a permanent deformation is produced. The depth of the penetration or the area of the indent and the required compressive force are easily measured and provide an indication of hardness. The resistance to permanent deformation is simply expressed in terms of load applied and area of the impression.

Indentors are made in various geometrical shapes such as spheres, cones and pyramids. The area over which the force acts increases with the depth of penetration.

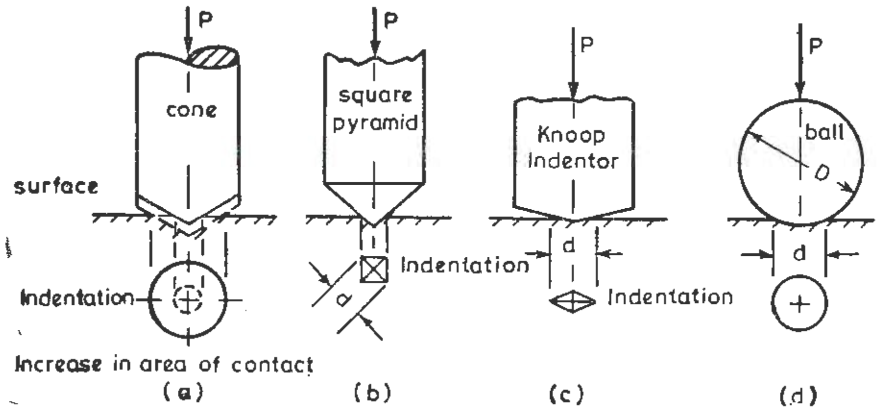


Fig. 21·26

In the Fig. 21·26 (*a*) to (*d*) indentations produced on surfaces by conical, square pyramid, knoop and ball indentors are shown. Around the indentation produced by a ball the stress distribution is highly complex. As the material is forced outward from the region of indentation, it is subjected to triaxial stresses which vary greatly from the centre to the edge of the indentation. Friction between the ball and the surface adds to the hydrostatic compression component. [Note that if the principal stresses are p, p, p each, equal in all the directions, it is said to hydrostatic component of stress]. In the case of pyramid indentors the sharp corners produce even more complex stress conditions.

Pyramid Hardness. Diamond points are ground in the shape of square or rhombus pyramids.

Hardness$=P/A$

Load, $P=\lambda\, d^2$

where d=diagonal of the square

Area, $A=\beta d^2$ where β is a constant

Hardness number $$H=\frac{\lambda d^2}{\beta d^2}=\frac{\lambda}{\beta},$$

independent of both the load and the size of the indentation.

The hardness number of a material for the given shape of the pyramid is the same regardless of the load used. The independence of hardness number and load makes it possible to use a wide range of loads for different purposes. Large loads for large indentation for measuring gross or average hardness and smaller loads for measuring local hardness are used.

It is easier to measure the diagonal of a pyramid indentation due to sharp edge than to measure the diameter of a circular impression.

In the case of Vicker's Pyramid Number (VPN) the angle between the opposite faces of the pyramid is 136°.

Surface of contact between indentor and impression

$$A=\frac{d^2}{2\sin\alpha/2}$$

where $\alpha = 136^\circ$

$$\text{VPN} = \frac{P}{d^2} \times 2 \sin \frac{\alpha}{2} = 1{\cdot}8544 \frac{P}{d^2}$$

where P = Load in kg

d = diagonal in mm.

Knoop Indentor is developed especially to study the microhardness *i.e.*, the hardness of microscopic areas as in the individual metallic grains. The Knoop Hardness number is computed from the projected area of the impression rather than the area of contact.

$$\text{Knoop hardness} = \frac{P}{0{\cdot}07028\, d^2}$$

where P = Applied load in kg

d = Long diagonal of the impression in mm

Brinell Hardness Number. J.A. Brinell used hardened steel ball to determine the hardness of the metals.

$$\text{BHN} = \frac{P}{A}$$

where P = load in kg

A = area of the indentation in mm^2

$$= \frac{2P}{\pi D\,(D - \sqrt{D^2 - d^2})}$$

where D = diameter of the ball

d = diameter of indentation

BHN is dependent on the load used. For this reason, it is necessary to use the same load for all meaurements with a given ball if a comparison of hardness of different materials is to be made.

Rockwell Hardness Test. This method is used to determine the hardness of a wide range of materials. Rockwell Hardness is measured by use of either a steel ball or a cone shaped diamond indentor. It differs from BHN and VPN as in this test depth of impression is measured in place of diameter or diagonal of the indentation. But depth and diameter are always geometrically related, the hardness measurement is the same in principle.

For metallic specimen 3 tests are used as

Rockwell A—For case hardened materials and thin metals such as safety razor blades.

Rockwell B—For soft or medium hard metals as mild steel, brass, copper etc.

Rockwell C—For hard metals such as high speed steel, high carbon steel, tool steels etc.

For Rockwell *A* and *C*, diamond indentor (120° cone angle) and for Rockwell *B* steel ball inductor (diameter 1·59 mm) are used. A minor load of 10 kg is applied initially to overcome the oxide film thickness on the metal which may have been formed in due course of time. Then additional load of 50, 90 and 140 kg is applied on the indentor in the case of Rockwell *A*, *B* and *C* tests respectively.

$$\text{Rockwell hardness} = H - \frac{t}{0{\cdot}002}$$

where H = a constant depending upon the type of the inductor used

= 130 for steel ball indentor

= 100 for diamond cone indentor

t = depth in mm

0·002 mm = 1 unit.

