ELEMENTS OF STRUCTURAL ANALYSIS

S.A. BARI

Srikant Tripathy

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(For Undergraduate Classes)

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Dedicated to
My Mentor and Guide
Late Dr. Ing. S.J. Bari
Formerly Prof. of Civil Engg.,
Delhi College of Engg., Delhi

PREFACE

The text of this book has been written for the benefit of Students preparing for Diploma, B.Sc. Engineering and A.M.I.E. Examinations. The book has been written in S.I. Units. A large number of well graded questions have been solved keeping in view the needs of average students. Important questions from various examining bodies have been included in the text. These have been approximately converted into S.I. Units.

The vast material available on the subject has been freely consulted. The author acknowledges with thanks authors of various standard treatises, which were consulted during the preparation of the text of this book.

The author would like to express his thanks to Messrs Shahid Akhtar, Chaman Lal Arora and Prof. Ravi Kapoor for their help at various stages of preparation of this book.

New Delhi
July 1997

S.A. BARI

CONTENTS

1

Simple Stresses And Strains

Elasticity

One of the most significant properties of a structural material is elasticity. You will observe that when a steel wire is suspended and gradually loaded along its axis up to a certain maximum load, the length of the wire increases and when the applied loads are gradually removed, the wire comes back to its original length.

A body which returns to its original shape and size and all traces of deformation disappear when the loads are removed is called an elastic body. This behaviour of the material is called elastic behaviour and the property by virtue of which it returns to its original dimensions is called elasticity. A perfectly elastic body shows 100% recovery i. e. deformation completely disappears. But in practice no material has been found to be perfectly elastic. Steel is supposed to be the best example of an elastic material. Copper, brass, aluminium, concrete etc. are all elastic materials of varying degrees.

All discussions in this chapter are based on the assumption that the material of which the structural member is made is homogenous and Isotropic.

Homogenous Material

A homogenous material is one which has the same modulus of elasticity (E) and Poisson's ratio μ at all points in the body. The material has the same physical and chemical composition throughout.

Isotropic Material

The second assumption usually made is that the material is Isotropic i.e. it possesses the same elastic properties in all directions at any one point of the body.

All materials are not isotropic. Materials having no symmetry in elastic property are called Anistropic or sometimes aeolotropic materials.

Mechanical Properties of materials

Ductility

If a material can undergo deformation without rupture, then it is called a ductile material. It is due to this property that a material may be drawn into a wire. Copper is an example of ductile material.

Brittleness

Brittle materials possess very little resistance to rupture. Such materials can not undergo deformation when external forces are applied and fail by rupture. Cast iron is an example of brittle material.

Malleability

The property of a material by virtue of which it can be rolled into plates is known as malleability. Wrought iron is an example of malleable material.

Plasticity

A material is said to be plastic when the deformation produced by the application of an external force does not disappear even after the removal of the external force. Lead is an example of plastic material.

Elasticity

As already explained the property by virtue of which a specimen of a material regains its original shape and size after the removal of the deforming forces is called elasticity. Mild steel is an example of elastic material.

Loads

When a structural member is subjected to external forces, their combined effect on the member is called load.

Loads are classified according to

(1) Their manner of application

(2) According to the effect they produce.

Dead Loads

The loads which do not change under any conditions are called dead loads. Self weight of a member is an example of dead load.

Live Loads

Loads which are applied with velocity and change their value are called live loads. Weight of the traffic crossing a bridge falls under this category.

Depending upon the effect produced on the member, loads are classified as

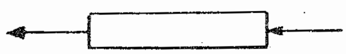
Tensile Loads

Tensile Loads
Fig. 1.1 (a)

These loads have a tendency to pull the member in the direction of their application.

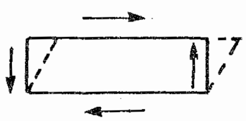
Compressive Loads

These loads try to compress the member on which they act. They shorten the dimensions of the member in the direction in which they act.



Compressive Load
Fig. 1.1 (b)

Shearing Loads

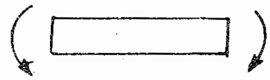


Shearing Load
Fig. 1.1 (c)

The loads which cause sliding of one face relative to the other of a body are called shearing loads.

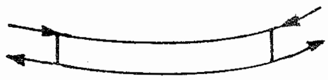
Twisting or Torsional Loads

When two couples are applied at opposite ends of a member, they tend to turn these ends in opposite directions in parallel planes. The loads produced by the couples are called twisting loads.



Twisting Load
Fig. 1.1 (d)

Bending Loads



Bending Load
Fig. 1.1 (e)

Loads causing a certain degree of curvature or bending in the member are called bending loads.

Stress

When a body is subjected to external forces the body deforms in shape, size or volume. The natural tendency of the body is to resist any deformation hence internal forces of resistance are developed within the body to resist the external forces. These internal forces of resistance per unit area of the body are called stresses.

Since internal forces developed within the body are equal to the applied forces, hence stress may be expressed as the applied force per unit area of the body.

Direct Stress or Normal Stress

When external forces are applied along the axis of a body, then the resulting stress is called *direct stress* or *axial stress* or *normal stress.*

$$\text{Normal stress} = \frac{\text{Axially applied load}}{\text{Area of cross–section}}$$

$$\sigma = \frac{P}{A}$$

Stress is measured in units of forces per unit area and expressed as N/mm^2 or MPa. Depending upon the nature of the applied force direct stress may be classified as

1. Tensile stress 2. Compressive stress

Tensile Stress

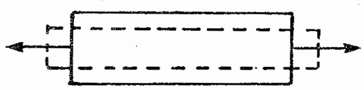


Fig. 1.2 (a)

When equal and opposite forces are axially applied on a body such that the length of the body increases, then the stress produced is called tensile stress.

Compressive Stress

When equal and opposite forces are axially applied on a body such that the body is compressed, the stress produced is called compressive stress.

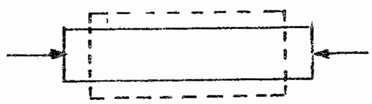


Fig. 1.2 (b)

Strain

When an axial load is applied on a body the body undergoes deformation. Strain is the measure of deformation of the body. It is the ratio of change in dimension of the body to its original dimension.

$$\text{Strain} = \frac{\text{Change in dimension}}{\text{Original dimension}}$$

$$\textit{Longitudinal Strain Or Linear Strain} = \frac{\text{Change in length}}{\text{Original length}}$$

$$\varepsilon = \frac{\delta l}{l}$$

Tensile Strain – When the stress induced is tensile in nature the corresponding strain is called tensile strain.

$$\text{Tensile strain} = \frac{\text{Increase in length}}{\text{Original length}}$$

Compressive Strain

When the body is compressed and a shortening in length takes place due to compressive *stress*, the corresponding strain is called compressive strain.

$$\text{Compressive Strain} = \frac{\text{Decrease in length}}{\text{Original length}}$$

Since strain is a ratio of two dimensions hence it is a pure member. It is a dimensionless quantity. It has no units.

Hooke's Law

Sir Robert Hooke in 1678 observed that the relation between stress and strain is linear for comparatively small values of strain. Hooke's law states that within elastic limit, stress is proportional to strain. This ratio of stress and strain is always constant and depends on the nature of the material only.

$$\text{Stress} \propto \text{Strain}$$

$$\frac{\text{Stress}}{\text{Strain}} = \text{Constant}$$

$$\frac{\sigma}{\varepsilon} = E$$

Hooke's law holds good both in tension and compression. This constant E is called *modulus of elasticity* or *young's modulus.*

Modulus of Elasticity

The quantity E is the ratio of unit stress to unit strain, within elastic limit. It indicates how much stress accompanies a given strain in the material of a given structural member. As strain is merely a number, the units of modulus of elasticity are same as those of the stress.

E is measured in GN/m^2 or KN/mm^2.

Modulus of elasticity E for some structural materials.

TABLE 1.1

S.No.	*Name of material*	*Value of modulus of elasticity E in GN/m^2 or KN/mm^2*
1.	Steel	200—220
2.	Wrought iron	190—200
3.	Cast iron	100—160
4.	Copper	90—110
5.	Brass	80—90
6.	Aluminium	60—80
7.	Timber	10

Stress-strain curve for mild steel

When a specimen of mild steel is gradually loaded in a tension testing machine and a graph is plotted between the stress and the corresponding strain, a curve is obtained as shown in fig. - 1.3

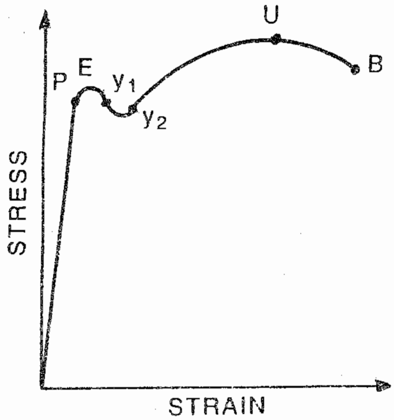


Fig. 1.3

Limit of Proportionality

It is observed that with a gradual increase in loading there is a proportional increase in strain as well. The maximum stress value upto which this relationship is maintained is called the limit of proportionality. Point P on the curve shows the limit of proportionality.

Elastic Limit

It is the maximum stress upto which the material behaves as an elastic material. There is no permanent or residual deformation left when the load is entirely removed. Point E on the curve represents elastic limit. These two points are very close to each other. But in most cases elastic limit is higher than limit of proportionality.

Permanent Set

It is the permanent dimensional change which persists after all the loads are removed. If the specimen is stressed beyond the elastic limit it will not regain its original size and shape when the deforming forces are removed. This small permanent deformation is called permanent set.

Yield Point

It is the point Y on the stress strain curve. It will be observed that at a point just above the limit of proportionality a considerable increase in strain takes place in ductile materials with little increase in stress. The stress value at which this large increase in strain takes place is termed as 'yield point' of the material. In some materials there are two yield points on the stress-strain curve at which there is an increase in strain without an increase in stress. These are known as upper and lower yield points. Stress at yield point is called **yield stress.**

Ultimate Strength

The maximum stress that the specimen under test can bear without breaking is termed as the ultimate strength or the tensile strength of the material. It is shown as point U on the curve.

Breaking Strength

If the specimen is loaded beyond the point of ultimate strength the material will break and the graph will show a downward trend. It is shown as point B on the graph.

Elastic Range

The region on the stress-strain curve extending from origin to the point of proportional limit is called Elastic Range.

Plastic Range

The region of the stress-strain curve extending beyond the limit of proportionality to the breaking point is called Plastic Range.

Percentage reduction in area

When tensile forces act on a bar, the cross-sectional area decreases, but for calculation of normal stresses, original area is considered. If original area is A_1 and A_2 is the cross-sectional area at the plane of failure of a bar then

$$\text{Percentage reduction in area} = \frac{A_1 - A_2}{A_1} \times 100$$

Percentage elongation

If the increase in length of the specimen after fracture is L_1 and the original length is L, then percentage elongation is calculated as

$$\frac{\text{Increase in length}}{\text{Original length}} \times 100 = \frac{L_1}{L} \times 100$$

Proof Stress

Some of the structural materials such as cast iron, concrete, timber etc. do not show a firm or well defined yield point limit. For such materials proof stress corresponding to yield point is generally specified. Proof stress is defined as the limiting stress which produces a permanent set not exceeding 0.5% of the original length.

Working stress and factor of safety

The maximum stress for which a structural member is designed is always less than the ultimate strength of the material. Working stress is generally 2 to 5 times less than the ultimate stress.

Working stress is determined by dividing the ultimate stress by a number called factor of safety.

$$\text{Working stress} = \frac{\text{ultimate stress}}{\text{factor of safety}}$$

$$\text{or Factor of safety} = \frac{\text{ultimate stress}}{\text{workingstress}}$$

Units

The following nomenclature are adopted to express quantities of various magnitudes.

Kilo-- 10^3	MILLI – 10^{-3}
MEGA-- 10^6	MICRO – 10^{-6}
GIGA – 10^9	MANA – 10^{-9}
TERRA – 10^{12}	PICA – 10^{-12}

In S. I. units force is generally expressed in Newtons.

Kilo Newton (KN) means 1000 Newtons.

Stress intensity is expressed in various forms like.

Newtons/mm^2, Kilo newton/mm^2 GIGA Newton/m^2

One N/m^2 = 10^{-6} N/mm^2 = One Pascal

1 N/mm^2 = 10^6 N/m^2 = 1 Mega Newton/m^2

One Mega Newton /m^2 = One Mega Pascal

One Newton /mm^2 = One Mega Pascal

N/mm^2 = MPa

Change in length of a bar due to an axial load.

Let A = Area of cross-section of the bar

l = length of the bar

P = Axial load acting on the bar

E = Modulus of elasticity of the material

δl = Change in length of the bar

σ = Stress induced due to force P

Direct or Normal stress

$$\sigma = \frac{\text{Axial load}}{\text{Area of cross-section}}$$

$$\sigma = \frac{P}{A}$$

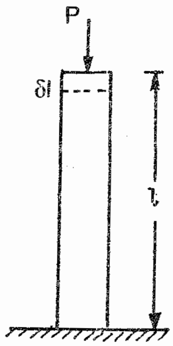


Fig. 1.4

Since the applied load is compressive the direct stress will be compressive and shortening in the length will take place.

$$\text{Strain } \varepsilon = \frac{\text{Stress}}{\text{modulus of elasticity}}$$

$$\varepsilon = \frac{\sigma}{E}$$

$$\text{Strain Produced} = \frac{\text{Change in length}}{\text{Original length}}$$

$$\varepsilon = \frac{\delta l}{l}$$

Change in length $\delta l = \varepsilon \times l$ = Strain $\times$ Original length

$$\delta l = \frac{P}{A.E} \times l$$

If the applied load is tensile, then tensile stress and elongation in length can be similarly calculated.

Example 1.1

Determine the elongation of a steel rod 2 metres long and 40 mm in diameter, when subjected to an axial tensile force of 6 KN. The modulus of elasticity of steel may be taken as 200 GN/m².

Solution

Axial load on the rod = 6KN = 6000 Newtons

Area of cross-section of the rod $= \frac{\pi}{4}(40)^2 = 400\,\pi$ sq.mm

$$\text{Tensile stress} = \frac{\text{Axial load}}{\text{Area of cross-section}}$$

$$\sigma = \frac{6000}{400\,\pi} = 4.77\,\text{N/mm}^2$$

$$\sigma = \mathbf{4.77\ MPa}$$

$$\text{Strain} = \frac{\text{Stress}}{\text{modulus of elasticity}} \text{ and } E = 200\,\text{GN/m}^2 = \frac{200\times10^9}{10^6}\,\text{N/mm}^2$$

$$\varepsilon = \frac{4.77}{200\times\frac{10^9}{10^6}} = 0.0238\times10^{-3}$$

$$\text{Strain} = \frac{\text{Change in length}}{\text{Original length}}$$

$$\varepsilon = \frac{\delta l}{l}$$

or $\delta l = \varepsilon \times l = 0.0238\times10^{-3}\times2\times10^3$

$= .0478\text{mm}$

Elongation = **.0478 mm** **Answer**

Example 1.2

A straight bar of uniform cross-section is subjected to an axial tensile force of 40 KN. The cross-sectional area of the bar is 500 mm² and its length is 5 metres. Find the modulus of elasticity of the material if the total elongation of the bar is 2 mm.

Solution

Sectional area of the bar = 500 mm²

Applied Load = 40 KN

$$\text{Tensile tress } \sigma = \frac{\text{Load}}{\text{Cross-sectional area}}$$

$$= \frac{40\times10^3}{500}$$

$$= 80\ \text{MP}$$

$$\text{Strain } \varepsilon = \frac{\delta l}{l} = \frac{2}{5\times10^3} = .4\times10^{-3}$$

$$\text{Modulus of elasticity} = E = \frac{\sigma}{\varepsilon}$$

$$= \frac{80.0}{0.4\times10^{-3}}$$

$E = 200$ KN/mm^2 Ans.

Example 1.3

Determine the change in the length of the rod AB as shown in fig 1.5. The length of the rod is 4 metres and diameter 30 mm. Take $E = 210$ KN/mm^2

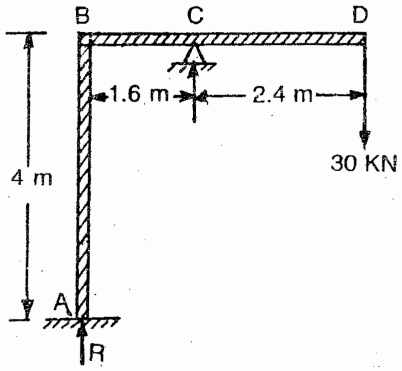


Fig. 1.5

Solution

The 30 KN load will produce a reaction R in the rod.

Taking moments about the hinge c, we get

Reaction in the rod $R \times 1.6 = 30 \times 2.4$

$$\text{Reaction in the rod} = \frac{30 \times 2.4}{1.6} = 45 \text{ KN}$$

$$\text{Stress induced in the rod} = \frac{45 \times 10^3}{\frac{\pi}{4}(30)^2} = 63.66 \text{ MPa}$$

$$\therefore \text{Strain caused in the rod } \varepsilon = \frac{\sigma}{E}$$

$$\varepsilon = \frac{63.66}{210\times10^3} = 0.302 \times 10^{-3}$$

$\therefore$ Change in the length of the rod

$$\delta l = \varepsilon \times l$$

$$= 0.302 \times 10^{-3} \times 4 \times 10^3$$

$$= 1.208 \text{ mm}$$ **Ans.**

Example 1.4

Calculate the diameter of the rod AB in the system shown in fig 1.6, if the stress is not to exceed 150 MPa.

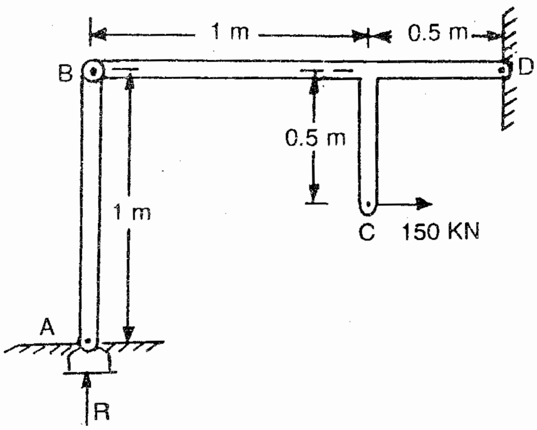


Fig. 1.6

Solution

Taking moments about D

$$R \times 1.5 = 150 \times 0.5$$

$$\text{or} \quad R = \frac{150 \times 0.5}{1.5} = 50\text{KN}$$

$\therefore$ Cross-sectional area of the member

$$A = \frac{50 \times 10^3}{150} = 333.3 \text{ mm}^2$$

Diameter of the rod

$$\frac{\pi}{4} (d)^2 = 333.3 \text{ mm}^2$$

$$d = 20.60 \text{ mm} \quad \textbf{Answer}$$

Example 1.5

The diameter of the piston of a diesel engine is 300 mm and the maximum compressive pressure in the cylinder is 40 N/mm². The cylinder is held by 4 bolts whose effective diameter is 20 mm and Length is 400 mm. Estimate the elongation of each bolt if E = 210 KN/mm²

Solution

Total pressure on the piston

$$= \text{Area of piston} \times \text{pressure in cylinder}$$

$$= \frac{\pi}{4} (300)^2 \times 40 \text{ Newtons}$$

$$= 9\pi \times 10^5 \text{ N}$$

Total area of 4 bolts $= \frac{\pi}{4} (20)^2 = 100\,\pi$ sq. mm

Stress produced in 4 bolts $= \dfrac{9\pi \times 10^5}{100\pi}$

$$= 9 \times 10^3 \text{ Mpa}$$

$$\text{Strain produced in 4 bolts} = \frac{9 \times 10^3}{210 \times 10^3} \doteq .0428$$

$$\text{Hence strain produced in one bolt} = \frac{.0428}{4}$$

$$= .01071$$

$$\text{Elongation of each bolt} = 400 \times .01071$$

$$= 4.282 \text{ mm.}$$

Example 1.6

A square tie bar 50mm × 50mm in cross-section is attached to a bracket by means of 8 bolts and carries a load P. If the permissible stresses in tie bar and bolts are 25 MPa and 15 MPa respectively, find the diameter of the bolts.

$$\text{Cross-sectional area of the tie bar} = 50 \times 50 = 2500 \text{ mm}^2$$

$$\text{Load carried by the tie bar } P = \sigma \times A$$

$$P = 25 \times 2500 = 62500 \text{ Newton}$$

$$\text{Load carried by one bolt} = \frac{62500}{8} = 7812.5 \text{ N}$$

$$\text{Cross-sectional area of one bolt} = \frac{\text{Load carried by one bolt}}{\text{stress in the bolt}}$$

$$= \frac{7812.5}{15} = 520.83 \text{ mm}^2$$

$$\text{Area of one bolt} = \frac{\pi}{4}(d)^2 = 520.83$$

$$\text{Diameter of bolt } d = \sqrt{\frac{520.83 \times 4}{\pi}}$$

or $d = 25.75$ mm **Answer.**

Elongation of a bar due to self weight

A bar AB of length L hanging freely is shown in fig. 1.7

Let L = Length of the bar

A = Cross-sectional area

γ = Weight density of the material of the bar

E = Modulus of elasticity of the material of the bar

Consider a small length dx of the bar at a distance x from the base. The downward force acting on this element of length dx is the weight of the bar that lies below this element and is equal to $A.x.\gamma$. The portion of the bar of length dx may be considered to be subjected to the weight of the material below this section

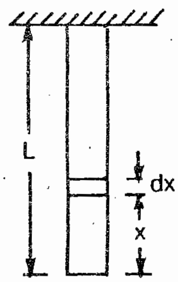


Fig. 1.7

$$\text{Stress in the small element} = \frac{A.x.\gamma}{A} = x\gamma$$

$$\text{Strain of the small element} = \frac{x.\gamma}{E}$$

Elongation of the element $= \frac{x.\gamma}{E} . dx$

Total elongation of the bar $= \int_o^L \frac{x.\gamma}{E} dx$

$= \frac{\gamma L^2}{2E}$

Total weight of the bar W = A. L.γ

Total elongation due to self weight $\delta l = \frac{W.L}{2AE}$

Example 1.7

A uniform steel rope 250 metres long is hanging freely down a vertical mine shaft. Determine the elongation of top 125 meter length of the rope due to the self weight. Weight of steel may be taken $7.5\times10^4 N/m^3$ and modulus of elasticity as 200 GN/m^2.

Solution

The total extension of the upper 125 meter length of the steel rope is caused partially by the weight of lower 125 meter length of the rope and partially due to its own weight.

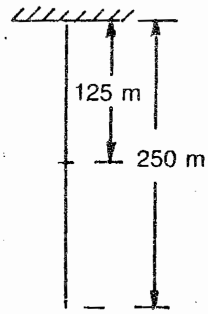


Fig. 1.8

The weight of lower 125 meter length which can be assumed to be acting at the end of upper 125 meter length of the rope is

$= 125 \times \frac{\pi}{4} D^2 \times 7.5\times10^4$ Newtons

$= 937.5 \times \frac{\pi}{4} D^2\times10^4$ Newton

Where D is the diameter of the rope

The elongation due to this load is

$$\delta_1 = \frac{PL}{AE} = \frac{937.5\times\frac{\pi}{4}D^2\times10\times125}{\frac{\pi}{4}D^2\times200\times10^9}$$

or $\delta_1 = .005859$ meters.

The elongation due to the weight of the upper 125 meter length of the rope is equal to half of this extension

$$\delta_2 = \frac{PL}{2AE} = \frac{.005859}{2} \text{ m}$$

$=.002929$m

Hence total elongation of the steel rope is $\delta = \delta_1 + \delta_2$

$=.005859 + .002929$

$= .008788$ metres

$= 8.788$ mm **Answer**

Example 1.8

A solid conical bar of circular cross-section is suspended vertically as shown in fig-1.9. If the length of the bar is L, the diameter of the base D, the modulus of elasticity E and the weight per unit volume is γ, determine the total elongation of the bar due to its own weight. *(Poona Univ.)*

Solution

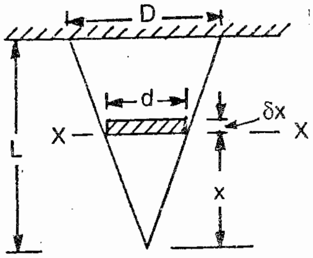


Fig. 1.9

Consider a section of length δx at a distance x from the free end

Diameter of the conical bar at the section x–x

$$d = D.\frac{x}{L}$$

Weight supported at the section xx

$$= \frac{\pi}{4}d^2 \times \frac{x}{3}.\gamma = \frac{\pi d^2}{12}x.\gamma$$

$$\text{Stress at } xx = \frac{\pi d^2 x.\gamma}{12 \times \frac{\pi d^2}{4}} = \frac{\gamma.x}{3}$$

Due to this stress, the elongation of the elementary length

$$= \frac{\gamma x}{3E}\,\delta x$$

Total elongation of the bar

$$= \int_o^L \gamma.\frac{x}{3E}\,\delta x$$

$$= \frac{\gamma.L^2}{6E} \quad \textbf{Ans.}$$

Principle of Superposition

According to the principle of superposition when an elastic body is simultaneously subjected to two or more forces then their effect at a point on a given plane is the algebraic sum of the individual effect of each load. The total strain in the body will be the algebraic sum of the strains caused by all the forces separately. Principle of superposition is valid only if

(i) The structural stability of the body is not affected.

(i)The stresses are within the elastic limit.

(ii) Deflection does not affect the applied loads.

Free-body diagrams

If a small portion of a structure is separated which is in equilibrium, then the separated portion will also be in a state of equilibrium under the combined action of the external forces acting on this portion and the internal forces acting on the cut part. The diagram showing such an isolated portion of the structure together with the forces acting on it, external as well internal is known as free body diagram.

When two or more loads are acting on a body at different sections, the deformation of individual sections can be determined by drawing the free body diagram for each section. The total deformation of the body can then be found by algebraically adding the deformations of individual sections under the given system of loads.

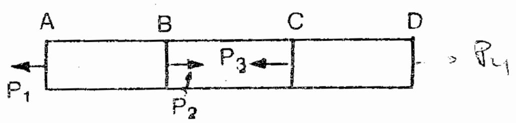


Fig. 1.10

Consider a bar ABCD with axial forces P_1, P_2, P_3 and P_4 acting at various sections of the bar as shown in Fig 1.10. The total strain of the bar will be the algebraic sum of the strains in sections AB, BC and CD. Now draw free body diagrams for each portion as shown in fig 1.10 (a), (b) and (c)

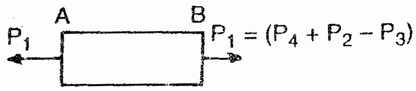


Fig. 1.10 (a)

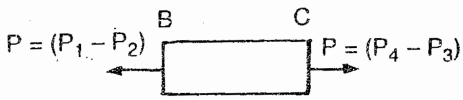


Fig. 1.10 (b)

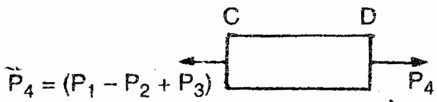


Fig. 1.10 (c)

Fig. 1.10 (a) Shows the free body diagram for portion AB of the bar. A tensile force P_1 is acting at section A and a tensile force $(P_4 + P_2 - P_3)$ is acting to the right of section B. The sum of all the forces acting to the right of the section must be equal to $P_1 = (P_4 + P_2 - P_3)$ Hence the portion AB is subjected to a tensile force P_1 and the strain of this portion will be

$$\varepsilon_{AB} = \frac{P_1}{AE} = \frac{(P_4 + P_2 - P_3)}{AE}$$

Where A and E are the area of cross-section and modulus of elasticity of the material of the bar

Similarly for portion BC

$$\varepsilon_{BC} = \frac{P}{AE} = \frac{P_1 - P_2}{AE} = \frac{P_4 - P_3}{AE}$$

and strain for the third portion CD

$$\varepsilon_{CD} = \frac{P_4}{AE} = \frac{P_1 - P_2 + P_3}{AE}$$

The total strain of the bar

$$\varepsilon = \varepsilon_{AB} + \varepsilon_{BC} + \varepsilon_{CD}$$

Example 1.9

A mild steel bar of uniform section having an area of cross-section of $1000\ mm^2$ is subjected to axial forces as shown in fig 1.11. Calculate the total elongation or contraction of the bar. Take $E = 200\ KN/mm^2$.

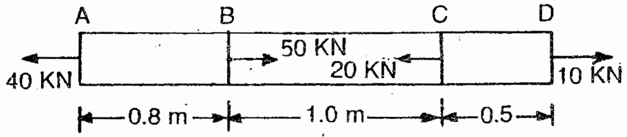
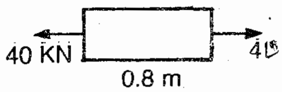


Fig. 1.11

Solution :

Draw free body diagrams for each portion and calculate the change in length of each portion of the bar

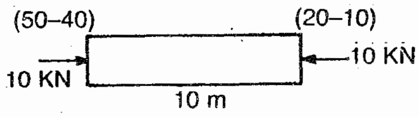
Portion AB



$$\delta l_1 = \frac{P}{AE} \times l_1$$

$$= \frac{40 \times 10^3 \times 800}{1000 \times 200 \times 10^3} = 0.16 \text{ mm (elongation)}$$

Portion BC



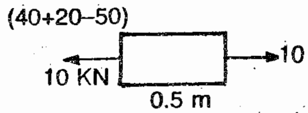
$$\delta l_2 = \frac{P}{AE} \times l_2$$

$$= \frac{10 \times 10^3 \times 1000}{1000 \times 200 \times 10^3}$$

$= .05$ mm (shortening)

Portion CD

$$\delta l_3 = \frac{P}{AE} \times l_3$$



$$= \frac{10 \times 10^3 \times 500}{1000 \times 200 \times 10^3}$$

$= .005$ (elongation)

$\delta l = \delta l_1 + \delta l_2 + \delta l_3$

$= 0.16 - .05 + .005$

$= 0.115$ mm (elongation) **Answer**

Bars of Varying Sections

When a bar is made up of different lengths having different cross-sectional areas, then the total elongation of the bar is the algebraic sum of the elongation of each portion of the bar.

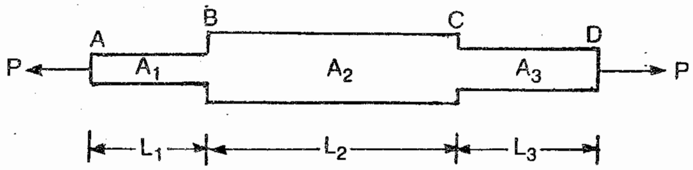


Fig. 1.12

Consider a bar consisting of three portions of lengths l_1, l_2 and l_3 and cross-sectional areas A_1, A_2 and A_3 respectively as shown in fig 1.12 Let the bar be subjected to an axial load P. Let E be the modulus of elasticity of the material of the bar.

The total change in length of the bar will be

$$\delta l = \delta l_1 + \delta l_2 + \delta l_3$$

$$= \frac{P}{A_1 E} l_1 + \frac{P}{A_2 E} \times l_2 + \frac{P}{A_3 E} \times l_3$$

$$= \frac{P}{E} \left\{ \frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3} \right\}$$

Sometimes the modulus of elasticity may be different for different portions of the bar. In such cases the total deformation

$$\delta l = P \left\{ \frac{l_1}{A_1 E_1} + \frac{l_2}{A_2 E_2} + \frac{l_3}{A_3 E_3} + ---- \right\}$$

Example 1.10

(a) Define Hooke's law

(b) Find the elongation of the bar shown in the figure. Take E = 210 GN/m^2 A.M.U.

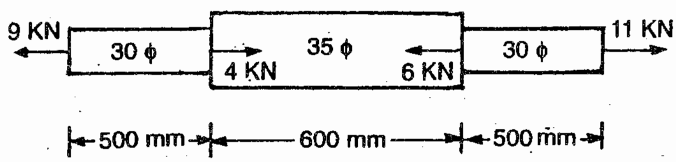


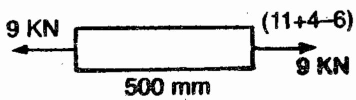
Fig. 1.13

Slution

Draw the free body diagrams for the three portions and calculate δ_1 δ_2 and δ_3

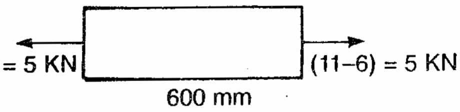
Portion (1)

elongation $\delta l_1 = \frac{P_1}{A_1 E} \times l_1$



$$= \frac{9 \times 10^3 \times 500}{\frac{\pi}{4}(30)^2 \times 210 \times 10^3} = 0.032 \text{ mm}$$

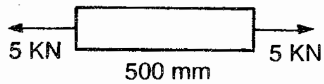
Portion (2)



$$\delta l_2 = \frac{P_2}{A_2 . E} \cdot l_2$$

$$= \frac{5 \times 10^3 \times 600}{\frac{\pi}{4}(35)^2 \times 210 \times 10^3} = 0.01495 \text{ mm}$$

Portion (3)



$$\delta l_3 = \frac{P_3}{A_3 E} \times l_3$$

$$= \frac{5 \times 10^3 \times 500}{\frac{\pi}{4}(30)^2 \times 210 \times 10^3} = .0152$$

$\delta l = (.032 + .0149 + .0152) = .0621$ mm **Answer**

Example 1.11

A bar ABCD is subjected to forces P_1, P_2, P_3 and P_4 as shown in figure 1.14. Calculate the force P_3 necessary for equilibrium if $P_1 = 100$ KN, $P_2 = 200$ KN and $P_4 = 150$ KN. Find the net change in the length of the bar taking modulus of elasticity E = 200 KN/mm^2.

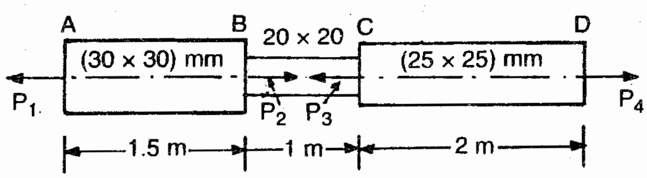


Fig. 1.14

Solution

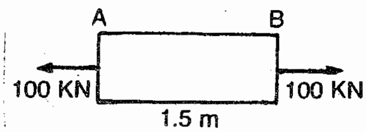
For equilibrium of bar

$$P_1 + P_3 = P_2 + P_4$$

$$100 + P_3 = 200 + 150 \text{ or } P_3 = (350 - 100) = 250 \text{ KN}$$

Portion AB

Force on the section 100 KN (Tensile)



$$\delta l_1 = \frac{P_1 \times l_1}{A_1 E}$$

$$\text{Elongation} = \frac{100 \times 10^3 \times 1.5 \times 10^3}{(30)^2 \times 200 \times 10^3}$$

$= 0.83$ mm (elongation)

Portion BC

Force on the section = 100 KN (Comp)

Shortening in the length of portion BC

$$\delta l_2 = \frac{100\times 10^3 \times 1\times 10^3}{(20)^2\times 200\times 10^3} = 1.25 \text{ mm} \quad \text{(Shortening)}$$

Portion CD

Force on the section = 150 KN (Tensile)

Elongation in the length of portion CD

$$\delta l_3 = \frac{150\times 10^3 \times 2\times 1000}{(25)^2\times 200\times 10^3} = 2.4 \text{ mm (elongation)}$$

Therefore net change in the length of the bar

$$\delta l = \delta l_1 + \delta l_2 + \delta l_3$$
$$= (+.83 - 1.25 + 2.4) \text{ mm}$$
$$= 1.98 \text{ mm} \quad \textbf{Answer}$$

Example 1.12

A bar one metre long is of 30mm diameter for a length of 0.6 m and the remaining portion has a diameter of 40mm. The bar is loaded as shown in figure 1.15. Determine the total elongation of the bar. Take E = 200 KN/mm².

Solution

Draw the free body diagram for portion (1) as shown in fig 1.15 (a)

Area of cross-section

$$A_1 = \frac{\pi}{4}(30)^2 = \frac{900\pi}{4}$$
$$= 225\,\pi \text{ mm}^2$$

Stress on Section (1)

$$\sigma = \frac{P}{A} = \frac{50\times 1000}{225\,\pi}$$

Elongation of Section (1)

$$\delta l_1 = \frac{P}{AE}\times l_1$$
$$= \frac{50\times 1000\times 0.6\times 1000}{225\,\pi\times 200\times 10^3} = .19\text{mm}$$

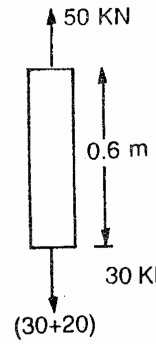


Fig 1.15 (a)

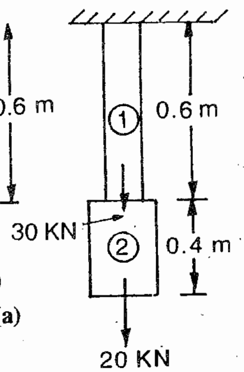


Fig. 1.15

from the free body diagram of portion (2) fig. 1.15 (b)

$$A_2 = \frac{\pi}{4}(40)^2 = 400\pi$$

$$\delta l_2 = \frac{P}{A_2 E}\times l_2$$

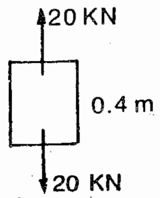


Fig. 1.15 (b)

$$= \frac{20 \times 1000 \times 0.4 \times 1000}{400\pi \times 200 \times 10^3}$$

$$= 0.127 \text{mm}$$

Total elongation of the bar

$$\delta l = \delta l_1 + \delta l_2$$

$$= 0.19 + 0.127 = 0.317 \text{ mm}$$ **Answer**

Example 1.13

A Prismatic bar fixed at both the ends is loaded axially at a distance 'a' from one of the supports as shown in figure 1.16. Determine the reactions at the supports. (Engg. services)

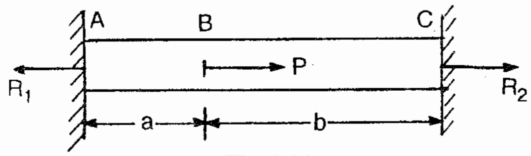


Fig. 1.16

Solution :

The application of force P at B as shown in the figure, will cause tension in portion AB and compression in portion BC. Reactions R_1 at A and R_2 at C will be in opposite direction to applied force P

$$\therefore \; R_1 + R_2 = P \qquad \ldots\ldots\; (i)$$

Since the ends are fixed at A and C hence there will be no change in the length of the bar. Elongation of portion A B will be equal to the reduction in the length of portion BC.

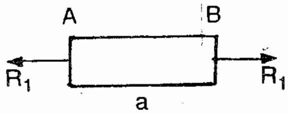


Fig. 1.16 a

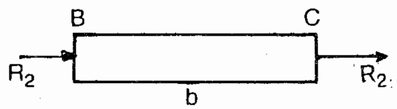


Fig. 1.16 b

Now draw the free body diagrams for the two portions as shown in fig 1.16 (a) and 1.16 (b)

$$\delta l_{AB} = \frac{P}{AE} l_{AB} = \frac{R_1 \times a}{AE}$$

$$\text{and } \delta l_{BC} = \frac{P}{AE} \cdot l_{BC} = \frac{R_2 \times b}{AE}$$

now Since $\delta l_{AB} = \delta l_{BC}$

$$\therefore \; \frac{R_1 \times a}{AE} = \frac{R_2 \times b}{AE}$$

$$\text{or } \; R_1 = \frac{R_2 \times b}{a} \qquad \ldots\ldots\ldots\ldots\; (ii)$$

Putting R_1 in equation (i) we have

$$R_1 + R_2 = P$$

$$\frac{R_2 \times b}{a} + R_2 = P$$

$$\text{or} \qquad R_2 = \frac{P \times a}{(a+b)}$$

Now substituting R_2 in equation (ii)

$$R_1 = \frac{R_2 \times b}{a} = \frac{P \times a}{(a+b)} \cdot \frac{b}{a} = \frac{Pb}{(a+b)}$$

Hence $R_1 = \dfrac{P.b}{(a+b)}$ and $R_2 = \dfrac{P.a}{(a+b)}$ **Answer**

Bar of Tapering Section

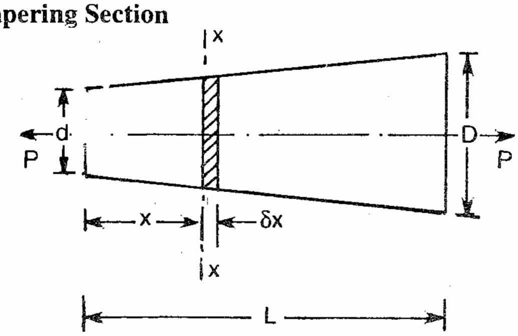


Fig. 1.17

Let a bar of length L taper uniformly from a diameter D at one end to a diameter d at the other.

Consider a Section of length δx at a distance x from A

Diameter of the bar at section xx

$$dx = d + (D - d) \cdot \frac{x}{L}$$

Extension of the small length $\delta x = \dfrac{p\,\delta x}{\dfrac{\pi}{4}\left[d + (D-d)\cdot\dfrac{x}{L}\right]^2 E}$

For whole length of the bar the extension will be

$$\delta L = \int_o^L \frac{4\,p\,\delta x}{\pi\left[d + (D-d)\cdot\dfrac{x}{L}\right]^2 .E}$$

$$\text{Let}\left(\frac{D-d}{L}\right) = \text{K}$$

$$\therefore \text{ or } \qquad \delta L = \int_o^L \frac{4Pdx}{\pi(d+k.x)^2 E}$$

$$= \frac{-4P}{\pi E} \cdot \frac{1}{K}\left[\frac{1}{(d+kx)}\right]_0^L = \frac{-4PL}{\pi E\,(D-d)}\left[\frac{1}{d+D-d} - \frac{1}{d}\right]$$

$$= \frac{4PL}{\pi\,E\,D.d}$$

Now when $D = d$, We have

$$\delta L = \frac{4.PL}{\pi Ed^2} = \frac{PL}{AE}$$

Example 1.14

A steel bar tapers uniformly from a diameter of 50mm at one end to a diameter of 30mm at the other end. The Length of the bar is one metre. If an axial force of 90KN is applied at each end of the bar. Determine the elongation of the bar. Take E = 200 KN/mm²

Solution

Elongation of the bar

$$\delta l = \frac{4\,PL}{\pi\, ED.d}$$

$$= \frac{4 \times 90 \times 10^3 \times 1 \times 10^3}{\pi \times 200 \times 10^3 \times 50 \times 30}$$

$$\delta l = .38 \text{ mm.} \qquad \textbf{Ans.}$$

Example 1.15

A flat steel plate is of trapezoidal form and uniform thickness of 10mm. The plate tapers uniformly from a width of 150mm to 100 mm over a length of 500mm. Determine the elongation of the plate under an axial pull of 100 KN. Take E = 200 KN/mm²

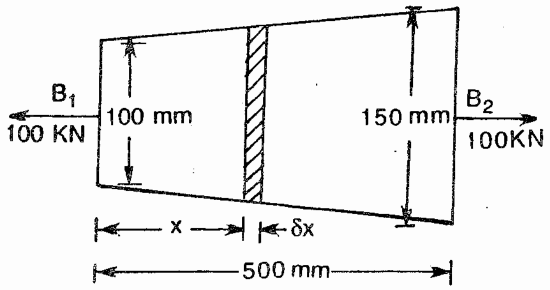


Fig. 1.18

Solution :-

Consider a Small Section δx at a distance x from A

The width at the section $= B_1 + (B_2 - B_1) \cdot \frac{x}{L}$

$= (B_1 + K.x)$ where $K = \frac{B_2 - B_1}{L}$

Area of cross section $= (B_1 + K.x)\, t$

Elongation of the section $\delta l = \frac{P(\delta x)}{(B_1 + Kx) \times t \times E}$

Total Elongation $\delta L = \int_0^L \frac{Pdx}{(B_1 + Kx)tE}$

$$\delta L = \frac{P}{tE} \times \frac{1}{K}\left[Log_e(B_1 + Kx)\right]_0^L$$

$$= \frac{P}{K\,t\,E}\left(\frac{Log_e\,B_1 + K.L.}{B_1}\right) = \frac{P}{K\,t\,E}\log_e\frac{B_2}{B_1}$$

$$\text{Where K} = \frac{150-100}{500} = 0.1$$

$$\delta L = \frac{100\times 1000}{0.1\times 10\times 200\times 10^3}\,\text{Log}_e\frac{150}{100}$$

$$= \frac{1}{2}\,\text{Log}_e\frac{150}{100} = \frac{1}{2}\,\text{Log}_e\,1.5 = \frac{0.4054}{2}$$

$$= 0.2027\text{mm} \qquad \textbf{Ans.}$$

Stresses in Composite Sections

When two or more bars of different materials are rigidly connected such that when subjected to loads, each bar undergoes equal change in length, the system is known as composite system.