Mechanism of Indentation. When the indentor is pressed into the surface under a static compressive load, large amount of plastic deformation takes place locally. The material thus deformed flows out in all directions. The region affected extends to a distance approximately 3 times the radius of indentation. Taking into account the principle of constant volume during the plastic deformation, the surface surrounding the impression bulges out slightly to account for the volume of the metal displaced under the indentor. In some cases, the metal bulges out around the indentation as shown by Fig. 21·27 (*a*) this is called *Ridging*. This is

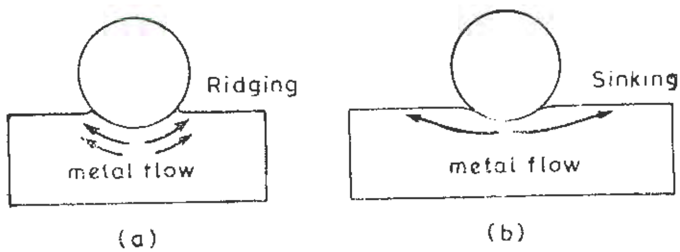


Fig. 21·27

generally obtained in cold worked alloys. While in same cases, the metal bulges out at the ends resulting in *sinking* at the impression shown by Fig. 21·27 (*b*). Sinking takes place in the case of annealed metals.

In the case of ridging type impression, the diameter of the indentation is greater than the true value, whereas with sinking type impression, the diameter of the impression is slightly less than the true value.

Time is an important factor in the process of hardness measurement as large plastic deformations are accompanied by large amount of transient creep which varies with the characteristics of the material.

With the harder materials, the time required to reach the maximum deformation is nearly 15 seconds. Such as for iron and steel. Soft materials like magnesium may require unreasonably long time, sometimes 2 minutes.

Rebound hardness. Hardness measurements are sometimes made by dropping a hard object as on the surface and observing the height of the rebound. Usually a diamond point is used to strike the surface. As it falls its potential energy is converted into the kinetic energy. A part of this kinetic energy is stored in the form of recoverable elastic strain energy in the surface and a part is dissipated in producing plastic deformation. The amount of strain energy stored depends upon the yield point, stiffness and damping capacity of the material. All the elastic strain energy is not recovered in the form of rebound of indentor due to the internal friction of the metal. So the rebound hardness measures a combination of hardness, stiffness and damping capacity.

In the Shore seleroscope tests, a pointed hammer is allowed to fall from a height of 25·4 cm, within a glass tube, which has graduated scale inscribed on it. The standard hammer is approximately 6·35 mm diameter, 1·9 cm long and 2·4 gm weight with a diamond striking tip of radius 0·25 mm. The scale is graduated in 140 divisions. A rebound of 100 is approximately equivalent to the hardness of a martensitic high carbon steel.

2·7. FATIGUE BEHAVIOUR OF MATERIALS

Materials subjected to fluctuating loads or repeated load cycles tend to develop characteristics different from their behaviour under steady loads. This behaviour is called fatigue and is characterised by (*i*) loss of strength (*ii*) loss of ductility (*iii*) increased uncertainty in strength and service life. The inhomogeneity of the material is responsible for all these 3 features of fatigue behaviour.

The fatique of the materials is primarily an effect of the repititions of the loads and not simply a time effect. The rate of application of the load is not an important factor in fatigue. A STM defines fatigue as "A general term used to describe the behaviour of materials under repeated cycles of stress or strain which cause a deterioration of the material that results in a progressive fracture.

Fatigue occurs at stresses well within the ordinary elastic range as measured in a static tensile test on the material. Fatigue occurs under all kinds of loadings and at high and low stresses.

Deterioration resulting from fatique consists primarily in the formation of cracks in the material. These cracks originate from visible discontinuities which act as stress raisers. These discontinuities include design details such as holes, fillets, keyways etc ; imperfections in the material such as inclusions, blowholes or fabrication cracks.

The progress of simple fatique can be traced in 3 stages (*i*) nucleation (*ii*) crack propagation (*iii*) fracture as shown in Fig. 21·28. In short, localised changes in the atomic structure begin within the first few cycles at scattered points in the material. These changes in atomic structure soon develop into submicroscopic cracks which grow as the loading cycles continue into the microscopic size and eventually become large cracks which are visible. Finally when the cracks have grown to some critical size, the member becomes weak and it breaks.

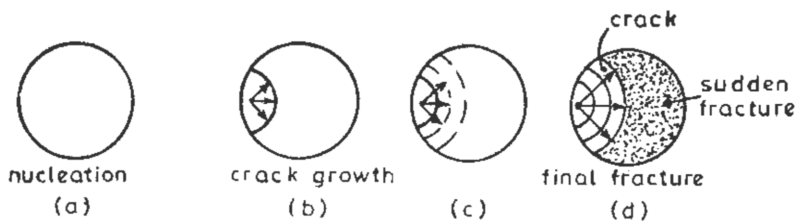


Fig. 21·28

(*a*) The mechanism of nucleation and cracks growth for metals can be explained as follows :

Fatigue in metals begins with highly localised yielding. In polycrystalline metals in simple tension, there always exist a few crystals which are so oriented that slip can easily start in these crystals. As the load is increased, these weak crystals yield first, but since they are surrounded by elastic material, they do not affect the static stress-strain diagram noticeably. Nevertheless they do yield and at an overall stress that is within the elastic range of the material. If the material is loaded only for once, the effect of the localised yielding is insignificant. But if the load is repeated, each repitition produces additional localised yielding which eventually results in the formation of submicroscopic cracks in the yielded region, due to the strain hardening effect produced by repeated loading cycles.

Fig. 21·29 (*a*) shows a cantilever type cylindrical specimen rotating at w radian/sec- and subjected to a vertical load at the free end. The critical section of the specimen is subjected to a bending moment $M=Wl$ resulting in maximum stress $f_{max}=\frac{32\,M}{\pi d^3}$. If a point A is considered on the periphery of the section, then it is subjected to a stress cycle 0, f_{max}, 0, $-f_{max}$ as shown in the Fig. 21·29 (*b*). Say in a particular crystal near the outer surface of the specimen, at the critical section, the stress has exceeded f_{yp}, as shown in the Fig. (*c*). With each stress reversal, yield stress goes on increasing and when the yield stress reaches the ulti-

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mate stress of the material *submicroscopic crack* is developed at the point, which has acted as a stress raiser. As load cycles continue new submicroscopic cracks are formed mostly in the