The Strains induced in all the bars are equal and the total load on the Composite section is shared by all the bars.

Let three bars of length L each and cross-sectional area A_1, A_2 and A_3 be subjected to a load P as shown in the fig. Let E_1, E_2, E_3 be the modulii of elasticity of the materials of the three bars. Let P_1, P_2, P_3 be the loads taken by the three bars.

then $P = P_1 + P_2 + P_3$

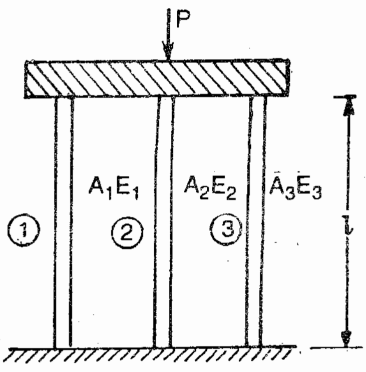


Fig. 1.19

The bars will undergo equal change in length, hence strains will be equal

$$\text{or } \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{\delta L}{L}$$

$$\text{Stress in each bar} = \frac{\delta L}{L} \times E$$

$$\text{Load taken by bar no (1) } P_1 = \frac{\delta L}{L} \times E_1 . A_1$$

$$\text{Load taken by bar no (2) } P_2 = \frac{\delta L}{L} \times E_2 . A_2$$

$$\text{Load taken by bar no (3) } P_3 = \frac{\delta L}{L} \times E_3 . A_3$$

$$\text{or} \qquad P = \frac{\delta L}{L}\{A_1E_1 + A_2E_2 + A_3E_3\}$$

$$\text{or} \qquad \delta L = \frac{P.L}{(A_1E_1 + A_2E_2 + A_3E_3)}$$

$\therefore$ Load taken by bars

$$P_1 = \frac{P \cdot A_1 E_1}{(A_1 E_1 + A_2 E_2 + A_3 E_3)}$$

$$P_2 = \frac{P A_2 E_2}{(A_1 E_1 + A_2 E_2 + A_3 E_3)}$$

$$P_3 = \frac{P \cdot A_3 E_3}{(A_1 E_1 + A_2 E_2 + A_3 E_3)}$$

Stress in each bar

$$\sigma_1 = \frac{P E_1}{(A_1 E_1 + A_2 E_2 + A_3 E_3)}$$

$$\sigma_2 = \frac{P E_2}{(A_1 E_1 + A_2 E_2 + A_3 E_3)}$$

$$\sigma_3 = \frac{P E_3}{(A_1 E_1 + A_2 E_2 + A_3 E_3)}$$

If there are only two bars one of steel and other of copper making the compound section then

$$\sigma_s = \frac{P.E_s}{(A_s E_s + A_c E_c)} \quad \text{and} \quad \sigma_c = \frac{P.E_c}{(A_c E_c + A_s E_s)}$$

Example 1.16

A steel tube surrounding a solid aluminium cylinder is compressed between infinetely rigid cover plates by a centrally applied force of 200 KN. If the aluminium cylinder is 75 mm inside diameter and the outside diameter of the steel tube is 90mm, determine the load taken by the rod and the tube. $E_s = 210 KN/mm^2$ *and* $E_{al} = 70\ KN/mm^2$

Solution

Shortening of the tube and the cylinder will be equal.

Strain in the tube = strain in the cylinder

$$\varepsilon_s = \varepsilon_{al}$$

$$\text{or} \quad \frac{\sigma_s}{E_s} = \frac{\sigma_{al}}{E_{al}}$$

$$\text{or} \quad \sigma_s = \sigma_{al} \frac{E_s}{E_{al}} = \sigma_{al} \times \frac{210 \times 10^3}{70 \times 10^3} = 3\,\sigma_{al}$$

Area of steel tube $A_s = \frac{\pi}{4}(90^2 - 75^2) = 1943.86\ mm^2$

Area of aluminium cylinder $A_{al} = \frac{\pi}{4}(75)^2 = 4417.86\ mm^2$

Total Load will be shared by the tube and the cylinder.

$$\therefore\ P = P_s + P_{al}$$

$$200 \times 10^3 = \sigma_s . A_s + \sigma_{al} . A_{al}$$

$$200 \times 10^3 = 3\sigma_{al} \times 1943.86 + \sigma_{al} \times 4417.86$$

$$= (10248.44)\,\sigma_{al}$$

$$\text{or } \sigma_{al} = \frac{200 \times 10^3}{10248.44} = 19.6 \text{ MPa}$$

$$\therefore \sigma_s = 3\sigma_{al} = 3 \times 19.6 = 58.8 \text{ MPa}$$

Load on the tube $P_s = \sigma_s . A_s = 58.8 \times 1943.86$

$= 114$ KN

Load on the cylinder $= \sigma_{al}.A_{al} = 19.6 \times 4417.86$

$= 86$KN **Answer**

Example 1.17

An aluminium tube of 40 mm external diameter and 20 mm internal diameter is fitted on a solid steel rod of 20 mm diameter. The composite bar is loaded in Compression by an axial load P. Find the stress in steel when the load is such that the stress induced in aluminium is 70 N/mm². What is the value of P? E_s = 210 KN/mm² E_{al} = 70 KN/mm². JMI

Solution

Strain in both materials will be equal

Strain in the tube = Strain in the rod

$$\varepsilon_{al} = \varepsilon_s$$

$$\frac{\sigma_{al}}{E_{al}} = \frac{\sigma_s}{E_s}$$

$$\text{or } \sigma_s = \frac{E_s}{E_{al}} \times \sigma_{al} = \frac{210}{70} \sigma_{al} = 3\,\sigma_{al}$$

$$= 3 \times 70 = 210 \text{ MPa}$$

Area of the tube $= \frac{\pi}{4}(40^2 - 20^2) = 300\ \pi \text{ mm}^2$

Area of the rod $= \frac{\pi}{4}(20)^2 = 100\,\pi \text{ mm}^2$

Total load will be shared by the tube and the rod

$$P = P_s + P_{al}$$

$$= \sigma_s . A_s + \sigma_{al} . A_{al}$$

$$= 3\,\sigma_{al}.A_s + \sigma_{al}.A_{al}$$

$$= 3 \times 70 \times 100\,\pi + 70 \times 300\pi$$

$= 132$ KN. **Answer**

Example 1.18

A reinforced concrete column 400 mm in diameter is reinforced with 6 steel bars of 30mm diameter. The column carries a load of 200 KN. Determine the stresses induced in steel and concrete.

Take E_s = 210 KN/mm² and E_c = 14 KN/mm²

Solution

Cross-sectional area of the column

$$A = \frac{\pi}{4}(400)^2 = 125.66 \times 10^3 \text{ mm}^2$$

Area of 6 steel bars $A_s = 6 \times \frac{\pi}{4}(30)^2 = 4.24 \times 10^3 \text{ mm}^2$

Area of concrete $A_c = A - A_s$

$$A_c = (125.66 - 4.24) \times 10^3 = 121.42 \times 10^3 \text{ mm}^2$$

Since steel and concrete will act as a composite unit, the strain in the two materials will be same,

$$\text{Strain} = \varepsilon_s = \varepsilon_c$$

$$\text{or} \quad \frac{\sigma_s}{E_s} = \frac{\sigma_c}{E_c} \text{ or } \sigma_s = \sigma_c \times \frac{E_s}{E_c} = \frac{210}{14}\,\sigma_c = 15\,\sigma_c$$

$$\text{or} \quad \sigma_s = 15\,\sigma_c$$

Load carried by the column

$$= \text{Load on conc.} + \text{Load on steel bars}$$

$$W = A_c.\sigma_c + A_s.\sigma_s$$

$$= 121.42 \times 10^3\,\sigma_c + 4.24 \times 10^3 \times 15\,\sigma_c$$

$$200 \times 10^3 = (121.42 + 63.60) \times 10^3\,\sigma_c = 185.02 \times 10^3\,\sigma_c$$

$$\text{or} \quad \sigma_c = \frac{200}{185.02} \times \frac{10^3}{10^3} = 1.08 \text{ MPa}$$

$$\sigma_s = 15 \times 1.08 = 16.2 \text{ MPa}$$ **Answer**

Example 1.19

A reinforced concrete column 400 mm × 400 mm is reinforced with 4 steel bars of 22mm dia one at each corner. Calculate the safe Load that the Column can carry if the allowable stress in concrete is 5N/mm² and the modulus of elasticity of steel is 18 times that of concrete.

Solution :

Cross Sectional area of the column

$$A = 400 \times 400 = 16 \times 10^4 \text{ mm}^2$$

Area of steel reinforcement

$$A_s = 4 \times \frac{\pi}{4}(22)^2 = 1520.53 \text{ mm}^2$$

Area of concrete in the column

$$A_c = (A - A_s) = (160000 - 1520.53)$$
$$= 158479.47 \text{ mm}^2$$

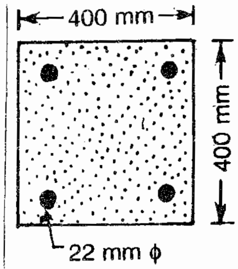


Fig. 1.20

Now $\sigma_c = 5\text{N/mm}^2$ and $\frac{E_s}{E_c} = 18$

$\therefore\ \sigma_s = 5 \times 18 = 90 \text{ N/mm}^2$

$\therefore$ Total Load on the column

$$= \text{load on concrete} + \text{Load on steel bars}$$
$$= A_c.\,\sigma_c + A_s.\,\sigma_s$$
$$= 158479.47 \times 5 + 1520.53 \times 90$$
$$= 792377.35 + 136847.776$$
$$= 929245.122 \text{ N}$$
$$= 929.245 \text{ KN}$$ **Answer.**

Example 1.20

A steel rod 20 mm diameter is passed through a brass tube 25 mm internal diameter and 30mm external diameter. The tube is 1 meter long and is closed by thin rigid washers and fastened by nuts, screwed to the rod. The nuts are tightened until the compressive force in the tube is 40KN. Determine the stresses induced in the rod and the tube. Take $E_s = 200\ KN/mm^2$ *and* $E_b = 80 KN/mm^2$

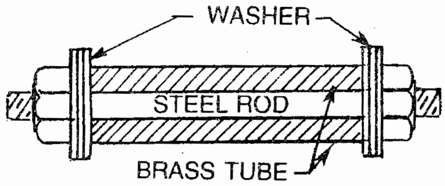


Fig. 1.21

Solution :

Area of steel rod $A_s = \frac{\pi}{4}(20)^2 = 100\,\pi$ Sq.mm

Area of brass tube $A_c = \frac{\pi}{4}(30^2 - 25^2)$

$= 275\ \pi$ Sq.mm

Since the rod and tube are rigidly fixed

Therefore strains in both are the same.

Let δl be the common decrease in length

$\therefore$ Strain in the rod = Strain in the tube

$$\varepsilon_s = \varepsilon_b = \frac{\delta l}{l} = \frac{\delta l}{1000}$$

Stress in the steel rod $\sigma_s = \varepsilon_s . E = \frac{\delta l}{1000} \times 200 \times 10^3$

$$\sigma_s = 200\,\delta l \text{ MPa}$$

Stress in the brass tube

$$\sigma_b = \frac{\delta l}{1000} \times 80 \times 10^3 = 80\,\delta l \text{ MPa}$$

Force in the rod $P_s = \sigma_s . A_s = (200\,\delta l)(100\,\pi)$ Newton

Force in the tube $P_b = \sigma_b . A_b = (80\delta l)(275\pi)$ Newton

Total Compressive force is 40000 Newton

$$P = P_s + P_b$$

$$40000 = (200 \times \delta l)(100\,\pi) + (80\delta l)(275\pi)$$

$$\text{or} \quad \delta l = \frac{42000\ \pi}{40000} = 1.3297\text{mm}$$

Stress in steel rod $\sigma_s = 200\,\delta l = 200 \times .3297$ Newton/mm^2

$$\sigma_s = 65.84 \text{ MPa}$$

Stress in brass tube $\sigma_b = 80\ \delta l = 80 \times .3297$ Newton/mm^2

$\sigma_b = 26.37$ MPa **Answer**

Temperature Stresses

A body expands or contracts with rise or fall in temperature. If the change in the dimensions of the body is prevented then internal stresses are induced within the body. These stresses which are induced in the body due to change in temperature are called thermal stresses or temperature stresses.

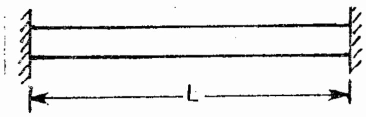


Fig. 1.22

Let a bar of length L be heated through t°c. The bar will expand, which is prevented by providing restrains at both ends. Let α be the coefficient of linear expansion. If the bar was free to expand the change in length of the bar $= \alpha\ L\ t$

Hence strain due to rise in temperature $= \dfrac{\text{Change in length}}{\text{Original length}}$

$$\varepsilon = \frac{\alpha L t}{L} = \alpha\ t$$

Temperature stress induced $\sigma = \alpha t E$

When the ends yield by an amount δ

Net expansion prevented $= \alpha\ Lt - \delta$

Strain in the bar $= \left(\dfrac{\alpha t L - \delta}{L}\right) = \left(\alpha\ t - \dfrac{\delta}{L}\right)$

Stress induced in the bar $= \left(\alpha\ t - \dfrac{\delta}{L}\right) \times E$

Example 1.21

A Copper bar 3 meters long having a cross-sectional area of 1200 mm^2 is rigidly attached to the walls as shown in fig 1.23. At a temperature of 35°C the bar is stress free. Determine the stress in the bar when the temperature falls to 20°C. Assume that the supports do not yield. Take E_c = 120 GN/m^2 and $\alpha_c = 20 \times 10^{-6}$/°C.

Solution

Assuming that the ends are not rigidly attached and the bar is free to contract due to fall in temperature of (35 – 20) = 15°C

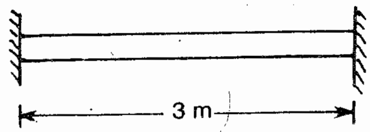


Fig. 1.23

Shortening of the bar

$$= \alpha t l = 20 \times 10^{-6} \times 15 \times 3 \times 10^3 = 0.9 \text{ mm}$$

But Since the ends do not yield, a force P is required to prevent the bar from shortening by an amount 0.9 mm

$$\delta l = \frac{P.L}{AE}$$

$$0.9 = \frac{P \times 3 \times 10^3}{1200 \times 120 \times 10^3} \qquad \text{or } P = 43.2 \text{ KN}$$

Stress due to this force $\sigma = \frac{P}{A}$

$$\sigma = \frac{43.2 \times 10^3}{1200} = 36 \text{ MPa} \quad \textbf{Answer.}$$

Example 1.22

A railway track 20 meters long is to be laid so that the rails are stress-free at a temperature of 80°C. If the temperature rises to 140°C, Calculate

(a) The stress if there is no allowance for expansion

(b) If expansion allowance is 5 mm

(c) The expansion allowance if the stress in the rails is zero at 140°C

(d) The maximum temperature to have no stress for an expansion allowance of 10 mm.

Take $E = 200 \text{ KN/mm}^2$ *and* $\alpha = 12 \times 10^{-6}/°C$.

Solution

(a) Stress = $\alpha t E$

$$= 12 \times 10^{-6} \times (140 - 80) \times 200 \times 10^3$$

$$\sigma = 144 \text{ MPa}$$

(b) Expansion allowance is 5 mm.

$$\text{Stress} = \left(\frac{L\alpha t - x}{L}\right) \times E$$

$$= \left(\frac{20 \times 10^3 \times 12 \times 10^{-6} \times 60 - 5}{20 \times 10^3}\right) \times 200 \times 10^3$$

$$= (14.4 - 5) \times 10 = 94 \text{ MPa}$$

(c) If the stress is to be zero at 140°C then the expansion

allowance = $L\alpha t$

$$= 20 \times 10^3 \times 12 \times 10^{-6} \times 60 = 14.4 \text{ mm.}$$

(d) If the stress is to be zero for an allowance of 10 mm

then $L\alpha t - 10 = 0$

or $20 \times 10^3 \times 12 \times 10^{-6} \times t = 10$

or $t = \dfrac{10}{20 \times 10^3 \times 12 \times 10^{-6}} = 41.6°C$

Hence Maximum temperature = (80 + 41.6) = 121.6°C **Ans.**

Example 1.23

A thin circular ring of steel is heated and slipped over a rigid wooden wheel of 1 meter external diameter. If the permissible stress in steel is 40 MPa, find the exact internal diameter of the steel ring and the temperature through which the ring is required to be heated before slipping on the wheel. $E_s = 200 \text{ KN/mm}^2$ *and* $\alpha = 11 \times 10^{-6}/°C$. (Tech Board Punjab)

Solution

Temperature stress = 40 MPa

$$\sigma = \alpha t E$$

$$40 = 11 \times 10^{-6} \times t \times 200 \times 10^3$$

$$t = \frac{40}{11 \times 10^{-6} \times 200 \times 10^3} = 18.18°C$$

Let the internal diameter be 'd'

$$\text{Strain} = \frac{\text{Contraction prevented}}{\text{Original circumference}}$$

$$\delta l = \frac{\pi 1000 - \pi \times d}{\pi d} = \frac{\pi(1000 - d)}{\pi\ d}$$

But Strain = αt

or $\dfrac{\pi(1000-d)}{\pi\ d} = \alpha t$

or $1000 - d = d\alpha t$ or $d = \dfrac{1000}{1 + \alpha t}$

or $d = \dfrac{1000}{1 + 11 \times 10^{-6} \times 18.18} = 999.8$ mm **Answer.**

Example 1.24

Two parallel walls 5 metres apart are stayed together by a steel rod of 25 mm diameter at a temperature of 80°C passing through washers and nuts at each end. Calculate the stress in the rod when it has cooled down to a temperature of 20°C.

(i) If the ends do not yield

(ii) If the total yield at the two ends is 1.2 mm

Take E = 200 KN/mm^2 and $\alpha = 12 \times 10^{-6}$/°C. (Punjab Univ.)

Solution

(i) When the ends do not yield, stress in the rod

$$\sigma = \alpha.t.E$$
$$= 12 \times 10^{-6}(80 - 20) \times 200 \times 10^3 \text{ N/mm}^2$$
$$= 144 \text{ MPa}$$

(ii) When the ends yield by 1.2 mm, stress in the rod is found by using the relation

$$\sigma = \left(\alpha t - \frac{\delta}{L}\right)E$$

$$= \left(12 \times 10^{-6} \times 60 - \frac{1.2}{5 \times 1000}\right) \times 200 \times 10^3$$

$= 96$ MPa **Answer**

Example 1.25

A 30 meter steel tape 20 mm × 1 mm in section was found to be correct at a temperature of 40°C and under a pull of 160 newtons. Find the error in the tape when used at a temperature of 60°C and under a pull of 80 newtons. Take E_s = 200 KN/mm^2 and $\alpha_s = 12 \times 10^{-6}$ /°C.

Solution

Increase in the length of the tape when

Temperature rises by 20°C = αlt

$$= 12 \times 10^{-6} \times 30 \times 1000 \times 20$$
$$= 7.2 \text{ mm}$$

Decrease in the pull on the tape = (160 – 80) = 80 N

Decrease in length due to a push of 80 N

$$= \frac{P}{AE} \times l$$

$$= \frac{80 \times 30 \times 10^3}{20 \times 1 \times 200 \times 10^3} = 0.6 \text{ mm}$$

Hence the tape will be too long by (7.2 – 0.6) = 6.6 mm **Answer.**

Example 1.26

A 40 mm × 20 mm copper flat is brazed to a steel flat 40 mm × 40 mm as shown in figure 1.24.

The combination is then heated through 100°C. Calculate the stress produced in each flat and the shearing force at the plane of brazing. Take $E_s = 200$ *KN/mm*^{2} $E_c = 100$ *KN/mm*^{2}, $\alpha_c = 18.5 \times 10^{-6}$ */°C and* $\alpha_s = 12 \times 10^{-6}$ / °C.

Solution

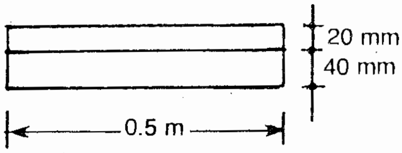


Fig. 1.24

Let ε be the Common strain

Compressive strain in copper flat

ε_c = Strain when free to expand – Common strain

$$= (\alpha_c t - \varepsilon) \quad \ldots \quad \ldots \quad \text{(i)}$$

Similarly tensile strain in steel flat

ε_s = Common strain – Strain when free to expand

$$\varepsilon - \alpha_s . t \quad \ldots \quad \ldots \quad \text{(ii)}$$

From equations (i) and (ii)

$$\varepsilon_c + \varepsilon_s = \alpha_c t - \varepsilon + \varepsilon - \alpha_s t = (\alpha_c - \alpha_s)\, t$$

But $\varepsilon_c = \dfrac{\sigma_c}{E_c}$ and $\varepsilon_s = \dfrac{\sigma_s}{E_s}$

$$\therefore \quad \frac{\sigma_c}{E_c} + \frac{\sigma_s}{E_s} = (\alpha_c - \alpha_s)\, t$$

$$\frac{\sigma_c}{100 \times 10^3} + \frac{\sigma_c}{200 \times 10^3} = (18.5 \times 10^{-6} - 12 \times 10^{-6}) \times 100$$

$$\sigma_c + 0.5\, \sigma_s = 65 \quad \ldots \quad \ldots \quad \text{(iii)}$$

Now pull on steel flat = Push on copper flat

$$\sigma_s . A_s = \sigma_c . A_c$$

$$\sigma_s \times 40 \times 40 = \sigma_c \times 40 \times 20$$

$$\sigma_c = 2\ \sigma_s \quad \ldots \quad \ldots \quad \text{(iv)}$$

From equation (iii) and (iv) We have

$$2\,\sigma_s + 0.5\ \sigma_s = 65$$

$$\sigma_s = (65/2.5) = 26 \text{ Mpa}$$

and $\sigma_c = 52$ Mpa

Shearing force $= \sigma_c \times A_c$

$= 52 \times 40 \times 20 = 41.6$ KN

$$\text{Shear stress} = \frac{Force}{Shearing\ area} = \frac{41.6 \times 10^3}{0.5 \times 10^3 \times 40}$$

$= 2.08$ MPa **Ans.**

Example 1.27

Two steel rods each 50 mm diameter are connected end to end by means of a turn buckle as shown in fig (1.25). The other end of each rod is rigidly fixed with a little initial tension in the rods.

The length of each rod is 4 meter and there are 0.2 threads per mm on each rod. Calculate the increase in the initial tension when the turn buckle is tightened by one quarter of a turn. Take E = 200 KN/mm^2.

State with reasons, whether further effect of temperature rise, would nullify the increase in tension or add to it more. (Banglore University)

Solution

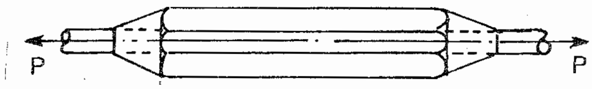


Fig. 1.25

Cross-sectional area of each rod $= \frac{\pi}{4}(50)^2 = 1963$ mm^2

When the turn buckle is turned by one quarter of a turn, extension of each rod.

$$= \frac{1}{4} \times \frac{1}{0.2} = 1.25 \text{ mm}$$

Total extension of both the rods $= 2 \times 1.25 = 2.5$ mm

If t be the increase in tension in each rod, then elongation of the two rods

$$\delta l = 2 \times \frac{t \times l}{AE} = \frac{2 \times t \times 4 \times 1000}{1963 \times 200 \times 10^3}$$

But total elongation of the two rods is 2.5 mm

$$\therefore \quad \frac{2 \times t \times 4 \times 1000}{1963 \times 200 \times 10^3} = 2.5$$

$$\text{or} \quad t = \frac{1963 \times 200 \times 10^3 \times 2.5}{2 \times 4 \times 1000} = 122.7 \text{ KN}$$

Further rise in temperature would cause increase in length of each rod and when rise in temperature has caused an increase in length of 2.5 mm the tension would be totally nullified.

Example 1.28

A weight of 150 KN is supported by three short pillars each of 500 sq.mm in section. The outer pillar are of copper and the central pillar is of steel. The adjustment of pillar is such that at a temperature of 25°C each carried an equal Load. After this the temperature is raised to 125°C.

Estiimate the stress in each pillar at 25°C and 125°C. Take $E_s = 200 KN/mm^2$, $E_c = 80\ KN/mm^2$, $\alpha_s = 12 \times 10^{-6}/°C$ and $\alpha_c = 18.5 \times 10^{-6}/°C$ AMIE

Solution

Initially at 25°C the Load shared by each pillar will be equal

$$\text{Load on each pillar} = \frac{150}{3} = 50 \text{ KN}$$

$$\text{Compressive stress} = \frac{50 \times 1000}{500} = 100 \text{ MPa}$$

When the temperature rises to 125°C, the extension in length of each pillar will be α t.L. But due to the Load of 150 KN each pillar will be compressed. Let x be the shortening in length of each pillar due to compressive load. Hence net change in length will be $(x - \alpha t L)$

$$\text{Strain in each pillar} = \left(\frac{x - \alpha t.L}{L}\right) = \left(\frac{x}{L} - \alpha t\right)$$

$$\text{Stress produced} = \left(\frac{x}{L} - \alpha t\right). E.A.$$

Total Load = Load carried by steel pillar + Load carried by two copper pillars

$$= \left[\left(\frac{x}{L} - \alpha_s.t\right) E_s A_s\right] + 2\left[\left(\frac{x}{L} - \alpha_c.t\right) E_c A_c\right]$$

$$150 \times 10^3 = \left[\left(\frac{x}{L} - 12 \times 10^{-6} \times 100\right)\left(200 \times 10^3 \times 500\right)\right]$$

$$+ 2\left[\left(\frac{x}{L} - 18.5 \times 10^{-6} \times 100\right]\right. \left[(80 \times 10^3 \times 500)\right]$$

$$150 \times 10^3 = \left[\frac{x}{L} - 12 \times 10^{-4}\right] \times 10^8 + 2\left[\left(\frac{x}{L} - 18.5 \times 10^{-4}\right) 40 \times 10^6\right]$$

$$= \frac{x}{L} \times 10^8 - 12 \times 10^4 + \frac{2.x}{L} \times .4 \times 10^8 - 18.5 \times 2 \times 0.4 \times 10^8 \times 10^{-4}$$

$$= \frac{1.8x}{L} \times 10^8 - 12 \times 10^4 - 14.8 \times 10^4$$

$$15 \times 10^4 = \frac{1.8x}{L} \times 10^8 - (26.80)(10^4)$$

or $\dfrac{1.8x}{L} \times 10^4 = 41.8$ or $\dfrac{x}{L} = \left(\dfrac{41.8}{1.8}\right) 10^{-4} = 23.22 \times 10^{-4}$

or $x/L = 23.22 \times 10^{-4}$

$$\text{Load carried by steel pillar} = \left(\frac{x}{L} - \alpha s.t\right) E_s.A_s$$

$$= (23.22 \times 10^{-4} - 12 \times 10^{-6} \times 100)\ 200 \times 10^3 \times 500 = 112.2 \text{ KN}$$

$$\text{Load carried by each copper pillar} = \left(\frac{x}{L} - \alpha_c.t\right) E_c.A_c$$

$$= (23.22 \times 10^{-4} - 18.5 \times 10^{-6} \times 100)\ 80 \times 10^3 \times 500$$

$$= 18.88 \times 10^3 \text{ N} = 18.88 \text{ KN}$$

$$\text{Stress in steel pillar} = \frac{112.2 \times 10^3}{500} = 224.4 \text{ Mpa}$$

$$\text{Stress in copper pillar} = \frac{18.88 \times 10^3}{500} = 37.76 \text{ MPa}$$

Statically indeterminate Problems

Statically indeterminate problems involve determination of more than three unknown forces in a system.

Such problems can not be solved by the three equations of static equilibrium. $\Sigma H = 0$, $\Sigma V = 0$ and $\Sigma M = 0$. Hence some more equations are formed considering the deformations of the structure. This helps in getting the required number of equations equal to the number of unknown forces. Thus all the unknown forces can be determined.

Example 1.29

Two identical steel bars are pin-connected and support a load of 500 KN as shown in figure 1.26. Determine the cross-sectional area of the bar so that the direct stress in bar does not exceed 250 MPa, Also determine the vertical displacement of the point B. Take E = 200 KN/mm² and length of each bar, 4 meters.

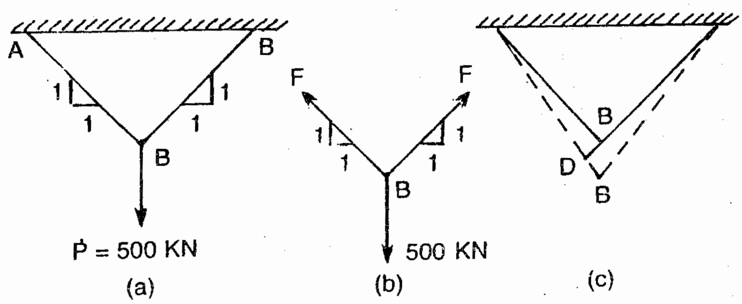


Fig. 1.26

Free body diagram of point B is shown above F represents the Force in Newtons in each bar

Resolving Vertically

$$F \sin 45 + F \sin 45 = 500$$

$$2F\left(\frac{1}{\sqrt{2}}\right) = 500 \text{ or } F = \left(\frac{500}{\sqrt{2}}\right) = 353.6\text{KN}$$

Hence the required area of each bar

$$A = \frac{Force}{Stress} = \frac{353.6 \times 10^3}{250} = 1414 \text{ mm}^2$$

The elongation of AB is represented by the distance DB' *and* BB' is the displacement of B

$$\text{Hence } DB' = \frac{Stress}{E} \times L = \frac{353.6 \times 10^3}{1414 \times 200 \times 10^3} \times 4 \times 10^3 = 5 \text{ mm}$$

$$\therefore\ BB' = \frac{5}{Cos\ 45^\circ} = 7.07 \text{ mm} \quad \textbf{Ans.}$$

Example 1.30

A rigid bar AB is supported by three equally spaced rods of length 1.5 meter each. The two outer rods are of steel having a cross-sectional area of 200 mm^2 each and the central rod is of copper of cross-sectional area 800 mm^2. If two Loads 40 KN each are applied midway between the rods, determine the load shared by each rod. The bar AB remains horizontal after the loads have been applied. Take $E_{st} = 200$ KN/mm^2 and $E_{cu} = 120$ KN/mm^2

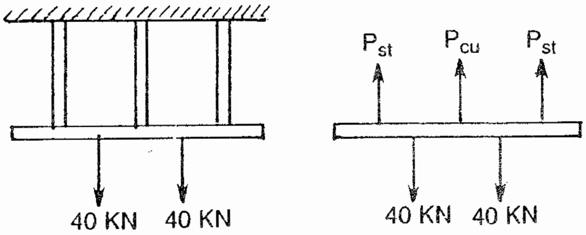


Fig. 1.27

Solution

Since the bar AB is in static equilibrium hence sum of all vertical forces must be equal to Zero

$$2P_{st} + P_{cu} - 80 = 0 \quad \ldots \quad \ldots \quad \text{(i)}$$

The elongation of each bar due to the applied load is also equal hence strain in steel rod is equal to strain in copper rod

$$\varepsilon_{ST} = \varepsilon_C$$

$$\frac{P_{st}}{A_{st} \times E_{st}} = \frac{P_{cu}}{A_{cu} \cdot E_{cu}}$$

$$\text{or} \quad P_{st} = P_{cu} \times \frac{A_{st} \times E_{st}}{A_{cu} \times E_{cu}}$$

$$= \frac{P_{cu} \times 200 \times 200 \times 10^3}{800 \times 120 \times 10^3}$$

$$= P_{cu} \times \frac{40}{96} = .416\, P_{cu}$$

$$\text{or} \quad P_{st} = .416\, P_{cu} \quad \ldots \quad \ldots \quad \text{(ii)}$$

Substituting P_{st} in terms of .416 P_{cu} in equation (i) we get

$$2\, P_{cu} \times .416 + P_{cu} = 80 \text{ KN}$$

$$\text{or} \quad P_{cu}\,(2 \times .416 + 1) = 80 \text{ KN}$$

$$\text{or} \quad P_{cu} = \frac{80}{1.832} = 43.66 \text{ KN}$$

$$\therefore \quad P_{st} = .416\, P_{cu} = .416 \times 43.66 = 18.16 \text{ KN}$$

Hence Load taken by copper rod is 43.66 KN and Load taken by each steel rod is 18.16 KN **Answer**

Example 1.31

A steel rod of cross-sectional area 1600 mm² and two brass rods each of cross-sectional area 1000 mm support a load of 50 KN uniformity distributed as shown in figure 1.28. Find the stresses in the rods

Take $E_s = 200$ KN/mm² and $E_b = 100$ KN/mm² (Alig.University)

Solution

Shortening in all the three rods will be equal

$$\delta l_s = \delta l_b$$

$$\frac{\sigma_s}{E_s} \cdot l_s = \frac{\sigma_b}{E_b} \cdot l_b$$

$$\text{or } \sigma_s = \frac{E_s}{E_b} \cdot \frac{l_b}{l_s} \times \sigma_b$$

$$= \frac{200 \times 10^3}{100 \times 10^3} \times \frac{4000}{3000} \sigma_b$$

$$\sigma_s = \frac{8}{3} \sigma_b$$

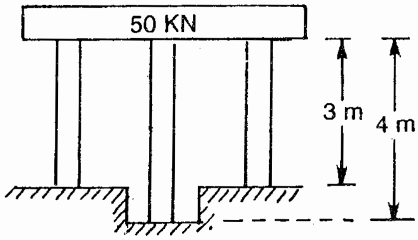


Fig. 1.28

Total Compressive Load

= Load shared by steel rod + Load shared by 2 brass rods

$$P = P_s + 2\,P_b$$

$$= A_s\,\sigma_s + 2A_b.\sigma_b$$

$$50 \times 10^3 = 1600 \times \frac{8}{3} \sigma_b + 2 \times 1000 \times \sigma_b$$

$$= \sigma_b\,(4.26 + 2) \times 10^3$$

$$\text{or} \quad \sigma_b = \frac{50 \times 10^3}{6.26 \times 10^3} = 7.97 \text{ MPa}$$

$$\sigma_s = \frac{8}{3} \times 7.97 = 21.25 \text{ MPa}$$ **Ans.**

Example 1.32

Two rods L meter long and 90 sq. mm cross-sectional area are fastened rigidly to a level support at distance of 1.20 m from each other. A horizontal cross-bar is provided at lower ends as shown in figure 1.29. Find the position of a 50 KN load on the cross-bar so that the bar remains horizontal after loading. Also calculate the stresses in the two rods. Take E_s = 200 GN/m² and E_b = 90 GN/m² AMIE

Solution

Since the cross-bar remains horizontal after loading strain in both bars will be equal

$$\varepsilon_s = \varepsilon_b$$

$$\text{or} \quad \frac{\delta l_s}{l} = \frac{\delta l_b}{l}$$

$$\therefore \quad \frac{\sigma_s\, l_s}{E_s} = \frac{\sigma_b}{E_b}.l_b$$

$$\text{or} \quad \frac{E_s}{E_b}.\sigma_b = \frac{200}{90}\sigma_b$$

$$\sigma_s = 2.22\,\sigma_b$$

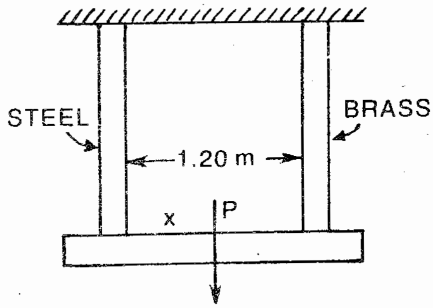


Fig. 1.29

$A_s = A_b = 90$ mm^2

$200 \text{ GN/m}^2 = \frac{200\times 10^9}{10^6}$ N/mm2

$= 200$ KN/mm^2

$90 \text{ GN/m}^2 = 90$ KN/mm^2

Total Load = Load on steel rod + Load on brass rod

$$P = A_s.\,\sigma_s + A_b.\sigma_b$$

$$50 \text{ KN} = 90\times\sigma_s + 90\times\sigma_b$$

$$50 = 90\,(\sigma_s + \sigma_b)$$

$$= 90\,(2.22 + 1)\,\sigma_b$$

$$\text{or} \quad \sigma_b = \frac{50}{90\times 3.22} = 0.1724 \text{ KN/mm}^2 = 172.4 \text{ MPa}$$

Load on brass rod = $A_b\,\sigma_b = 90\times 172.4 = 15.52$ KN

Stress in steel rod = (2.22) (172.4) = 383.07 MPa

Load on steel rod = $A_s\,\sigma_s$

$= 90\times 383.07 = 34476.3$ N

$= 34.47$ KN

Taking Moments about A

$$P.x = P_b\times 1.20$$

$$50\times x = 15.52\times 1.20$$

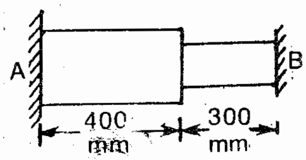
$$x = .372 \text{ metres} \qquad \textbf{Ans.}$$

Example 1.33

A Composite bar is rigidly attached to two supports as shown in figure 1.30. The left portion is a cooper bar of 7000 mm^2 sectional area and 400 mm length. The right portion is of aluminium of uniform sectional area 1500 mm^2 and 300 mm length. At a temperature of 300°C the entire assembly is stress free. When the temperature falls down the right support yields 0.5 mm in the direction of the contracting metal. Determine the minimum temperature in order that the stress in aluminium does not exceed 150 MPa. Take $E_c = 120$ KN/mm^2 $\alpha_c = 20\times 10^{-6}$/°C and = $E_a = 70$ KN/mm^2 and $\alpha_a = 25\times 10^{-6}$/°C

Solution

Consider that the bar is just cut to the left of B and is free to contract due to drop in temperature t°c



1.30

Total shortening of the composite section = Shortening of copper bar + Shortening of aluminium bar

$= (\alpha_c L_c t) + (\alpha_A L_A t)$
$= (20 \times 10^{-6} \times 400)\, t + (25 \times 10^{-6} \times 300)\, t$

The force required to prevent this shortening of the composite bar

$$= \frac{P \times 400}{7000 \times 120 \times 10^3} + \frac{P \times 300}{1500 \times 70 \times 10^3}$$

Since the right Support Yields by 0.5 mm due to fall in temperature

$$\frac{P \times (400)}{7000 \times 120 \times 10^3} + \frac{P \times (300)}{1500 \times 70 \times 10^3}$$
$$= [(20 \times 10^{-6} \times 400)t + (25 \times 10^{-6} \times 300)t - 0.5]$$

As the max^m stress allowed in aluminium bar is 150MPa

The max^m value of P is obtained from $P = \sigma.A$

$= 150 \times 1500 = 225000$N

Putting the Value of P in the above equation we obtain the value of t

$$\frac{225 \times 10^3 \times 400}{7000 \times 120 \times 10^3} + \frac{225 \times 10^3 \times 300}{1500 \times 70 \times 10^3}$$
$$= [(8 \times 10^{-3})t + (7.5 \times 10^{-3}) \times t - 0.5]$$

or $\quad 0.107 + 0.642 + 0.5 = (15.5 \times 10^{-3})t$

$$t = \frac{1.250}{15.5} \times 10^3 = 80.6°\text{C}$$

Fall in temperature = 80.6°C

Min ^{m} temperature = (300 – 80.6) = 219.4°C **Ans.**

Shear Stresses

When two equal and opposite forces act tangentially on any cross sectional plane of a body tending to slide its one part over the other at that plane, the body is said to be in a state of shear and the corresponding stress is called shear stress.

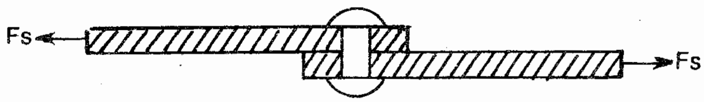


Fig. 1.31

If F_s is the tangential force and A is the resisting area then

$$\text{Shear stress} = \frac{\textit{Shearing Force}}{\textit{Resisting area}}$$

$$\tau = \frac{Fs}{A}$$

Shear stress is measured in N/mm^2 or MPa

The figure 1.32 shows a bar cut by a plan x–x perpendicular to its axis. Shear stress τ is acting along the plane where as normal stress σ is acting at right angles to the plane as shown.

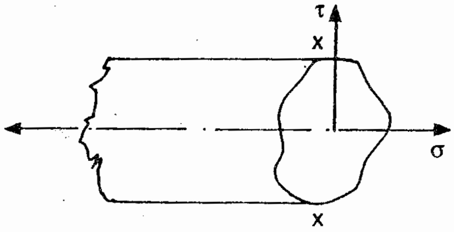


Fig. 1.32

Shear Strain

A rectangular element under the action of shear forces is shown in figure 1.33 (a)

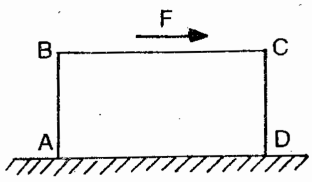


Fig. 1.33 (a)

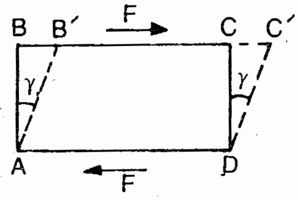


Fig. 1.33 (b)

Fig 1.33 (b) shows the distorted shape of the rectangular element. The length of the sides of rectangular element do not undergo any change but there will be an angular movement of the corners. This change of angle γ at the corners is the shear strain produced due to he shear force *F*. Shear Strain γ is expressed in radians.

Modulus of Rigidity or shear modulus

The ratio of Shear Stress to shear strain is called modulus of rigidity or shear modulus and represented by the symbol *G*

$$G = \frac{\tau}{\gamma}$$

Units of *G* are GN/m^2 or KN/mm^2

Values of modulus of rigidity for Various materials are given in the table.

TABLE 1.2

Name of material	Value of modulus of rigidity *G* in GN/m^2 or KN/mm^2
Steel	80 – 100
Wrought iron	80 – 90
Cast iron	40 – 50
Copper	30 – 50
Brass	30 – 50
Timber	10

Example 1.34

Two Steel Plates A and B are connected to each other by means of a rivet 25 mm in diameter. If a force of 20 KN is applied as Shown in figure 1.34 determine the average Shearing stress developed in the rivet.

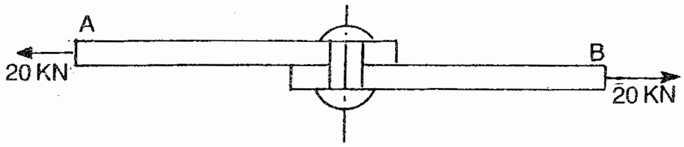


Fig. 1.34

Solution

The average shear stress = $\dfrac{Fs}{A}$

Where A is the area of the rivet hole.

Diameter of the rivet = 25 mm

Diameter of the rivet hole = 25 + 1.5 = 26.5 m

Area of the rivet hole $= \dfrac{\pi}{4}(26.5)^2$

Average shearing stress $= \dfrac{20 \times 10^3}{\frac{\pi}{4}(26.5)^2}$ N/mm^2

$\tau = 36.26$ MPa

Example 1.35

A hole of 20 mm diameter is to be punched in a plate 30 mm thick. Determine the force required for punching the hole and the stress in the punch if the shear stress is not to exceed 40 MPa

Solution

Area to be sheared $= \pi.\ d.t. = \pi\ 20 \times 30 = 600\ \pi$ mm^2

Punching force = shear stress$\times$ area to be sheared

$= 40 \times 600\ \pi$ N = 75.38 KN

The punch is subjected to a compressive stress

$$\sigma \text{ comp} = \frac{\text{Punching force}}{\text{Area of the hole}} = \frac{75.38 \times 10^3}{\frac{\pi}{4}(20)^2}$$

Stress in the punch = 240 MPa **Ans.**

Example 1.36

A load of 40 KN is acting on the horizontal surface of an angle bracket which is tranfixed to a vertical column as shown in fig. 1.35. If two 15 mm diameter rivets resist this force, find the average shearing stress in each of the rivets.

Solution

Total force acting on the bracket = 40 KN = 40000 N

Area of each rivet, $A = \frac{\pi}{4}(d)^2$

$A = \frac{\pi}{4}(15)^2 = 176.7\text{mm}^2$

Total resisting area $= 2 \times 176.71 = 353.42\text{mm}^2$

Average Shearing Stress $= \frac{40000}{353.42} = 113.1\text{MPa}$

$= 113\text{MPa}$ **Ans.**

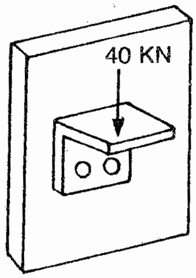


Fig. 1.35

Example 1.37

A lever is keyed to a shaft of 120 mm diameter. The width of the key is 20 mm and the length is 75 mm as shown in figure 1.36. If the shear stress in the key is not to exceed 85 MPa. Find the load that can be applied at a radius of 1.5 meters.

Solution

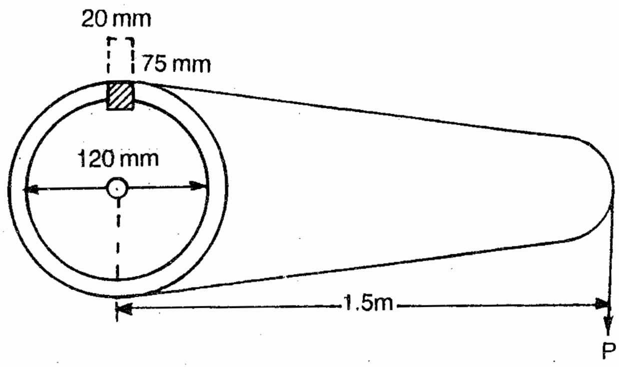


Fig. 1.36

Cross-Sectional area of the key $= 20 \times 75 = 1500\text{ mm}^2$

Allowable Shear Stress $\tau = 85\text{MPa}$

Shear Strength of the key = Area $\times \ \tau = 1500 \times 85\text{N}$

Taking moments about 0

$P \times 1.5 \times 1000 = (1500 \times 85) \times 60$

or $P = \frac{1500 \times 85 \times 60}{1.5 \times 1000} = 5100$ Newton

$= 5.1$ KN **Answer**

Example 1.38

A pulley is keyed to a circular shaft of diameter 60 mm. Two unequal belt pulls T_1 and T_2 on the two sides of the pulley give rise to a net turning moment of 120 N-m. The key is 10 mm × 15mm in cross-section and 75 mm long as shown in figure 1.37. Determine the average shearing stress acting on a horizontal plane through the key.

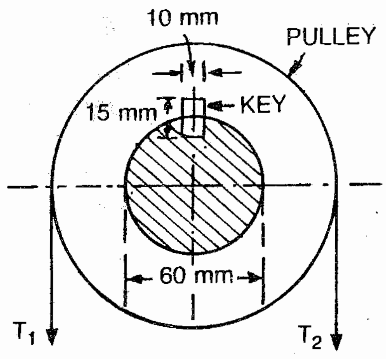


Fig. 1.37

Turning moment on the pulley = 120 N-m Let F be the horizontal force exerted by the key on the pulley. Then for equilibrium the moment of the force F about the centre of the pulley must be equal to the applied turning moment..

$F \times 30 = 120 \times 100$ or $F = 4000$ Newtons

Let τ = Shear Stress in key

Area of Cross-Section of the key in Shear = 75×10

Shear Strength of the key = $\tau \times 75 \times 10$

This is the horizontal shear Force F_s

or $Fs = \tau \times 75 \times 10$ or $\tau = \dfrac{4000}{75 \times 10} = 5.33$ MPa

Example 1.39

Two length of a tie bar, each of diameter 'D' are connected by a pin joint. The end of one part is forked, in which is fitted the end of the other and both are secured by a pin of diameter 'd' passing at right angles to theaxis of the bar as shown in figure 1.38 If σ_t and τ are the tensile and shear stresses in the bars and the pin respectively, establish a relation ship between their diameters and stresses, assuming that both offer equal resistance.

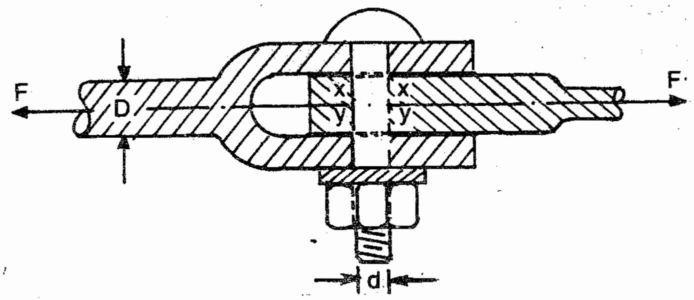


Fig. 1.38

Solution

Tensile strength of the bar $P = \sigma_t \times \frac{\pi}{4} (D)^2$ this force F tends to shear the pin at two sections x-x and y-y and the pin is thus under double shear.

Strength of the pin against shearing $= 2\left[\tau \times \frac{\pi}{4} d^2\right]$

Since both are to offer equal resistance, hence their strengths must be equal.

$$\therefore \sigma_t \times \frac{\pi}{4} D^2 = 2 \times \tau \frac{\pi d^2}{4} \quad \text{or} \quad \frac{\sigma_t}{\tau} = \frac{2d^2}{D^2} \qquad \textbf{Answer.}$$

Poisson's Ratio

You will observe that when a specimen of an elastic material is subjected to tensile forces along its horizontal axis, the length of the specimen increases and the thickness and breadth decrease. Similarly when a compressive force in applied shortening in length is accompanied by an increase in the lateral dimesions (Thickness and width). This effect is called poisson's effect.

Therefore every longitudinal strain is accompanied by a lateral strain in a direction at right angles to the linear strain.

$$\textbf{Lateral strain} = \frac{\text{Change in Lateral dimension}}{\text{Original Lateral dimension}}.$$

Within elastic limit the ratio of lateral strain to liner strain is constant. This ratio is called poisson's ratio and denoted by μ

$$\textbf{Poisson's ratio } \mu = \frac{\text{Lateral strain}}{\text{Linearstrain}}$$

μ Varies between 0 and 0.5 for all materials for metals the value of μ lies between 0.2 and 0.45

The Value of μ for some materials are given in the table.

TABLE 1.3

Name of material	Value of μ
Steel	0.25 – 0.35
Cast iron	0.23 – 0.27
Copper	0.31 – 0.34
Brass	0.32 – 0.42
Aluminium	0.32 – 0.36
Concrete	0.08 – 0.18
Ply wood	0.07

Volumetric strain

When a specimen of a material is acted upon by stresses in three mutually perpendicular directions, the volume of the specimen changes.

$$\textbf{Volumetric strain} = \frac{\text{Change in Volume}}{\text{Original Volume}}$$

$$\varepsilon_v = \frac{\delta v}{V}$$

Bulk modulus

The ratio of stress and volumetric strain is called Bulk modulus of elasticity

$$\text{Bulk modulus} = K = \frac{\text{stress}}{\text{Volumetric strain}}$$

$$K = \frac{\sigma}{\varepsilon_v}$$

Relation Between Elastic Constants

(i) Relation between E and G

Consider a solid cube ABCD subjected to shear stress τ along the faces AB and CD. Complementary shear stress will be induced in the faces BC and AD. Let ABC′D′ be the deformed shape of the cube Draw a perpendicular CL on AC′. As the deformation is very small, angle AC′C may be taken as 45 degrees.

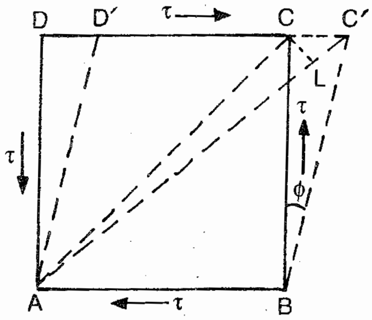


Fig. 1.39

$$\text{Now } \phi = \frac{CC'}{BC} = \frac{C'L}{Cos\,45\,BC}$$

$$= \frac{C'L}{Cos\,45\,AC\,Cos\,45} = \frac{2\,C'L}{AC}$$

Since AC is very nearly equal to AL. there fore C′L is the elongation of diagonal AC

$$\therefore \quad \text{Linear strain of the diagonal} = \frac{C'L}{AC} = \frac{\phi}{2}$$

$$\text{But } \phi = \frac{\tau}{G} \qquad \text{or} \quad \varepsilon_{Ac} = \frac{1}{2} \cdot \frac{\tau}{G}$$

It means that strain in the diagonal is equal to half the shear strain.

The diagonal AC elongates, where as diagonal BD is subjected to compressive stress. Therefore the strain of the diagonal AC

$$\varepsilon_{Ac} = \frac{\tau}{E} + \mu \frac{\tau}{E} = \frac{\tau}{E}\,(1+\mu)$$

$$\text{or} \quad \frac{\tau}{2G} = \frac{\tau}{E}\,(1+\mu) \quad \text{or} \quad E = 2G\,(1+\mu)$$

$$\mathbf{E = 2G\,(1+\mu)} \qquad \ldots \qquad \ldots \qquad (1)$$

Relation between *E* and *K* Let the solid cube be subjected to a tensile stress σ on each face.

$$\text{Direct strain along each axis} = \frac{\sigma}{E} \text{ (Tensile)}$$

Lateral strain along an axis due to the tensile stress along any other axis = $\frac{\mu\sigma}{E}$ (Compressive)

Net tensile strain $= \frac{\sigma}{E} - \frac{\mu\sigma}{E} - \frac{\mu\sigma}{E}$

$$= \frac{\sigma}{E}\ (1 - 2\,\mu)$$

Volumetric strain = 3 × Linear strain

$$\frac{\sigma}{K} = \frac{3\sigma}{E}\ (1 - 2\,\mu)$$

$$\text{or}\quad \mathbf{E = 3\,K\,(1 - 2\,\mu)} \qquad \ldots \qquad \ldots \qquad (2)$$

(iii) Relation between E, G, and K

$$E = 2\,G\,(1 + \mu) \qquad \ldots \qquad \ldots \qquad (i)$$

$$E = 3\,K\,(1 - 2\,\mu) \qquad \ldots \qquad \ldots \qquad (ii)$$

From equation (i) $\mu = \left(\frac{E}{2G} - 1\right)$

Putting this in equation (ii)

$$E = 3\,K\left[1 - 2\left(\frac{E}{2G} - 1\right)\right] = 3K\left[\left(1 - \frac{E}{G} + 2\right)\right]$$

$$E = 3K\left(3 - \frac{E}{G}\right) \quad \text{or} \quad E = \left(9K - \frac{3KE}{G}\right)$$

$$E = \frac{9KG - 3KE}{G} \quad \text{or} \quad EG + 3\,KE = 9\,KG$$

$$\text{or}\quad E\,(G + 3\,K) = 9\,KG$$

$$\boldsymbol{E = \frac{9KG}{G + 3K}} \qquad \ldots \qquad \ldots \qquad (3)$$

Example 1.40

If the modulus of elasticity of a material is 200 GN/m^2 and modulus of rigidity is 80 GN/m^2 determine the poisson's ratio and bulk modulus.

Solution

Using the relation

$$E = 2G\,(1 + \mu)$$

$$200 = 2 \times 80\,(1 + \mu)$$

$$\text{or}\quad (1 + \mu) = \frac{200}{2 \times 80} = 1.25$$

Poission's ratio $\mu = 1.25 - 1 = 0.25$

To find bulk modulus, use the relation

$$E = 3\,K\,(1 - 2\,\mu)$$

$$200 = 3\,K\,(1 - 2 \times 0.25)$$

$$K = \frac{200}{3 \times 0.50} = 133.3 \text{ GN/m}^2$$

Answer

Principal strain

Strains in the direction of principle stresses are called principle strains. Every principal stress produces a strain in its own direction and a strains apposite in nature in all directions at right angles to the principal stress. This is because of poisson's effect. Thus principal stress σ_x along $\times$ – axis will produce a principal strain $\frac{\sigma_x}{E}$ in its own direction and $-\frac{\mu\sigma_x}{E}$ and $\frac{-\mu\sigma_z}{E}$ along Y and Z axis.

Volumetric strain of a rectangular Block

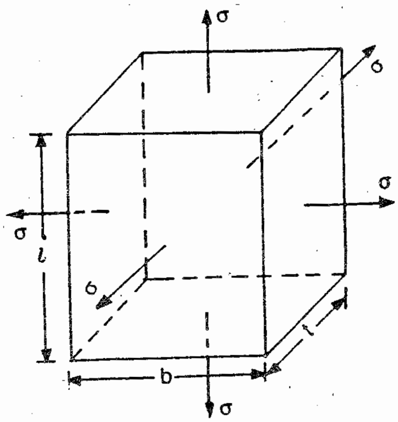


Fig. 1.40

Let l = length, b = breadh and t = thickness of a rectangular block shown in fig. 1.40

Let δl, δb, and δt be the increase in the dimensions of the block.

Increase in the volume of the block $\delta v = (l + \delta l)(b + \delta b)(t + \delta t) - V$

Neglecting higher powers of small quantities

$$\delta_V = lb.\,\delta t + l \times t\,\delta b + b.t.\delta l$$

$$= l \times b \times t\left(\frac{\delta t}{t} + \frac{\delta b}{b} + \frac{\delta l}{l}\right)$$

$$= V\ (\text{sum of three strains}) \qquad = V\,(\varepsilon_t + \varepsilon_b + \varepsilon_l)$$

$$\frac{\delta v}{v} = \text{Volumetric strain} = \text{Sum of three principal strains}$$

Volumetric strain $= (\varepsilon_x + \varepsilon_y + \varepsilon_z)$

Volumetric strain of a cylindrical rod

Let l be length and d be the diameter of a cylindrical rod

Let δl and δd be the change in length and diameter of the cylindrical rod.

The Changed Volume of the rod

$$V + \delta v = \frac{\pi}{d}(d + \delta d)^2 \times (l + \delta l)$$

$$V + \delta v = \frac{\pi}{4}(d^2.l + 2d.l.\delta d + d^2.\delta l)$$

(Neglecting the product of smaller quantities)

$$\text{Now } V = \frac{\pi}{4} d^2.l$$

$$\delta_v = \frac{\pi}{4} \ (2d.\ l.\delta d + d^2.\delta l)$$

$$\text{or} \quad \frac{\delta v}{v} = \frac{2d.l.\delta d + d^2.\delta l}{d^2.l} = 2\frac{\delta d}{d} + \frac{\delta l}{l}$$

$$\varepsilon_v = \frac{\delta v}{v} = (2\ \varepsilon_d + \varepsilon_l)$$

Hence volumetric strain in case of a cylindrical rod is the sum of strain in length and twice the strain in diameter.

Example 1.41

A metal bar 50 mm × 50 mm section is subjected to an axial compressive force of 500 KN. The contraction of a 200 mm guage length was found to be 0.5 mm and increase in thickness as 0.04 mm. Find the Values of Young's modulus and poisson's ratio. (J.M.I.)

Solution

$$\text{Normal Stress} = \frac{\text{Axial load}}{\text{Area of cross–section}}$$

$$\sigma = \frac{500 \times 10^3}{50 \times 50} = 200 \text{ MPa}$$

$$\text{Linear Strain} = \frac{0.5}{200} = 0.0025$$

$$\text{Young's Modulus} = E = \frac{\sigma}{\varepsilon} = \frac{200}{0.0025} = 80 \text{ KN/mm}^2$$

$$\text{Lateral Strain} = \frac{0.04}{50} = 0.0008$$

$$\text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Linear strain}}$$

$$\mu = \frac{0.0008}{0.0025} = 0.32 \quad \textbf{Answer.}$$

Example 1.42

A flat made of elastic material is subjected to two mutually perpendicular stresses of 100 MPa tensile and 80 MPa compressive. If there is no stress in any other direction, determine the strains in the directions f applied stresses take E = 200 KN/mm² and K = 170 KN/mm²

Solution

Using the relation

$$E = 3\text{ k}\,(1 - 2\,\mu)$$

$$200 \times 10^3 = 3 \times 170 \times 10^3\,(1 - 2\,\mu)$$

$$(1 - 2\,\mu) = \frac{200 \times 10^3}{3 \times 170 \times 10^3} = \frac{20}{51}$$

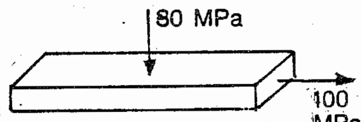


Fig. 1.41

$$\text{or} \qquad \mu = 0.304$$

Strain in the direction of 100 MPa (Tensile Stress)

$$= \frac{\sigma_x}{E} + \frac{\mu\sigma_y}{E} = \frac{1}{E}\ (100 + 0.304 \times 30)$$

$$= \frac{124.32}{200 \times 10^3} = 0.00062 \text{ (Tensile)}$$

Strain in the direction of 80 MPa (Compressive)

$$= \frac{80}{E} + \mu \times \frac{100}{E} = \frac{1}{E}(80 + .304 \times 100)$$

$$= \frac{110.4}{200 \times 10^3} = .000552 \text{ (Compressive)}$$

Example 1.43

A steel block 200 mm × 20 mm × 20 mm is subjected to a tensile force of 40 KN in the direction of its length. Determine the change in volume of the block f E = 205 KN/mm² and poission's ratio μ = 0.3 (Roorkee Univ.)

Solution

$$\text{Direct stress} = \frac{\text{axial load}}{\text{cross–sectional area}}$$

$$\sigma = \frac{40 \times 10^3}{20 \times 20} = 100 \text{ MPa}$$

$$\text{Linear strain } \varepsilon_x = \frac{\sigma}{E} = \frac{100}{205 \times 10^3} = + 4.878 \times 10^{-4}$$

Poisson's ratio $= 0.3$

Lateral strain $\varepsilon_y = \mu \times$ Linear strain

$$\varepsilon_y = -0.3 \times 4.878 \times 10^{-4} = -1.463 \times 10^{-4}$$

$$\varepsilon_z = -1.463 \times 10^{-4}$$

$$\frac{\delta v}{v} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$= (+4.878 - 1.468 - 1.463) \times 10^{-4} = +1.951 \times 10^{-4}$$

$$\delta_v = 1.951 \times 10^{-4} \times \text{Volume of the block}$$

$$= 1.951 \times 10^{-4} \times (200 \times 20 \times 20)$$

$$\delta_v = 15.609 \text{ mm}^3$$ **Answer.**

Example 1.44

A rectangular block 240 mm× 80 mm × 60 mm is subjected to axial loads on each of the face as shown in figure.1.42. Assuming Poisson's ratio as 0.3 determine the change in volume of the block and the values of modulus of rigidity and bulk modulus. Take E = 200 KN/mm²

(AMIE)

Solution

$$\sigma_x = \frac{80 \times 10^3}{240 \times 60}$$

$$= + 555 \text{ MPa (Tension)}$$

$$\sigma_y = \frac{120 \times 10^3}{240 \times 80}$$

$$= - 625 \text{ MPa (Comp)}$$

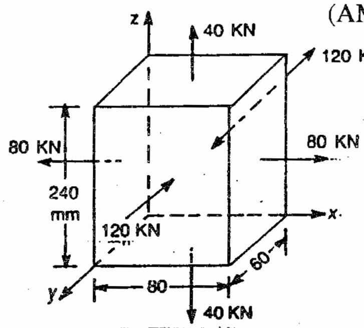


Fig. 1.42

$$\sigma_z = \frac{40 \times 10^3}{80 \times 60} = +833 \text{ MPa (Tension)}$$

Strain in the direction of each force

$$\varepsilon_x = \frac{1}{E}[+555 - 0.3(-625) - 0.3(833)]$$

$$= +\frac{494.6}{E}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \mu(\sigma_x + \sigma_z)]$$

$$= \frac{1}{E}[-625 - 0.3(555 + 833)]$$

$$= -\frac{1041.4}{\text{E}}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_y + \sigma_x)]$$

$$= \frac{1}{E}[+833 - 0.3(-625 + 555)]$$

$$= +\frac{854}{\text{E}}$$

Volumetric Strain

$$\frac{\delta_v}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$= \frac{494.6}{E} - \frac{1041.4}{E} + \frac{854}{E} = \frac{307.2}{E}$$

Now V $= 240 \times 80 \times 60 = 1152 \times 10^3 \text{ mm}^3$

Hence $\delta_v = \frac{307.2}{E} \times 1152 \times 10^3 = 1771.2 \text{ mm}^3$

Change in Volume = 1771.2 mm^3

Using the relation

$$E = 2G(1 + \mu)$$

$$200 \times 10^3 = 2G(1 + 0.3)$$

or $$G = \frac{200 \times 10^3}{2 \times 1.3} = 76.9 \text{ KN/mm}^2$$

Again

$$E = 3K(1 - 2\mu)$$

$$K = \frac{E}{3(1 - 2\mu)}$$

$$= \frac{200 \times 10^3}{3(1 - 2 \times 0.3)} \text{KN/mm}^2$$

$$= 166.6 \text{ KN/mm}^2$$ **Answer**

SUMMARY

1. Stress is load per unit area

   $$\text{Normal Stress } \sigma = \frac{P}{A}$$

   Units of stress KN/mm^2 or N/mm^2 Mpa (Mega Pasal)

2. Change in Length per unit length is strain

   $$\varepsilon = \frac{\delta l}{L} \quad \text{Strain has no units}$$

3. Hooke's Law states that within elastic limit stress is proportional to strain

   $$\frac{\text{Stress}}{\text{Strain}} = \text{Constant}$$

   or $\frac{\sigma}{\varepsilon} = E$, Where E is called Modulus of elasticity

   or Young's modulus. Unit of E are GN/m^2 or KN/mm^2

4. Change in length of a bar $\delta l = \frac{P}{AE} .L$

5. Bars of Varying section

   $$\text{Total change in length } \delta l = \frac{P}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3} + ...\right)$$

6. Change in length of a bar due to self weight

   $$\delta l = \frac{W.L}{2AE}$$

7. Change in length of a loaded tapering rod

   $$\delta l = \frac{4Pl}{\pi E\, D.d}$$

8. Compound Bars

   $$\sigma_s = \frac{P.Es}{(As\,Es + Ac\,Ec)} \text{ and } \sigma_c = \frac{P.Ec}{As\,Es + Ac\,Ec}$$

9. Change in length due to temperature variation

   $$L_t = L_0\,(1 + \alpha\, t)$$

10. For bars totally restrained at ends and subjected to rise or fall in temperature.

    Temp. Stress = $E\,\alpha\, t$.

11. Poission's ratio $\mu = \frac{\text{Lateral strain}}{\text{Linear strain}}$

12. Shear stress $\tau = \frac{\text{Shearing Force}}{\text{Resisting area}}$

    Modulus of rigidity $G = \frac{\tau}{r}$

13. $E = 3K(1 - 2\mu) = 2G(1 + \mu) = \dfrac{9KG}{3K + G}$

14. Volumetric strain of a rectangular block $\dfrac{\delta_v}{V} = (\varepsilon_x + \varepsilon_y + \varepsilon_z)$

15. Volumetric strain of a cylindrical rod $\varepsilon_v = \dfrac{\delta_v}{V} = (2\varepsilon_d + \varepsilon_L)$

QUESTIONS

(1) What is elasticity ? explain. In order of descending elasticity, rewrite the following materials

(a) Rubber (b) Cast iron (c) Timber (d) Copper (e) Steel

(2) Define direct stress, compressive stress, tensile stress and young's modulus of elasticity.

(3) State Hooke's Law, Explain elastic limit.

(4) Draw the stress-strain curve for mild steel and explain various points on it..

(5) Explain Lateral strain, Linear strain and poisson's ratio.

(6) What do you understand by shear stress, shear strain and shear modulus of elasticity?

(7) Explain bulk modulus. Establish a relationship between the three modulus of elasticity E, G and K.

EXERCISES

(8) A surveyor's steel tape 30 meters long has a cross-sectional area 8 mm^2. A force of 60 N is axially applied on the tape. If the modulus of elasticity is 200 KN/mm^2, determine the elongation of the tape. (δl = 1.125 m **Ans.**)

(9) A 25 mm × 25 mm bar 6 meters long is fixed at ends. An axial load of 60 KN is applied at section 2.5 meter from the top. Determine the stresses in the bar above and below the section. (σ_t = 56 Mpa, σ_c = 40 Mpa)

(10) A steel rod 20 mm diameter and 2 meters long is subjected to an axial pull of 20 KN. Determine the nature and magnitude of the stress produced and the elongation of the rod. Take E = 200 GN/m^2

(σ_t = 63.6 Mpa, δl = .636 mm)

(11) A circular punch 25 mm diameter is used to punch a hole through a steel plate 12 mm thick. If the force required to punch this hole is 360 KN, determine the maximum shearing stress developed. (τ = 382 MPa)

(12) Three pieces of wood 40 mm × 40 mm square cross-sectional area are glued together and to the foundation. If a horizontal force of 40 KN is applied as shown in figure, 1.43 determine the average shearing stress in each of the glued joints (τ_{av} = 50 MPa)

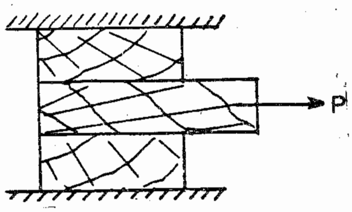


Fig. 1.43

(13) A metal bar 30 mm in diameter was subjected to a tensile load of 54 KN and the measured extension on 300 mm guage length was 0.112 mm and change in diameter was 0.00366 mm. Calculate the poissons ratio and value of the three modulus ($\mu = 0.32$, $E = 206.4$ KN/mm^2, $K = 191.7$ KN/mm^2 and $G = 78.2$ KN/mm^2).

(14) A rectangular block 250 mm$\times$ 100 mm $\times$ 75 mm is subjected to axial loads as follows.

(a) 48 KN tensile in the direction of its length

(b) 90 KN tensile on the 250 mm $\times$ 75 mm face

(c) 100 KN Compressive on the 250 mm $\times$ 100 mm face

Assuming poisson's ratio as 0.25 find in terms of modulus of elasticity E, the strains in the direction of each force. If $E = 200$ KNmm^2, find the values of K and G, Also calculate the change in the volume of the block.

Ans $\left[\varepsilon_x = \dfrac{420}{E}, \varepsilon_y = -\dfrac{680}{E}, = \varepsilon_z = \dfrac{620}{E}, K = 133 \text{ KN/mm}^2\right]$

G = 80 KN/mm^2, δ_v = **3375** mm^3

(15) A steel bar 4 meters Long is made up of three portion, $AB = 1.5$ meter long and 20 mm dia, *BC = 1 m long and 40 mm dia and portion CD* = 1.5 meter long and 30 mm dia is loaded as shown in figure 1.44. Determine the total elongation of the bar. (**Ans** $\delta l = 17.62$ mm)

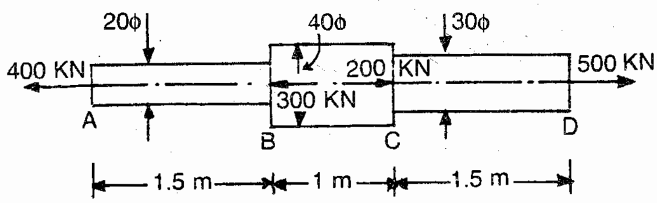


Fig. 1.44

(16) A round bar shown in figure 1.45 is subjected to a pull of 16 KN. Determine the diameter of middle portion if the stress there is not to exceed 25 MPa. What must be the length of the middle portion if the total extension of the bar under the given load is to be 0.362 mm. Take $E = 200$ KN/mm^2

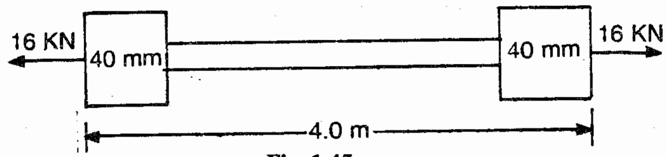


Fig. 1.45

(17) Two prismatic bars are rigidly fastened together and support a vertical load of 75 KN. The upper bar is of steel, 10 meters Long and 6000 mm^2 in cross-sectional area. The lower bar is of brass 6 meters Long and 5000 mm^2 cross-sectional area. Determine the stress in each material. Take Es = 200 GN/m^2 and E$_b$ = 90 GN/m^2. Weight of steel and brass may be taken as 7.7×10^4 N/m^3 and 8.25×10^4 N/m^3 respectively.

Ans - ($\sigma_s = 15.75$ MPa, $\sigma_b = 14.25$ MPa.)

(18) A Composite bar consists of two timber sections 500 mm$\times$ 250 mm and a steel plate 200 mm$\times$ 20 mm is symmetrically placed between them. If E$_s$ = 200

KN/mm^2 and E_t =10 KN/mm^2 · determine the maximum tensile stress in steel plate when maximum tensile stress in timber is 80 MPa. The bar is subjected to direct tension.

(19) A reinforced concrete column 500 mm diameter has four steel rods of 30 mm diameter, one at each corner. If the column supports a Load of 1000 KN determine the stresses in steel and concrete. Take $\frac{E_s}{E_c} = 15$

Ans - (σ_s = 63.11 MPa, σ_c = 4.22 MPa)

(20) A uniform rope 10 meters Long hangs vertically. Find the extension of the first 4 meters of its length from the top due to the weight of the rope itself. Find also the total extension of the rope. Take = E = 200 GN/m^2 and $\rho = 3.2 \times 10^4$ N/m^3.

Ans - (0.08 mm; .00763 mm).

(21) A steel rod meters Long and 25 mm diamter is connected to two grips one at each end at a temperature of 130°C. Find the pull exerted when the temperature falls to 60°C.

(i) If the ends do not yield

(ii) If the ends yield by 1.2 mm

Take E = 200 KN/mm^2 and $\propto = 12 \times 10^{-6}$/ C°

Ans - (82.46 KN; 58.90 KN)

(22) A flat bar of aluminium 30 mm wide and 8 mm thick is placed between two steel bars each 30 mm wide and 11 mm thick. The three bars are fastened together at their-ends when the temperature is 20°C. (a) Find the stress in each bar when the temperature rises to 70°C. (b) If at the new temperature a tensile load of 50KN is applied to the composite bar, what are the final stresses in steel and aluminium. Take E_s = 210 KN/mm^2; E_{al} = 70 KN/mm^2, $\propto_s = 12 \times 10^{-6}$/°C and $\propto_{al} = 24 \times 10^{-6}$/°C

Ans - [(*a*) σ_s = 14.8 MPa, σ_{al} = 40.7 MPa]

(*b*) $\sigma_s = 52.69\,MPa, \sigma_{al} = 63.22\,MPa$]

(23) Explain the following

(a) (i) Hooke's law

(ii) Poissions ratio

(iii) Yield Stress

(b) A reinforced Concrete Column 500 mm × 500 mm in Section is reinforced with 4 steel bars of 25 mm diameter one in each corner. The Column is Carrying an axial load of 200 KN. Determine the stresses in concrete and steel. Take E_s = 210 KN/mm^2 and E_c = 14 KN/mm^2.

σ_s = 108 MPa and σ_c = 7.2 MPa. J.MI.1995

(24) (a) Define three modulii and Poisson's ratio

(b) A steel bar *ABCD* of Varying section is subjected to the axial forces as shown in fig (1.46). Find the Value of *P* neccessery for equilibrium. If *E* = 210 KN/mm^2, determine total elongation of the bar. (A.M.U. 1993)

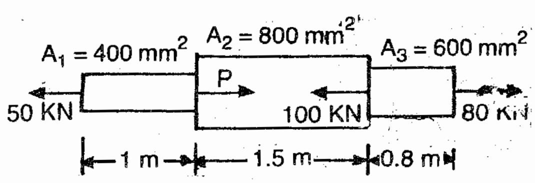


Fig. 1.46

(25) A metal bar 50 mm $\times$ 50 mm Section is subjected to an axial Compressive load of 500 KN. The Contraction for 200 m guage length is found to be 0.5 mm and increase in thickness 0.04 mm. Find the Value of Young's modulus and Poission's ratio. A.M.U. 1992

(26) A steel rod of 20 mm diameter passes centrally through a tight fitting copper tube of external diameter 40 mm. the tube is closed with the help of rigid washers of negligible thickness and nuts threaded on the rod. the nuts are tightened till the compressive load on the tube is 50 KN. Determine the stresses in the rod and the tube, when the temperature of the assembly falls by 50°C. Take E for steel and Copper as 200 GN/m^2 and 100 GN/m^2 and $\propto$ for steel and copper as 12×10^{-6} /°C and 1.8×10^{-6}/°C . A.M.U. 1992 and Cambridge

Answer - (σ_s = 123.15 MPa

σ_c = 41.05 MPa)

□ □ □

2

Analysis of Complex Stresses

So far we have analysed the stresses produced in an elastic body subjected to one loading at a time. Normal stresses produced due to axial loading or shearing stresses caused due to Shearing force have been discussed. But when an elastic body is subjected simultaneously to several loadings then it gives rise to a complex system of stresses. The aim of the present discussions is to determine the normal and shearing stresses on an arbitrary plane passing through a point in an elastic body when subjected to several loadings simultaneously.

Two Dimensional Stress

When a plane element is separated from a body it will be subjected to normal stresses as well as shearing stresses.

Stresses on An Inclined Plane

If σ_x and σ_y are the normal stresses acting on two mutually perpendicular planes accompanied by a shearing stress τ_{xy} as shown in figure 2.1, then normal stress σ and shearing stress τ on a plane inclined at an angle θ to the x-axis are given by the expression.

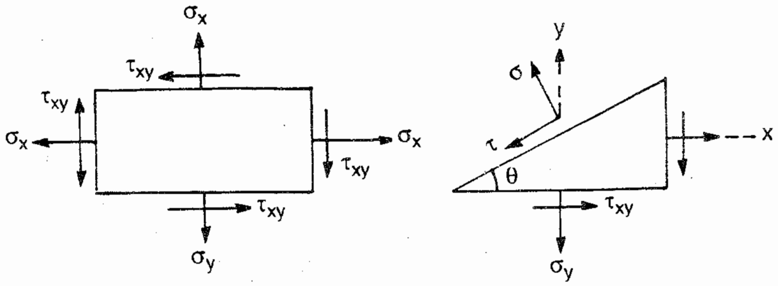


Fig. 2.1

$$\text{Normal Stress } \sigma = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \text{Cos} 2\theta + \tau_{xy} \text{Sin} 2\theta \quad \ldots \quad \ldots (1)$$

$$\text{Shearing Stress } \tau = \frac{\sigma_x - \sigma_y}{2} \text{Sin } 2\theta + \tau_{xy} \text{Cos } 2\theta \quad \ldots \quad \ldots (2)$$

Principal Stresses

The maximum and minimum values of normal stress depends upon the angle θ or the direction of the inclined plane.

When normal stress assumes maximum and minimum values at a particular inclination, then these stresses are termed as Principal Stresses.

Principal Planes.

The planes perpendicular to which the principal stresses act are called Principal Planes. the shearing stress at which will be zero.

Sign Convention

(i) Normal stress is considered positive if it is a tensile stress.

(ii) Normal stress is considered negative if it is a compressive stress

(iii) Shearing stresses are considered positive if they tend to rotate the element in a clock wise direction.

(iv) Shearing stresses are considered negative if they tend to rotate the element in a counter clock wise direction.

Law Of Complementary Shears.

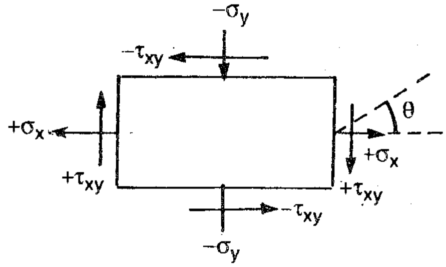


Fig. 2.2

A state of shear stress along a plane must be accompanied by a balancing shear stress of the same intensity along a plane at right angles to it. The directions of these shearing stresses are such that if one tends to rotate the element in a clock wise direction, the complementary shear must rotate it in a counter clock wise direction.

Normal And Shearing Stresses On An Inclined Plane

Let an axial force P be applied on a bar of uniform Cross-Sectional area A as shown in figure 2.3 (a)

Fig. 2.3 (a)

Normal Stress $\sigma_x = \dfrac{P}{A}$

Now consider a plane inclined at an angle θ to the x - axis of the bar.

Cross-sectional area of the inclined plane $AB = \dfrac{A}{\sin\theta}$

Stress on the plane AB,

$$\sigma' = \frac{P}{A/\sin\theta} = \frac{P\sin\theta}{A}$$

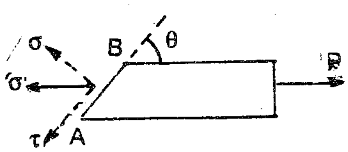


Fig. 2.3 (b)

The component of σ' which is normal to the inclined plane represents the normal stress on the inclined plane.

$$\therefore \sigma = \sigma' \sin\theta$$

$$= \frac{P}{A}\sin\theta \times \sin\theta = \frac{P}{A}\sin^2\theta$$

From trignometry, we can write $\sin^2\theta = \left(\frac{1 - Cos\,2\theta}{2}\right)$

$$\therefore \sigma = \frac{P}{A}\left(\frac{1 - Cos\,2\theta}{2}\right)$$

But $\frac{P}{A} = \sigma_x$

$$\therefore \sigma = \frac{1}{2}\sigma_x\,(1 - \text{Cos}\,2\theta)$$

The component of σ' along the plane AB gives the tangential stress τ on the inclined plane

$$\tau = \sigma' \text{Cos}\,\theta$$

$$= \frac{P}{A}\sin\theta\,\text{Cos}\,\theta$$

From trignometry we know that $\sin 2\theta = 2\sin\theta\ \text{Cos}\,\theta$

$$\therefore \quad \tau = \frac{1}{2}\sigma_x \sin 2\theta$$

Shearing stress will be maximum when $\sin 2\theta$ is maximum i.e. when $2\theta = 90°$ or $\theta = 45°$

Maximum Shearing Stress

$$\tau_{max} = \frac{1}{2}\sigma_x \sin 90° = \frac{1}{2}\sigma_x$$

Example 2.1

A bar of uniform cross-section 30 mm × 25 mm is subjected to axial tensile forces of 50 KN applied at each and of the bar. Determine the normal and shearing stresses on a plane inclined at 30° to the direction of loading. Also determine the maximum shearing stress in the bar.

Solution

Area of cross-section = 30 × 25 = 750 mm^2

Normal Stress on a cross -section perpendicular to the axis of the bar.

$$\sigma_x = \frac{p}{A} = \frac{50 \times 10^3}{30 \times 25}$$

$= 66.6$ MPa

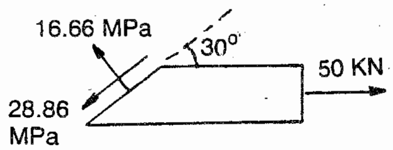


Fig. 2.4

The Normal Stress on a plane at an angle θ with the direction of the loading is given by

$$\sigma = \frac{1}{2}\sigma_x\,(1 - \text{Cos}2\,\theta)$$

$$= \frac{1}{2} \times 66.6\,(1 - \text{Cos}\,60°) = \frac{1}{2} \times 66.6\,(1 - 0.5)$$

$= 16.66$ MPa

Shear Stress on a plane at an angle θ with the direction of loading

$$\tau = \frac{1}{2}\ \sigma_x\ \sin 2\,\theta$$

$$= \frac{1}{2} \times 66.6 \times \sin 60 = 28.86\ \text{MPa}$$

Shearing Stress will be maximum when $2\,\theta = 90°$

or $$\tau_{max} = \frac{1}{2}\,\sigma_x \sin 90°$$

$$= \frac{1}{2} \times 66.66 \times 1 \qquad = 33.33\ \text{MPa}$$

Example 2.2

A bar of uniform cross-sectional area 625 mm² is subjected to axial compressive forces of 60 KN at each and of the bar. Determine the normal and shearing stresses acting on a plane inclined at 30° to the line of action of the axial load. The bar is so short that the possibility of buckling as a column may be neglected.