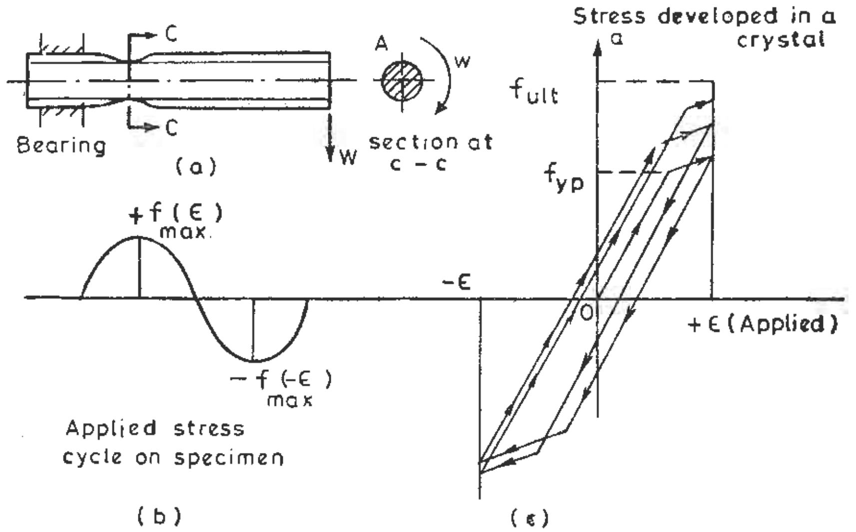


Fig. 21·29

same crystal and these submicroscopic cracks join together to make a microscopic crack. The microscopic cracks have been observed after only 0·1 per unit of the total number of cycles endured before failure. In general, the fatigue cracks start in the surface of the member possibly because the crystals adjacent to the surface are less restricted by the surrounding crystals.

The first microscopic cracks appear in slip planes in certain unfavourably oriented crystals : *i.e.* crystals whose orientation is such that slip planes coincide with the planes of maximum shear. So the microscopic cracks grow in these planes which are at 45° to the axis of the member, and these cracks usually originate in more than one such planes. The intersection and joining of a number of such microscopic cracks produce a zig zag crack in a direction at right angle to the axis of the member. Growth beyond the crystal of origin to adjoining crystals brings about slight changes in direction to accomodate the planes of easy slip.

A *notch effect* accompanies the crack and increases its tendency to grow in the general direction at right angle to the tensile stress Fig. 21·28 (*b*) shows the gradual crack growth. When the remaining cross sectional area becomes small enough, final fracture occurs in which again the notch effect is the controlling factor. Thus the failure in fatigue is a brittle failure. Sometimes the members are subjected to high maximum stresses and gross yielding of the entire section takes place. In such a case strain hardening in general plays dominant role than the localised slip.

A majority of fatigue failures start at visible discontinuities which act as stress raisers, such as shown in Fig. 21·30 (*a*) and (*b*). In such instances the initial yielding is caused not by an unfavourably oriented crystal but by a local increase in stress resulting from stress concentration. Nucleation is still highly localised. The weak points are now the small regions affected by the stress concentration. Fig. 21·30 (*a*) shows the fatigue failure of a shaft subjected to load cycles, the shaft is having a large fillet radius and therefore low stress concentration. Cracks nucleate from a few points on the surface and cracks grow and propagate

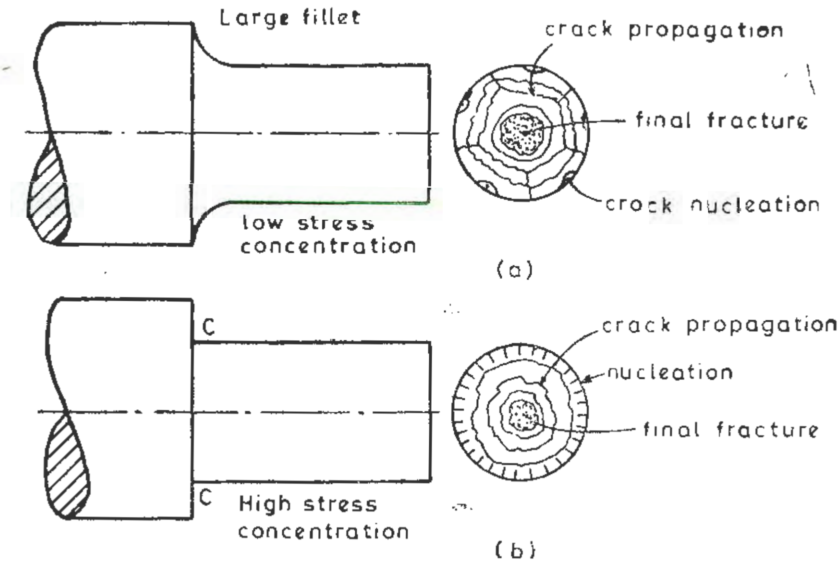


Fig. 21·30

in more or less in the radial direction and final fracture takes place at the central section shown. Fig. 21·30 (*b*) shows the fatigue failure of a shaft having high stress concentration around the circular corner CC. All around the periphery cracks are nucleated due to high localised stress and all these cracks join together to form an irregular crack all near the circumference. The crack progresses radially till the final fracture takes place.

(*b*) **Statistical Nature of Fatigue.** In fatique, fracture depends on a random distribution of weak points and the whole chain of events preceding fatigue fracture depends on a series of random processes and varies widely from one member to another. Therefore the scatter in observed values is considerable. Consequently neither a single observation nor an average of several observations can give a measure of fatigue life. So the fatigue life of a material can only be truly depicted as a distribution of values for individual specimens. With the use of statistical methods, the distribution of values can be used in a much more rational manner than the individual values with a suitable factor of safety. With the statistical analysis of data, a machine member can be designed for a low percentage of failures or a high percentage of survivals.

(*c*) **Fatigue Properties.** The total number of cycles required to bring about the final fractures under the given conditions (of stress amplitude, maximum stress and rate of cycling) is the basic fatigue property. This is directly measured from experiments for individual specimen.