Normal stress on a cross-section perpendicular to the axis of the bar

$$\sigma_x = \frac{P}{A} = -\frac{60 \times 1000}{625} = -\ 96\ \text{MPa}$$

The Normal Stress on a plane at on angle θ with the direction of loading

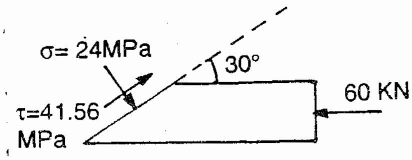


Fig. 2.5

$$\sigma = \frac{1}{2}\,\sigma_x\,(1 - \text{Cos}\ 2\,\theta\,)$$

$$= \frac{1}{2} \times 96\,(1 - \text{Cos}\ 60\,)$$

$$= 24\ \text{MPa}$$

Shearing Stress on a plane at an angle θ with the direction of loading

$$\tau\ = \frac{1}{2}\ \sigma_x\ \sin 2\,\theta$$

$$= \frac{1}{2} \times 96 \times \sin 60$$

$$= \frac{1}{2} \times 96 \times .866 = \quad 41.56\ \text{MPa}$$

Stresses On An Inclined Plane Of A Body Subjected To Normal Stress In One Plane Accompanied By A Simple Shear Stress

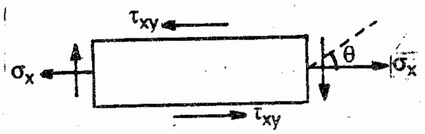


Fig. 2.6 (a)

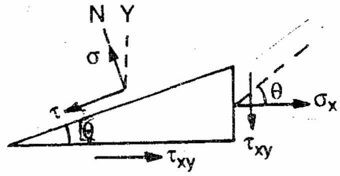


Fig. 2.6 (b)

Consider a rectangular block subject to normal stresses σ_x and shear stress τ_{xy} as shown in fig. 2.6 (a) on a plane inclined at an angle θ to the x-axis, the normal stresses and shearing stresses are required to be determined.

Fig. 2.6 (b) represents the free body diagram of a triangular element separated from the block by a plane inclined at an angle θ to the x-axis. Let A be the area of the inclined face.

Now applying the conditions of equilibrium along the axes chosen parallel and perpendicular to the inclined plane as shown in fig 2.6 (c)

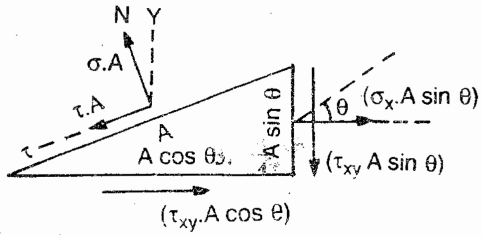


Fig. 2.6 (c)

$\Sigma F_N = o$

$(\sigma.A = (\sigma_x . A \ \sin\theta) \sin\theta + (\tau_{xy}.A \sin\theta) \cos\theta + (\tau_{xy}.A \cos\theta) \sin\theta$

$\sigma = \sigma_x \ \sin^2\theta + 2 \ \tau_{xy} \sin\theta . \cos\theta$

$$\sigma = \frac{1}{2} \sigma_x (1 - \cos 2\theta) + \tau_{xy} \sin 2\theta \qquad \ldots \qquad \ldots \qquad (1)$$

Now consider the equilibrium of forces along an axis T parallel to the inclined plane

$\Sigma F_T = 0$

$\tau.A = \sigma_x . A \sin\theta \cos\theta - \tau_{xy} . A . \sin^2\theta + \tau_{xy} . A . \cos^2\theta$

$$\tau = \frac{1}{2} \sigma_x . \sin 2\theta + \tau_{xy} \cos 2\theta \qquad \ldots \qquad \ldots \qquad (2)$$

To determine the maximum Value of normal stress, differentiate equation (1) with respect to θ and equate the derivative to Zero

$$\frac{d\sigma}{d\theta} = \sigma_x \sin 2\theta + 2 \tau_{xy} \cos 2\theta = 0$$

$$\tan 2\theta_p = -\frac{2\tau_{xy}}{\sigma_x} \qquad \ldots \qquad \ldots \qquad (3)$$

Here θ_p defines the planes of maximum and minimum normal stresses. These planes are called **Principal Planes.**

The normal stresses that exist on these planes are called **Principal Stresses**

There are two solutions to (3), consequently two values of $2\theta_p$ differ by 180° and also two value of θ_p differ by 90°.

Putting the values of sin 2 θ and Cos 2 θ as obtained from equation (3) in equation (1) We get the following results.

Maximum normal stress

$$\sigma_{max} = \frac{1}{2}\sigma_x + \sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} \qquad \dots \qquad \dots \qquad (4)$$

Minimum normal stress

$$\sigma_{min} = \frac{1}{2}\sigma_x - \sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} \qquad \dots \qquad \dots \qquad (5)$$

Maximum Shearing Stress

To obtain the maximum value of the shearing stress we enotc differentiate equation (2) and set the derivative equal to zero.

$$\frac{d\tau}{d\theta} = \sigma_x \text{ Cos } 2\theta - 2\tau_{xy} \text{ Cos } 2\theta = 0$$

$$\tan 2\theta_s = \frac{\sigma_x}{2\tau_{xy}} \qquad \dots \qquad \dots \qquad (6)$$

θ_s defines the planes on which shearing stress is maximum or minimum.

Now putting the values of sin 2 θ and Cos 2 θ as obtained from equation (6) in equation (2) we obtain the following expression for maximum and minimum shearing stress.

$$\tau_{\substack{max\\min}} = \pm\sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2} \qquad \dots \qquad \dots \qquad (7)$$

Example 2.3

A plane element in a body is subjected to a normal stress in the x-direction of 80 MPa, as well as a shearing stress of 20 MPa as shown in figure 2.7

(a) *Determine the normal and shearing stress intensities on a plane inclined at an angle of 30° to the normal stress.*

(b) *Determine the maximum and minimum values of the normal stress that may exist on inclined planes and the directions of these stress.*

(c) *Determine the magnitude and direction of the maximum shearing stress on an inclined plane.*

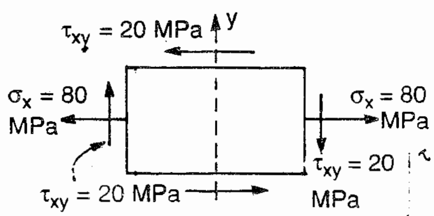


Fig. 2.7 (a)

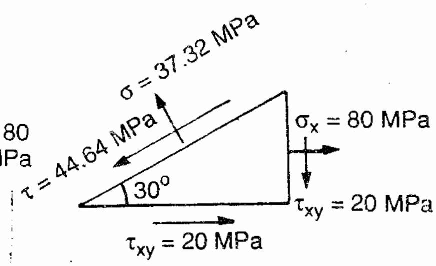


Fig. 2.7 (b)

Solution

(a) Normal stress on a plane inclined at an angle θ to the x-axis is given by the equation.

$$\sigma = \frac{1}{2}\sigma_x - \frac{1}{2}\sigma_x \text{Cos } 2\theta + \tau_{xy} \sin 2\theta$$

$$= \frac{1}{2}(80) - \frac{1}{2}(80) \text{ Cos } 60° + 20 \text{ Sin } 60°$$

$$= 40 - 40 \times 0.5 + 20 \times .866 = 40 - 20 + 17.32$$

$$= 37.32 \text{ MPa}$$

Shearing Stress on a Plan inclined at an angle θ tothe x-axis

$$\tau = \frac{1}{2}\sigma_x \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$= \frac{1}{2} \times 80 \text{ Sin } 60 + 20 \text{ Cos } 60 = 40 \times .866 + 20 \times 0.5$$

$$= 34.64 + 10 = 44.64 \text{ Mpa}$$

(b) Maximum normal stress

$$\sigma_{max} = \frac{1}{2}\sigma_x + \sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}$$

$$= 40 + \sqrt{(40^2) + (20^2)} = 40 + 44.72$$

$$= 84.72 \text{ Mpa}$$

Minimum normal Stress

$$\sigma_{mim} = \frac{1}{2}\sigma_x - \sqrt{(\frac{1}{2}\sigma_x)^2 + (\tau_{xy})^2}$$

$$= 40 - \sqrt{(40^2) + (20^2)} = 40 - 44.72$$

$$= -4.72 \text{ Mpa.}$$

The direction of the planes on which these principal stresses occur are

$$\tan 2\theta_p = -\frac{\tau_{xy}}{\frac{1}{2}\sigma_x} = -\frac{20}{40} = -\frac{1}{2}$$

Since the tangent of the angle $2\theta_p$ is negative the values of $2\theta_p$ lie in II and IV quadrant. Hence $2\theta_p = -153° 26'$ in the 2nd quadrant and $2\theta'_p = 333° 26'$. in the fourth quadrant. Consequently the principal planes are defined by $\theta_p = 76° 43'$ and $\theta'_p = 166° 43'$.

∴ Principal stress on the principal plane oriented at 76° 43′ to the x-axis

$$\sigma = \frac{1}{2}\sigma_x - \frac{1}{2}\sigma_x \text{ Cos } 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma = 40 - 40 \cos 153° 26' + 20 \sin 153° 26'$$

$$= 40 - 40(-0.893) + 20(0.449)$$

$$= 40 + 35.722 + 8.998 = 84.720 \text{ MPa}$$

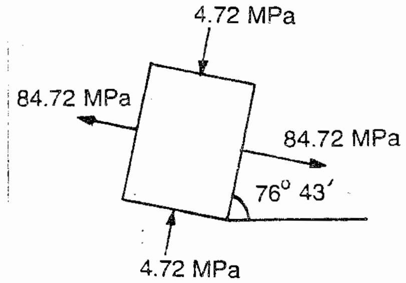


Fig. 2.7 (c)

(c) Maximum Shearing stress

$$\tau_{\substack{max\\min}} = \pm\sqrt{\left(\frac{1}{2}\sigma_x\right)^2 + (\tau_{xy})^2} \quad = \pm\sqrt{(40^2) + (20)^2}$$

$$= \pm\ 44.72 \text{ MPa}$$

The direction of the planes on which these stresses occur is given by

$$\tan 2\theta_s = \frac{\frac{1}{2}\sigma_x}{\tau_{xy}} = \frac{40}{20} = 2$$

The angles $2\theta_s$ will be in the first and third quadrant since the tamgent is positive Thus $2\theta_s = 63°\ 26'$ and $2\theta's = 343°\ 26'$, $\theta = 31°\ 43'$ and $\theta'_s = 121°\ 43'$. The shearing stress on any plane inclined at an angle θ with the axis of x is given by

$$\tau = \frac{1}{2}\sigma_x \sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2} \times 80 \times \sin 63°\ 26' + 20\cos 63°\ 26' = 44.72 \text{MPa}$$

Hence Shearing stress on the $31°43'$ plane is positive the normal stress on the planes of maximum Shearing stress is

$$\sigma = \frac{1}{2}\sigma x = \frac{1}{2}(80) = 40 \text{ MPa}$$

This normal stress acts on each of the planes of maximum shearing stress as shown in the figure 2.7 (d)

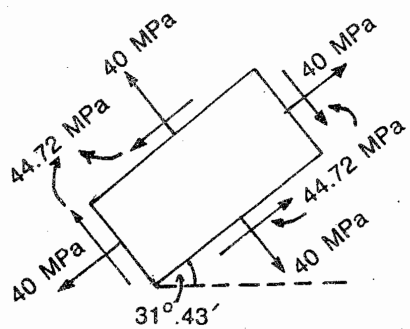


Fig. 2.7 (d)

Example 2.4

A plane element in a body is subjected to a normal compressive stress in the x– direction of 60 MPa and a shearing stress of 15 MPa as shown in figure 2.8 (a). Determine

(a) The normal and Shearing stress intensities on a plane inclined at an angle of 30° to the normal stress

(b) The maximum and minimum values of normal stress on the inclined planes and the direction o these stresses

(c) The magnitude and direction of the maximum shearing stress on inclined plane.

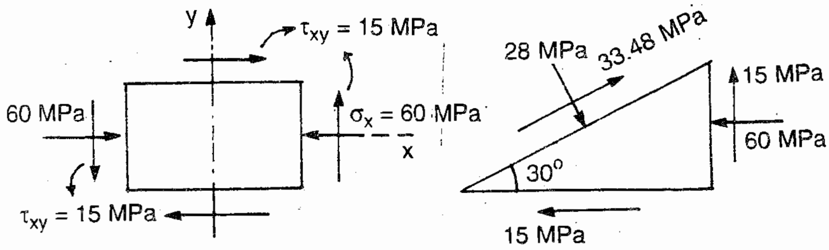


Fig. 28 (a) **Fig. 2.8 (b)**

(a) $\sigma_x = -60$ MPa and $\tau_{xy} = -15$ MPa

Normal stress on a plane inclined at 30°

$$\sigma = \frac{1}{2}\sigma_x - \frac{1}{2}\sigma_x \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= \frac{1}{2}(-60) \frac{-1}{2}(-60)\cos 60 + (-15)\sin 60°$$

$$= -30 + 15 - 15 \times .866 = -30 + 15 - 12.99$$

$$= -27.99 \text{ Mpa} = 28\text{Mpa}$$

Shearing Stress on a plane inclined at 30°

$$\tau = \frac{1}{2}\sigma_x \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$= \frac{1}{2}(-60)\sin 60 - 15\cos 60$$

$$= -25.98 - 7.5 = -33.48 \text{ MPa}$$

Normal and Shearing Stress on a plane inclined at 30° are shown fig.– - 2.8 (b)

(b) Maximum normal stress

$$\sigma_{max} = \frac{1}{2}\sigma_x + \sqrt{\left(\frac{1}{2}\sigma_x\right)^2 + (\tau_{xy})^2}$$

$$= -\frac{60}{2} + \sqrt{(-\frac{60}{2})^2 + (-15)^2}$$

$$= -30 + \sqrt{900 + 225} = -30 + 33.54$$

$$= 3.45 \text{ MPa}$$

$$\sigma_{\min} = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + (\tau_{xy})^2}$$

$$= -30 - \sqrt{(-\frac{60}{2})^2 + (-15)^2}$$

$$= -30 - 33.54 = -63.54 \text{ MPa}$$

The direction of the planes on which these principal stresses occur are given by

$$\tan 2\theta_p = \frac{\tau_{xy}}{\frac{1}{2}\sigma_x} = \frac{-15}{-\frac{1}{2} \times 60} = -\frac{1}{2}$$

The angles defined by $2\theta_p$ lie in second and fourth quadrants since the tangent is negative. Hence $2\theta_p$ = 153°26′ and $2\theta'_p$ = 333°26′ Thus the principal planes are defined by θ_p= 76° 43′ and θ'_p = 166° 43′. Substituting these values in the equation

$$\sigma = \frac{1}{2}\sigma_x - \frac{1}{2}\sigma_x \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= -\frac{60}{2} - \frac{1}{2}(60) \cos 153^\circ 26' - 15 \times \sin 153^\circ 26'$$

$$= -30 - 26.83 - 6.70 = -63.54 \text{ Mpa}$$

Thus the principal stress of 63.54 MPa occurs on the Principal plane oriented at 76° 4′3 to the x–axis as shown in the figure 2.8 (c)

(c) Maximum Shear stress

$$\tau_{\substack{\max \\ \min}} = \pm\sqrt{\left(\frac{1}{2}\sigma_x\right)^2 + (\tau_{xy})^2}$$

$$= \pm\sqrt{\left(\frac{-1}{2}60\right)^2 + (-15)^2} = \pm\sqrt{(-30)^2 + (-15)^2}$$

$$= \pm\sqrt{900 + 225} = \pm 33.54 \text{ Mpa}$$

The direction of the planes on which these shearing stresses occur are

$$\tan 2\theta_s = \frac{\frac{1}{2}\sigma_x}{\tau_{xy}} = \frac{-60/2}{-15} = 2$$

Therefore $2\theta_s$ = 63°26′ and $2\theta'_s$ = 243° 26′

or θ_s = 31° 43′ and θ'_s = 121° 43′

The shearing stress on any plane inclined at an angle θ with the x– axis is given by

$$\tau = \frac{1}{2}\sigma_x \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$= \frac{1}{2}(-60) \sin 63^\circ 26' - 15 \cos 63^\circ 26' = -33.54 \text{ Mpa}$$

Therefore shearing stress on the 31°43′ plane is negative. The normal stress on the planes of maximum shearing stress is given by

$$s = \frac{1}{2}\,sx = \frac{-60}{2} = -30 \text{ MPa}$$

This normal stress acts on each of the planes of maximum shearing stress as shown in the figure 2.8 (d)

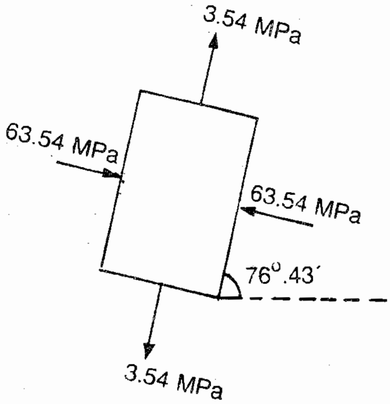


Fig. 2.8 (c)

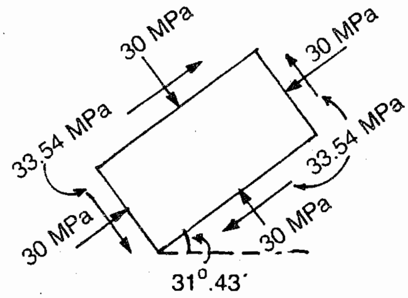


Fig. 2.8 (d)

General Case Of Plane State Of Stress

A rectangular block subjected to normal stresses σ_x and σ_y in two perpendicular directions as well as shearing stress τ_{xy}.as shown in fig. 29 (a)

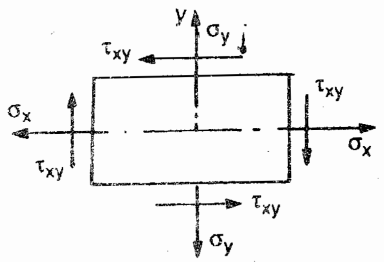


Fig. 2.9 (a)

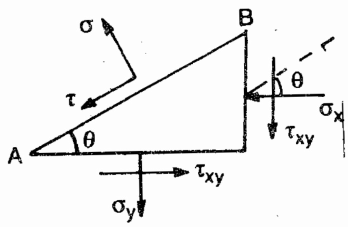
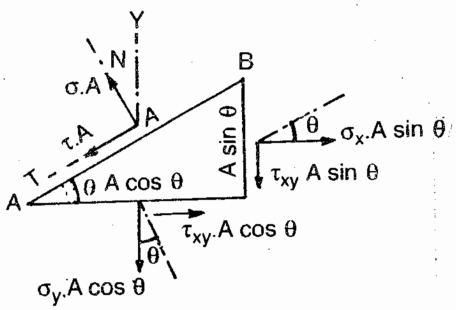


Fig. 2.9 (b)

Determine the normal and shearing stresses on a plane inclined at an angle θ to the*X*–axis.

Pass a plane *AB* inclined at an angle θ to the *X*–axis. Fig. 2.9 (b) represents the free body diagram of the triangular element separated from the block. Let *A* be the area of the inclined face *AB*. Fig 2.9 (c) shows the equilibrium forces on this element. Now applying the conditions of equilibrium along an axis *N*–perpendicular to the inclined plane and an other axis *T*– parallel to the inclined plane *AB*.



2.9 (c)

$\Sigma F_N = 0$

$\sigma . A = (\sigma_x . A \sin\theta) \sin\theta + (\tau_{xy}. A \sin\theta) \cos\theta + (\sigma_y . A\cos\theta) \cos\theta + (\tau_{xy}. A\cos\theta) \sin\theta$

$\sigma = \sigma_x \sin^2\theta + \sigma_y . \cos^2\theta + 2\,\tau_{xy} . \sin\theta \cos\theta$

From trigometry we know that

$$\sin^2\theta = \frac{(1 - \cos 2\theta)}{2},\ \cos^2\theta = \frac{(1 + \cos 2\theta)}{2} \text{ and } \sin 2\theta = 2\sin\theta \cos\theta$$

$$\therefore \quad \sigma = \sigma_x \frac{(1 - \cos 2\theta)}{2} + \sigma_y \frac{(1 + \cos 2\theta)}{2} + \tau_{xy} \sin 2\theta$$

$$\sigma = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta \qquad \ldots \quad \ldots \quad (1)$$

This is the normal stress on any plane inclined at an angle θ to the X–axis

Similarly resolving all forces on the element along the inclined plane

$\Sigma F_T = 0$

$\tau . A = (\sigma_x . A \sin\theta) \cos\theta - (\tau_{xy} . A \sin\theta) \sin\theta + (\tau_{xy} . A \cos\theta) \cos\theta$ $(\sigma_y . A \mathrm{Cs}\theta) \sin\theta$

$\tau = (\sigma_x - \sigma_y) \sin\theta \cos\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta)$

But from trignometry we know

$\sin 2\theta = 2\sin\theta \cos\theta$ and $\cos 2\theta = (\cos^2\theta - \sin^2\theta)$

$$\therefore \tau = \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta + \tau_{xy} \cos 2\theta \qquad .. \quad \ldots \quad (2)$$

Therefore the above equation gives the shearing stress on any plane inclined at an θ to the X– axis

Principal Stresses

Often it is required to determine the maximum values of normal stress and the plane on which such stesses will occur. For this differentiate equation (1) with respect θ and set this derivative equal to zero

$$\frac{d\sigma}{d\theta} = (\sigma_x - \sigma_y) \sin 2\theta + 2\,\tau_{xy} \cos 2\theta$$

Hence thevalues of θ leading to maximum and minimum values of the normal stress are given by

$$\tan 2\theta_p = -\frac{2\tau_{xy}}{(\sigma_x - \sigma_y)} \quad \ldots \quad \ldots \quad (3)$$

The planes defined by the angle θ_p are called "**Principal planes**". The normal stresses that exist on these planes are called "**Principal stresses**"

Equation (3) has two roots. Since the value of the tangent of an angle in the diametrically opposite quadrant is same. Hence these roots are 180° apart for double the angle. Therefore roots of θ are 90° apart. On one plane the normal stress will be maximum, on the other corresponding plane normal stress will be minimum.

The magnitude of these principal streses can be obtained by substituting the value of 2θ from equation (3) into equation (1)

$$\sigma_{max} = \frac{1}{2}(\sigma_x + \sigma_y) + \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + (\tau_{xy})^2} \quad \ldots \quad \ldots \quad (4)$$

$$\sigma_{min} = \frac{1}{2}(\sigma_x + \sigma_y) - \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + (\tau_{xy})^2} \quad \ldots \quad \ldots \quad (5)$$

Maximum Shearing Stress

To obtain the maximum or minimum shearing stress and the corresponding planes on which these stresses act, equation (2) is differentiated with respect to θ and the derivative is set equal to zero.

$$\frac{d\tau}{d\theta} = (\sigma_x - \sigma_y)\,\mathrm{Cos}2\theta - 2\,\tau_{xy}\,\mathrm{Sin}2\theta = 0$$

$$\tan 2\theta_s = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad \ldots \quad \ldots \quad (6)$$

Here θ_s defines the planes on which shearing stress is maximum or minimum. Like equation (3), equation (6) has also two values of the angle $2\theta_s$ giving two planes mutually perpendicular to each other.

Substituting the value of $2\theta_s$ from equation (6) into equation (2), we get the maximum and minimum shearing stress

$$\tau_{\substack{max\\min}} = \pm\sqrt{\left[\frac{1}{2}(\sigma_x - \sigma_y)\right]^2 + (\tau_{xy})^2} \quad \ldots \quad \ldots \quad (7)$$

Comparing equation (3) and (6) we find that the angles $2\theta p$ and $2\theta s$ differ by 90° Since the tangents of these angles are negative reciprocals of one an other. Hence the planes defined by angles θp and θ_s differ by 45° i.e. the planes of maximum shearing stress are oriented 45° from the planes of maximum normal stress.

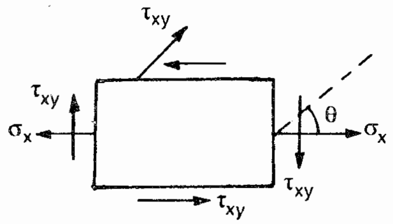
To determine the normal stresse on the planes of maximum shearing stress substitute the values of sin $2\theta_s$ and cos $2\theta_s$ from equation (6) into equation (1)

$$\sigma = \frac{1}{2}(\sigma_x + \sigma_y) \quad \ldots \quad \ldots \quad (8)$$

Thus on each of the planes of maximum shearing stress acts a normal stress of magnitude $\frac{1}{2}(\sigma_x + \sigma_y)$

Derivation of specific cases from the general case.

1. Stresses on an inclined plane of a body subjected to normal stresses in one plane accompanied by simple shear stress.

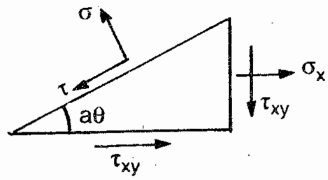


2.10 (a)

In this case $\sigma_y = 0$, hence equation (1) will be reduced to

$$\text{Normal stress } \sigma = \frac{\sigma_x (1 - Cos\, 2\theta)}{2} + \tau_{xy} \text{ Sin } 2\theta$$

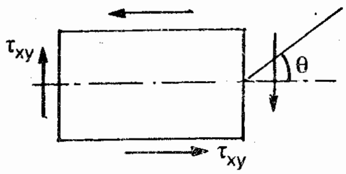
$$\text{and Shearing Stress } \tau = \frac{1}{2} \sigma_x \,.\, \text{Sin } 2\theta + \tau_{xy} \text{ Cos } 2\theta$$



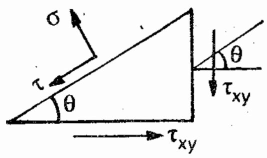
2.10 (b)

2. A state of simple shear

The normal stresses σ_x and σ_y are both equal to zero. Hence stresses on an inclined plane will be



2.11 (a)



2.11 (b)

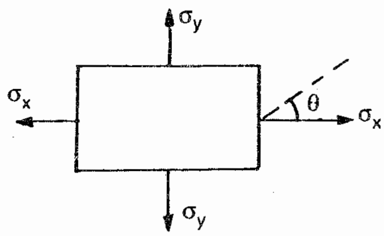
Normal Stress $\sigma = \tau_{xy}$ Sin 2θ

Shearing stress $\tau = \tau_{xy}$Cos 2θ

3. Tension and compression in two directions Here $\tau_{xy} = 0$

Normal Stress

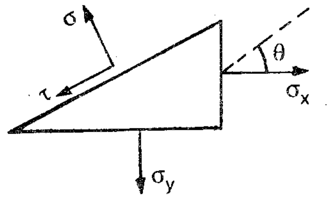
$$\sigma = \frac{\sigma_x (1 - Cos\, 2\,\theta\,)}{2} + \frac{\sigma_y (1 + Cos 2\theta)}{2}$$



2.12 (a)

Shearing Stress

$$\tau = \frac{1}{2}(\sigma_x - \sigma_y)\ \mathrm{Sin}2\theta$$



2.12 (b)

(4) Stresses on an inclined plane of a body subjected to tensile stress in one plane only.

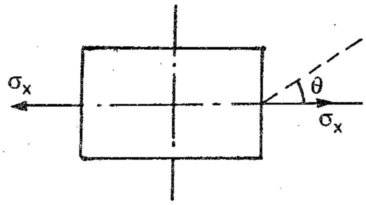
In this case $\sigma_y = 0$ and $\tau_{xy} = 0$

∴ Normal Stress

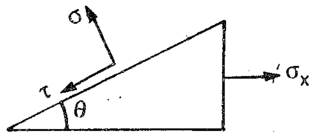
$$\sigma = \frac{1}{2}\sigma_x\,(1 - \mathrm{Cos}2\theta)$$

Shearing Stress

$$\tau = \frac{1}{2}\ \sigma_x\ \mathrm{Sin}2\theta$$



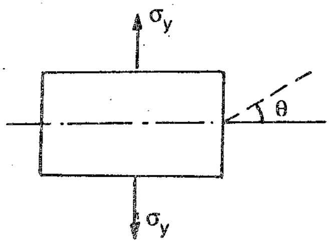
2.13 (a)



5. When $\sigma_x = 0$ and $\tau_{xy} = 0$, Then

Normal Stress

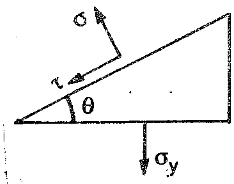
$$\sigma = \sigma_y\,(1 + \mathrm{Cos}2\theta)$$



2.14 (a)

Shearing stress

$$\tau = -\frac{1}{2}\sigma_y\ \mathrm{Sin}\ 2\theta$$



2.14 (b)

Example 2.5

A Plane element is subjected to the stresses shown in fig. 2.15 Determine the following stresses

(a) The principal stresses and their directions

(b) The maximum shearing stresses and the directions of the planes on which they occur.

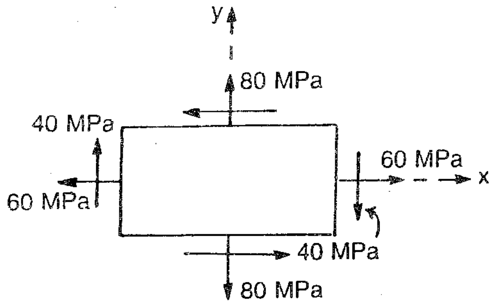


Fig. 2.15

The maximum normal stress is given by

$$\sigma_{max} = \frac{1}{2}(\sigma_x + \sigma_y) + \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + (\tau_{xy})^2}$$

$$\sigma_{max} = \frac{1}{2}(60 + 80) + \sqrt{\left\{\frac{1}{2}(60 - 80)\right\}^2 + (40)^2}$$

$$= 70 + \sqrt{100 + 1600} = 70 + 41.2 = 111.23 \text{ MPa}$$

The minimum normal stress in given by

$$\sigma_{min} = \frac{1}{2}(\sigma_x + \sigma_y) - \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + (\tau_{xy})^2}$$

$$= \frac{1}{2}(60 + 80) - \sqrt{\left\{\frac{1}{2}(60 - 80)\right\}^2 + (40)^2} = 70 - 41.23 = 28.77 \text{ MPa}$$

The directions of the principal planes on which these stresses are induced is given by the equation

$$\tan 2\theta_p = \frac{\tau_{xy}}{\frac{1}{2}(\sigma_x - \sigma_y)} = \frac{-40}{\frac{1}{2}(60 - 80)} = 4$$

$\therefore \quad 2\theta_p = 76°$ and $256°$ $\qquad$ ανδ $\theta_p = 38°, 128°$

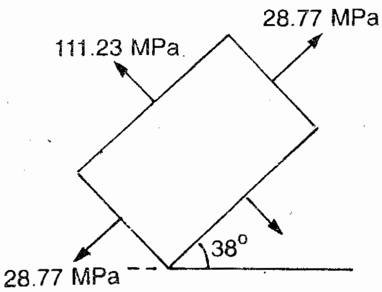
Substituting $\theta_p = 38°$ in the equation

$$\sigma = \frac{1}{2}(s_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma = \frac{1}{2}(60 + 80) - \frac{1}{2}(60 - 80)\cos 76° + 40 \sin 76°$$

$$= 70 + 2.41 + 38.81 = 111.23 \text{ MPa}$$

The element oriented along the principal planes at 38° and subjected to the above principal stress are shown in the figure fig. 2.15 (a) the shearing stresses on these planes are zero.



Principal stresses
Fig. 2.15 (a)

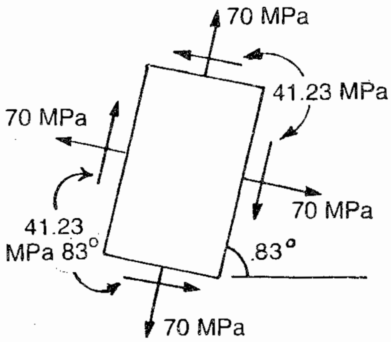


Fig. 2.15 (b)

(b) The maximum and minimum shearing stresses are given by the formula.

$$\tau_{\substack{max \\ Min}} = \pm \sqrt{\left\{\frac{1}{2}(\sigma x - \sigma_y)\right\}^2 + (\tau_{xy})^2}$$

$$= \pm \sqrt{\left\{\frac{1}{2}(60-80)\right\}^2 + (40)^2}$$

$$= \pm \sqrt{100 + 1600} = 41.23 \text{ MPa}$$

The planes on which these shearing stresses occur are obtained from the equation

$$\tan 2\theta_s = \frac{\frac{1}{2}(\sigma_x - \sigma_y)}{\tau_{xy}} = \frac{\frac{1}{2}(60 - 80)}{40} = -0.25$$

Hence $2\theta_s = 166°, 346°$

and $\theta_s = 83°$ and $173°$. These planes are located 45° from the planes of maximum and minimum normal stresses

Now to determine whether the shearing stress is positive or negative on the 83° plane

$$\tau = \frac{1}{2}(\sigma_x + \sigma_y) \sin 2\theta + \tau_{xy} \cos 2\theta$$

Putting $\theta = 83°$in the above equation, we get

$$\tau = \frac{1}{2}(60 - 80) \sin 166° + 40 \cos 166°$$

$$= -2.41 - 38.81 = -41.2 \text{ MPa}$$

Normal Stresses on these planes of maximum Shearing Stresses are found from the equation.

$$\sigma = \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(60+80) = 70 \text{ MPa}$$

The orientation of the element for which the Shearing Stresses are maximum are shown in figure 2.15 (b)

Example 2.6

A Plane element is subjected is the stresses shown in fig. 2.16 (a) Calculate the Principal Stresses and their direction (b) the Maximum shearing stresses and the direction of the planes on which they occur.

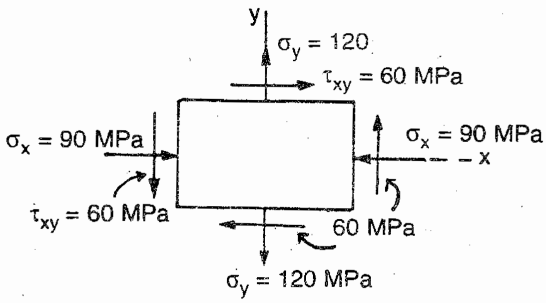


Fig. 2.16 (a)

Solution

$\sigma x = -90$ MPa, $\sigma y = 120$ MPa and $\tau xy = -60$ Mpa

The Maximum normal stress

$$\sigma_{max} = \frac{1}{2}(\sigma x + \sigma y) + \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + (\tau_{xy})^2}$$

$$\sigma_{max} = \frac{1}{2}(-90 + 120) + \sqrt{\left\{\frac{1}{2}(-90 - 120)\right\}^2 + (-60)^2}$$

$$= 15 + \sqrt{11025 + 3600}$$

$$= 15 + 120.93 = 135.93 \text{ MPa}$$

The minimum normal stress is given by

$$\sigma \min = \frac{1}{2}(\sigma x + \sigma y) - \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + (\tau_{xy})^2}$$

$$= 15 - 120.93 = -105.93 \text{ MPa}$$

The directions of the principal planes on which these normal stresses occur are obtained from the equation

$$\tan 2\theta_P = \frac{\tau_{xy}}{\frac{1}{2}(\sigma_x - \sigma_y)} = \frac{-60}{\frac{1}{2}(-90 - 120)} = -0.571$$

$$\tan 2\theta_P = -0.571$$

$$\therefore 2\,\theta_P = 150^\circ\,15', 330^\circ\,15' \text{ and } \theta_P = 75^\circ 8', 165^\circ\,8'$$

Now determine the planes on which these principal stresses occur by using the relation

$$\sigma = \frac{1}{2}(\sigma_x \times \sigma_y) - \frac{1}{2}(\sigma x - \sigma y) \text{ Cos } 2\theta + \tau_{xy} \sin 2\theta$$

and putting $\theta = 75° \ 8'$

$$\sigma = \frac{1}{2}(-90 + 120) - \frac{1}{2}(-90 - 120) \cos 150°15' - 60 \sin 150°15'$$

$$= 105.93$$

An element oriented along the principal and planes subjected to the above principal stresses is show in figure 2.16 (b) the shearing stresses on there planes are zero.

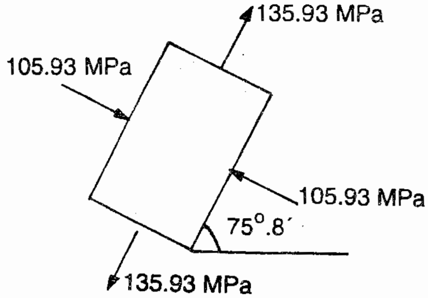


Fig. 2.16 (b)

(b) The maximum and minimum shearing stresses are found from the equation

$$\tau_{\substack{max \\ min}} = \pm \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + (\tau_{xy})^2}$$

$$= \pm \sqrt{\left\{\frac{1}{2}(-90 - 120)\right\}^2 + (-60)^2}$$

$$= \pm \ 120.93 \text{ MPa}$$

The plane on which these maximum shearing stresses occur are given by

$$\tan 2\theta_s = \frac{\frac{1}{2}(\sigma_x - \sigma_y)}{\tau_{xy}}$$

$$= \frac{\frac{1}{2}(-90 - 120)}{-60}$$

$$= \frac{-105}{60} = 1.75$$

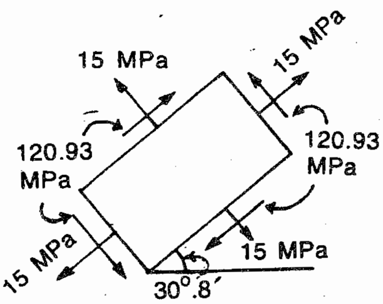


Fig. 2.16 (c)

$\therefore \ 2\theta_s = 60° \ 15', 240° \ 15'$

$\theta_s = 30°8'$ and $120°8'$, these plane sare located 45° from the planes of maximum and minimum normal stresses.

To determine whether the shearing stress is positive or negative on 30° 8′ plane, put $\theta = 30° 8'$ in the following equation

$$\tau = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}(-90 - 120)\text{ Sin } 60°16' - 60\cos 60°16'$$

$$= -120.93 \text{ MPa}$$

The normal stresses on these planes of maximum shearing stress are found from the equation

$$\sigma = \frac{1}{2}(\sigma_x + \sigma_y)$$

$$= \frac{1}{2}(-90 + 120) = 15 \text{ MPa}$$

The Orientation of the element for which shearing stresses are maximum is shown in figure 2.16 (c)

MOHR'S CIRCLE METHOD

The graphical approach to the two dimensional stress problem was first presented by Otto Mohr in the year 1882. In this representation a circle is used, accordingly the construction is called Mohr's Circle.

The rules for the construction of Mohr's circle are summarised as follows.

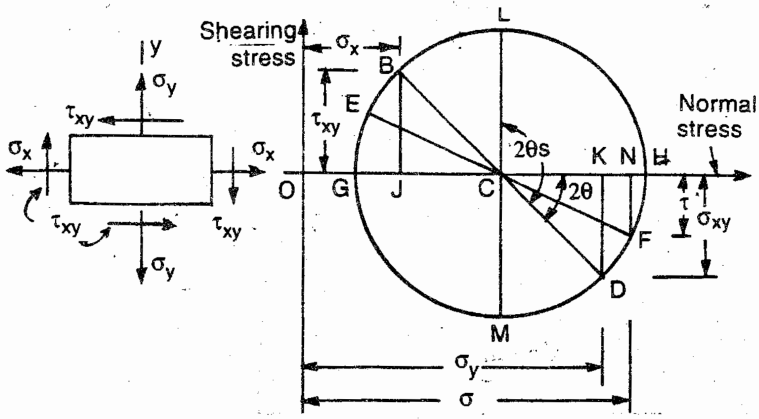


Fig. 2.17

(i) Normal stresses σ_x and σ_y are plotted along x– axis to a suitable scale.

(ii) Shearing stresses τ_{xy} are plotted to the same scale along the vertical axis

(iii) Tensile stresses are plotted to the right of the origin O. Compressive stressed are considered negative and plotted on the left side of the origin O.

(iv) Shearing stresses which rotate the element in clockwise direction are considered positive and negative if they rotate the element in an anti clock wise direction

(v) Point B and D are thus located and joined. mark the mid point of the diameter $B D$ as C.

(vi) Now with C as centre and $BC = CD$ as radius draw a circle. This is the Mohr's circle.

(vii) Now measure an angle 2θ from the diameter BD in a counter clock wise direction and mark points E and F on the circle.

(viii) The cordinates of point F represent the normal and shearing stresses on the plane inclined at an angle θ to the x – axis. In the above diagram ON represents σ and NF represents the shearing stresses τ.

(ix) CL represents the maximum shearing stresses.

Example . 2.7

A bar of uniform cross-sectional area 750 mm² is subjected to axial tensile forces of 50 KN applied at each end of the bar. Determine the normal and shearing stresses on a plane inclined at 30° to the direction of loading with the help Mohr's circle.

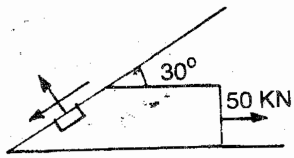


Fig. 2.18 (a)

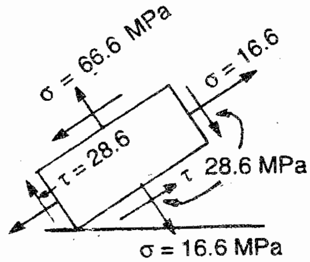


Fig. 2.18 (b)

$$\sigma x = \frac{50 \times 10^3}{750} = 60 \text{ Mpa}$$

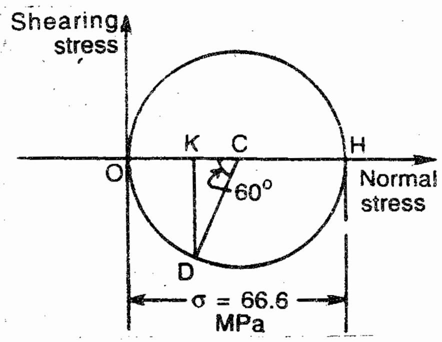


Fig. 2.18 (c)

Represent Normal stress on the horizontal axis, lay off OH= 66.6 MPa and mark its mid point C. Now draw a circle with center C and radius $OC = CH$. This is the Mohr's circle. Measure angle $2\theta = 60°$ counter clock wise from OC. The coordinate of point D are

$$DK = \tau = \frac{1}{2} \times 66.6 \times \text{S in } 60° = 28.86 \text{ MPa}$$

$$OK = \sigma = OC{-}KC = \frac{1}{2}66.6 - \frac{1}{2}66.6 \text{ Cos } 60° = 16.6$$

Since the value of shearing stresses in negative it indicates that shearing stresses on this plane of 30° tends to rotate the element in a counter clock wise direction

Example 2.8

A plane element is subjected to the stresses shown in fig. 2.19 (a) using Mohr's circle determine

(a) The principle stresses and their directions

(b) The maximum shearing stresses and the direction of the planes on which they occur.

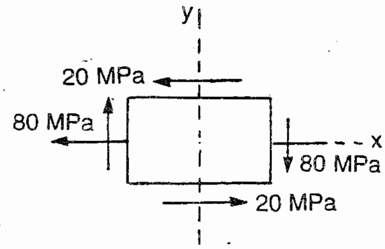


Fig. 2.19 (a)

Solution

Given σ_x = 80MPa (Positive) Tensile

τ_{xy} = 20MPa (Positive) on vertical faces

τ_{xy} = – 20 MPa (Negative) on horizontal faces

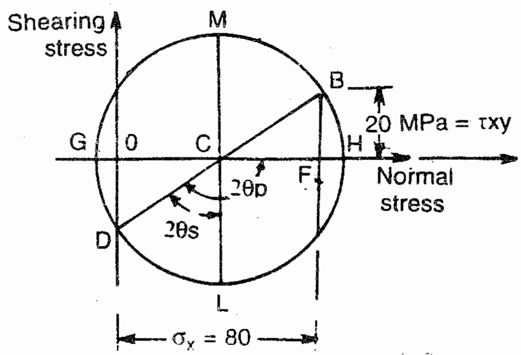


Fig. 2.19 (b)

(1) Principal stresses are represented on the horizontal axis.

(2) Shearing stresses are represented on the vertical axis to.

(3) Locate point *B* by laying out $OF = \sigma_x$ = 80 MPa and

$FB = \tau_{xy}$ = 20 MPa to a suitable scale.

Locate point *D* by laying OD = - 20 MPa on the negative side of the vertical axis to the same scale

(4) Now draw the line BD and Locate its centre C

(5) Draw a circle with C as its Centre and $CB = CD$ as its radius. Now this is the Mohr's Circle. The end point of BD represent the stress Conditions existing in the element in its original orientation.

Principal Stresses

Point G and H represent the principal stresses. Principal stresses can now be measured from the Mohr's Circle

$\sigma_{max} = OH = 84.72$ Mpa

$\sigma_{min} = OG = -4.72$ Mpa

The angle $2\theta_P = \dfrac{20}{40} = \dfrac{1}{2}$ or $\theta_P = 76° \, 43'$

The principal stress represented by point H acts on a plane oriented at 76° 43′ from the original x – axis as shown in figure 2.19 (c) The shearing stresses on these planes are zero, since points G and H lie on the horizontal axis of the Mohr's circle

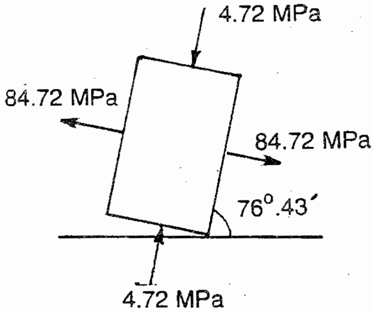


Fig. 2.19 (c)

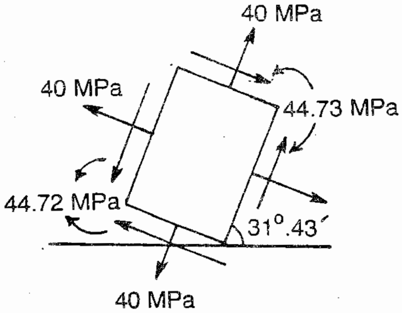


Fig. 2.19 (d)

The maximum shearing stress is represented by CL on the Mohr's circle, $CL = 44.72$ Mpa, The angle $DCL=2\theta_s$ is found to be 63°26′ or θ_s = 31°43′. Hence on this plane the shearing stress tends to rotate the element in a counter clock wise direction as shown in the figure 2.19 (d)

Example 2.9

A plane element is subjected to stresses as shown is fig 2.20 (a) Using Mohr's circle determine

(a) The principal stresses and their directions

(b) The maximum and minimum shearing stresses and the directions of the planes on which they occur.

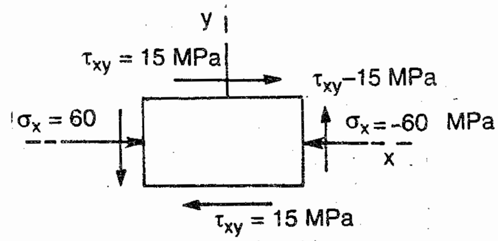


Fig. 2.20 (a)

Solution

Given $\sigma_x = -60$ Mpa (Compressive)

$\tau_{xy} = -15$ Mpa (Negative) on vertical faces

$\tau_{xy} = 15$ Mpa (Positive) on horizontal faces

Represent principal stresses on the horizontal axis to a suitable scale

Represent shearing stresses on the vertical axis to the same scale

(i) Plot $\sigma_x = OF = -60$ Mpa on the horizontal axis and $\tau_{xy} = FB$ – 15 Mpa on the positive side of the vertical axis and obtain Point B.

(ii) Plot Point D $\tau_{xy} = 15$ Mpa on the positive side of the vertical axis.

(iii) Join BD and mark its mid point c.

(iv) Draw a circle with C as centre and $CB = CD$ as radius. This is the Mohr's circle. The end points of the diameter BD represent the stress conditions existing in the element if it has the original orientation as shown in fig. 2.20 (b)

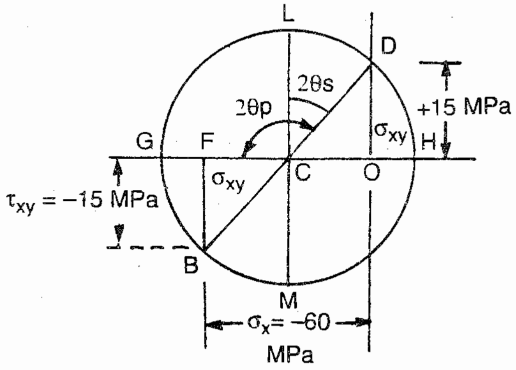


Fig. 2.20 (b)

$$CD = \sqrt{(30)^2 + (15)^2} = 33.54 \text{ Mpa}$$

$$\sigma_{max} = OH + CH - CO = 33.54 - 30 = 3.45 \text{ Mpa}$$

$$\sigma_{min} = OG = OC + CG = -30 - 33.54 = -63.54 \text{ Mpa}$$

$$\tan 2\theta_P = \frac{-15}{30} = -\frac{1}{2} = \text{or } 2\theta_P = 153° 26' \text{ and } \theta_P = 76°43'$$

The Principal stresses on a plane oriental at 76°43′ from the original X–axis are shown in fig 2.20 (c) since G and H lie on X– axis, hence the shearing stresses on these planes are zero.

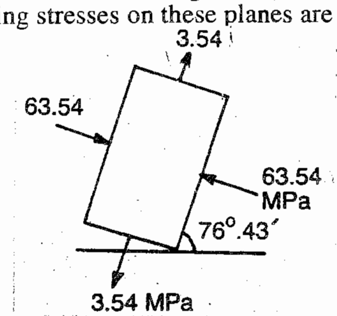


Fig. 2.20 (c)

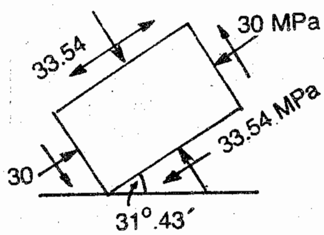


Fig. 2.20 (d)

(b) The maximum shearing stress in represented by CL = 33.54 Mpa. The angle $2\theta s = (2\ \theta_p - 90) = (153°2'6 - 90)$

$= 63°\text{-}26'$ or $\theta_s = 31°43'$ the shearing stress represented by point L is positive. Hence on the 31°43′ *plane the shearing stress tends to rotate the element in a clock wise direction. This is represented in fig 2.26 (d),*

Example 2.10

A plane element is subject ed to the stresses shown in figure Determine

(a) The principal stresses and their directions

(b) The maximum shearing stresses and the directions of the planes on which the occur

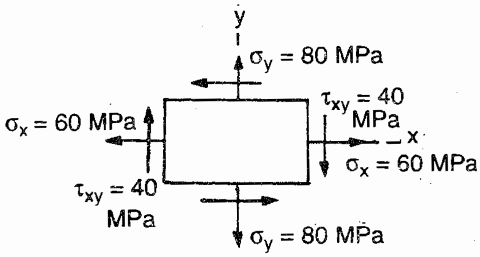


Fig. 2.21 (a)

Given

σ_x = 60 Mpa (Positive) Tensile

σ_y = 80 Mpa (Positive) Tensile

τ_{xy} = – 40 Mpa (Positive) on the vertical faces

τ_{xy} = – 40 Mpa (Negative) on the horizontal faces

Solution

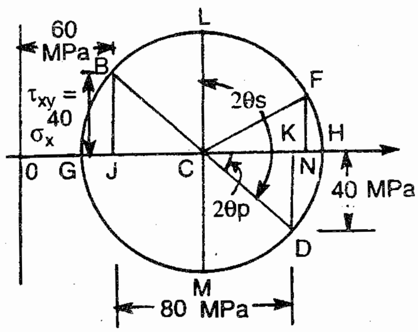


Fig. 2.21 (b)

Draw $OJ = \sigma_x = 60$ Mpa and Ok =+ σ_y = 80 Mpa on the horizontal axis. Draw J B = 40 Mpa, KD, = –40 Mpa on the Vertical axis Point B represents the stress conditions of σ_x = 60 Mpa and τ_{xy} = 40 Mpa existing on the vertical faces of the element.

Point D represents the stress conditions of σ_y = 80 Mpa and τ_{xy} = – 40 Mpa existing on the horizontal faces of the element. Join BD and find its mid point C. Now draw a circle with centre C and radius CB = CD. This is the Mohr's circle.

(a) Point G and H represent the principal stresses.

$$CD = \sqrt{(CK)^2 + (KD)^2} \text{ or } CD = \sqrt{(10)^2 + (40)^2} = 41.23 \text{ Mpa}$$

Maximum principal stress

$$\sigma_{max} = OH = OC + CH = 70 + 41.2 = 111.23 \text{ Mpa}$$
$$\sigma_{min} = OG = OC - CG = 70 - 41.23 = 28.77 \text{ Mpa}$$
$$\text{Tan } 2\theta_p = \frac{40}{10} = 4 \text{ or } 2\theta_p = 76° \text{ or } \theta_p = 38°$$

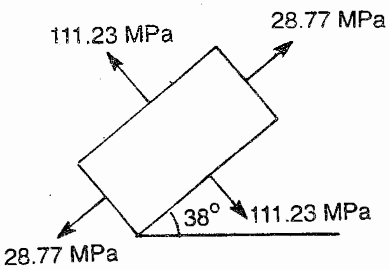


Fig. 2.21 (c)

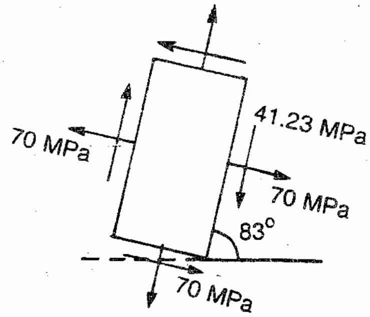


Fig. 2.21 (d)

The Principal stress represented by point H acts on a plane inclined at 38° from the X– axis as shown in fig 2.21 (c). Shearing stress on these planes are zero because G and H lie on the horizontal axis of the Mohr's circle.

(b) The maximum shearing stress in represented by CL = 41.23 Mpa

$$2\theta_s = 166° \text{ or } \theta_s = 83°$$

The shearing stress represented by point L is positive, hence on this plane the shearing stress tends to rotate the element in clockwise direction. Also from Mohr's circle the abscissa of point L is 70 Mpa and this represents the normal stress occuring on the planes of maximum shearing stress.

SUMMARY

1. The planes mutually perpendicular to each other on which shear stress or tangential stress is zero are called principal planes. The corresponding values of normal stresses are called principal stresses.
2. The planes which will have only shear stress but no normal stress are said to be in pure shear.
3. The planes at 45° and 135° carry normal stresses tensile and compressive in nature and each of the same magnitude but do not carry any shear or tangential stress.
4. Normal and shearing stresses on a plane inclined at an angle θ to the direction of loading.

$$\sigma = \frac{1}{2}\sigma_x (1 - \text{Cos}\theta) \qquad \tau = \frac{1}{2}\sigma_x \text{Sin}2\theta$$

5. Normal stress on an inclined plane subjected to direct stresses σ_x and σ_y in two mutually perpendicular planes accompanied by a shearing stress τ_{xy}

Normal stress $\sigma = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \text{Cos}2\theta + \tau_{xy} \text{ Sin}2\theta$

Shearing stress $\tau = \frac{\sigma_x + \sigma_y}{2} \text{ Sin}2\theta + \tau_{xy} \text{ Cos}2\theta$

Principal stresses

$$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

$$\sigma_{mim} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

$$\tan 2\theta_p = \frac{-\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)}$$

Stresses on an inclined plane of a body subjected to normal stress in one plane accompanied by a simple shear stress.

$$\sigma = \frac{1}{2}\sigma_x (1 - \text{Cos}2\theta) + \tau_{xy} \text{ Sin}2\theta$$

$$\tau = \frac{1}{2} \sigma_x \text{ Sin}2\theta + \tau_{xy} \text{ Cos}2\theta$$

$$\tan 2\theta_p = -\frac{2\tau_{xy}}{\sigma_y}$$

$$\sigma_{max} = \frac{1}{2}\sigma_x + \sqrt{\left(\frac{1}{2}\sigma_x\right)^2 + \left(\tau_{xy}\right)^2}$$

$$\sigma_{min} = \frac{1}{2}\sigma_x - \sqrt{\left(\frac{1}{2}\sigma_x\right)^2 + (\tau_{xy})^2}$$

$$\tau_{\substack{max \\ min}} = \pm \sqrt{\left(\frac{1}{2}\sigma_x\right)^2 + \left(\tau_{xy}\right)^2}$$

EXERCISES

(1) A bar of cross-sectional area 600 mm^2 is acted upon by axial tensile forces of 50 KN applied at each end of the bar. Determine the normal and shearing stresses on a plane inclined at 30° to the direction of loading.

(σ = 20.83 MPa, τ = 35.8 MPa)

(2) A bar of cross-sectional area 800 mm^2 is subjected to axial compressive load of 80 KN at each end of the bar. Determine the normal and shearing stresses on a plane inclined at 35° to the direction of loading

(σ = 32.9 MPa , τ = 49. MPa)

(3) Solve the above problems using Mohr's circle method.

(4) A plane element is subjected to stresses shown in the figure . Determine

(a) the principal stresses and their directions

(b) The maximum shearing stresses and the directions of the planes on which they occur.

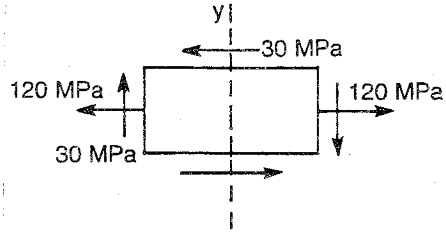


Fig. 2.22

(5) Solve the above problem Graphically.

(6) A plane element in a body is subjected to the stresses $\sigma_x = 40$ MPa, $\sigma_y = 0$ and $\tau_{xy} = 60$ MPa. Determine the normal and shearing stresses on a plane inclined at 45° to the horizontal axis. ($\sigma = 80$ MPa, $\tau = 20$ MPa)

(7) A plane element is subjected to the stresses $\sigma_x = 100$ MPa and $\sigma_y = 100$ MPa. Determine the maximum shearing stress existing in the element. (**Ans.** Zero)

(8) Draw Mohr's circle for a plane element subjected to stresses $\sigma_x = 100$ MPa and $\sigma_y = 100$ MPa. Determine the stresses acting on a plane inclined at 45° to the x – axis. (σ = zero and $\tau = 100$ MPa)

(9) A plane element is subjected to the stresses shown in fig. 2.23 Determine (a) the principal stresses and their directions. (b) The maximum shearing stresses and their directions.

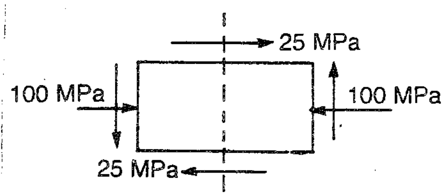


Fig. 2.23

(10) Solve the above problem using Mohr's circle.

(11) A plane element is subjected to the stresses shown in figure 2.24. Determine analytically or Graphically (a) the principal stresses and their directions (b) the maximum shearing stresses and the direction of the planes on which they occur.

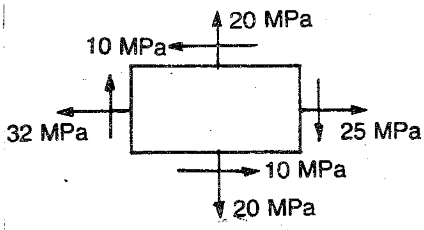


Fig. 2.24

$\sigma_{max} = 37.66$ MPa, $\sigma_{min} = 14.34$, $\theta_p = 29° - 30$

$\tau_{\substack{max \\ min}} = \pm 11.66$ Mpa, $\theta_s = 130° - 48$

□ □ □

3

Strain Energy

When external forces are applied on an elastic body the body gets deformed. The work done on the body by the applied forces is stored within the body in the form of strain energy.

When the external forces are removed the stored energy is released and the body returns to its original dimensions. The internal strain energy stored within the body is equal to the amount of work done on it by the applied force. Strain energy is always a positive scalar quantity.

Resilience

When a body is stressed within elastic limit the amount of internal energy stored is called resilience or strain energy

Proof resilience

When a body is stressed upto the elastic limit the maximum amount of strain energy stored is called Proof resilience.

Modulus of resilience

Proof resilience per unit volume is called modulus of resilience.

Modes of Loading

Strain energy stored in a body depends upon the mode of loading. Loads can be applied in three different ways

(i) Gradual loading

(ii) Sudden loading

(ii) Impact loading

(i) Gradual Loading

A gradually applied load means starti ng from zero, the applied load gradually increases to the maximum value.

(ii) Sudden loading

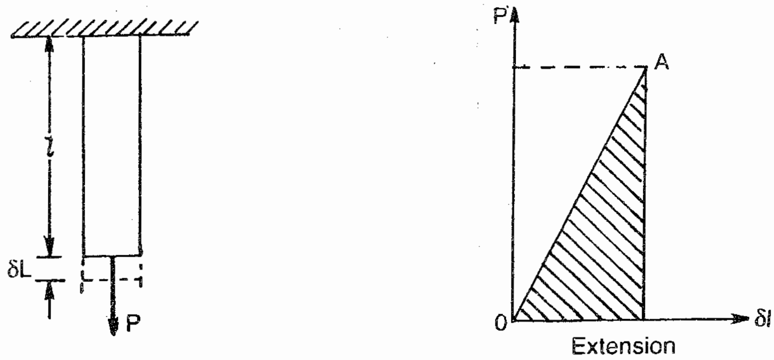
A suddenly applied load means that the total load is applied at once on the body

(ii) Impact loading

When the load falls from a height causing strain, the loading is called impact loading. The maximum stress induced in the body by the three different modes of application of the load will be different.

Strain energy due to gradual loading

Let an axial load P be gradually applied to a bar of length l and cross-sectional area A. Let δl be the extension of the bar.



(a) **Fig. 3.1** (b)

Energy stored in the bar

$$= \text{Work done by the gradually applied load } P$$
$$= \text{Average load} \times \text{extension of bar}$$
$$= \frac{1}{2} . P.\delta l$$
$$= \frac{1}{2} \ \sigma.A . \frac{\delta l . l}{l}$$
$$= \frac{1}{2} \sigma. A . l . \frac{\sigma}{E} = \frac{1}{2} \frac{\sigma^2}{E} \times \text{Volume of bar}$$
$$U = \frac{\sigma^2}{2E} . \times \text{Volume of bar}$$

If the value of stress at the elastic limit is σ_e then Proof resilience $U_p = \frac{\sigma_e^2}{2E} \times$ Volume of bar and Modulus of resilience $= \frac{\sigma_e^2}{2E}$, while $\frac{\sigma_e^2}{E}$ is called coefficient of resilience, which may be looked upon as the property of the material.

Example 3.1

A steel rod 1 metre long and 12 mm diameter is subjected to a gradually applied load till elastic limit is reached. If the safe stress for steel is 150 MPa ad modulus of elasticity is 200 KN/mm^2, determine

(a) Proof resilience

(b) Modulus of resilience

(c) The coefficient of resilience

Solution

(a) Proof resilience

$$U_p = \frac{\sigma_e^2}{2E} \times \text{Volume}$$

$$= \frac{(150)^2}{2 \times 200 \times 10^3} \times \frac{\pi}{4} (12)^2 \times 1 \times 1000$$

$= 6361.725$ N–mm.

(b) Modulus of resilience

= Resilience Per unit Volume

$$= \frac{\sigma^2}{2E} = \frac{(150)^2}{2 \times 200 \times 10^3}$$

$= 0.562$ MPa

(c) Coefficient of resilience

$$= \frac{\sigma^2}{E}$$

$$= \frac{(150)^2}{200 \times 10^3} = .112 \text{ Mpa}$$

Answer.

Example. 3.2

Calculate the strain energy stored in a bar 3 metre long and 40 mm in diameter when subjected to a tensile load of 80 KN. What will then be the modulus of resilience of the material of the bar ? Take E = 210 KN/mm^2.

Solution

Area of Cross-section of the bar

$$A = \frac{\pi}{4} (40)^2 = 400\,\pi \text{ mm}^2$$

Volume of the bar $= A\,.\,l$

$= 400\,\pi \times 3 \times 10^3 = 12\,\pi \times 10^5$ mm^3

Load applied on the bar $= 80 \times 10^3$ Newton.

Strain energy stored in the bar

$$U = \frac{\sigma^2}{2E} \times \text{Volume}$$

$$\text{and } \sigma^2 = \left(\frac{P}{A}\right)^2 = \left(\frac{80 \times 10^3}{400\pi}\right)^2 = 4052.84 \text{ N/mm}^2$$

$$\therefore \quad U = \frac{4052.84}{2 \times 210 \times 10^3} \times 12\pi \times 10^5 \text{ N-mm}$$

$$= 363.77 \times 10^2 \text{ N–mm}$$

$$\text{Modulus of resilience} = \frac{\text{Strain energy}}{\text{Volume}}$$

$$= \frac{363.77 \times 10^2}{12\pi \times 10^5}$$

$$= 9.649 \times 10^{-3} = .00964 \text{ N-mm/mm}^3$$

Example. 3.3

A bar of uniform section hangs vertically as shown in fig. 3.2 . Determine the strain energy stored with in the bar.

Solution

The bar wll be subjected to its self weight only and strain energy due to this weight will be stored in the bar.

Let A = Area of cross-section of the bar.

l = Length of the bar.

γ = Weight density of the material of the bar.

Now consider an element of length d_x at a distance x from A.

Force acting on this element is the wt of the portion below it. $P = A \,.\, x \,.\, \gamma$

Strain energy stored in the shaded element

$$dU = \frac{(A\,x\,\gamma)^2}{2AE}\,.dx$$

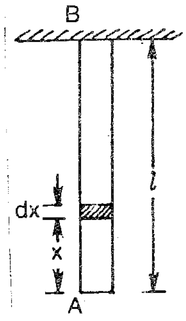


Fig. 3.2

Strain energy stored in the whole bar

$$U = \int_0^l \frac{(A\,.x\,.\gamma)^2}{2\,A\,E}\,dx = \frac{A\,\gamma^2 l^3}{6E}$$

$$U = \frac{A\,\gamma^2 l^3}{6E} \qquad \textbf{Answer}$$

Example. 3.4

Two bars A and B each 2 metre long are shown in fig 3.3. The maximum tensile stress in each bar is 150 MPa. Compare the strain energies of the two bars assuming that they are made of theame material. Take E = 200 KN/mm^2.

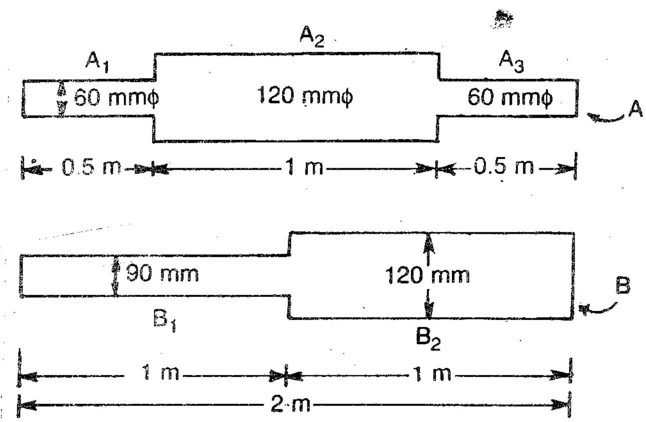


Fig. 3.3

Solution

For bar A strain energy stored

$U_A = U_{A1} + U_{A2} + U_{A3}$

Since U_{A1} and U_{A3} will be equal

$$\therefore \quad U_A = 2U_{A1} + U_{A2}$$

$$= 2\left[\frac{\sigma^2}{2E} \times V\right] + \left[\frac{\sigma^2}{2E} \times \text{Volume of midde portion}\right]$$

$$U_A = 2\left[\frac{(150)^2}{2 \times 200 \times 10^3} \times \frac{\pi}{4}(60)^2 \times 0.5 \times 1000\right] + \left[\frac{(150)^2}{2 \times 200 \times 10^3} \times \frac{\pi}{4}(120)^2 \times 1 \times 1000\right]$$

$$= 159043.12 + 636.72.51 = 795215.63 - \text{N -- mm}$$

$$U_B = U_{B1} + U_{B2}$$

$$= \left[\frac{\sigma^2}{2E} \times \text{Volume of 1st portion}\right] + \left[\frac{\sigma^2}{2E} \times \text{Volume of 2nd portion}\right]$$

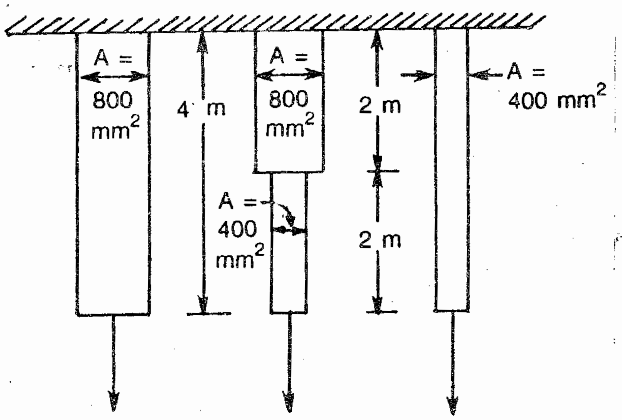
$$= \left[\frac{(150)^2}{2 \times 200 \times 10^3} \times \frac{\pi}{4}(90)^2 \times 1.0 \times 1000\right] + \left[\frac{(150)^2}{2 \times 200 \times 10^3} \times \frac{\pi}{4}(120)^2 \times 1.0 \times 1000\right]$$

$$= 357847.03 + 63617.51 = 994019.63 \text{ N-mm}$$

$$\therefore \quad \frac{U_A}{U_B} = \frac{795215.63}{994019.63} = 0.799 \quad \textbf{Answer.}$$

Example 3.5

Compare the strain energy stored in each of the three steel bars shown in figure 3.4 subject to the condition that the axial stress in the lower portion of the second bar is equal to that in the first and the third bars namely 100 MPa.



Fiig. 3.4

Solution

Strain energy in each bar

$$U = \frac{\sigma^2}{2E} \times \text{Volume of the bar}$$

U_1 = Strain energy stored in the first bar

$$= \frac{\sigma^2}{2E} \cdot A \cdot l$$

$$= \left[\frac{(100)^2}{2E}(800)(4 \times 10^3)\right] = \frac{32}{2E} \times 10^9$$

$$U_2 = \left[\frac{(50)^2}{2E}(800)(2 \times 10^3) + \frac{(100)^2}{2E}(400)(2 \times 10^3)\right]$$

$$= \left[\frac{4 \times 10^9}{2E} + \frac{8 \times 10^9}{2E}\right] = \frac{12}{2E} \times 10^9$$

$$U_3 = \frac{(100)^2}{2E}(400)(4 \times 10^3) = \frac{16}{2E} \times 10^9$$

Ratio of strain energies in the three bars

$U_1 : U_2 : U_3 = 32 : 12 : 16$

$= 8 : 3 : 4$ **Answer**

Sudden Loading

Let a force ***P*** be suddenly applied on a bar of length l and cross-sectional area A. Let δ_l be the change in the length of the bar. Let σ be the instantaneous stress in the bar when the load P has just been applied then equating the strain energy in the bar to the work done by the applied load we have

U = Work done by the load

or $\left(\frac{\sigma}{2} \cdot A\right) \cdot \delta l = P \cdot \delta l$

or $\sigma = \frac{2P}{A}$

Instantaneous stress developed in a bar subjected to sddenly applied load is thus twice the stress produced by the same load applied gradually.

Instantaeous elongation

$$\delta l = \frac{\sigma}{E} \cdot l = \frac{2P}{AE} \cdot l$$

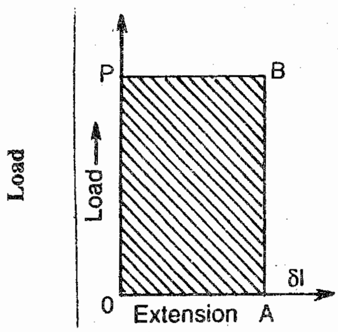


Fig. 3.5

Thus, Instantaneous elongation of a bar subjected to suddenly applied load is twice the extension produced by the same load applied gradually.

Example. 3.6

A compressive Load of 40 KN is placed all of a sudden on a bar of length 4 metres and diameter 40 mm. Determie the amount by which the length of the bar shortens and the amount of work done ? Take E = 200 KN/mm²

Solution.

Since is load is suddenly applied the stress produced in the bar will be instantaneous.

$$\therefore \text{ Instantaneous stress } = \frac{2P}{A} = \frac{2 \times 40 \times 1000}{\frac{\pi}{4}(40)^2} = 63.66 \text{ MPa}$$

$$\text{Strain in the bar } = \frac{63.66}{200 \times 10^3} = 0.318 \times 10^{-3}$$

$$\text{Shortening in length } \delta l = \varepsilon \times l$$
$$= 0.318 \times 10^{-3} \times 4 \times 1000$$
$$= 1.273 \text{ mm}$$

$$\text{Work done on the bar } = P \times \delta l$$
$$= 40 \times 1000 \times 1.273 \text{ N–mm}$$
$$= 50.92 \times 10^3 \text{ N–mm}$$ **Answer**

Example 3.7

An axial load of 50 KN is suddenly applied on a bar of T-section 100 mm × 100 mm. If the Length of the bar is 5 metres, calculate

(a) The maximum instantaneous stress produced

(b) The elongation in the length of the bar

(c) The work stored in the bar at the instant of maximum elongation. Take E = 210 GN/m²

Solution

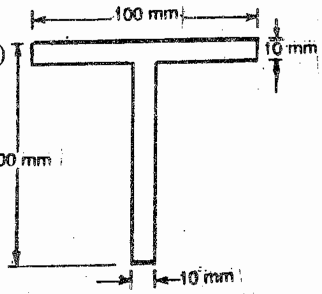


Fig. 3.6

Area of resisting section

$$= (100 \times 10) + (90 \times 10)$$
$$= 1900 \text{ mm}^2$$

(a) Instantaneous stress

$$\sigma = \frac{2P}{A} = \frac{2 \times 50 \times 10^3}{1900}$$
$$= 52.63 \text{ MPa}$$

$$\text{Strain } \varepsilon = \frac{52.63}{210 \times 10^3} = 0.25 \times 10^{-3}$$

(b) Elongation of the bar.

$$\delta l = \varepsilon \times l = 0.25 \times 10^{-3} \times 5 \times 10^3$$
$$= 1.25 \text{ mm}$$

(c) Work stored in the bar = Energy stored

$$U = \frac{\sigma^2}{E} \times \text{ Volume of the bar}$$

$$= \frac{(52.63)^2}{210 \times 10^3} \times 1900 \times 5 \times 1000$$

$$= 1253.05 \times 100 \text{ N–mm}$$

$$= 0.125 \text{ KN–m}$$

Example 3.8

A bar of copper one metre long and 80 mm diameter is subjected to a shock of 0.50 KN-m. Determine the instantaneous stress and the change in the length of the bar. Take E = 110 KN/mm^2

Solution

Shock energy

$$U = \frac{\sigma^2}{2E} \; . Al$$

$$0.5 \times 10^6 = \frac{\sigma^2}{2 \times 110 \times 10^3} \times \frac{\pi}{4}(80)^2 \times 1 \times 10^3$$

or
$$\sigma^2 = \frac{0.5 \times 10^6 \times 2 \times 100 \times 10^3 \times 4}{\pi \times (80)^2 \times 1 \times 10^3}$$

$$\sigma = 141 \text{ MPa}$$

Change in length of the bar

$$\delta l = \frac{\sigma}{E} \times l$$

$$= \frac{141}{110 \times 10^3} \times 1 \times 1000 = 1.28 \text{ mm}$$ **Answer**

Example 3.9

The material of a bar of length1.5 meter and cross-sectional area 600 mm^2 *has an elastic limit of 150 MPa. What is its proof resilience ? Determine the maximum suddenly applied load which may be applied without exeeding the elastic limit. What gradually applied load will produce the same extension as that produced by the sudden load ? Take E = 210 GN/m*^{2}.

Solution

$$\text{Proof resilience } = \frac{\sigma_e^2}{2E} \times \text{ Volume}$$

$$= \frac{(150)^2}{2 \times 210 \times 10^3} \times 600 \times 1.5 \times 1000$$

$$= 48214.28 \text{ N–mm}$$

$$= .0482 \text{ KN–m}$$

Let P be the maximum suddenly applied load within the elastic limit

$$\therefore \frac{2P}{A} = 150$$

$$\text{Suddenly applied load } P = \frac{150 \times 600}{2} = 45000 \text{ N} = 45 \text{ KN}$$

The effect of gradually applied load will be one half of the effect of suddenly applied load, therefore the extension produced by a suddenly applied load of 45 K.N., will be produced by a gradually applied load of $2 \times 45 = 90$ KN.

$\therefore$ Gradually applied load = 90 KN

Strain Energy Due To Impact

Figure 3.7 represents a rod of length l and cross-sectional area A. The upper end of the rod is fixed and a collar is provided at the lower end. Let a load P fall from a height h on to the collar and δl be the extension of the rod. let σ be the stress induced.

$$\text{Energy stored } U = \frac{\sigma^2}{2E} \times \text{Volume of the rod} \quad \text{--- (i)}$$

$$\text{Work done} = P \times \text{distance moved}$$
$$= P(h + \delta l) \quad \text{--- (ii)}$$

Equating (i) and (ii)

$$\frac{\sigma^2}{2E} \times V = P(h + \delta l)$$

or $$\sigma^2 = \frac{P(h + \delta l) \times 2E}{V}$$

When δl is very small as compared to h it may be neglected

$$\therefore \quad \sigma^2 = \frac{2PhE}{\text{Volume of the rod}}$$

$$\sigma^2 = \frac{2PhE}{A.l}$$

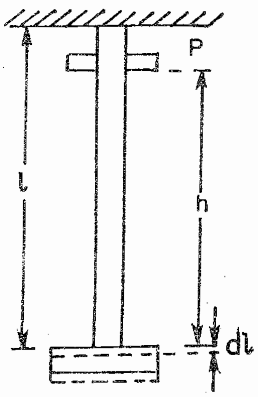


Fig. 3.7

Example 3.10

A mild steel bar 2 metre long and 25 mm diameter hangs freely and has a collar firmly fixed with the lower end. Determine the instantaneous elongation of the bar, if a load of 250 Newtons falls on the collar from a height of 100 mm. Take E = 200 KN/mm^2.

Solution

Since elongation will be very small as compassed to the height of fall, δl can be neglected.

$$\text{Strain energy} = \frac{\sigma^2}{2E} \times \text{Volume of the bar}$$

$$\text{Work done} = P(h + \delta l)$$

Neglecting δl and equating work done to strain energy

$$P \times h = \frac{\sigma^2}{2E} \times V$$

$$\sigma^2 = \frac{P \times h \times 2E}{V} = \frac{250 \times 100 \times 2 \times 200 \times 10^3}{981.74 \times 10^3}$$

$\sigma^2 = 1.018 \times 10^4$ or $\sigma = 100.9$ MPa

$$\text{Instantaneous strain} = \frac{100.9}{E} = \frac{100.9}{200 \times 10^3} = .504 \times 10^{-3}$$

$$\begin{aligned}\text{Elongation of the bar} &= 0.504 \times 10^{-3} \times \text{Length of the bar}\\ &= 0.504 \times 10^{-3} \times 2 \times 10^3\\ &= 1.009 \text{ mm}\end{aligned}$$

Answer.

Example 3.11

Determine the maximum load P that can be dropped 250 mm on to the flange at the end of a steel bar. The bar is 25 mm × 50 mm in cross-section and 2 metre long. The axial stress is not to exceed 150 MPa. Take E = 200 GN/m^2.

Solution

Area of cross-section = 25 × 50 = 1250 mm^2

Volume of the bar = $1250 \times 2 \times 10^3 = 250 \times 10^4$ mm^3.

Stress induced due to falling load

$$\sigma^2 = \frac{2 \times P \times hE}{\text{Volume of the bar}}$$

or $$(150)^2 = \frac{2 \times P \times 250 \times 200 \times 10^9}{250 \times 10^4 \times 10^6} = 40P$$

or $$P = \frac{150 \times 150}{40} = 562.5 \text{ Newtons}$$ **Answer**

Example 3.12

A rod of 12.5 mm diameter streches 3.2 mm under a steady load of 10 KN. What stress would be produced in the rod by a weight of 700 N which falls through 75 mm before commencing to stretch the rod, the rod being initially un stressed. Take E = 200 KN/mm^2.

Solution

$$\text{Cross-sectional are of the rod} = \frac{\pi}{4}(12.5)^2 = 122.65 \text{ mm}^2.$$

$$\text{Stress in the rod} = \frac{10 \times 10^3}{122.65} = 81.53 \text{ MPa}$$

$$\text{Hence strain in the rod} = \frac{81.53}{200 \times 10^3}$$

$$\text{Length of the rod} = \frac{\delta l}{\text{strain}} = \frac{3.2 \times 200 \times 10^3}{81.53} = 7.85 \text{ metres}$$

Stress induced by a weight of 700 N

$$\sigma^2 = \frac{2PhE}{\text{Volume of the rod}}$$

$$= \frac{2 \times 700 \times 75 \times 200 \times 10^3}{122.65 \times 7.85 \times 10^3} = 21785.76$$

$$\sigma = 147.6 \text{ MPa} \qquad \textbf{Answer}$$

Example 3.13

A crane chain whose sectional area is 900 mm² carries a load of 15 KN which is being lowered at a uniform rate of 60 metres/minute. When the length of the chain unwound is 10 metres, the chain jams suddenly on the pulley; Estimate the stress induced in the chain due to sudden stoppage. Neglect the weight of chain. Take E = 200 KN/mm².

Solution

Volume of the chain $= 900 \times 10 \times 10^3 = 9 \times 10^6 \text{ mm}^3$.

Velocity of the load $= \dfrac{60 \times 1000}{60} = 1000$ mm/sec.

Kinetic energy $= \dfrac{1}{2} m v^2 = \dfrac{1}{2} \dfrac{P}{g} \times v^2$

$$= \frac{1}{2} \times \frac{15 \times 10^3 \times (1000)^2}{9.81 \times 10^3} = .7652 \times 10^6 \text{ N–mm}$$

Strain energy $= \dfrac{\sigma^2}{2E} \times$ Volume of the chain

$$= \frac{\sigma^2 \times 9 \times 10^6}{2 \times 200 \times 10^3} = 22.5\,\sigma^2$$

Since there is no loss of energy therefore kinetic energy is converted into strain energy.

$$22.5\,\sigma^2 = .7652 \times 10^6$$

$$\sigma^2 = \frac{.7652 \times 10^6}{22.5} = 33900$$

$$\sigma = 184.33 \text{ MPa} \qquad \textbf{Answer.}$$

Example 3.14

A brass rod 30 mm diameter is enclosed in a steel tube of 30 mm internal and 50 mm external diameter. The composite bar is vertically suspended and held rigidly at the upper end of a collar provided at the lower end. A weight of 80 KN falls freely on the collar from a height of 150 mm. If the length of the bar is 3 meters, find the maximum stress produced in each material. Take Es = 200 KN/mm² and Eb = 80 KN/mm².

Solution

Area of brass rod $= \dfrac{\pi}{4}(30)^2 = 706.8$ sq. mm

Area of steel tube $= \dfrac{\pi}{4}(50^2 - 30^2) = 1256.6$ sq. mm.

Let δl be the extension produced by the falling load

Strain in steel tube = Strain in the bar.

$$\varepsilon s = \varepsilon b$$

$$\frac{\sigma_s}{E_s} = \frac{\sigma_b}{E_b} \quad \text{or} \quad \sigma_s = \sigma_b \cdot \frac{E_s}{E_b}$$

or $$\sigma_s = \sigma_b \times \frac{200 \times 10^3}{80 \times 10^3} = 2.5\,\sigma_b$$

Work done by the falling load

= Strain energy in the tube + strain energy in the bar,

$$P\,(h + \delta l) = \frac{\sigma_s^2}{2Es} \cdot As.\,l + \frac{\sigma_b^2}{2E_b} \times A_b.l$$

$$80 \times 10^3 \left(150 + \frac{\sigma b \times 3 \times 10^3}{80 \times 10^3}\right) = \frac{(2.5\,\sigma b)^2}{2Es} \cdot A_s..\,l + \frac{\sigma_b^2}{2E_b} \times A_b \times l$$

Simplyfying we get

$72.12\,\sigma_b^2 - 3 \times 10^3\,\sigma_b - 12000 \times 10^3 = 0$

Solving the quadratic equation

$\sigma_b = 429$ MPa and $\sigma_s = 2.5\ \sigma_b = 1072.5$ MPa **Answer**

Strain Energy due to Shear

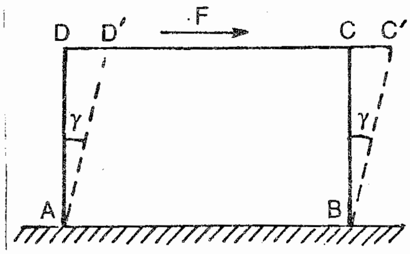


Fig. 3.8

Consider a rectangular block *AB C* D, subjected to a shear force *F* and fixed at the base *A B*. Let the thickness of the block perpendicular to the plane of the paper be unity. Under the action of the shearing force *F* the edge *DC* takes up the position *D′C′*. Let the force *F* be applied gradually increasing from zero to the value *F*, then the work done by the force in displacing the point *D* to *D′*. will be $\frac{F}{2} \times DD'$

Now $DD' = AD\ \tan\gamma$

If γ is small then $DD' = AD\,.\gamma$

External work done $= \frac{F}{2} \times DD' = \frac{F}{2}\ AD.\gamma$

Shear strain $\gamma = \frac{\tau}{G}$

$$\therefore \text{ External work done} = \frac{F}{2} \times AD \cdot \frac{\tau}{G}$$

$$\text{Shear Force } F = DC \times 1 \times \tau$$

$$\therefore \text{ External work done} = \frac{1}{2}(DC \times l \times \tau \times AD \cdot \frac{\tau}{G})$$

$$= \frac{\tau^2}{2G} \times DC \times AD \times 1$$

$$\text{Therefore strain energy} = \frac{\tau^2}{2G} \cdot \text{Volume} = \frac{\tau^2}{2G} \cdot V$$

$$\text{Strain energy per unit Volume} = \frac{\tau^2}{2G}$$

Pulling $\tau = G.\gamma$, we get strain energy per unit volume $= \frac{1}{2}\gamma^2$

Example 3.15

Calculate the total strain energy at a point in a material subjected to a shearing stress of 20×10^3 MPa. Take modules of rigidity for the material as $80 KN/mm^2$.