Fig. 21·31 (*a*) shows the stress vs. number of cycles (upto failure) curve for a phosphor bronze strip subjected to reversed bending, where mean stress is zero, (*b*). The vertical axis represents the maximum stress, f_{max}, the horizontal axis represents the number of cycles to failure or fatigue life N. The range of N becomes large in comparison to f_{max} and there is considerable curvature at all points of the range except for very large N. But if N is plotted on a logarithmic scale, the first part of the curve often becomes nearly a straight line and it is possible to fit most of the observations quite well by two straight lines intersecting at a point, as shown in Fig. 21·31 (*b*). If is interesting to note in this example that the point where the two lines meet is near the proportional limit stress for the material.

When the f-N curve approaches a horizontal asymptote, the corresponding stress is called the endurance limit f_e, and the fatigue life at stresses lower than f_e is assumed to be

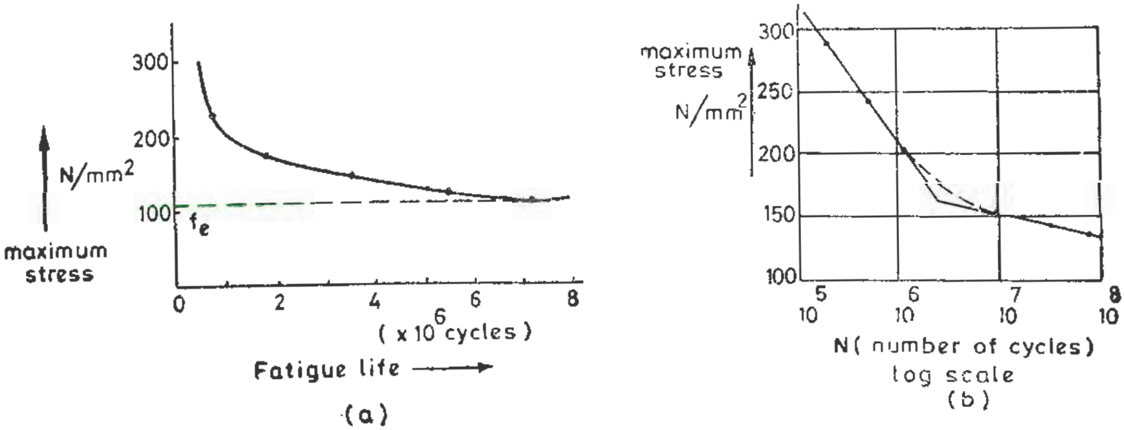


Fig. 21·31

infinite. Ferrous metals usually have a fatigue limit whereas non ferrous metals often do not have fatigue limit, f_e.

Fatigue Strength (f_n). In a general way fatigue strength is defined as the stress which a material can withstand respectively for N cycles, and is developed by interpolation from graph of stress versus fatigue life.

(*d*) **Factors Affecting Fatigue.** Fig. 21·32 (*a*) shows a general stress-cycle, with f_{max}=maximum stress ; f_{min}=minimum stress, f_m=mean stress=$\frac{f_{max}+f_{min}}{2}$ and f_a=alternating stress=$\frac{f_{max}-f_{min}}{2}$. Most of the fatigue data in the literature have been determined for

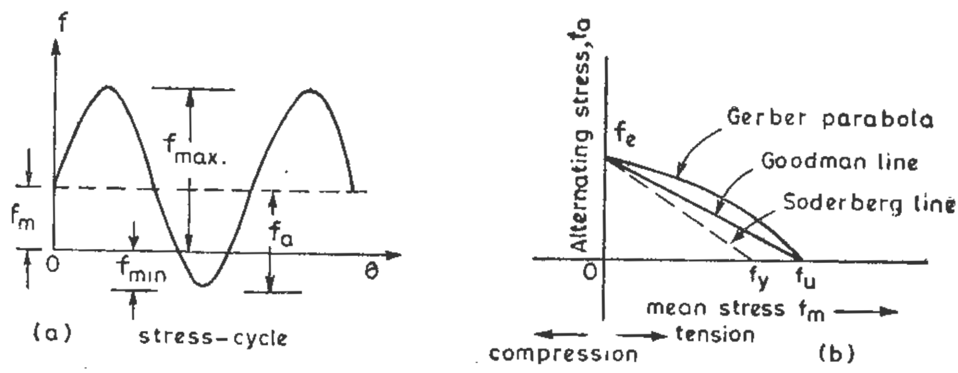


Fig. 21·32

completely reversed cycle of stress *i.e.*, $f_m=0$, because this type of cycle produces the worst type of effect. However conditions are frequently met in industrial applications where the stress cycle consists of an alternating stress and a superimposed mean or steady stress. For each value of mean stress, there is a different value of range *i.e.*, $f_{max}-f_{min}$ which can be withstood by the material without failure. Fig. 21·32 (*b*) shows the variation of f_a with f_m (mean stress). As the mean stress becomes more tensile, the alternating stress f_a, is reduced, until at the tensile strength f_u, the stress range is zero. However for practical purposes testing is usually stopped when the yield stress f_y is reached. A straight line relationship follows the suggestion of Goodman, while the parabolic curve was proposed by Gerber. Test data for ductile metals generally fall closer to the parabolic curve, but the tests data on notched specimens fall closer to the Goodman line, the linear relationship is usually preferred in engineering design. Relationship between stresses can be expressed as

$$f_a = f_e \left[1 - \left(\frac{f_m}{f_u} \right)^x \right]$$

where $x=1$ for Goodwan straight line

$x=2$ for Gerbar parabola

f_e=endurance stress or fatigue limit for completely reversed loading.

But if the design is based on yield strength, then dashed straight line given by Soderbeg can be used and in the above expression f_u is replaced by f_y (yield strength).