Solution.

$$\text{Strain energy per unit volume} = \frac{1}{2}\frac{\tau^2}{G}$$

$$= \frac{1}{2}\frac{(20 \times 10^3)^2}{80 \times 10^3}$$

$$U = 2.5 \text{ KN–mm.}$$ **Answer**

Example. 3. 16

Calculate the total 3 strain energy stored in a rectangular block 600 mm × 120 mm × 50 mm. When subjected to a shear stress of 100 MPa. Take $G = 84 \text{ kN/mm}^2$.

Solution.

Volume of rectangular block $= 600 \times 120 \times 50 = 36 \times 10^5 \text{ mm}^3$

$$\text{Strain energy stored } U = \frac{1}{2}\frac{\tau_2}{G} \times \text{Volume}$$

$$U = \frac{1}{2} \times \frac{(100)^2}{84 \times 10^3} \times 36 \times 10^5 = 214 KN\text{–}mm$$ **Answer**

SUMMARY

1. Strain energy $U = \frac{\sigma^2}{2E} \times$ Volume Where σ is the instantaneous stress and E the modulus of elasticity
2. Proof resilience $U_p = \frac{\sigma_e^2}{2E} \times$ Volume
3. Modulus of resilience $= \frac{\sigma_e^2}{2E}$

4. Instantaneous stress when the load is suddenly applied

$$\sigma = \frac{2P}{A}$$

5. For impact loads

$$\sigma^2 = \frac{2PhE}{A.l}$$

6. Strain energy due to shear

$$U = \frac{\tau^2}{2G}.\text{ Volume}$$

7. Modulus of resilience $= \dfrac{\tau^2}{2G}$

QUESTIONS

(1) What is strain energy ? Explain. From the first principle, derive an expression for the energy stored in a bar subjected to a gradually applied load.

(2) Explain the following
 (a) Resilience
 (b) Proof resilience
 (c) Modulus of resilience

(3) Show that in a bar subjected to an axial load the instantaneous stress due to suddenly applied load is twice the stress caused by the gradual application of the same load.

(4) Obtain an expression for the stress induced in a body if a load is applied with an impact.

(5) Calculate the strain energy in a bar 2.5 m long and 50 mm diameter when it is subjected to a tensile load of 100 KN. What will be the modulus of resilience of the material of the bar ? Take $E = 200$ KN/mm^2

(31831N–mm, .0065 N–mm/mm^3)

(6) A Copper bar 80 mm diameter and 1.5 meter long has to bear a shock of 640 KN/mm. Determine the instantaneous stress and the change in length of the bar. $E = 200 \times$ KN/mm)

(1600 MPa, 12 mm)

(7) A mild steel rod 4 meter long and 25 mm diameter is subjected to a pull of 45 KN. Find the elongation of the rod, when the load is applied (a) gradually (b) suddenly Take $E = 200$ KN/mm^2 . (1.75 mm, 3.5 mm)

(8) A crane chain whose sectional area is 625 mm^2, carries a load of 10 KN; which is being lowered at a uniform rate of 40 m/minute when the length of the chain unwound is 10 meters, the chain jams suddenly on the pulley. Estimate the stress induced in the chain due to sudden stoppage. Neglect the weight of the chain. Take $E = 210$ KN/mm^2 (123.3 MPa)

(9) A hammer weighing 100 N falls 2 on a 100mm cube mild steel block before coming to rest. Find the instantaneous stress and the compression of the block. Also determine the velocity with which the hammer will strike the block. Take $E = 200\ KN/mm^2$

(20.0 MPa, .001 mm, 6.26 m/sec)

(10) A Vertical tie, fixed rigidly at the top end consists of a steel rod 2.5 metres long and 20 mm dia. encased throughout in a brass tube 20 mm internal dia. and 30 mm. external diameter. The rod and the casing are fixed together at both ends. The compound rod is suddenly loaded in tension by a weight of 10 KN. falling freely through 3 mm. before being arrested by the tie. Calculate the maximum stresses in steel and brass. Take E_s = 200 KN/mm^2 and E_b = 100 KN/mm^2. (118.5 MPa, 59.25 MPa)

(11) An unknown weight falls through 10 mm on a collar rigidly attached to the lower end of a vertical bar, 3 metre long and 600 mm^2 in section. If the maximum instantaneous extension is known to be 2 mm, what is the corresponding stress and the value of unknown weight? Take E = 200 KN/mm^2. (133.3 MPa, 6.66 KN)

□□□

4

Thin Walled Pressure Vessels

Thin cylindrical and spherical shells have very small thickness of wall plates as compared to their cross-sectional dimensions. The wall thickness is generally less than $\frac{1}{20}^{th}$ of the internal diameter.

Water pipes, steam boilers, air vessels storing fluids have to with stand internal fluid pressure. A uniform fluid pressure acts on the internal surface and the direction of the pressure at any point is normal to the surface of contact. since the walls of these pressure vessels are very thin the stresses induced across them is assumed to be uniformly distributed. Two principal tensile stresses acting on the walls of these vessels are

(i) Circumferential or Hoop stress

(ii) Longitudinal stress.

Circumferential or Hoop Stress σ_h

Hoop stresses are induced at right angles to the Longitudinal axis of the cylinder. These stresses along the circumference of the cylinder may brea the cylinder into two traughs. The stresses acting tangentially to the circumference are known as hoop stresses and represented by σ_h

Longitudinal Stress σ_L

Stresses that are set up parallel to the length of the cylinder are called Longitudinal stresses. These stresses may break the cylinder into two cylindrical parts.

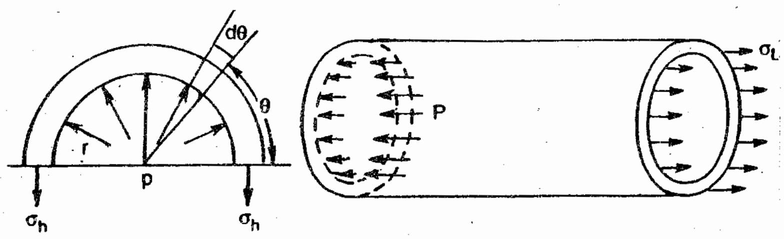


Fig. 4.1 (a) **Fig. 4.1 (b)**

Determination of stresses

The following assumptions are made while determining the hoop and longitudinal stresses.

(a) The rdial stresses in the cylinder walls are negligible.

(b) There are no longitudinal stays in the cylinder.

(c) The sresses are uniformly distributed through the wall of the pressure Vessels.

Circumferential or Hoop Stress σ_h

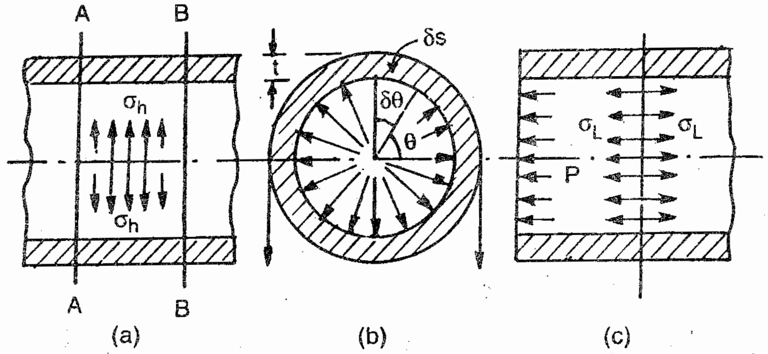


Fig. 4.2

Consider a thin cylinder of internal radius r. Let *p be the intensity of internal fluid pressure. Consider the equilibrium of an elementary length l* between the sections AA and BB. Let a very small strip of this shell subtend an angle δ_θ at the centre and let it be inclined at an angle θ to the horizontal axis $x - x$

width of the strip $\delta_s = r.\ \delta_\theta$

Area of the strip $= l.\ r.\ \delta_\theta$

Radial force acting on the strip. $= p \times l \times r \times \delta\theta$

Vertical componant of this radial force.

$$= p \times l\, r \times \delta\ \theta\ \sin\theta \qquad \text{---} \qquad \text{---} \qquad \text{(i)}$$

Total vertical force perpendicular to the diameter

$$= \int_o^\pi p.l.\ r.\sin\theta\ \ \delta\theta = 2p.l.r$$

It σ_h is the intensity of Hoop stress, then the resisting force.

$$= 2\sigma_h \times l \times t \qquad \text{---} \qquad \text{---} \qquad \text{(ii)}$$

Hence for equilibrium of the material, equating (i) & (ii)

$$2\ \sigma_h \times l \times t = 2p.l.r$$

$$\sigma_h = \frac{p.r}{t}$$

Longitudinal Stress (σ_L)

Consider the thin cylinder closed at both ends by cover plates and subjected to uniform internal pressure p. Let r be the internal radius and t be the thickness of the walls of the cylinder.

Total force on the ends acting axially due to the internal fluid pressure.

$= \text{Area} \times$ intensity of fluid pressure.

$= \pi r^2 \times p$ --- --- --- (i)

Resisting force $= 2\pi r t \times \sigma_L$ --- --- --- (ii)

For equilibrium of the material equating (i) & (ii)

$$2\pi . r . t . \sigma_L = \pi r^2 p$$

$$\sigma_L = \frac{p . r}{2t}$$

Hence the Hoop stress σ_h is half the longitudinal stress σ_L

Maximum shear stress

Hoop stress σ_h and longiludinal stress σ_L act on two mutually perpendicular planes. Hence at any point on the circumference of a cylindrical shell subjected to internal fluid pressure, these are the principal stresses. The maximum shear stress is therefore given by the relation.

$$\tau_{max} = \frac{\sigma_h - \sigma_L}{2}$$

Example 4.1

Find the Longitudinal and the circumferential stress induced in the walls of a cylindrical Boiler 1.5 meter diameter if subjected to an internal fluid pressure of 2.5 MPa. The walls of the cylinder are 30 mm thick.

Solution

Diameter of the cylindrical shell = 1.5 meters

Radius of the shell $= \frac{1.5 \times 10^3}{2} = 750\ mm$

Wall thickness = 30 mm

Internal pressure = 2.5 MPa

$$\sigma_h = \frac{2.5 \times 750}{30} = 62.5 \text{ MPa}$$

Longiludinal stress $\sigma_L = \frac{p . r}{2t} = \frac{2.5 \times 750}{2 \times 30} = 31.25$ MPa.

Answer

Example 4.2

A Compressed air cylinder is subjected to an internal pressure of 15 MPa. The outside diameter of the cylinder is 250 mm. If steel has a yield point of 250 MPa and a factor of safety of 2.5 is used, Calculate the required thickness of the walls.

Solution

Working stress $= \frac{250}{25} = 100$ MPa

Hoop Stress $\sigma_h = \frac{p . r}{t}$

or $t = \frac{p . r}{\sigma_h} = \frac{15 \times 125}{100} = 18.75$ mm

Required wall thickness = 18.75 mm **Answer**

Example 4.3

The tank of an air compressor consists of a cylinder closed by hemispherical ends. The cylinder is 500 mm internal diameter and the internal pressure acting on the internal surface is 3 MPa. If the yeild point of the material is 250 MPa; and a factor of safety of 2.5 is used. Calculate the wall thickness of the cylinder ?

Solution

Radius of the cylinder = 250 mm

Intensity of pressure p = 3 MPa.

$$\text{Hoop Stress } \sigma_h = \frac{250}{2.5} = 100 \text{ MPa}$$

$$\sigma_h = \frac{p.r}{t}$$

or $$t = \frac{p.r}{\sigma_h} = \frac{3 \times 250}{100} = 7.5 \text{ mm.}$$

Example 4.4

A vertical cylinderical gasoline storage tank is 30 m in diameter and is filled to a depth of 15 m with gasoline whose relative density is 0.74. If the yield point to the shell plate is 250 MPa and factor of safety is 2.5. Calculate the required wall thickness at the bottom of the tank neglecting anylocalised bending effect ?

Solution -

$$\text{Pressure intensity} = (15 \times 0.74) \times \frac{10^4}{10^6} \text{ N/mm}^2$$

$$= \frac{15 \times 0.74}{100} \text{ N/mm}^2$$

Internal radius $= 15 \times 1000$ mm = 15000 mm

$$\text{Working Stress} = \frac{250}{2.5} = 100 \text{ MPa}$$

$$\text{Hoop Stress } \sigma_h = \frac{p.r}{t}$$

$$\text{Required wall thickness } t = \frac{p.r}{\sigma_h} = \frac{15 \times 0.74 \times 15000}{100 \times 100} = 16.6\, mm$$

Example 4.5

A Vertical stand pipe stands 25 meters high and has a diameters of 4 meters. Determine the wall thickness if the pipe is filled with water. The yield point of the material is 250 MPa and a factor of safety of 2 is used. Water weighs 10 KN/m^3.

Solution -

Head of water = 25 meters = 25×10^3 mm

Radius of pipe = 2 meters = 2×10^3 mm

$$\text{Allowable working stress } \sigma_h = \frac{250}{2} = 125 \text{ MPa}$$

$$\text{Weight of water per mm}^3 = \frac{10\times10^3}{10^9} = 10^{-5}\ \text{N/mm}$$

Water pressure $p = wh$

$$p = 10^{-5}\times 25\times 10^3 = 250\times 10^{-3}\ \text{N/mm}^2$$

Now hoop stress $\sigma_h = \frac{p.r}{t}$

$$\text{or} \quad t = \frac{p.r}{\sigma_h} = \frac{250\times 10^{-3}\times 2\times 10^3}{125}$$

Wall thickness = 4 mm **Answer**

Change in Volume of thin cylindrical shells

Let l and r be the length and radius of the cylinder.

p = intensity of internal pressure

μ = Poisson's ratio

E = Modulus of elasticity of the material then

$$\text{Circumferential stress} \quad \sigma_h = \frac{p.r}{t}$$

$$\text{Longitudinal Stress} \quad \sigma_L = \frac{p.r}{2t}$$

$$\text{and} \quad \text{Circumferential Strain} = \frac{1}{E}(\sigma_h - \mu\sigma_L)$$

$$\varepsilon_h = \frac{1}{E}\left(\frac{p.r}{t} - \mu.\frac{p.r}{2t}\right)$$

$$= \frac{p\,r}{tE}\left(1 - \frac{1}{2}\mu\right)$$

$$\text{Longitudional Strain} = \frac{1}{E}(\sigma_L - \mu\,\sigma_h)$$

$$\varepsilon_L = \frac{1}{E}\left(\frac{p\,r}{2t} - \mu.\frac{p\,r}{t}\right)$$

$$= \frac{p\,r}{t\,E}\left(\frac{1}{2} - \mu\right)$$

Volume of the cylinder $V = \pi\, r^2 l$

$$\therefore \quad \delta_V = 2\pi\ r\,l\,\mathrm{d}\,r + \pi\, r^2 dl$$

$$\varepsilon_V = \frac{\delta_V}{v} = \frac{2\pi\, r\, l\, dr}{\pi.r^2.l} + \frac{\pi\, r^2 dl}{\pi\, r^2 l}$$

$$\text{Volumetric Strain} = \frac{2\,dr}{r} + \frac{dl}{l}$$

$= 2\times$ Hoop strain + Longitudinal strain

$$= 2\varepsilon_h + \varepsilon_l$$

$$= 2\frac{p\,r}{t\,E}\left(1 - \frac{\mu}{2}\right) + \frac{p\,r}{t\,E}\left(\frac{1}{2} - \mu\right)$$

$$= \frac{pr}{tE}\left(\frac{5}{2} - 2\mu\right)$$

Volumetric Strain $\varepsilon_V = \frac{\delta V}{V} = \frac{p.r}{tE}\left(\frac{5}{2} - 2\mu\right)$

Change in the Volume of the Cylinder

$$\delta v = V \times \varepsilon_V$$

$$= \frac{p.r}{tE}\left(\frac{5}{2} - 2\mu\right).\pi r^2 \times l$$

$$\delta v = \frac{p.r}{tE}\left(\frac{5}{2} - 2\mu\right) \times \text{ Volume of Cylinder}$$

Example 4.6

Calculate the change in diameter and length of an air vessel 400 mm in diameter and 12 mm thick when subjected to an internal pressure of 16 MPa. Take the modulus of elasticity of the material as 200 KN/mm^2 and μ = 0.3. The Length of the vessel is 1.5 meters.

Solution

Diameter of the Vessel = 400 mm

Radius = 200 mm

Internal pressure = 16 MPa

Thickness of plate = 12 mm

Circumferential stress $\sigma_h = \frac{p.r}{t}$

$$\sigma_h = \frac{16 \times 200}{2 \times 12} = 266.16 \text{ MPa}$$

Longitudinal stress $\sigma_L = \frac{p.r}{2t}$

$$\sigma_L = \frac{16 \times 200}{2 \times 12} = 133.3 \text{ MPa}$$

Circumferential Strain $\varepsilon_h = \frac{\sigma_h}{E} - \frac{\mu\,\sigma_L}{E}$

$$\varepsilon_h = \frac{1\,(266.6 - 0.3 \times 133.3)}{200 \times 10^3} = \frac{226.6}{200 \times 10^3}$$

$$= 1.133 \times 10^{-3}$$

$\therefore$ Change in diameter $\delta d = 1.333 \times 10^{-3} \times 400$

$$\delta d = 0.533 \text{mm}$$

Longitudinal Strain $= \frac{\sigma_L}{E} - \frac{\mu\,\sigma_h}{E}$

$$\varepsilon_L = (133.3 - 0.3 \times 266.6) \times \frac{1}{200 \times 10^3} = \frac{53.32}{200 \times 103}$$

$$= 0.2\,666 \quad 10^{-3}$$

Change in Length $\delta_l = 0.2666 \times 10^{-3} \times 1.5 \times 10^3$

$\delta_l = 0.3999 = .4$ mm **Answer**

Example 4.7

Calculate the increase in volume of a boiler 8 meter Long and 1 meter diameter when subjected to an internal pressure of 1.5 MPa. The wall thickness is such that the maximum tensile stress in the shell is 30 MPa. Take E = 200 KN/mm^2 and μ = 0.3

Solution

Wall thickness when maximum tensile stress is 30 MPa.

$$\sigma_h = \frac{p.r}{t} \; or \; 30 = \frac{1.5 \times (500)}{t}$$

or
$$t = 1.5 \times \frac{500}{30} = 25 \text{ mm}$$

$$\text{Volume of boiler} = \frac{\pi}{4} (d)^2 \times l$$

$$= \frac{\pi}{4} (1000)^2 \times 8 \times 1000$$

$$= 2\pi \times 10^9 \text{ mm}^3$$

Increase in Volume

$$\delta_V = \frac{p.r}{2t.E} (5 - 4\mu) \times \text{Volume of Cylinder}$$

$$= \frac{1.5 \times 500(5 - 4 \times 0.3)}{2 \times 25 \times 200 \times 10^3} \times 2\pi \times 10^9 \text{ mm}^3$$

$$= \frac{750 \times 3.8 \times 2\pi \times 10^9}{10^7}$$

$$= 179.07 \times 10^4 \text{ mm}^3$$

$= .00179$ m^3 **Answer.**

Example 4.8

A cylindrical shell 800 mm internal diameters and 10 mm wall thickness is subjected to an internal pressure of 20 MPa. Calculate the maximum intensity of shear stress induced in the shell. If the length of cylinder is 2 m, calculate changein in the volume of the shell. Take E = 200 KN/mm^2 and poisson's ratio 0.3

Solution

Radius of the shell = 400 mm

Wall thickness = 10 mm

Intensity of internal pressure = 20 MPa.

$$\text{Hoop Stress } \sigma_h = \frac{p.r}{t} = \frac{20 \times 400}{10} = 800 \text{ MPa.}$$

Longitudinal Stress $\sigma_L = \dfrac{p.r}{2t} = \dfrac{20 \times 400}{2 \times 10} = 400$ MPa

Maximum intensity of shear stress

$$= \frac{\sigma_h - \sigma_L}{2} = \frac{800 - 400}{2} = 200 \text{ MPa}$$

Volume of the cylindrical shell

$$= \pi (r)^2 \times l = \pi (400)^2 \times 2 \times 10^3 \text{ mm}^3$$

Changein Volume $\delta_v = \pi V \times \dfrac{p.r}{2tE}(5 - 4\mu)$

$$\delta v = \pi (400)^2 \times 2 \times 10^3 \left[\frac{20 \times 400}{2 \times 10 \times 200 \times 10^3} (5 - 4 \times 0.3) \right]$$

$= 47.728 \times 10^4$ mm^3 **Answer**

Built-up Thin Cylindrical Shells

Large size thin cylindrical and spherical shells can not be made of one single piece of metal hence joints are necessary for making such pressure vessels. This is done by joining different plates usually by means of rivets. Sometimes plates may be welded as well. The plates are bent to required diameters and butt joints are provided. Individual fabricated shells are joined by Lap joints.

Built-up shells are not as strong as seamless shells or shells without joints. These joints reduce the resisting strength of the shell plates both against bursting and tearing. Dependin g upon the efficieney of the jointsthe circumferential stress and Longitudinal stress are calculated from the modified formula as under .

Hoop Stress $\sigma_h = \dfrac{p.r}{t\eta}$

Longitudinal Stress $\sigma_L = \dfrac{p.r}{2t.\eta}$ where η is the efficiency of the joints.

From the above expressions for σ_h and σ_L it is to noted that the effect of providing joints is that the hoop and Longitudinal Stresses are increased.

Example 4.9

An air vessel provided with an air compressor is 12 mm thick 2 metres long and of 1200 mm diameter. It is designed for a maximum working pressure of 3.5 MPa. Determine the maximum and the minimum stresses induced in the material of the vessel when.

(i) It is seamless and

(ii) It is built-up with longitudinal and Circumferential joint efficiencias as 70% and 65% respectively.

Solution

(i) Maximum stress induced is the hoop stress

$$\sigma_h = \frac{pr}{t} = \frac{3.5 \times 600}{12} = 175 \text{ MPa}$$

Minimum stress induced is the longitudinal stress

$$\sigma_L = \frac{p.r}{2t} = \frac{3.5 \times 600}{2 \times 12} = 87.5 \text{ MPa}$$

(ii) For built up shell

$$\sigma_h = \frac{p.r}{t.\eta} = \frac{3.5 \times 600}{12 \times 0.7} = 250 \text{ MPa}$$

$$\sigma_L = \frac{p.r}{2t \times \eta} = \frac{3.5 \times 600}{2 \times 12 \times 0.65} = 134.6 \text{ MPa}$$ **Answer.**

Example 4.10

A cast iron pipe is required to carry water at a pressure of 4 MPa. If the permissible longitudinal stress and hoop stresses are 40 MPa and 60 MPa and efficiencies of longitudinal and circumferential joints are 60% and 70% respectively. Determine the thickness of the metal if the diameter of the pipe is 180 mm.

Solution

Permissible Longitudinal Stress

$$\sigma_L = \frac{p.r}{2t.\eta}$$

∴ Thickness of the metal $t = \dfrac{p.r}{2\sigma_L.\eta}$

$$t = \frac{40 \times 90}{2 \times .60 \times 40} = 7.5 \text{ mm}$$

Maximum permissible hoop stress $\sigma_h = \dfrac{p.r}{t..\eta}$

Thickness of metal required $t = \dfrac{p.r}{\eta \times \sigma_h} = \dfrac{4 \times 90}{.70 \times 60} = 8.57$ mm

Minimum thickness of metal required for the pipe will be the larger of the two values. Hence

$t = 8.57$ mm **Answer.**

Thin Spherical Shells

Thin spherical shells when subjected to internal fluid pressure are likely to burst into two hemispheres along the centre line of the sphere.The tensile stress developed at all points of the shell is same therefore for equilibrium total bursting force must be equal to the resisting strength of the plate.

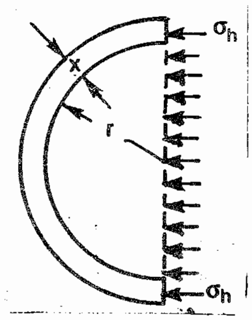


Fig. 4.3

Let p = intensity of internal fluid pressure.

r = radius of the shell

t = thickness of the shell plate

σ_h = stress induced in the shell material

Then

total bursting force = Area × intensiy of pressure

$= \pi r^2 \times p$ --- --- (i)

Resisting Strength of the shell plate

$= 2\pi r \times t \times \sigma_h$ --- --- (ii)

Equating (i) and (ii) we get

Hoop stress in the wall $\sigma_h = \dfrac{p.r}{2t}$

From symmetry this circumferential stress is the same in all directions at any **point** in the wall of the sphere.

If η **is** the efficiency of the joint in the spherical shell, then the hoop stress in the shell will be

$$\sigma_h = \frac{p.r}{2t..\eta}$$

Change in Volume of thin spherical shell

Strain in the diameter of the shell $= \dfrac{1}{E}\,(\sigma_h - \mu\,\sigma_h)$

$$= \frac{\sigma_h}{E}(1-\mu)$$

$$= \frac{p.r}{2t.E}(1-\mu)$$

Since the hoop stress σ_h is same in all directions of X - axis, Y - axis and Z - axis, therefore strains in all the three planes will be same

Volumetric strain $= \varepsilon_x + \varepsilon_y + \varepsilon_z = 3\varepsilon$

$$\varepsilon_v = 3.\frac{p.r}{2t\,E}(1-\mu)$$

$$\varepsilon_v = \frac{\delta V}{V} = \frac{3}{2}\,\frac{p.r}{t\,E}\,(1-\mu)$$

or $\quad \delta V = \dfrac{3}{2}\dfrac{p\,r}{t\,E}\,(1-\mu) \times$ Volume of spherical shell

or $\quad \delta V = \dfrac{3}{2}\,\dfrac{p.r}{t\,E}(1-\mu) \times \dfrac{4}{3}\,\pi\,r^3$

$$= \frac{2\,\pi\,p\,r^4}{t\,E}(1-\mu)$$

Example 4.11

A spherical tank of steel is used to store gas under pressure. The diameter of the shell is 25 meters and wall thickness 15 mm. If the yeild point of the metal is 250 MPa and a factor of safety of 2.5 is adequate, determine the maximum internal pressure. If the joint efficiency is 75% determine the permissible pressure.

Solution

Diameter of the shell = 25 m

Radius = 12.5 m = 12.5×10^3 mm

Wall thickness = 15 mm

Working stress $= \dfrac{2,50}{2.5} = 100$ MPa

Hoop Stress $\sigma_h = \dfrac{p.r}{2t}$

$\therefore$ Maximum internal pressure $p = \dfrac{\sigma_h \times 2t}{r}$

$$p = \frac{100 \times 2 \times 15}{12.5 \times 10^3} = 0.24 \text{ MPa}$$

Joint efficiency = 75%

$\therefore$ Permissible internal pressure

$$p = 0.24 \times \frac{75}{100} = 0.18 \text{ MPa}$$ **Answer.**

Example 4.12

Calculate the increase in volume of a spherical shell one meter diameter and 10 mm thick, when it is subjected to an internal pressure of 1.2 MPa. Take E 200 KN/mm^2 and $\mu = 0.3$

Solution

Volume of the shell $= \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi (50)^3$

$= \dfrac{4}{3}\pi \times 125 \times 10^3 = 523.59 \times 10^3$ Cu. mm

Increase in Volume

$$\delta_V = \frac{3pr}{2tE}(1 - \mu) \times \text{ Volume of Shell}$$

$$\delta_V = \frac{3 \times 1.2 \times 50}{2 \times 10 \times 200\ 10^3}(1 - 0.3) \times 523.59 \times 10^3$$

$= 16.5 \text{ mm}^3$ **Answer.**

Example 4.13

For a thin cylindrical shell and a thin spherical shell subjected to same internal pressure and having the same diameter/thickness ratio compare (a) the maximum tensile stresses and (b) the proportional increase in volume. Take $\mu = 0.3$

Solution

Hoop stress for the cylinder $\sigma_{h_1} = \dfrac{pr}{t}$

Hoop stress for the sphere $\sigma_{h_2} = \dfrac{pr}{2t}$

$$\therefore \frac{\sigma_{h_1}}{\sigma_{h_2}} = 2$$

Increase in the Volume of the cylinder

$$\delta_{V_1} = \frac{pr}{2tE}(5 - 4\mu)$$

$$\delta_{V_1} = \frac{pr}{2tE}(5 - 4\mu) = \frac{pr}{2tE}(5 - 4 \times 0.3)$$

$$= \frac{1.9pr}{tE}$$

$$\delta V_2 = \frac{3}{2}\frac{p.r}{tE}(1-\mu)$$
$$= \frac{3}{2}\frac{p.r}{tE}(1-0.3)$$
$$= \frac{1.05}{tE}pr$$

$$\therefore \frac{\delta V_1}{\delta V_2} = \frac{1.9pr}{tE} \times \frac{tE}{1.05\,pr} = \frac{1.9}{1.05} = 1.809 \qquad \textbf{Answer.}$$

SUMMARY

1. For cylindrical shells

   Hoop stress $\sigma_h = \frac{p.r}{t}$

   Longitudinal Stress $= \sigma_L = \frac{p.r}{2}t$

   Hoop Stres is also called circumferential stress

2. Hoop stress is twice the Longitudinal stress in case of thin cylindrical shell $\sigma_h = 2\,h_L$

3. Maximum shear stress $\tau_{max} = \frac{\sigma_h - \sigma_L}{2} = \frac{p.r}{2t}$

4. When a thin cylindrical shell is to withstand an internal fluid pressure p and tensile stress in the material of the shell does not exceed σ_h then the thickness of the plates is given by

   $$t > \frac{p.r}{\sigma_h}$$

5. Hoop Strain $\varepsilon_h = \frac{pr}{tE}\left(1 - \frac{1}{2}\mu\right)$

6. Longitudinal Strain $\varepsilon_L = \frac{pr}{tE}\left(1 - \frac{1}{2} - \mu\right)$

7. Volumetric strain $\varepsilon_v = 2\varepsilon_h + \varepsilon_L$

   $$= \frac{p.r}{tE}\left(\frac{5}{2} - 2\mu\right)$$

8. For built-up cylindrical shells

   $\sigma_h = \frac{p.r}{t.\eta}$ when η is the efficiency of the joint.

   $\sigma_L = \frac{p.r}{2t..\eta}$

9. For spherical stress

   $\sigma_h = \frac{p.r}{2t}$ and $\sigma_L = \frac{p.r}{2t}$

10. If η is the efficiency of the joint.

then $\sigma_h = \dfrac{p.r}{2t.\eta}$

11. Volume of thin spherical shell

$$V = \frac{\pi d^3}{6}$$

Circumferential stress $= \dfrac{pr}{2t}$

Strain in the diameter of the shell

$$= \frac{p.r}{2tE}(1-\mu)$$

Volumetric strain $\varepsilon_v = \dfrac{3pr}{2tE}(1-\mu)$

QUESTIONS

(1) Explain with sketches the following
  (a) Longitudinal Stress
  (b) Circumferential Stress

(2) A thin cylindrical shell is subjected to an internal fluid pressure p, show that the tendency to burst length wise is twice as great as at transverse section

(3) Derive an expression for the hoop stress in a thin cylindrical shell closed at both ends and subjected to an internal fluid pressure.

(4) How the efficiency of a thin cylindrical shell is affected by providing joints ? Explain.

EXERCISES

(5) A cylinderical boiler 1 meter diameter and 20 mm wall thickness is subjected to an internal fluid pressure of 2 MPa. Determine the longitudinal and circumferential stress induced. (250 MPA, 500 MPa.)

(6) A water main one meter diameter contains water at a pressure head of 100 meters. If the weight of water per cubic metre is 10,000 N. Find the thickness of the metal requried, if the permissible stress in the metal is 20 MPa.

(25 mm)

(7) A 20 m diameter spherical tank is to be used to store gas. The shell plating is 10 mm thick and the working stress of the material is 125 MPa.. What is the maximum permissible gas pressure. (0.25 MPa)

(8) Calculate the increase in volume per unit volume of a thin circular cylinder closed at both ends subjected to a uniform internal pressure of 0.5 MPa. Radius of the cylinder is 350 mm, wall thickness 1.5 mm and $\mu = 0.33$, Take $E = 200$ KN/mm^2

$$\left(\frac{\delta_v}{v} = 10^{-3}\right)$$

(9) The air vessel of a torpedo is 530 mm external diameter and 10 mm thick, the length being 1.83 m. Find the change in external diameter and the length when

charged to 10.5 MPa internal pressure. Take $E = 210$ KN/mm^2 and $\mu = 0.3$ (AMIE) (0.574 mm, 0.466 mm)

(10) A thin spherical shell 1.2 mm diameter is subjected to an internal pressure of 2 MPa. If the maximum permissible stress in the plate material is 160 MPa and the joint efficiency is 60% Find the minimum thickness. (6..25 mm)

(11) A seamless spherical shell of 1 meter diameter and 5 mm thick is filled with fluid pressure untill its volume increases by 200×10^3 mm^3 . Calculate the pressure exerted by the fluid on the shell. $E = 205$ KN/mm^2, u = 0.3 (0.75 MPa)

(12) Show that in the case of a thin cylindrical shell subjected to same internal fluid pressure the tendency to burst length wise is twice as great as at transverse section.

□□□

5

Shearing Force And Bending Moment

When the applied loads are vertical or inclined to the longitudinal axis of a beam they produce the following two effects.

(i) They produce forces which tend to shear one portion of the beam with respect to an other portion.

(ii) Moments are developed in the beam which try to bend the beam the beam should be strong enough to resist both these actions. Hence it is essential to calculate such forces and moments at every point along the longitudinal axis of the beam.

Beam.

Beams are structural members which are designed to support all types of load coming on to a floor supported on them.

Supports. supports may be classified into the following types.

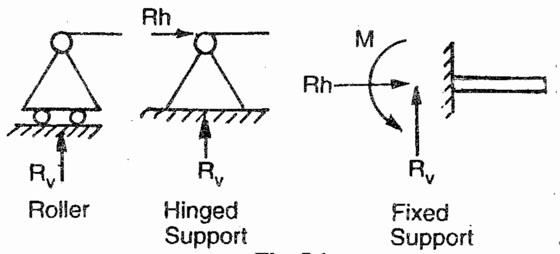
(a) Roller Supports

(b) Hinged Supports

(c) Fixed Supports.

Roller Support

A support in which beam is free to move to the right or left of it. Roller support develops only one support reaction which is perpendicular to the axis of the beam and the roller.



Roll

Support **Fig. 5.1**

2. Hinged Support

The structural member supported on hinged support can not slide side ways i.e. the position is fixed. The structure is allowed to rotate. Reactions developed at the hinge are two. One perpendicular and the other in lateral direction.

3. Fixed Support

Fixed support does not allow either lateral movements or the rotation of the structural member. Two reactions, one horizontal, the other vertical and a moment which prevents rotation, develope at the fixed support.

Classification Of Beams

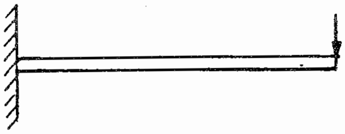
Beams are classified into the following types :

(i) Statically determinate beams (ii) Statically indeterminate beams.

Statically determinate beams

Beams in which the support reactions can be easily determined by the three equations of static equilibrium $\Sigma V = 0$ $\Sigma H = 0$ and $\Sigma M = 0$ are termed as statically determinate beams.

(a) Cantilever



A cantilever is a beam which is fixed at one end and free at the other

Fig. 5.2

(b) Simply supported beam

A simply supported beam rests freely on supports at both the ends.

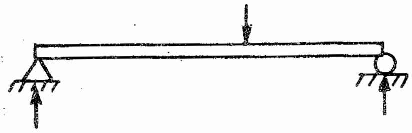


Fig. 5.3

(c) Over hanging beam

If a beam extends beyond its supports it is called an over hanging beam

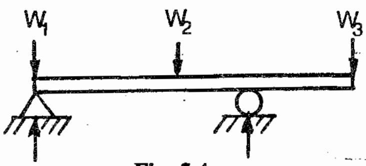


Fig. 5.4

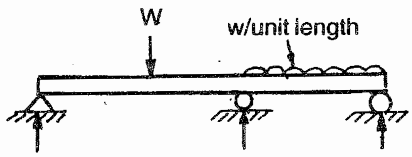
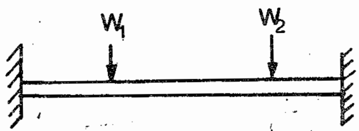


Fig. 5.5

Statically indeterminate beams

Beams in which the support reactions can not be determined by using the three equations of static equilibrium are known as statically indeterminate beams These beams are classified as



A fixed beam has both ends rigidly fixed into supporting walls.

Fig. 5.6

(b) Continuous beam

A beam which rests on more than two supports is called a continuous beam.

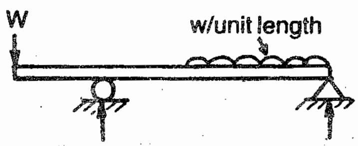
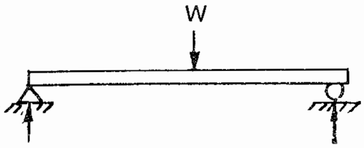


Fig. 5.7

Types of Loading

1. Concentrated Or Point Load



A concentrated load is assumed to be a load concentrated at one point.

Fig. 5.8

2. Uniformly Distributed Load

These loads are uniformly applied over the entire length of the beam.

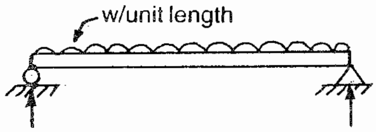


Fig. 5.9

3. Uniformly Varying Load

Triangular or trapezoidal loads fall uner this category. The variation in intensities of such loads is constant.

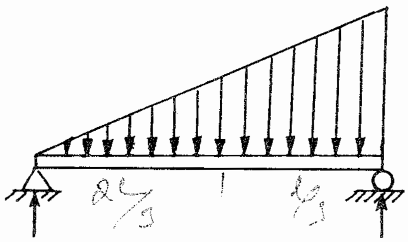


Fig. 5.10

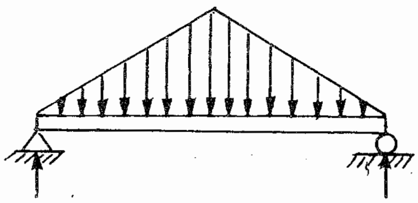


Fig. 5.11

Shear Force

Definition - Shear force at a section of a loaded beam may be defined as the algebraic sum of all vertical forces acting on any one side of the section.

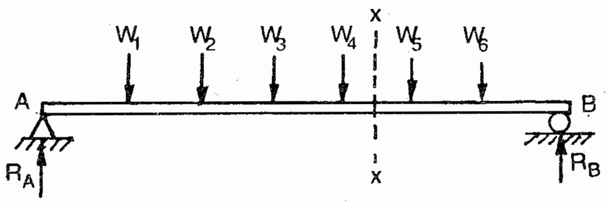


Fig. 5.12

The Shear force at section $x–x$ of the beam shown in figure; 5.12 when forces to the left of $x–x$ are considered.

$$S.F_{x-x} = R_A - W_1 - W_2 - W_3 - W_4$$

When the forces on the right hand side of the section are considered.

$$S.F_{x-x} = R_B - W_5 - W_6$$

Sign Convention

When external forces acting on the portion of the beam to the left of the section tend to push that part up, the shear is positive or when the external forces acting on the portion of the beam to the right of the section tend to push that part down the shear force is positive.

Positive Shear Force

Negative Shear Force

Bending Moment

Bending moment at a section of a loaded beam is the algebraic sum of the moments of all the force on any one side of the section.

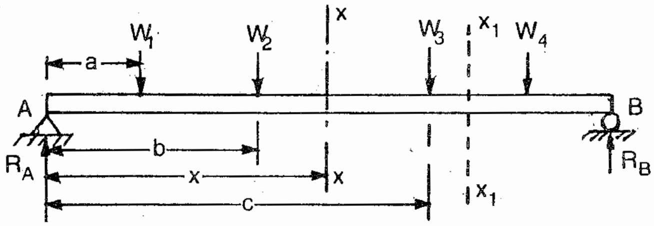


Fig. 5.13

Bending moment at section x–x of the beam shown in the figure can be written as

$$M_{x-x} = R_A.x - W_1(x-a) - W_2(x-b)$$

Simitarly at a section $x_1 - x$ at distance x_1 from A

$$M_{x1-x1} = R_A.x_1 - W_1(x_1 - a) - W_2(x_1 - b) - W_3(x_1 - c)$$

Sign Convention

Moments producing compression in the top fibre and tension in the bottom fibre are positive. These moments try to bend the beam down wards.

Positive Bending

Negative Bending

Moments which bend the beam upwards and Cause Compression in the bottom and tension in the top fibre are taken negative.

Bending Moment And Shear Force Diagrams

This is graphical representation of bending moments acting simultaneously at various sections of the beam under a given system of loading.

Similarly the graphical representation of shearing forces at various section of a beam under a given system of loading is called shear force diagram.

Relation Between Bending Moment And Shear Force

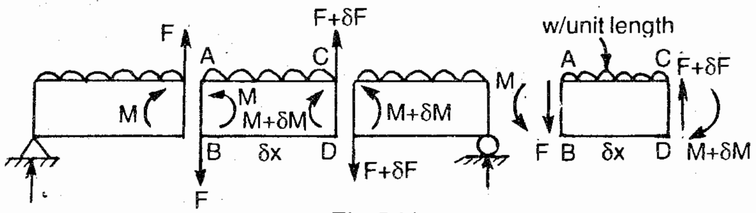


Fig. 5.14

Consider a small length δx of a simply supported beam carrying uniformly distributed load *w*/unit length. Let M and F be the *B.M.* and *S.F* at *AB* and $(M+\delta M)$ and $(F+\delta F)$ be the bending moment and shearing force at *CD*. Since the element *ABCD* is in equilibrium, the sum of all vertical forces on it must be Zero.

Hence $F + w\ \delta x = F + \delta F$

or $\dfrac{dF}{dx} = w$ ... ... (i)

Thus the rate of change of shear force is equal to the intensity of loading on the beam. similarly equating all moments at *AB* to Zero

$$M + (F + \delta F)\,\delta x - \frac{\omega\,(\delta x)^2}{2} - (M + \delta M) = 0$$

Neglecting the products and squares of small quantities, we get

$F\delta x - \delta M = 0$ or $\dfrac{dM}{dx} = F$ ... ... (ii)

That is the rate of change of bending moment is equal to the shearing force.

Now integrating equation (i) we get

$$F = \int_o^x w d_x \quad \ldots \quad \ldots \quad \text{(iii)}$$

Integrating equation (ii) we get

$$M = \int_o^x F\,d_x = \int_o^x \int_o^x w\,d_x \quad \ldots \quad \ldots \quad \text{(iv)}$$

Hence the change of bending moment from o to x is proportional to the area of shear force diagram from o to$x$

For bending moment to be maximum $\dfrac{dM}{dx} = 0$

But $\dfrac{dM}{dx} = F$ from equatiom (ii)

Thus bending moment is maximum where shear force is Zero or changes sign.