(*e*) **Fatigue Damage.** The problem of design for variable loading spectrum is of primary importance in the design of rotary wing aircrafts. It has been observed that fatigue cracks are nucleated during the first few cycles of loading, therefore practically no phase of service lift is free from damage of some kind. Crack propagation involves many factors, out of which stress gradient is an important factor. Cracks propagate at different speeds in different materials, under different conditions. Experiments have been performed in which crack length was measured as a function of the number of cycles. Fig. 21·33 shows some typical curves of crack growths under various stress levels. Each curve ends with fracture at some critical crack length. Another effect of stress level is its effect on the character of crack; low stress level produces fine cracks and high stress levels produce course cracks. The order in which stress levels are applied has important effects on the progress of fatigue damage. A course crack started by high stress level will not propagate very rapidly under a subsequent low stress. On the other hand a fine crack started by a low stress level might propagate very rapidly under a subsequent high stress. At the same time, strain hardening at the tip of crack plays important part on how it behaves under subsequent higher or lower stress levels. All these variations tend to average out if stress levels are applied in a random order and cumulative damage theory has been developed on this very basis. According to this theory each series of stress cycles accounts for a certain fraction of total damage and when these fractions add upto unity failure occurs.

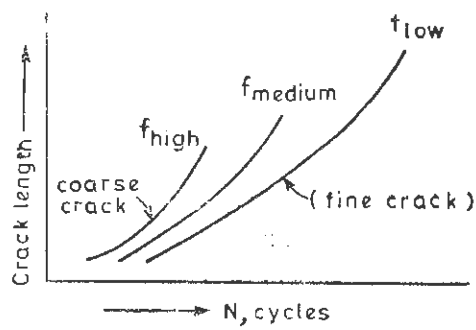


Fig. 21·33

$$\sum \frac{n_i}{N_i} = 1 \quad \text{or} \quad \frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} \ldots\ldots = 1$$

The value of N is observed from stress versus fatigue life curves for different stress levels.

(*f*) **Surface Effects.** It has been observed that most fatigue cracks are nucleated near the surface of members. Therefore the condition of the surface is very important. A rough surface can lower the fatigue strength by as much as 15—20 per unit. Therefore surface scratcher must be removed by slow grinding and polishing operation.

Electroplating a surfaee usually lowers the resistance to fatigue because electrically deposited metal layer contains microscopic cracks.

The most common surface treatments for improving the resistance to fatique and increase service lift are those which produce residual compressive stresses in the surface such as Peening.

Peening consists in striking the surface with a rounded hammer or ball, which makes a series of overlapping indentations covering the entire surface. The surface layer of the member is compressed which acts as a crack-resistant armor around the inner material and markedly improves the resistance to fatigue Metallic shots of diameter 0·2 mm to 4 mm in diameter are propelled against the surface at high velocity around 60 m/second. Cold surface rolling also introduces compressive residual stresses in the surface.

(*g*) **Understressing**. In some materials having well defined fatigue limit (f_e) it has been observed that application of stress cycles at stresses below f_e can strengthen the material. If these cycles are applied to materials in a series of increasing-stress cycles starting from just below f_e (say 10 million cycles at each level). These materials have found to withstand much higher stresses than f_e' without failure. This process of repeated cycling at successively higher levels, by which the fatigue properties of materials are improved, is called understressing.

(*h*) **Experimental Methods**. Fatigue tests are performed on members by applying cyclic load in (*a*) simple axial loading *i.e.*, tension-compression loading (*b*) rotating bending (*c*) twisting (*d*) combination of these loads.

There are two types of fatigue testing machings

(*i*) Constant load type, loading cycle remains the same throughout the experiment and deflection usually increases as specimen becomes weaker.

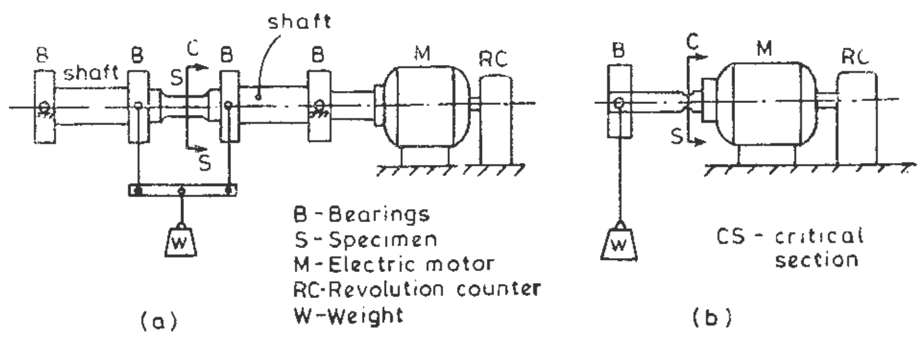
(*ii*) Constant deflection type, a fixed cycle of displacement is imposed on the specimen and the resulting stress may change as fatique progresses.

The machines which are most commonly used in laboratories are

(*i*) Rotating bending machine with pure bending

(*ii*) Rotating bending machine with load on specimen supported as cantilever.

Fig. 21·34 (*a*) shows the most popular type rotating beam fatigue testing machine the specimen is carried between two bearings and connected to shafts which are supported in bearings. The assembly is connected to an electric motor and a revolution counter. Load is supported at the ends of the specimen as shown in the Fig. This type of loading gives constant



F g. 21 34

bending moment throughout the length of the specimen. But the section of the specimen varies uniformly with minimum diameter at the centre. Therefore maximum bending stress occurs at the centre. In this case stress is always completely reversed with means stress $f_m=0$. The rotational speeds attainable are 3600—10,000 R.P.M.

Another variation of rotating bending machine uses a specimen which is mounted and loaded as a cantilever as shown in the Fig. 21·34 (*b*). In this case bending moment linearly varies along the length of the specimen and specimen has minimum diameter at the critical section shown.

21·8. CREEP

In many applications, materials are required to sustain steady loads for long periods of time for example blades of a turbine rotor, plastic mountings of electrical appliances, filaments in vacuum tubes, timber beams in roofs of building, steel reinforcement and concrete in prestressed concrete beams and lead sheathes of telephone cables. Under such conditions the material continues to deform until it is rendered useless.

ASTM defines creep as the "The time dependent part of the strain resulting from stress".

Because the creep is very much dependent on temperature it is generally thought of as *elevated temperature effect.*

Lead and plastic exhibit considerable creep at room temperature, while asphalt and tar creep even at temperatures far below room temperature. For materials like concrete and wood, temperature is not an important factor.

Mechanism of Creep. A constant stress or a constant load is applied on a member and strain is measured with respect to time.

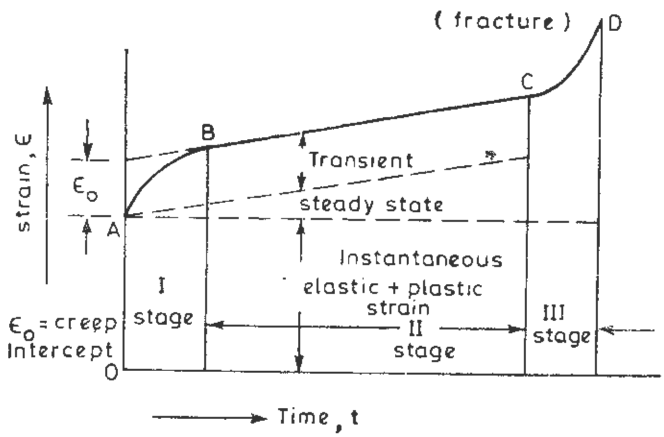


Fig. 21·35

Fig. 21·35 shows the strain-time curve with 3 distinct stages. Total strain at any time has following components

(*a*) Elastic plus the plastic strains (if the stress is high enough) occur almost instantaneously when the stress is applied,—represented by *OA*. This component is generally omitted in the creep curve

(*b*) Transient creep strain.

(*c*) Steady state creep strain.

The main characteristic of *transient creep* is its decreasing rate as is obvious from *AB* part of the curve. Deformation is rapid at first but gradually becomes slower and slower as it approaches the fixed strain rate—*i.e.*, steady strain rate.

The steady strain continues under constant stress which remains constant throughout deformation. So it is identical with viscous flow and sometimes referred to as *viscous creep*. Steady strain rate is also dependent on temperature, it is also called *hot creep.* Transient creep takes place at all temperatures even at zero degree temperature so it is also called *cold creep.*

The creep curve can be divided into 3 stages

1. Transient creep or the primary creep.
2. Steady state creep or the secondary creep
3. Creep fracture or the tertiary creep.

Transient creep. In crystaline materials, transient creep consists of a small additional yielding produced by thermal activation. (At higher temperature, the yield stress of the material is reduced). Application of stress is accompanied by initial plastic strain which ceases as soon as the stress is just balanced by the strain hardening effect. Thereafter impulses of thermal energy contribute to cause further small increase, in strain and each increment in strain causes strain hardening. Consequently each increment becomes a little more difficult and further increments less and less frequent. Thus transient creep gradually approaches to a minimum. This mechanism also operates at stresses in the upper elastic range, where thermal activation can sometimes induce localised yielding at scattered points.

In amorphous materials which do not strain hardden, transient creep is due to thermal activation only. Creep in concrete has been observed at stresses as low as one percent of the ultimate compressive strength. This is possibly due to (*i*) Flow of adsorbed water out of the cement get as a result of external pressure (*ii*) Closure of internal voids in the hardened cement paste.

Viscous Creep. In crystalline materials which strain harden, viscous creep takes place when the strain hardening effect is just balanced by the thermal softening effect. Each increment of plastic strain is accompanied by an increase in yield stress (due to strain hardening) which in turn is gradually lowered by thermal softening so that more plastic strain occurs and the cycle is repeated continuously. It is shown by the part *BC* of the creep curve or the secondary creep.

In amorphous or thermaplastics, viscous flow is the natural form of plastic deformations. The chain molecules slip past each other constantly breaking and reforming their bonds, but there is no strain-hardening. Therefore entire deformation can be classified as creep.

A secondary process in viscous creep of polycrystalline materials is the flow of the grains themselves as semi-rigid bodies. It is called *grain boundary shearing* and results in the rotation of grains during creep process. It ordinarily contributes only a small part of the

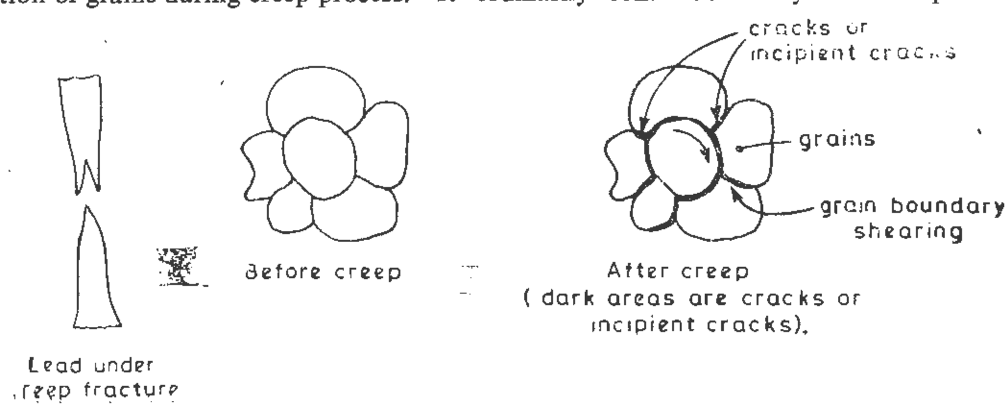


Fig. 21·36

total creep strain but plays an important part in fracture by creep, since the cracks are developed around the grains rotated during creep as shown in Fig. 21·36.

Creep Fracture. As the member under steady load continuously elongates, there is always an accompanying reduction in area and viscous creep in tension inevitably ends in fracture if allowed to continue long enough. This is shown by the part *CD* of the creep curve and at *D*, eventually the fracture occurs.

In the tertiary stage, at higher temperatures or under longer times, ductile metals begins to lose their ability to strain harden, when it occurs, more elongation is required to counteract the effects of thermal softening and the rate of elongation increases, and the fracture may occur without formation of a crack. If the elongation is large fracture is still ductile.