Standard Cases

Cantilever With A Concentrated Load at The Free End

A cantilever *A B* of span L with a point load W acting at the free end B is shown in figure 5.15

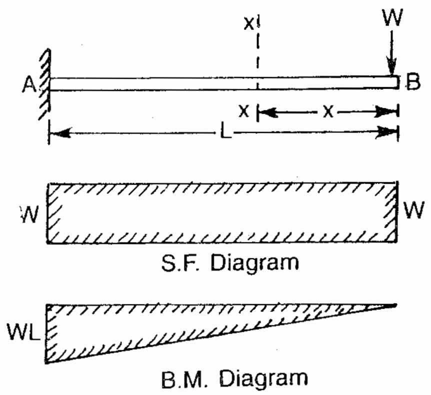


Fig. 5.15

Shear force at $B = W$

Consider a section *x-x* at a distance *x* from B.

At section *x–x* shear force is *W* and it remains constant as the value of *x* increases from Zero at *B* to *L at A*. Therefore *S.F.* is Constant throughout and represented by a rectangle.

$$B.M_{at} - B = 0$$
$$B\ M_{at}\ x\text{–}x = W\,x$$
$$B\ M_{at}\ A = W.L$$

Therefore *B.M.* is Zero at *B* and maximum at the fixed end *A* and represented by a triangle as shown.

Example 5.1

A cantilever AB of span 3 metres is fixed at A and carries a concentrated load of 5 KN at the free and B . Draw the shear force and bending moment diagrams.

Solution

Shear Force

Shear Force at $B =$ 5 KN

$$S.F_{XX} = 5\text{ KN}$$
$$S.F_{A} = 5\text{ KN}$$

Shear Force diagram will be a rectangle as shown in figure 5.16

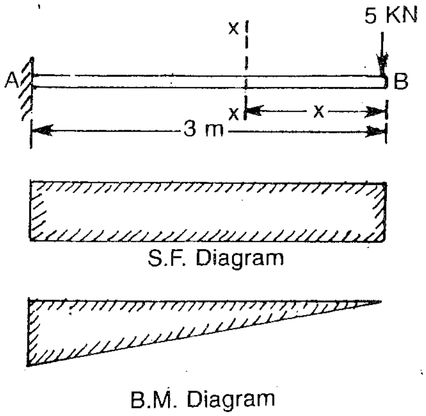


Fig. 5.16

Bending moment

Bending moment at A = 0

$B.M_{xx} = 5.x$

$B.M_A = 5 \times 3 = 15$ KN–m

Example 5.2

Draw the S.F. and B.M. diagram for the cantilever shown in figure 5.17.

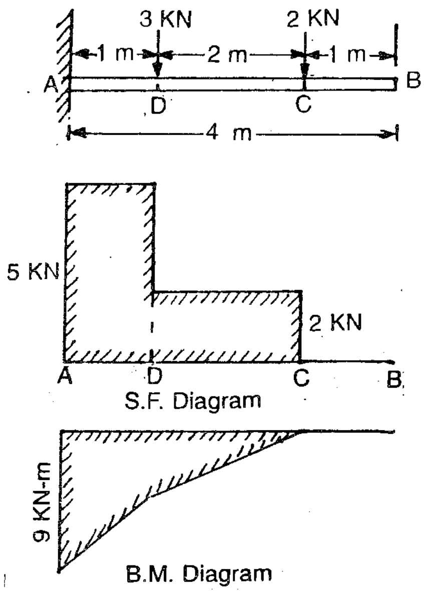


Fig. 5.18

Solution

Since there is no load between B and C, hence S .F. between B and C will be Zero.

Shear Force

$S\ F._{\text{at c}} = 2$ KN

$S\ .F_{\text{at D}} = 2 + 3 = 5$ KN

$S.\ F_{\text{at A}} = 5$ KN

Bending moment

Bending moment from B to C will be Zero

$B.\ M_B =$ Zero

$B.Mc =$ Zero

$B.\ M_D = 2 \times 2 = 4$ KN- m

$B.M$ at $A = 2 \times 3 + 3 \times 1$

$= 9$ KN–m

Example 5.3

A Cantilever A B 4 metres long is fixed at B and carries point loads of 2 KN, 4 KN, 6 KN and 8 KN as shown in figure. 5.18. Draw the S. F. and B. M. diagrams.

Solution

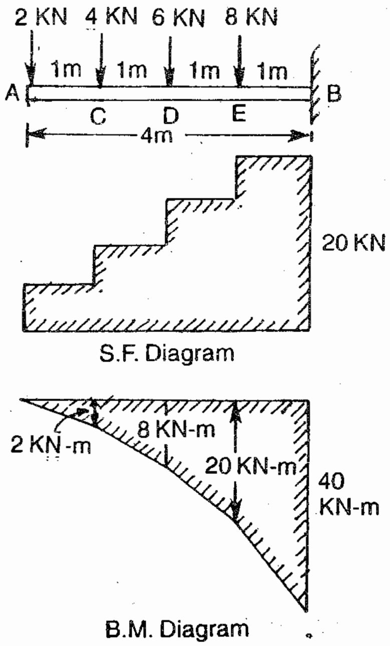


Fig. 5.18

Shear Force

$S.F_A = 2$ KN

$S.F_C = 2 + 4 = 6$ KN

$S.F_D = 2 + 4 + 6 = 12$ KN

$S.F_E = 2 + 4 + 6 + 8 = 20$ KN

$S.F_B = 20$ KN

Bending moment

$B.M_A = 0$

$B.M_c = 2 \times 1 = 2$ KN–m

$B.M_D = 2 \times 2 + 4 \times 1 = 8$ KN–m

$B.M_E = 2 \times 3 + 4 \times 2 + 6 \times 1$

$= 6 + 8 + 6 = 20$ *KN–m*

$B.M_B = 2\times 4 + 4\times 3 + 6 \times 2 + 8\times 1$

$= 8 + 12 + 12 + 8$

$= 40$ KN–m

Cantilever With Uniformly Distributed Load *w* Per Unit Length

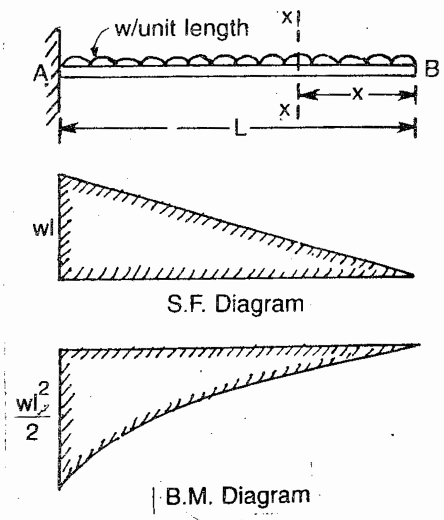


fig. 5.19

Consider a section $x–x$ at a distance x from the free end B.

Shear Force

$S.F_{at-B} =$ Zero

$S.F_{at-x-x} = w.x$

$S.F_{at\ A} = w.L$

Bending moment

For $B.M$ the load over the length x will be $(w.x)$ and act through its $C.G.$ Hence $(w.x)$ will act at $\frac{x}{2}$ from B.

$$B.M_{\text{at}-B} = \text{Zero}$$

$$B.M_{\text{at}-x-x} = (w.x)\left(\frac{x}{2}\right) = \frac{w.x^2}{2}$$

$$B.M \text{ at } A = (w.L)\left(\frac{L}{2}\right) = \frac{wL^2}{2}$$

Example 5.4

A cantilever of span 2 metres carries a uniformly distributed load of 1 KN per metre run throughout its length. Draw the S.F. and B. M. diagrams.

Solution

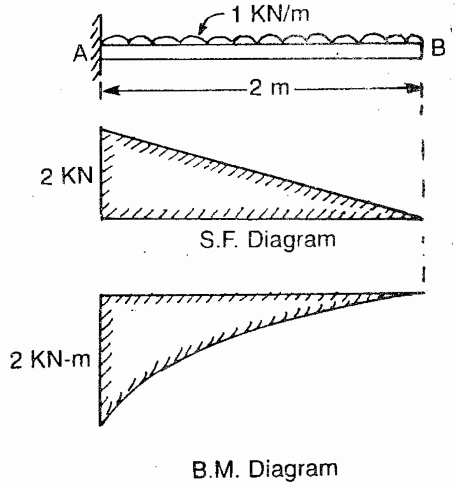


Fig. 5.20

Shear Force

$$S.F_{\text{at } B} = 0$$

$$S.F_{\text{at } x-x} = w.x = 1 \times x$$

$$S.F_{\text{at}-A} = 1 \times 2 = 2 \text{ KN}$$

Bending moment

$$\text{B.M}_{\text{at}-B} = 0$$

$$B.M._{\text{at } x-x} = (w.x)\ \frac{x}{2}$$

$$B.M._{\text{at}-A} = (1 \times 2)\left(\frac{2}{2}\right) = 2 \text{ KN-m}$$

Example 5.5

A cantilever A B of span 3 metres is loaded with a uniformly distributed load of 4 KN per metre run over half its span from the free end. Draw the š. F. and B.M diagrams.

Solution

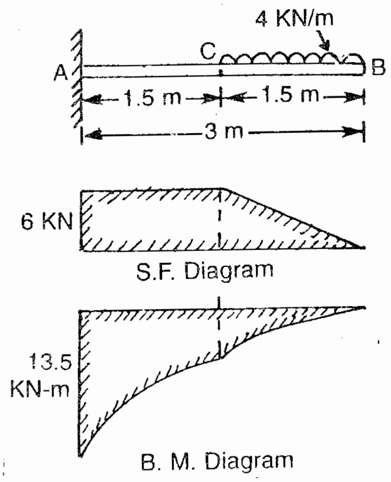


Fig. 5.21

Shear Force

$S.F_B = 0$

$S.F_C = 4 \times 1.5 = 6$ KN

$S.F_A = 6$ KN

Bending moment

$B.M_B = 0$

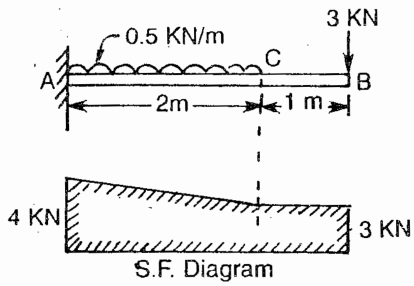
$B.M_C = (4 \times 1.5) \dfrac{(1.5)}{2} = 4.5$ KN–m

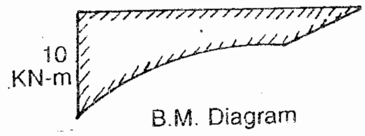
$B.M_A = (4 \times 1.5)\left(\dfrac{1.5}{2} + 1.5\right) = 13.5$ KN–m

Example - 5.6

A cantilever A B of span 3 metres is loaded with a concentrated load of 3 KN at the free end and a uniformly distributed load of 0.5 KN/metre run over a length of 2 metres from the fixed end. Draw the S.F. and B.M. diagrams

Solution





B.M. Diagram
Fig. 5.22

Shear Force

$S.F_B = 3$ KN

$S.F_C = 3$ KN

$S.F_A = 3 + 0.5 \times 2 = 4$ KN

Bending moment

$B.M._B =$ Zero

$B.M._C = 3 \times 1 = 3$ KN-m

$$B.M._A = 3 \times 3 + (0.5 \times 2)\left(\frac{2}{2}\right)$$

$$= 10 \ KN\text{–}m$$

Example 5.7

A cantilever A B 4 meters long is loaded as shown in figure 5.23. Draw the S. F. and B. M. diagrams.

Solution

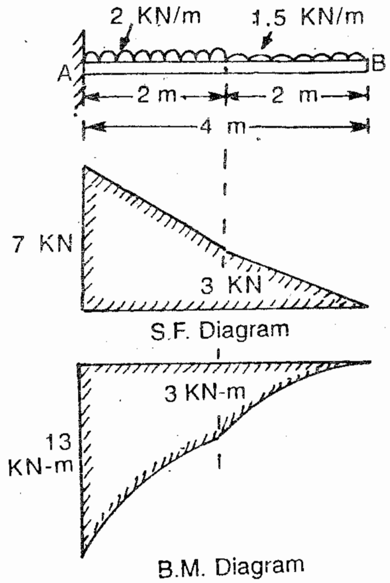


Fig. 5.23

Shear Force

$S.\ F_B = 0$

$S.\ F_C = 1.5 \times 2 = 3$ KN

$$S.F_A = 1.5 \times 2 + 2 \times 2 = 7 \text{ KN}$$

Bending moment

$$B.M_B = 0$$

$$B.M_C = (1.5 \times 2) \times \left(\frac{2}{2}\right) = 3 \text{ KN–m}$$

$$B.M_A = (1.5 \times 2)\left(\frac{2}{2} + 2\right) + 2 \times 2\left(\frac{2}{2}\right)$$

$$= 9 + 4 = 13 \text{ KN–m}$$

Example 5.8

Daw the shear force and B. M. diagrams for the cantilever shown in fig 5.24

Solution

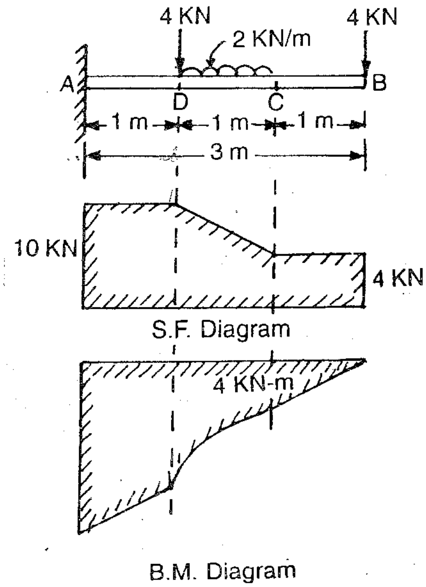


Fig. 5.24

Shear Force

$$S.F_B = 4 \text{ KN}$$

$$S.F_{B-C} = 4 \text{ KN}$$

$$S.F_D = 4 + 2 \times 1 + 4 = 10\text{KN}$$

$$S.F_{D-A} = 10 \text{ KN}$$

$$S.F_A = 10 \text{ KN}$$

Bending moment

$$B.M_B = 0$$

$$B.M_C = 4 \times 1 = 4 \text{ KN-m}$$

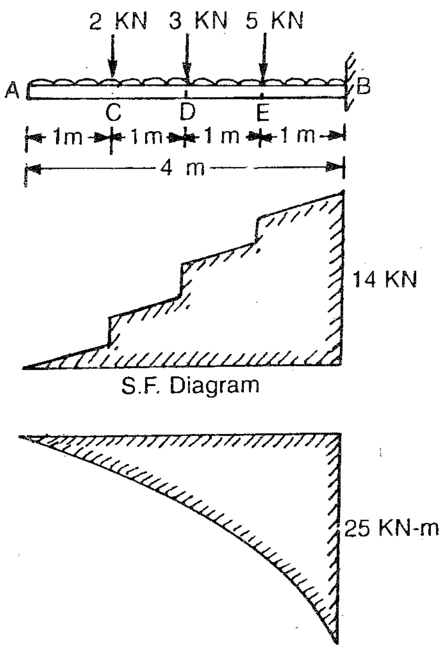
$$B.M_D = 4 \times 2 + 2 \times 1 \times \left(\frac{1}{2}\right) = 9 \text{ KN-m}$$

$$B.M_A = 4 \times 3 + (2 \times 1)\left(\frac{1}{2} + 1\right) + 4 \times 1 = 19 \text{ KN-m}$$

Example 5.9

A cantilever 4 meters long supports a u.d.l. of 1 KN per meter run on the whoe length and point loads of 2 KN, 3 KN and 5 KN at 1 meter, 2 metres and 3 meters from the free end A. Draw the S. F and B.M. diagrams.

Solution



B. M. Diagram

Fig. 5.25

Shear Force

$$S.F_A = 0$$

$$S.F_C = 1 \times 1 + 2 = 3 \text{ KN}$$

$$S.F_D = 1 \times 2 + 2 + 3 = 7 \text{ KN}$$

$$S.F_E = 1 \times 3 + 2 + 3 + 5 = 13 \text{ KN}$$

$$S.F_B = 1 \times 4 + 2 + 3 + 5 = 14 \text{ KN}$$

Bending moment

$$B.M_A = 0$$

$$B.M_C = (1 \times 1) \times \left(\frac{1}{2}\right) = 0.5 \text{ KN–m}$$

$$B.M_D = 1 \times 2 \left(\frac{2}{2}\right) + 2 \times 1 = 2 + 2 = 4 \text{ KN–m}$$

$$B.M_E = 1 \times 3 \left(\frac{3}{2}\right) + 2 \times 2 + 3 \times 1 = 4.5 + 4 + 3 = 11.5 \text{ KN–m}$$

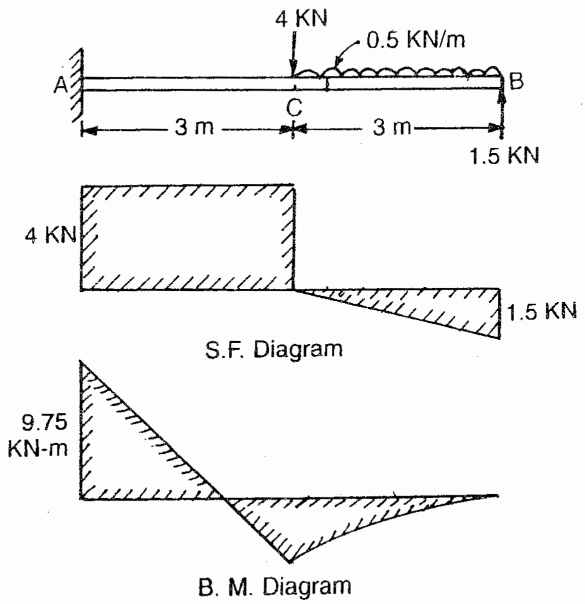
$$B.M_B = 1 \times 4 \left(\frac{4}{2}\right) + 2 \times 3 + 3 \times 2 + 5 \times 1 = 8 + 6 + 6 + 5$$

$$= 25 \text{ KN–m}$$

Example 5.10

A cantilever of 6 meters span has a central downward load of 4 KN. at c and an upward force of 1.5 KN at the free end. It also corties a u.d.l. of 0.5 KN/meter run between the two point loads as shown in the figure. Draw the S.F. and B.M diagrams.

Solution



B. M. Diagram
Fig. 5.26

Solution

Shear Forces

$S.F_B = \uparrow - 1.5$ KN

$S.F_C = -1.5 + 0.5 \times 3 = 0$

$S.F_A = -1.5 + 0.5 \times 3 + 4 = 4$ KN

Bending moments -

$B\,M_B$ = Zero

$B.\,M._C = -1.5 \times 3 + 0.5 \times 3 \times \frac{3}{2}$

$= -4.5 + 2.25 \quad = -2.25$ KN–m

$B.\,M._A = -1.5 \times 6 + 0.5 \times 3\left(\frac{3}{2} + 3\right) + 4 \times 3$

$= -9 + 1.5\ (4.5) + 12 \quad = -9 + 6.75 + 12$

$= 9.75$ KN– m.

A Cantilever with Uniformly Varying load

Consider a cantilever AB of span l with a uniformly varrying load zero at B increasing to w per unit run at the fixed end A.

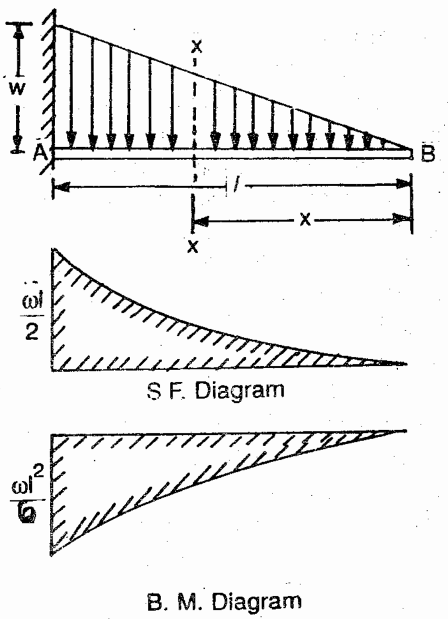


Fig. 5.27

At any section x–x at a distance x from B, intensity of loading $= \frac{w\,.\,x}{l}$

Total load on this portion $= \frac{1}{2}\,.\,x\,.\,\frac{w\,.\,x}{l} = \frac{w\,x^2}{2\,l}$ acting at $\frac{x}{3}$ from x–x

Shear force at B $F_{B\atop x=0} = 0$

Shear force at A $F_{A\atop x=l} = wl^2/2l = wl/2$

Bending moment at x–x $= \frac{w\,x^2}{2\,l} \times \frac{x}{3} = \frac{w\,x^3}{6\,l}$

Bending moment at B when $x = 0$, is Zero

Bending moment at A when $x = l$, $\frac{w\,l^2}{2\,l} \times \frac{l}{3} = \frac{w\,l^2}{6}$

Example 5.11

A Cantilever 5 metres long carries a uniformly varrying load, which increases from zero at the free end to 10 KN per metre at the fixed end. Determine the values of maximum shear force and Bending moment and draw the diagrams .

Solution

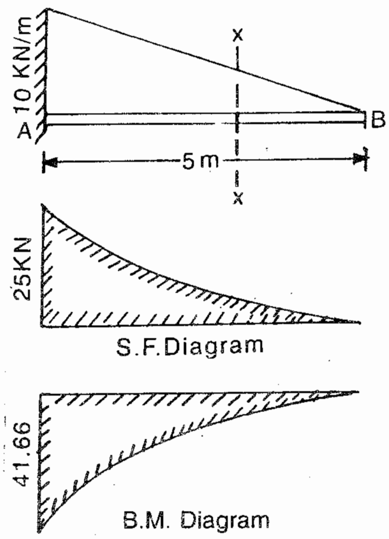


Fig. 5.28

At any section $x-x$ the rate of loading $= \frac{w\,x}{l} = 10 \times \frac{x}{5}$

Shear force

Shear force at $B = 0$

Shear force at Section $x\text{–}x = \frac{w\,.\,x}{l} \cdot \frac{x}{2}$

Shear force at $A = \frac{w\,l}{2} = \frac{10 \times 5}{2} = 25$ KN

Bending moment

Bending moment at B, when $x = 0$, is zero.

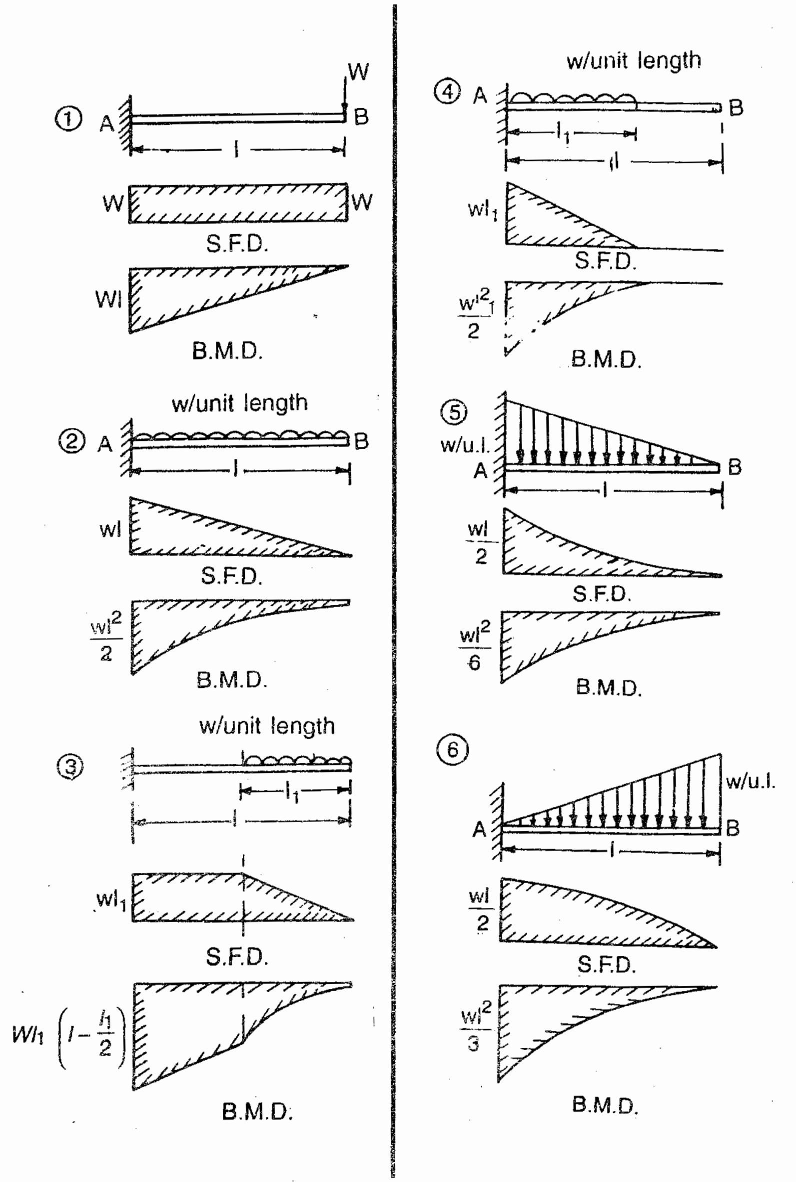
Bending moment at A, $B.M_A = \frac{w\,l^2}{6} = \frac{10 \times (5)^2}{6} = \frac{250}{6}$

$= 41.66$ KN–m.

TABLE - No. -5.1

S. F. and B. M. Diagrams

Standard Cases Of Cantilevers



Simply Supported Beam With a Point Load at Mid Span

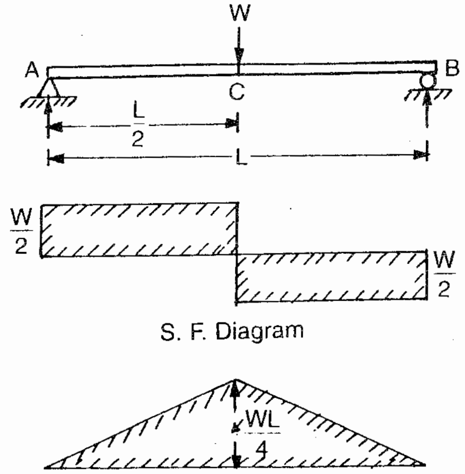


Fig. 5.29

A simply supported beam A B of span L with a point load W at its mid span is shown in the figure. since the load W is acting at the centre of the beam the reaction at the supports will be equal.

$$\therefore \quad R_A = R_B = \frac{W}{2}$$

Shear force between A and $C = R_A = \frac{W}{2}$

Shear force between C and $B = R_B = -\frac{W}{2}$

Bending moment at any section x–x between A and $C = R_{A}.x$

Bending moment at $C = R_A \cdot \frac{L}{2} = \frac{W}{2} \cdot \frac{L}{2} = \frac{WL}{4}$

Bending moment between C and B is

$$M_x = R_{A}.x - W\left(x - \frac{L}{2}\right)$$

Bending moment at $B = R_A \,.\, L - W\left(L - \frac{L}{2}\right) = 0$

Shear force and bending moment diagrams are shown in the figure.

Example 5.12

A simply supported beam of span 2.6 metres carries a concentrated load of 15 KN at its mid span. Draw the shear force and Bending moment diagrams.

Solution

Taking moments about A, $-R_B \times 2.6 + 15 \times 1.3 = 0$

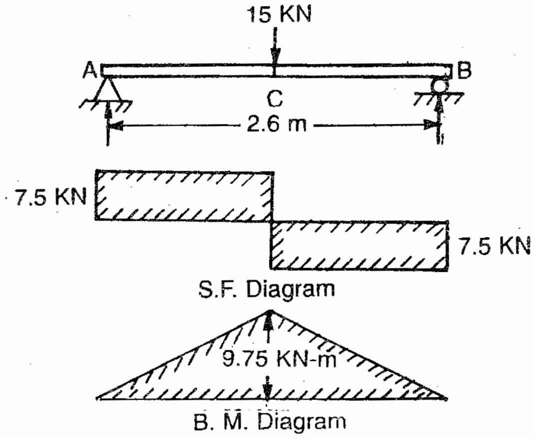


Fig. 5.30

$$R_B = \frac{15 \times 1.3}{2.6} = 7.5 \text{ KN}$$

Taking moments about B

$$R_A \times 2.6 - 15 \times 1.3 = 0$$

$$R_A = \frac{15 \times 1.3}{2.6} = 7.5 \text{ KN}$$

Shear Force

$S.F$ between A and $C = R_A = 7.5$ KN

$S.F$ between C and $B = R_B = -7.5$ KN

Bending moment

$B.M$ at A = Zero

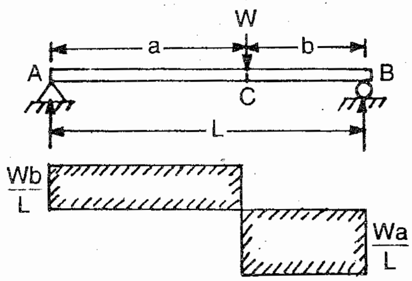
$$B\,M \text{ at } C = R_A \times \frac{L}{2} = \frac{W}{2} \times \frac{L}{2} = \frac{WL}{4}$$

$$= \frac{15 \times 2.6}{4} = 9.75 \text{ KN} - m$$

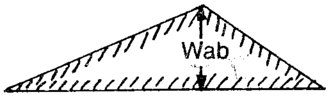
$$B.\,M \text{ at } B = R_A \times L - \frac{WL}{2}$$

$$= 7.5 \times 2.6 - 15 \times \frac{2.6}{2} = 0$$

A Simply Supported Beam With A Point Load Not At The Centre



S. F. Diagram



B. M. Diagram

Fig. 5.31

Taking moments about A

$$W \times a - R_B \times L = 0 \quad or \quad R_B = \frac{Wa}{L}$$

Taking moments about B

$$-W.b + R_A \times L = 0 \quad or \quad R_A = \frac{Wb}{L}$$

Shear Force

Shear force between A and $C = R_A = \dfrac{Wb}{L}$

Shear force between C and $B = R_A - W = -\dfrac{Wa}{L}$

Bending moment

$B.M$ at $A = 0$

$B.M$ between A and $C = M_{x-x} = R_{A}.x$

$B.M$ at $C = \dfrac{Wb}{L} \times a = \dfrac{Wab}{L}$

$B.M$ between C and $B = R_A.x - W(x-a)$

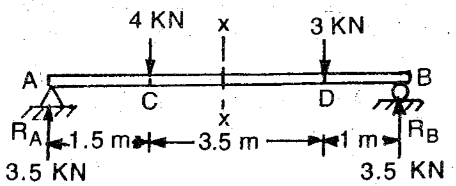
$$= \frac{Wb}{L}.x - W(x-a)$$

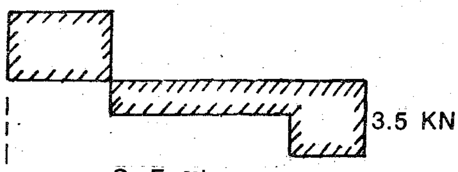
$B.M$ at $B = \dfrac{Wb}{L}(L) - W(L-a) = 0$

$B.F$ and $B.M.$ diagrams are shown in the figure.

Example. 5.13

Draw the shear force and bending moment diagrams for the beam shown in fig 5.32





S. F. Diagram

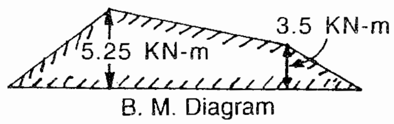


Fig. 5.32

Solution

Taking moments about A

$$R_B x 6 = 3\times\ 5 + 4\times 1.5$$
$$= 15 + 6 = 21$$
$$R_B \ = 21/6\ \ 3.5 \text{ KN}$$

Taking moments about B

$$R_A\, 6 = 4 \times 4.5 + 3 \times 1$$
$$= 18 + 3 = 21$$
$$R_A = \frac{21}{6} = 3.5 \text{ KN}$$

Shear Force

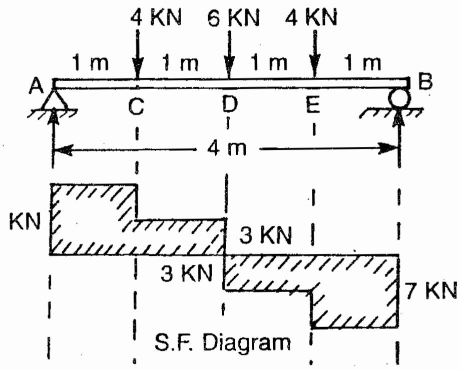
$$S.\,F._A = 3.5 \text{ KN}$$
$$S\,F_{A-C} = 3.5$$
$$S\,F_{C-D} = 3.5 - 4 = -0.5 \text{ KN}$$
$$S.F_{D-B} = -0.5 - 3 = -3.5 \text{ KN}$$
$$S.\,F_B = R_B = 3.5 \text{ KN}$$

Bending moment

$$B.\,M_A = 0$$
$$B.M_C = 3.5 \times 1.5 = 5.25 \text{ KN-m}$$
$$B.M_{xx} = R_A.\,x - 4\,(x - 1.5)$$
$$B.M_D = 3.5 \times 5 - 4 \times 3.5$$
$$= 17.5 - 14 = 3.5 \text{ KN–m}$$
$$B\,M_B = 3.5 \times 6 - 4 \times 4.5 - 3 \times 1 \text{ KN–m}$$
$$= 21 - 18 - 3 = 0$$

Example 5.14

A simply supported beam of span 4 metres is loaded as shown in figure 5.33. Draw the S. F. and B. M. diagrams.



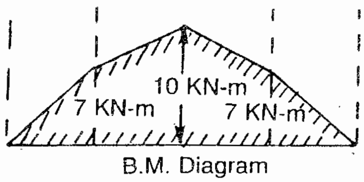


Fig. 5.33

Solution

Taking moments about B

$$R_A \times 4 - \times 3 - 6 \times 2 - 4 \times 1 = 0$$

$$R_A = 7 \text{ KN} = \text{RB}$$

Shear force

$$S.F_A = 7 \text{ KN}$$
$$S.F_{A-C} = 7 \text{ KN}$$
$$S.F_{C-D} = 7 - 4 = 3 \text{ KN}$$
$$S.F_{D-E} = 7 - 4 - 6 = -3 \text{ KN}$$
$$S.F_{E-B} = 7-4-6-4 = -7 \text{ KN}$$
$$S.F_B = -7 \text{ KN}$$

Bending moment

$$B.M_A = 0$$
$$B.M_C = R_A \times 1 = 7 \times 1 = 7 \text{ KN–m}$$
$$B.M_D = 7 \times 2 - 4 \times 1 = 10 \text{ KN–m}$$
$$B.M_E = 7 \times 3 - 4 \times 2 - 6 \times 1 = 21 - 14 = 7 \text{ KN–m}$$
$$B.M_B = 7 \times 4 - 4 \times 3 - 6 \times 2 - 4 \times 1 = \text{Zero.}$$

Example 5.15

Draw the shear force and bending moment diagram for a simply supported beam shown in figure 5.34

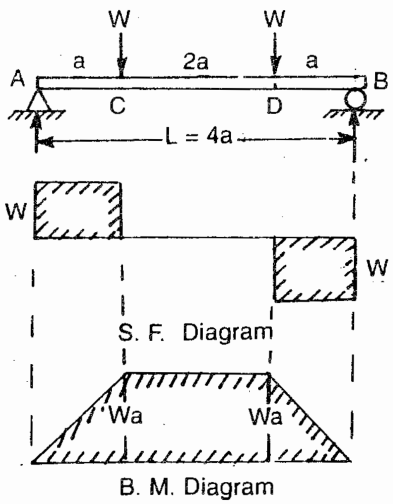


Fig. 5.34

Solution :

Since the loading is symmetnical

Hence $R_A = R_B = W$

Shear Force

$S.F_{A-C} = W$

$S.F_{C-D} = 0$

$S.F_{D-B} = W$

Bending moment

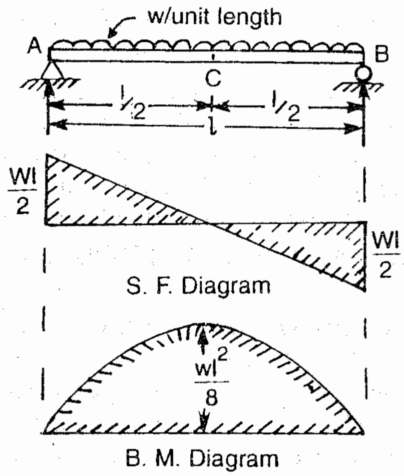
$B.M_A = 0$

$B.M_C = W \times a = Wa$

$B.M_D = W \times 3a - W \times 2a = Wa$

$B.M_B = 0$

Simply supported beam with *u.d.l.* on the whole span.



B. M. Diagram Fig. 5.34

Taking moments about A

$$R_B \times l - (w.l).\ l/2 = 0$$

$$R_B = \frac{w\,l}{2} = R_A$$

Shear force at $A = RA \quad = \dfrac{wl}{2}$

Shear force at section $x - x$ at a distance x from A

$S.F.\ x\text{-}x = R_A - w.\ x$

When $x = l/2 \quad S.F.\ x\text{-}x = R_A - \dfrac{wl}{2}$

$$= \frac{wl}{2} - \frac{wl}{2} = \text{Zero}$$

When $x = l$, $S.F_{.B} = R_A - wl = \frac{wl}{2} - wl = \frac{-wl}{2} = RB$

BM at A = zero

BM at $x - x = R_{A.}x - wx\ (x/2)$

When $x = l/2$, $M_{xx} = R_{A.}\ l/2 - \frac{wl}{2}\ (l/2 \times l/2)$

$$B.\ M_c = \frac{wl}{2} \times \frac{l}{2} - \frac{wl}{2} \times \frac{l}{4} = \frac{wl^2}{8}$$

When $x = l$, $M_B = R_A \times l - Wl \times \frac{l}{2}$

$$B.M_B = \frac{wl}{2} \times l - wl \times \frac{l}{2} = \text{Zero}$$

Since the general equation of B.M. is of second degree

i.e. $M_{xx} = R_{A.x} - \frac{wx^2}{2}$, Hence we obtain a parabolie curve.

The maximum B.M. will occur at midspan.

$M_C = \frac{wl^2}{8}$ and maximum shear force will occure at ends

$S.F.A = \frac{wl}{2} = S.F_B$

Example 5.16

A simply supported beam AB of Span 4 metres carries a uniformly distributed load of 6 KN per metre run over half the Span from the left end support A. Calculate the shear force and bending moment and draw the diagrams.

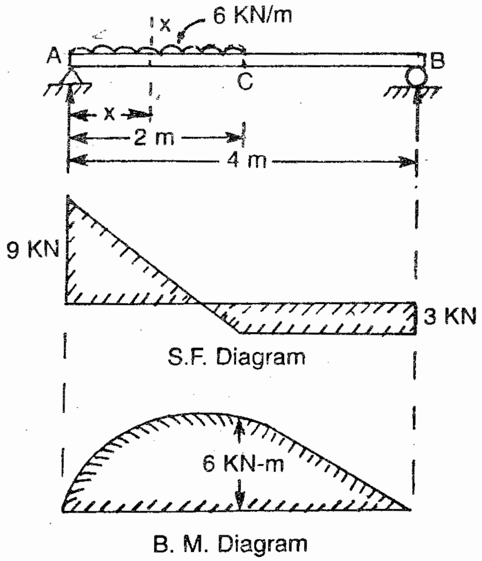


Fig. 5.35

Solution :

Taking moments about B

$$R_A \times 4 = 6 \times 2\left(\frac{2}{2} + 2\right)$$
$$R_A = 9 \text{ KN}$$

Taking moments about A

$$R_B \times 4 = 6 \times 2\left(\frac{2}{2}\right)$$
$$RB = 3 \text{ KN}$$

Shear Force

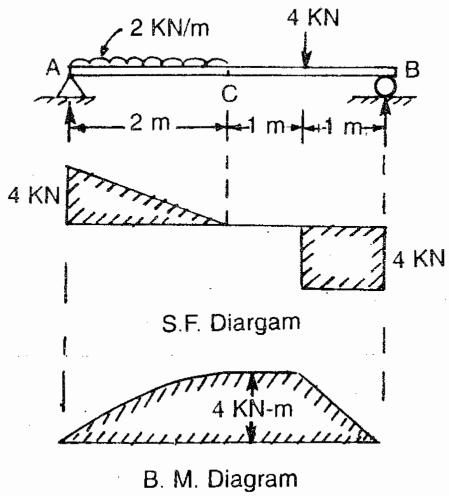
$$S.F_A = 9 \text{ KN}$$
$$S.F_{x\text{-}x} = R_A - w.x$$
$$S.F_C = 9 - 6 \times 2 = -3 \text{ KN}$$
$$S.F_B = -3 \text{ KN}$$

Bending moment

$$B.\,M_A = 0$$
$$B.M_{X-X} = R_A.x - w.\,x.\,\frac{x}{2}$$
$$B.\,M_C = 9 \times 2 - 6 \times 2 \times \frac{2}{2} = 18 - 12 = 6 \text{ KN–m}$$
$$B.M_B = 9 \times 4 - 6 \times 2\left(\frac{2}{2} + 2\right) = 36 - 36 = 0$$

Example 5.17

A freely supported beam of span 4 metres carries a u.d.l. of 2 KN per metre run over a length of 2 metres from the left end support and a point load of 4 KN at 1 metre from the right end. Draw the S. F. and B. M. diagrams.