Sometimes at high temperatures or after long periods of loading, metals fracture with very little plastic elongation. Under there conditions grain boundary shearing becomes important. The movement of whole grains relative to each other causes cracks to open up because of their irregular shapes, when one crack becomes large enough or several cracks join to form a larger crack, it spreads slowly across the member until fracture takes place.

At low stresses acting for a long time deformation is sometimes almost negligible and fracture tends to be brittle.

All the 3 stages of creep may not always appear

(*a*) If fracture is brittle, without appreciable reduetion in cross section, the third stage may be missing entirely.

(*b*) For high f or T, the second stage is reduced and at still highei values, second stage may be completed, missing.

(*c*) If the stress or temperature is low enough, the second stage increases to a considerable extent, as shown in Fig. 21·37.

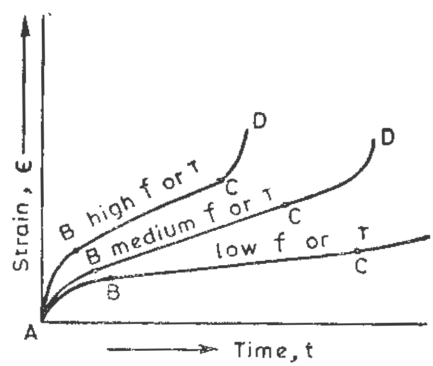


Fig. 21·37

Study of creep is complicated by the fact that four variables are involved : creep strain, time, stress and temperature. Generally the creep-time curve is taken as the primary variation and effects of temperature and stress on it are studied.

The creep tests are usually limited to 1000 hours or less. The extrapolation to service lives more than 10 times the duration of the test are sometimes necessary. The life of a steam power plant is 40 years, or 350,000 hours.

Creep Properties. The most important properties used in design for creep are

1. **Creep strength**—is defined as the highest stress that a material can withstand for a specified length of time without excessive deformation or rupture. The creep rupture strength is often referred to as the stress-rupture strength. These properties vary with temperature, a constant temperature is assumed and must be specified, for example the creep strength required for a steam turbine blade may be that stress which provides 0·20 per cent creep in 100,000 hours at 1500°F. In a jet turbine only a very small strain is permitted because of the elose tolerances (as 0·01 per cent strain in 2000 hours)

Creep strength is determined experimentally as follows

(*a*) Several specimens are simultaneously tested at the expected operating temperature but each under a different stress. The length of time required to produce the allowable strain is measured for each specimen. A curve of stress versus time can be plotted. From the results creep strengths can be tabulated on the basis of a specified amount of creep strain for various temperatures.

(*b*) Another method is based on creep rate.

A curve of creep rate versus stress is obtained for a series of creep time tests at the expected operating temperature. Each test is made at a different stress and is continued until the minimum creep rate appears to be well established.

$$V_0 = B f^n$$

where V_0 = creep rate

B = a constant

f = stress

$$ln\ V_0 = ln\ B + n\ ln\ f \quad \text{...(1)}$$

$ln\ f$ is plotted against $ln\ v_0$.

In using such a curve, total allowable strain ϵ is divided by the service life t to give an allowable V_0. Corresponding to V_0, the value of f *i.e.* creep strength is obtained.

Stress relaxation. Bolts and other members required to hold two or more rigid plates in tight contact are frequently found to have relaxed considerably after long periods of time as a result of creep. This is called stress relaxation and defined as the time dependent decrease in stress in a member which is constrained to a certain fixed deformation.

Say two plates are joined by a bolt and a nut and ϵ_i=initial strain in bolt. If this initial strain ϵ_i is maintained constant the elongation caused by creep is simply substracted from it, thereby reducing the elastic part of the total strain.

Elastic strain at any time $\epsilon_{el} = \epsilon_i - \epsilon_{cr}$ (creep strain)

The stress due to the reduction in elastic strain ϵ_{el} goes on decreasing with time as shown in the diagram 21·38.

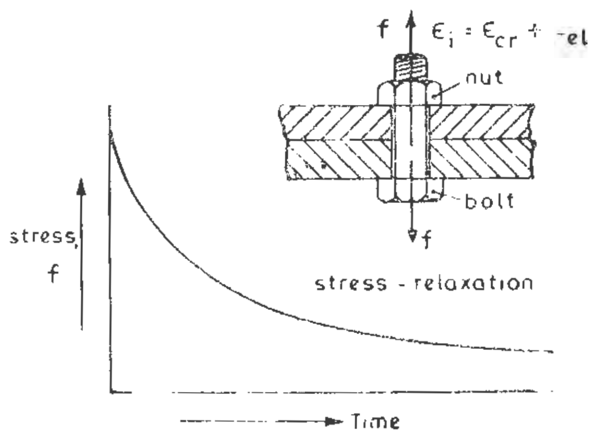


Fig. 21·38

Experimental Methods. Stress is applied by a testing machine which applies either a constant load or a constant stress. In a constant stress machine the load is adjusted continuously to conform to the changing cross sectional area.

$$P = f_c . A = f_c \left(\frac{A_0 . l_0}{l} \right)$$

where f_c=constant stress
A=cross sectional area at any instant
A_0, l_0=initial area and initial length
l=length at a particular instant.

Utmost care must be taken to avoid eccentricity of the loading.

The specimens for creep tests are usually the same as for the conventional tensile test. Elongated ends having a thermocouple well in each may be provided.

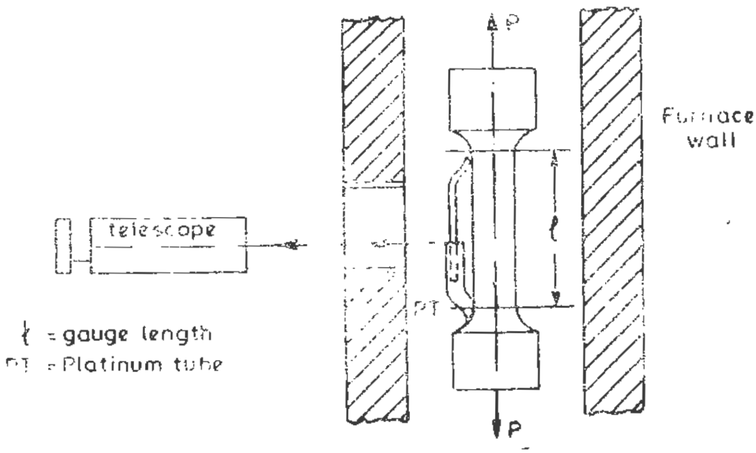


Fig. 21·39

Strains in creep tests can be measured by a travelling microscope. A platinum alloy wire is spot welded to the specimen at one end of the gauge length and a platinum alloy tube is spot welded to the specimen at the other end, as shown in Fig. 21·39. The wire slides inside the tube and reference marks on both are observed through a single telescope at the middle. Elongation is measured on a scale provided in the telescope. Temperature control is maintained by the furnace.

MULTIPLE CHOICE QUESTIONS

1. The Poisson's ratio for most of the metals is close to
   (a) $\frac{1}{5}$ (b) $\frac{1}{4}$
   (c) $\frac{1}{3}$ (d) $\frac{1}{2}$
2. A metallic specimen is loaded in tension beyond the yield point, then it is unloaded completely and reloaded again in tension. After unloading its yield point has
   (a) slightly decreased (b) slightly increased
   (c) considerably decreased (d) considerably increased
3. The most important, reason for Bauschinger's effect in ductile materials is
   (a) ductile material's weakness in shear (b) compressive residual stress
   (c) tensile residual stress (d) None of the above

4. The length between the supports of a Charpy Impact test specimen is
(*a*) 60 mm (*b*) 50 mm
(*c*) 40 mm (*d*) 30 mm

5. The notch-angle in the Izod Impact test specimen is
(*a*) 25° (*b*) 30°
(*c*) 35° (*d*) None of the above

6. The angle between the opposite faces of the diamond pyramid in the case of Vicker's Pyramid Hardness test is
(*a*) 120° (*b*) 128°
(*c*) 136° (*d*) 144°

7. For the measurement of microhardness, the indentor used is
(*a*) Vickers Diamond Pyramid (*b*) Brinell Ball
(*c*) Knoop Indentor (*d*) None of the above.

8. The depth of penetration of the hardened steel ball in the specimen is 0·140 mm. The Rockwell '*B*' hardness of the material is
(*a*) 70 (*b*) 60
(*c*) 50 (*d*) 40

9. The depth of penetration of diamond indentor in a specimen is 0·126 mm, the Rockwell *C* hardness number of the material is
(*a*) 63 (*b*) 50
(*c*) 37 (*d*) None of the above

10. The process which does not improve the fatigue strength of a material is
(*a*) shot peening of the surface (*b*) cold rolling of the surface
(*c*) electroplating the surface (*d*) understressing the surface

11. The clearance between the turbine rotor blade and the casing is reduced by 0·3 mm in 1000 hours. If the blade length is 300 mm, the creep strain rate per hour is
(*a*) 1 microstrain/hour (*b*) 2 microstrain/hour
(*c*) 5 microstrain/hour (*d*) 10 microstrain per hour

ANSWERS

1. (*c*)	2. (*b*)	3. (*b*)	4. (*c*)	5. (*d*)	6. (*c*)
7. (*c*)	8. (*b*)	9. (*c*)	10. (*c*)	11. (*a*)	

EXERCISE

1. Explain the process of yielding in polycrystalline materials.

2. Show that Poisson's ratio for most of the metals having crystalline structure is close to 1/3.

3. What is discontinuous yielding ?

4. Differentiate between the following :
(*i*) Elastic strain and plastic strain
(*ii*) Tangent modulus and secant modulus

5. What do you uuderstand by mechanical hysterisis loop ? Explain how repeated loading increases the yield stress of the material.

6. Why is the strength of cast iron more in compression, than that in tension, explain ?

7. Compare the type of fracture in tension for
   (*a*) Mild steel and wrought iron
   (*b*) High carbon steel and cast iron.

8. Compare the type of fracture in compression for wood, cast iron, concrete and brick.

9. With the help of a neat sketch, explain the Bauschinger's effect. What are the main reasons for this effect.

10. Explain the process of yielding in pure bending.

11. Explain the following in pure bending
    (*a*) Modulus of rupture
    (*b*) Shape factor.

12. What is the difference between pure bending and bending with shear ?

13. Mild steel ad cast iron are tested upto destruction in torsion. Compare their fractured surfaces.

14. Explain how triaxial stresses are developed at the root of the notch in a cylindrical specimum subjected to uniaxial tension.

15. Explain the temperature dependence of medium carbon steel for their impact behaviour.

16. Explain clearly the ductile, transition and brittle zones far a material under impact.

17. Discuss the effect of triaxiality, strain rate and temperature on the impact energy of a material.

18. What is notch sensitivity ?

19. What are the various types of indentors used for hardness measurement ?

20. Explain the principle of hardness measurement by Rockwell Hardness test.

21. Explain the principle of hardness measurement by Brinell's Hardness Test.

22. Explain the mechanism of indentation in hardness measurement and how ridge around the indentor is formed ?

23. Explain clearly the three stages which occur during a fatigue failure.

24. Explain how a submicroscopic crack is initiated during fatigue loading of a member.

25. What is the difference between fatigue strength and endurance limit in fatigue ?

26. Explain Gerber parabola and Goodman straight line law for the determination of stress amplitude.

27. What is cumulative fatigue damage ?

28. Explain how fatigue strength is improved by shot peening, cold rolling and under-stressing the surface of the machine member.

29. Explain how the strain rate goes on decreasing till it becomes constant during the transient creep.

30. What is the difference between hot creep and cold creep ?

31. Explain the temperature dependence of creep strain-time curve.

32. Explain the stress dependence of creep strain-time curve.

33. Explain how the following properties are determined experimentally
    (*i*) Creep strength
    (*ii*) Creep rate

34. What is the difference between creep and stress relaxation ?

35. Describe the procedure of performing a standard creep test on a specimen.

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