B. M Diagram Fig. 5.36

Solution :

Taking moments about B

$$R_A \times 4 - 2 \times 2\left(\frac{2}{2} + 1 + 1\right) - 4 \times 1 = 0$$

$$R_A = 4 \text{ KN}$$

Taking moments about A

$$-R_B \times 4 + 4 \times 3 + 2 \times 2\left(\frac{2}{2}\right) = 0$$

$$R_B = 4 \text{ KN}$$

Shear force

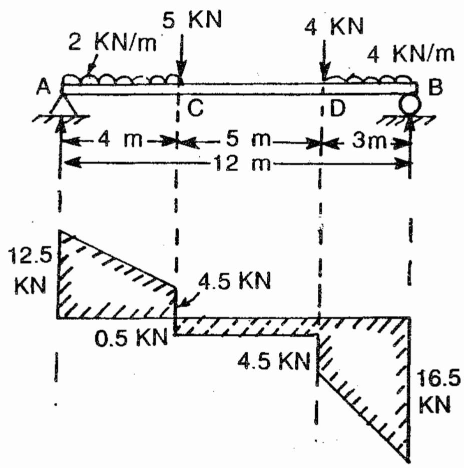
$$S.F_A = 4 \text{ KN}$$
$$S.F_C = 4 - 2 \times 2 = 0 \text{ KN}$$
$$S.F_D = 4 - 2 \times 2 = 0$$
$$S.F_B = 4 - 2 \times 2 - 4 = -4 \text{ KN}$$

Bending moment

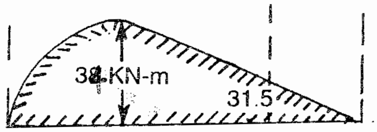
$$B.M_A = 0$$
$$B.M_{x\text{-}x} = R_A\, x - w \,.\, x \,.\, \frac{x}{2}$$
$$B.M_C = 4 \times 2 - 2 \times 2\left(\frac{2}{2}\right) = 4 \text{ KN–m}$$
$$B.M_D = 4 \times 3 - 2 \times 2\left(\frac{2}{2} + 1\right) = 4 \text{ KN–m}$$
$$B.M_B = 4 \times 4 - 2 \times 2\left(\frac{2}{2} + 2\right) - 4 \times 1 = 0$$

Example 5.18

Draw shear force and bending moment diagrams for the beam shown in figure. 5.37



S. F. Diagram



B.M. Diagram

Fig. 5.37

Solution :

Taking moments about B

$$R_A \times 12 = 2 \times 4\left(\frac{4}{2} + 8\right) + 5 \times 8 + 4 \times 3 + 4 \times 3\ (1.5)$$

$$R_A = 80 + 40 + 12 + 18 = \frac{150}{12}$$

$$R_A = 12.5 \text{ KN}$$

Taking moments about A

$$R_B \times 12 = 4 \times 3\left(\frac{3}{2} + 9\right) + 4 \times 9 + 5 \times 4 + 2 \times 4\left(\frac{4}{2}\right)$$

$$= 126 + 36 + 20 + 16 = 198$$

$$R_B = 16.5 \text{ KN}$$

Shear force

$S.\ F_A = 12.5$ KN

$S.\ F.$ just to left of C $= 12.5 - 2 \times 4 = 4.5$ KN

$S.\ F.$ just to the right of C $= 12.5 - 2 \times 4 - 5 = -0.5$ KN

$S.F.$ just to the right of D $= 12.5 - 2 \times 4 - 5 - 4 = -4.5$ KN

$S.F_B = 12.5 - 8 - 5 - 4 - 12 = -16.5$ KN

Bending moment

$$B.M_A = 0$$

$$B.M_C = 12.5 \times 4 - 2 \times 4\left(\frac{4}{2}\right) = 50 - 16 = 34 \text{ KN-m}$$

$$B.M_D = 12.5 \times 9 - 2 \times 4\left(\frac{4}{2} + 5\right) - 5 \times 5$$

$$= 112.5 - 56 - 25 = 31.5 \text{ KN-m}$$

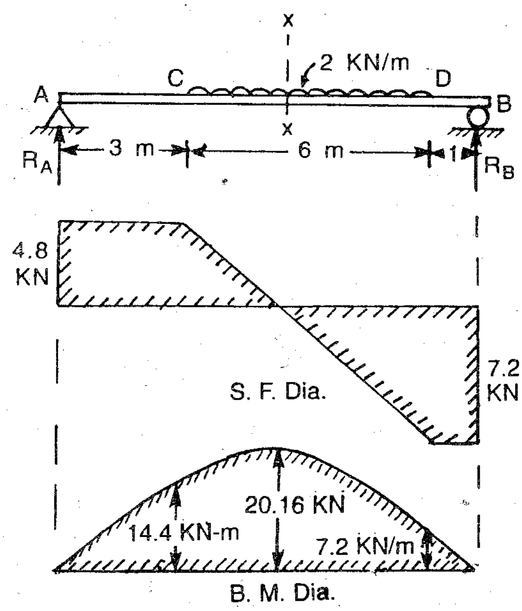
$$B.M_B = 12.5 \times 12 - 2 \times 4\left(\frac{4}{2} + 8\right) - 5 \times 8 - 4 \times 3 - 4 \times 3\left(\frac{3}{2}\right)$$

$$= 150 - 80 - 40 - 12 - 18 = \text{Zero}$$

Shear force and $B.M.$ diagrams are shown in the figure.

Example 5.19

A freely supported beam AB of span 10 metres carries a u.d.l. of 2 KN per metre run on portion CD over a length of 6 metre as shown in figure 5.38. Draw the shear force and Bending moment diagrams. Calculate the position and amount of maximum B.M.



B. M. Diagram
Fig. 5.38

Solution

Calculations for support reactions,

$$R_A \times 10 - 2 \times 6\ (6/2 + 1) = 0$$

$$R_A = \frac{48}{10} = 4.8 \text{ KN}$$

$$R_A + R_B = 12 \text{ KN}$$

Hence, $RB = 12 - 4.8 = 7.2\ KN$

Shear forces -

Shear force at $A = 4.8$ KN

Shear force at $C = 4.8$ KN

Shear force at $x - x$,

$$Fx - x = 4.8 - 2 \times x$$

Shear force at D

$$F_D = 4.8 - 2 \times 6 = 4.8 - 12 = -7.2 \text{ KN}$$

$$F_B = -7.2 \text{ KN}$$

Bending moment at A = Zero

Bending moment at $C = 4.8 \times 3 = 14.4$ KN-m

Bending moment at $M_{xx} = R_A\ (3 + x) - w.x.\frac{x}{2}$

Bending moment at D $M_D = 4.8\ (3 + 6) - 2 \times 6 \times 6/2 = 7.2$ KN–m

B.M. at $B = M_B = 4.8\ (3 + 6 + 1) - 2 \times 6\ (6/2 + 1)$

$= 4.8 \times 10 - 2 \times 6 \times 4 = \text{Zero}$

For maximum *B.M* :– It occur in the portion *CD* . To locate the point of Max *B.M.*, the differential of *B.M.* *i.e.* shear force must be zero.

$F_{xx} = 4.8 - 2 \times x = 0$

or $x = 4.8/2 = 2.4$ m from C

or 5.4 m from A.

$\therefore$ Put this value of x in the general equation of BM.

$$M_{x-x} = R_A\,(3 + x) - w\,.x\,.\frac{x}{2}$$

$B.M$ at $x = 2.4$, $= 4.8\ (3 + 2.4) - 2 \times (2.4)\ (2.4/2)$

$= 4.8\ (5.4) - 2 \times 2.4 \times 1.2$

$= 25.92 - 5.76 = 20.16$ KN–m.

Hence maximum B.M. 20.16 KN–m will occur at 5.4 m from A.

Uniformly Varying triangular Load :

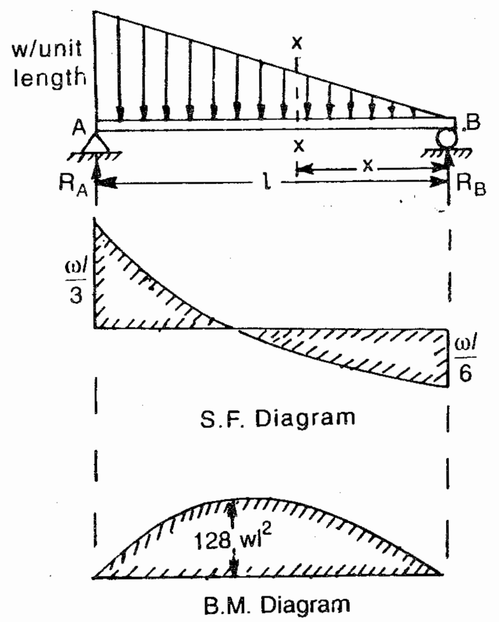


Fig. 5.39

Taking moments about A,

$$-R_B \times l + w.\ \frac{l}{2}\cdot\frac{l}{3} = 0$$

$$R_B = wl/6$$

Taking moments about B,

$$R_A \times l - w\,l/2 \times 2/3 \times l = 0$$

$$R_A = wl/3$$

Consider a section $x-x$ at distance x from B.

Intensity of loading at x–x = $w.\frac{x}{l}$

Shear force at x–x $= -R_B + w.\frac{x}{l}.\frac{x}{2}$

$$= -wl/6 + wx^2/2l$$

Shear force at $B = -\frac{wl}{6}$

Shear force at $x = \frac{l}{4}$ $= -R_B + w\left(\frac{l/4}{l}\right) l/4 \times 2$

$$= -\frac{wl}{6} + \frac{wl}{32} = -\frac{13}{96}wl$$

Shear Force at $x = \frac{l}{2}$ $= -R_B + w.\,(l/2)^2 \times \frac{1}{2l}$

$$= -\frac{wl}{6} + w.\frac{l^2}{4} \times \frac{1}{2l}$$

$$= -\frac{wl}{6} + \frac{wl}{8} = \frac{(-4+3)}{24}wl = -\frac{1}{24}wl$$

Shear force at $x = l$

$$F_{A\,x=l} = \frac{-wl}{6} + \frac{w}{2l}(l)^2 = -\frac{wl}{6} + \frac{wl}{2} = -\frac{wl+3wl}{6}$$

$$= \frac{1}{3}wl = R_A$$

B.M. at $B = 0$

Bending moment at x–x, $M_{x-x} = R_B \times x - \frac{wx}{l}.\frac{x}{2} \times \frac{x}{3}$

$$M_{xx} = \frac{wl}{6}.x - \frac{wx}{l}.\frac{x}{2}.\frac{x}{3}$$

Bending moment at $x = l/4$

$$\text{B.M.}_{x=l/4} = \frac{wl}{6}.\frac{l}{4} - \frac{w}{l}.\frac{l}{4}.\frac{1}{2} \times \frac{l}{4}.\frac{1}{3} \times \frac{l}{4}$$

$$= \frac{wl^2}{24} - \frac{wl^3}{l} \times \frac{1}{64 \times 6}$$

$$= \frac{wl^2}{6}\left(\frac{1}{4} - \frac{1}{64}\right) = \frac{wl^2}{6} \times \frac{15}{64} = \frac{5\,wl^2}{128}$$

Bending moment at $x = l/2$

$$BM._{\text{at } x = l/2} = \frac{wl}{6}.\frac{l}{2} - \frac{w}{l}.\frac{l}{2} - \frac{1}{2}.\frac{l}{2}.\frac{1}{3} \times \frac{l}{2}$$

$$= \frac{wl^2}{12} - \frac{wl^3}{l \times 6 \times 8} = \frac{wl^2}{12} - \frac{wl^2}{48} = \frac{wl^2}{16}$$

Bending moment at A when $x = l$

$$B.M._A = \frac{wl}{6}.l - \frac{w}{l}.l.\frac{l}{2}.\frac{l}{3} = \frac{wl^2}{6} - \frac{wl^3}{6l} = \text{Zero}$$

For calculating the maximum B.M., the differential of B.M. i.e. shear force must be zero.

$$F_{xx} = -R_B.x + \frac{w.x}{l}.\frac{x}{2} = 0$$

$$= -\frac{wl}{6}.x + \frac{wx^2}{2l} = 0$$

$$\text{or} \quad \frac{wx^2}{2l} = \frac{wl.x}{6}$$

$$\text{or} \quad x^2 = \frac{l^2}{3} \quad \text{or} \quad x = \frac{l}{\sqrt{3}} \quad \text{or} \quad x = 0.577l$$

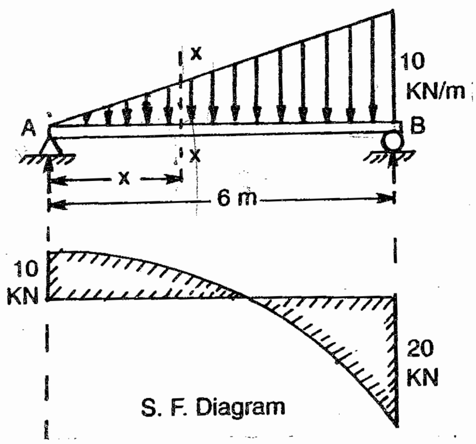
$$M_{max} = \frac{wl}{6}.x - w.\frac{x}{l}.\frac{x}{2}.\frac{x}{3}$$

$$= \frac{wl}{6}(0.577l) - w\frac{(0.577l)}{l}\frac{(0.577l)}{2}\frac{(0.577l)}{3}$$

$$= \frac{wl^2}{9\sqrt{3}} = 0.128wl^2$$

Example 5.20

A simply supported beam AB of span 6 metres carries a triangular load which varies from zero/m at A to 10 KN/m at the end B. Draw the shear force and bending moment diagrams. State the value of max. B.M.



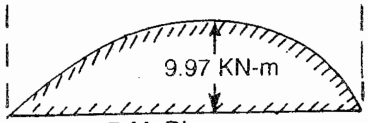


Fig. 5.40

Solution :

Taking moments about B

$$R_A \times 6 = \tfrac{1}{2} \times 10 \times 6 \times \frac{6}{3}$$

$$R_A = 10 \text{ KN}$$

$$\therefore\ R_A + R_B = 30 \text{ KN}$$

$$\therefore\ R_B = 30 - 10 = 20 \text{ KN.}$$

Consider a section $x - x$ at a distance x from A

$$S.F.\,xx = R_A - \frac{wx^2}{2l} = 10 - \frac{10\,x^2}{2 \times 6}$$

When $x = 0$

$$S.\,F_A = 10 \text{ KN}$$

$$S.\,F_B = 20 \text{ KN}$$

at $x = 6$ m,

Shear force will change sign

When $\frac{5}{6}x^2 = 10$ *or* $x = \sqrt{12}$

$$x = 3.46 \text{ m form } A.$$

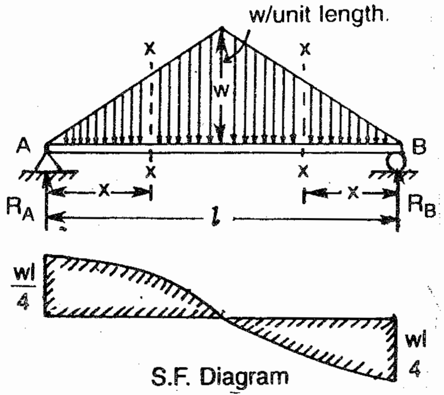
Bending moment at $x - x$

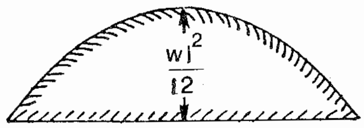
$$Mx - x = 10\,x - \frac{5}{6}\,x^3, \quad \text{When } x = 0, M_A = 0$$

$$\text{When } x = 6\,,\, M_B = 0$$

When $x = 3.46$ $\quad M_{max} = 10 \times 3.46 - \frac{5}{6}\,(3.46)^3 = 9.97$ KN – m.

A triangular load with a maximum at the centre :





B.M. Diagram

Fig. 5.41

AB is a freely supportd beam of span l. It supports a triangular load zero per metre run at ends increasing to w per metre run at the centre.

Average rate of loading $= \frac{0 + w}{2} = \frac{w}{2}$

Total load on the span; $= \frac{w}{2} . l$

Reaction at the supports $R_A = R_B = \frac{wl}{2} \times \frac{1}{2} = \frac{wl}{4}$

Consider a section $x - x$ at a distance x from B.

Shear force at $B = R_B = wl/4$

Shear force at $x - x$, $Fxx = - R_B + 1/2 . w \, (x/l/2) . x$

Shear force at mid span $F_c = -\frac{wl}{4} + \frac{w}{2} . \frac{2x^2}{l}$

at $x = l/2$

$$= - wl/4 + = \frac{wx^L}{l} = - \frac{wL}{4} + \frac{w\,(l/2)^2}{t}$$

$$= -\frac{wl}{4} + \frac{wl^2}{4l} = 0$$

Shear force at $A = + R_A = wl/4$

Bending moment at xx, $Mxx = \frac{wl}{4} . x - \frac{1}{2} . \frac{w}{l} . 2x \left(x - \frac{x}{3} \right)$

$$= \frac{wlx}{4} - \frac{wx^3}{3l}$$

Bending moment is maximum at mid span,

$$M_C = \frac{wl}{4}\,(l/2) - \frac{w}{3l}\,(l/2)^3 = \frac{wl^2}{8} - \frac{wl^2}{24}$$

when $x = l/2$ $\quad M_C = \frac{wl^2}{12}$

Example 5.21

A simply supported beam of span 4 metres carries a uniformly varying load whose intensity varies from Zero/m at each end to 20 KN/m at mid span. Calculate the maximum values of shear force and bending moment and draw the S.F. and B.M. diagrams.

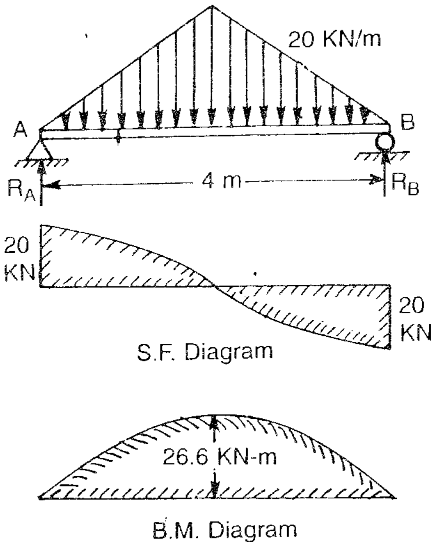


Fig. 5.42

Solution :-

Total load on the beam $= \frac{1}{2} \times 4 \times 20 = 40$ KN.

Hence reaction at each end, $RA = RB = 20$ KN.

Consider a section $x-x$ at a distance x from A,

Rate of loading at the Section $= w \cdot \frac{x}{l/2} = \frac{2wx}{l}$

$$S.F._{x-x} = R_A - \frac{2wx}{l} \cdot \frac{x}{2}$$

$$S.F. \text{ at } A = 20 - \frac{2 \times 20 \times (0)^2}{2 \times L} = 20 \text{ KN}$$

$S.F.$ at C when $x = l/2$

$$S.F_c = 20 - 2\times \frac{20}{4} \times \frac{(4/2)^2}{2}$$

$$= 20 - 20 = 0$$

$$S.F._B = -R_B = -20 \text{ KN}$$

$$M_{xx} = R_A \cdot x - \frac{wx^3}{3l},$$

$BM_A = 0, \quad BM_B = 0$

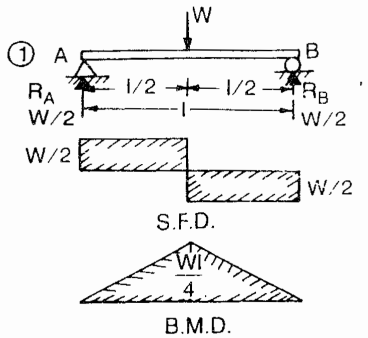
Max B.M. will occur at c when $x = l/2$

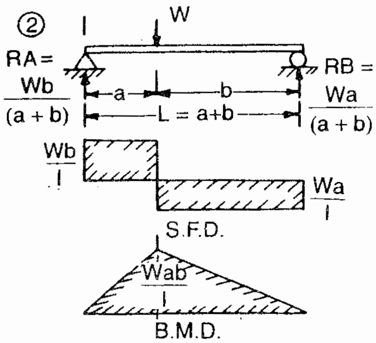
$$= 20 \times \frac{4}{2} - \frac{20\,(4/2)^2}{3 \times 4} = 26.6 \text{ KN-m}$$

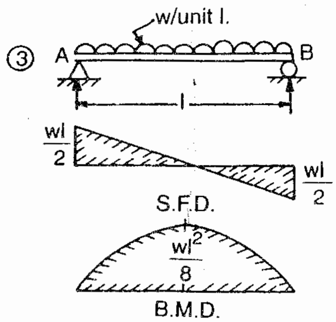
TABLE No. 5.2

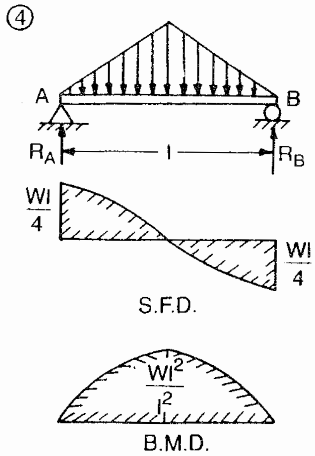
S.F. And B.M. Diagrams

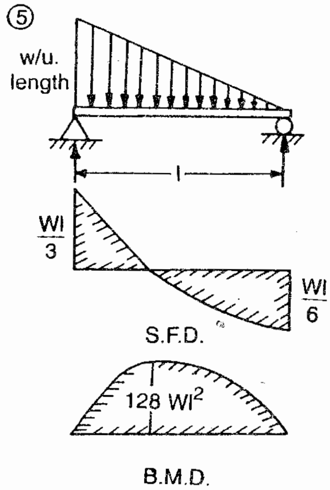
Standard Cases Of Beams











Overhanging Beams

A uniformly disributed load w per unit length on an overhanging beam :

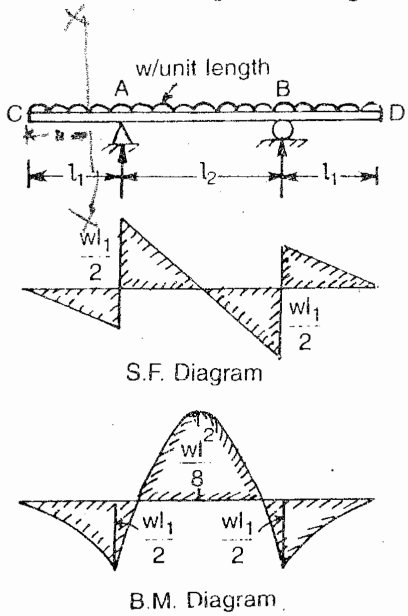


Fig. 5.43

$$R_A = R_B$$
$$= \frac{w}{2}(l_1 + l_2 + l_1) \qquad = \frac{w}{2}(l_2 + 2l_1)$$

Shear Forces –

$S.F.$ at $C = 0$

$S.F.$ to the left of $A = wl_1$ (– ve)

$S.F.$ to the right of $B = wl_1$ (+ ve)

$$S.F. \text{ at mid span} = R_A - \frac{w(l_2 + 2l_1)}{2}$$
$$= w\frac{(l_2 + 2l_1)}{2} - w\frac{(l_2 + 2l_1)}{2} = 0$$

Bending moment at $C = 0$

$$\text{Bending moment at } A = -wl_1 \times \frac{l_1}{2} = -\frac{w(l_1)^2}{2}$$

Bending moment at mid span;

$$= R_A \times \frac{l_2}{2} - w\left(l_1 + \frac{l_2}{2}\right) \times \frac{1}{2}\left(l_1 + \frac{l_2}{2}\right)$$
$$= +\frac{w}{2}(l_2 + 2l_1) \times \frac{l_2}{2} - \frac{w}{2}\left(l_1 + \frac{l_2}{2}\right)^2$$

$$= +w\left(l_1 + \frac{l_2}{2}\right) \times \frac{l_2}{2} - \frac{w}{2}\left(l_1 + \frac{l_2}{2}\right)^2$$

$$= \frac{wl_2^{\,2}}{8} - \frac{wl_1^{\,2}}{2}$$

Point of Contraflexure

From the bending moment diagram we can see that bending moment changes sign at two points. Point of contraflexure is the point where B.M. changes sign from positive to negative or Vice–Versa.

Example 5.22

An overhanging beam of span 30 metres is supported on two points A and B 20 metres apart. The beam supports a uiformly distributed load of 2 KN per metre on the portion AB and three point loads of 5 KN, 8 KN and 5 KN at C, E and D as shown in the Figure. Draw the S. F. and B.M. diagrams and locate the points of contraflexure .

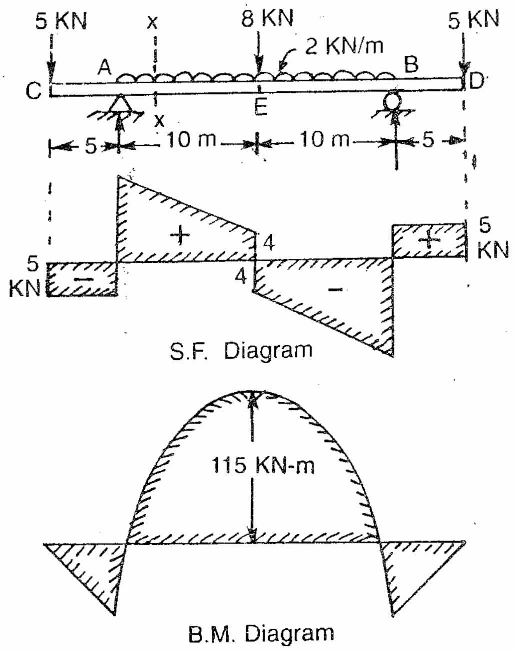


Fig. 5.44

Solution

As the loading is symmetrical, support reactions RA and RB will be equal

Taking moments about B.

$$R_A \times 20 - 5 \times 25 - 8 \times 10 - 2 \times 20 \times \frac{20}{2} + 5 \times 5 = 0$$

$20\,R_A - 125 - 80 - 400 + 25 = 0$

$20\,R_A = 125 + 480 - 25 = 580$

$R_A = 29$ KN.

Hence, $R_B = 29$ KN.

Shear Force,

$S.F_C = 5$ KN.

Shear Force just to the left of $A = -5$ KN.

Shear force just to the right of A = $5 + 29 = +24$ KN.

Shear force at section $x-x$, $F_{xx} = -5 + 29 - w.x$

Shear force just to the left of $B = -5 + 29 - 8 - 2 \times 20 = -24$ KN

Shear force just to the right of $B = -5 + 29 - 8 - 40 + 29 = 5$ KN

Shear force at $D = 5$ KN

Bending moment at C = Zero

Bending moment at $A = -5 \times 5 = -25$ KN–m

Bending moment at x–x

$$M_{x-x} = R_A.x - 5\ (5 + x) - 2\frac{x^2}{2}$$

Bending moment at E when $x = 10$ m.

$$M_E = 29 \times 10 - 5\ (5 + 10) - 2\ (10)^2 - 2\frac{(10)^2}{2} = 115 \text{ KN–m}$$

Bending moment at B, When $x = 20$

$$M_B = 29 \times 20 - 5\ (5 + 20) - \frac{2}{2}(20)^2 = -25 \text{ KN–m}$$

Bending moment at D = Zero

Point of Contraflexure

$$M_{xx} = R_A \times x - 5\ (5 + x) - 2\frac{x^2}{2} = 0$$

$$= 29\,x - 25 - 5\,x - x^2 = 0$$

$$x^2 - 24\,x + 25 = 0$$

or $x = 1.1$ m from A.

Example 5.23

A beam 8 metres long rests on two supports 2 metres from each end. It carries concentrated loads of 4 KN. at C, D & E. Draw the shear force and bending moment diagrams.

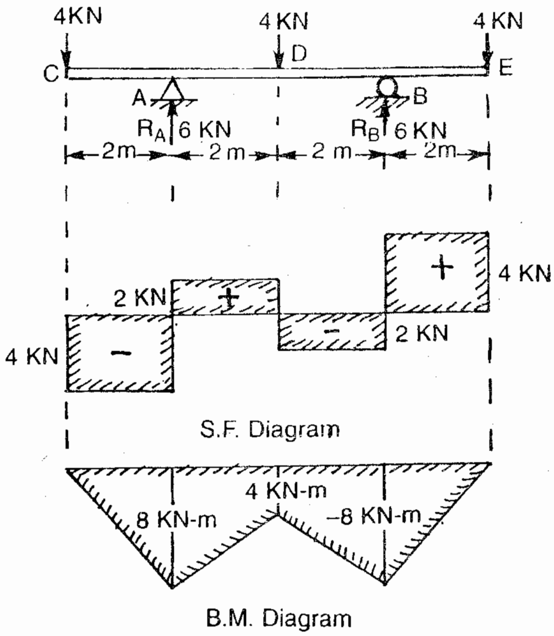


Fig. 5.45

Solution

Taking moments about B

$R_A \times 4 - 4 \times 6 - 4 \times 2 + 4 \times 2 = 0$

$R_A = 6 \text{ KN} = R_B$

Shear Force

Shear Force at $C = -4$ KN.

Shear Force just to the left of $A = -4$ KN.

Shear Force just to the right of $A = -4 + 6 = 2$ KN.

Shear Force to the left of $D = +2$ KN.

Shear Force to the right of $D = -2$ KN.

Shear Force just to the left of $B = -2$ KN.

Shear Force just to the right of $B = -4 + 6 - 4 + 6 = +4$ KN.

S .F. at E = 4 KN

Bending moments

Bending moment at $C = 0$

Bending moment at $A = -4 \times 2 = -8$ KN – m

Bending moment at $D = -4 \times 4 + 6 \times 2 = -4$ KN – m

Bending moment at $B = -4 \times 6 + 6 \times 4 - 4 \times 2 = -8$ KN-m

Bending moment at E = zero

Example 5.24

An overhanging beam of span 6 metres rests on two supports 5 metres apart. It carries a u. d. l. of 8 KN per metre run on the whole span. Construct the B.M and shear force diagrams. Also calculate the maximum B.M. and the point of contraflexure.

Solution

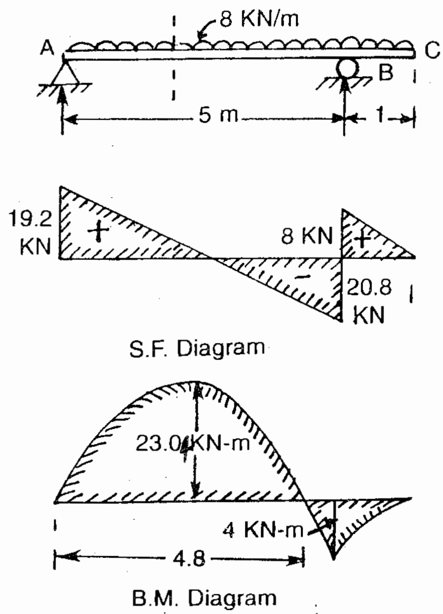


Fig. 5.46

Taking moments about B,

$$R_A . 5 - 8 \times 5 \times 5/2 + 8 \times 1 \times \frac{1}{2} = 0$$

$$5\,R_A - 100 + 4 = 0$$

$$R_A = 96/5 = 19.2 \text{ KN.}$$

Taking moments about A,

$$-R_B \times 5 + 8 \times 5 \times 5/2 - (8 \times 1)(1/2 + 5)$$

$$-5\,R_B + 100 + 44 = 0$$

$$R_B = 144/5 = 28.8 \text{ KN -}$$

Shear Force ;

Shear Force at A = + 19.2 KN

$S.\ F.$ just on the left of B = + 19.2 − 8× 5 = 19.2 − 40 = − 20.8 KN.

$S.\ F.$ just to the right of B = + 8 × 1 = 8 KN

$S.\ F.$ at C = Zero

$S.F.$ at xx $F_{xx} = R_A - w.x = 19.2 - 8\,xx$

Bending moment at A = zero

Bending moment at $x\text{-}x$ $M_{xx} = R_A.x - w.x.\,x/2$

Bending moment at B where $x = 5$

$$= 19.2 \times 5 - 8 \times \frac{5^2}{2} = 96 - 100 = -4 \text{ KN–m.}$$

Bending moment at $C = 0$

Maximum Bending moment occurs where $S.\ F.$ is zero.

$F_{xx} = R_A - w.x = 0$ or $19.2 - 8.x = 0$ or $x = 2.4$ m.

Now put this value of x in the general equation for $B.M.$

$$M_{xx} = R_A\,.\,x - w.x.\,\frac{x}{2}$$

When $x = 2.4$ m.

$$BM_{max} = 19.2\,(2.4) - 8\,(2.4)\left(\frac{2.4}{2}\right)$$

$$= 46.08 - 23.04 = 23.04 \text{ KN-m}$$

For point of contraflexure, equate the general equation of $B.M.$, M_{xx} to zero

$$M_{xx} = R_A\,.\,x - \frac{wx^2}{2} = 0$$

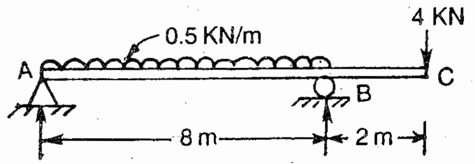
$$= 19.2\,.x - \frac{8x^2}{2} = 0$$

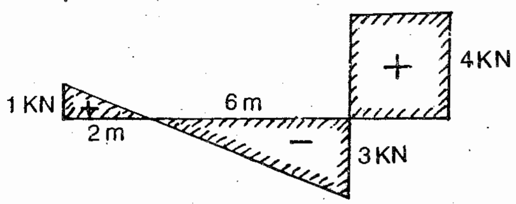
$$= 19.2\,x - 4\,x^2 = 0$$

$$x = \frac{19.2}{4} = 4.8 \text{ m from A.}$$

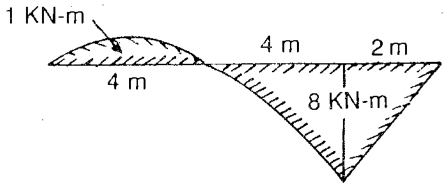
Example 5.25

A beam ABC of span 10 metres is hinged at A and supported at a point B at a distance of 8 m. from the hinge. The beam supports a concentrated load of 4 KN at C and a uniformly distributed load of 1/2 KN per metre from A to B. Draw the bending moment and shear force diagrams and locate the point of contraflexure, if any -





S.F. Diagram



B.M. Diagram

Fig. 5.47

Solution

Taking moments about B,

$$+R_A \cdot 8 - 0.5 \times 8 \times \frac{8}{2} + 4 \times 2 = 0$$

$$8\,R_A = 16 - 8 = 8$$

$$R_A = 1 \text{ KN.}$$

Taking moments about A..

$$-R_B \times 8 + 0.5 \times 8 \times 8/2 + 4 \times 10 = 0$$

$$-8\,R_B + 16 + 40 = 0$$

$$R_B = 56/8 = 7 \text{ KN.}$$

Shear Forces–

Shear Force at A =1 KN

Shear Force just on the left hand side of B $= 1 - 0.5 \times 8$
$= -3$ KN.

Shear Force just on the right hand side of B = 4 KN.

Shear Force at x-x $F_{x\text{-}x} = R_A - wx.$

Bending moments –

Bending moment at A = zero.

Bending moment at x-x, $M_{x-x} = R_A \cdot x - w.x.\frac{x}{2}$

Bending moment at B, $M_B = 1 \times 8 - 0.5 \times 8 \times 8/2$
$= 8 \times 1 - 16 = -8$ KN – m.

Maximum bending moment occurs where.

Shear Force is zero,

$$F_{x-x} = R_A - w.x = 0$$

$$1 - 0.5.x = 0 \quad \text{or} \quad x = \frac{1}{0.5} = 2 \text{ m.}$$

Now put this value of x in the general equation of Bending moment

$$M_{xx} = R_A \cdot x - w.x \cdot \frac{x}{2}$$

$$= 1 \times 2 - 0.5 \times 2 \times 2/2$$

$$= 2 - 1 = 1 \text{ KN - m}$$

Point of contraflexure –

For point of contraflexure, equate the general equation of $B.M.$ to zero

$$M_{x\text{-}x} = R_A \, . \, x - \frac{w\,x^2}{2} = 0$$

$$= 1 \, . \, x - 0.5\left(\frac{x^2}{2}\right) = 0$$

$$= x - \frac{x^2}{4} = 0 \qquad \text{or} \quad 4x - x^2 = 0$$

$$x\,(4 - x) = 0 \qquad \text{or} \qquad x = 4\text{m From } A.$$

Example 5.26

A beam of 4 metres span rests on supports 3 metres apart and over hangs one metre from support A.. the beam carries a uniformly distributed load of 4 KN per metre over a length of 2 metres as shown in the figure. A concentrated load of 9 KN acts at one metre from support B. Calculate the S.F. and B.M. and draw the diagrams.

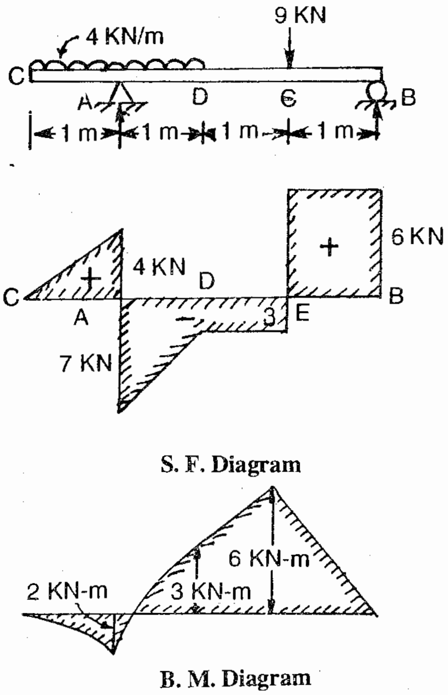


Fig. 5.48

Taking moments about B

$$R_A \times 3 - 4 \times 2\,(3) - 9 \times 1 = 0$$

or $\quad R_A = 11$ KN and $R_B = 17 - 11 = 6$ KN

Shear Force

Shear Force at $C = 0$

Shear Force just to left of $A = +4$ KN

Shear Force just to right of $A = +4 - 11 = -7$ KN

Shear Force at $D = +4 - 11 + 4 = -3$ KN

Shear Force just to the right of $D = -3$ KN

Shear Foce just to the left of $E = -3$ KN

Shear force just to the right of $E = -3 + 9 = +6$ KN

Shear Force just at $B = +6$ KN

Bending moment

$$B.M_c = 0$$

$$B.M_A = 4 \times 1 \times \frac{1}{2}$$

$$= -2 \text{ KN-m}$$

$$B.M_D = R_A \times 1 - 4 \times 2\left(\frac{2}{2}\right) = 11 \times 1 - 8 = 3 \text{ KN-m}$$

$$B.M_E = R_A \times 2 - 4 \times 2\left(\frac{2}{2} + 1\right) = 11 \times 2 - 16$$

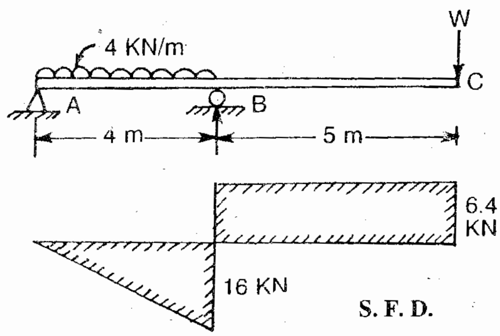
$$= 22 - 16 = 6 \text{ KN-m}$$

$$B.M_B = R_A \times 3 - 4 \times 2(3) - 9 \times 1$$

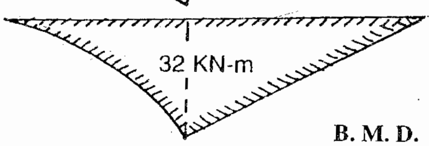
$$= 33 - 24 - 9 = 0$$

Example 5.27

An overhanging beam 9 metres long is supported on two supports 4 metres apart.. A uniformly distributed load of 4 KN per metre run is applied over the portion AB. Determine the magnitude of the load W applied at C. So that the reactions at A is zero.



S. F. D.



B. M. D.

Solution

Let a load w act at c, so that reaction at A is zero

Taking moments about B,

$$W \times 5 - 4 \times 4 \times 4/2 = 0$$

$$W = \frac{32}{5} = 6.4 \text{ KN.}$$

Reaction at $B = 4 \times 4 + 6.4$
$= 16 + 6.4$
$= 22.4$ KN.

Shear Force,

$S.F.$ at A = Zero since the reaction at A is zero.

Shear force just to the left of $B = 4 \times 4 = 16$ KN

Shear force just to the right of $B = 6.4$ KN.

Bending moment at A = zero

Bending moment at $B = 4 \times 4 \times \frac{4}{2} = 32$ KN – m

Bending moment at C = zero.

Example 5.28

A beam 8 metres long is hinged at A and freely supported at B and supports a.u.d. l of 10 KN/m over the entire length. A point load of 30 KN acts at C as shown in fig. 5.50 Draw the S.F. and B.M diagrams.

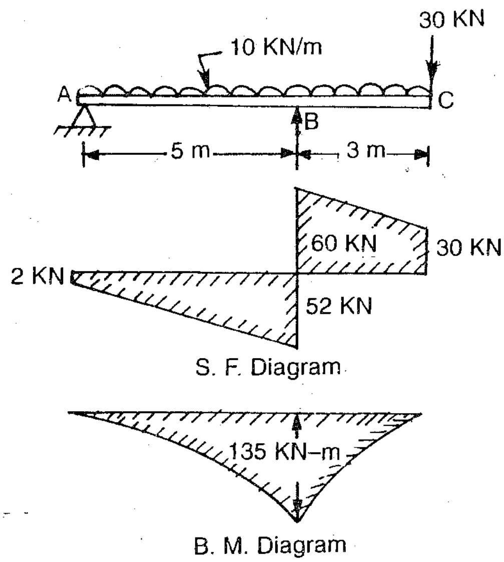


Fig. 5.50

Taking moments about A

$$R_B \times 5 = 30 \times 8 + (10 \times 8) \times \frac{8}{2}$$

$$R_B = \frac{560}{5} = 112 \text{ KN}$$

$$\therefore R_A = 10 \times 8 + 30 - 112 = -2 \text{ KN}$$

Reaction of hinge at A will be -2 KN $\downarrow$ (down word)

$$S.F_A = -\downarrow 2 \text{ KN}$$

$S.F.$ just to the left of B $= -(2 + 10 \times 5) = -52$ KN

$S.F$ just to the right of B $= -52 + 112 = 60$ KN

$S.F$ at $C = +60 - 30 = +30$ KN.

Bending moment at $A = 0$

$$B.M \text{ at }_B = 2 \times 5 + 10 \times 5 \times \frac{5}{2} = 10 + 125 = 135 \text{ KN} - \text{m}$$

$$B.M_C = 2 \times 8 + (10 \times 8)\left(\frac{8}{2}\right) - 112 \times 3$$

$$= 16 + 320 - 336 = \text{zero}$$

Shear force and bending moment diagrams are shown in fig 5.50

Example 5.29

*A beam AB 10 metres long overhangs 2 metre to the left of support A and carries a uniformly varying load as shown in fig 5.51 Draw the **B.M.** and S.F. diagrams.*

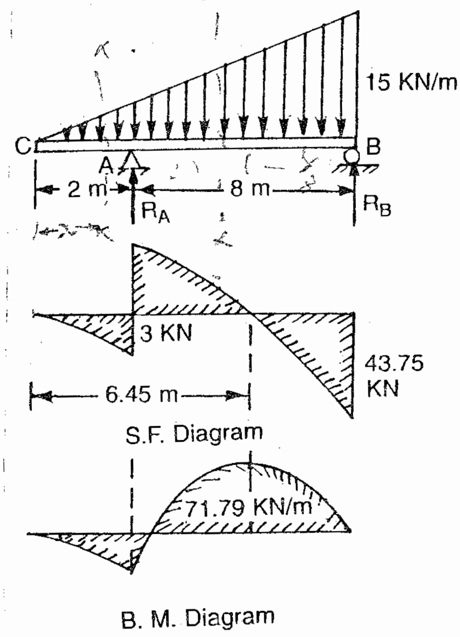


Fig. 5.51

Solution

Taking moments about B

$$R_A \times 8 = \frac{15 \times 10}{2} \times \frac{10}{3}$$

$$R_A = \frac{750}{24} = 31.25 \text{ KN}$$

$$R_B = 75 - 31.25 = 43.75 \text{ KN}$$

Shear Force

Consider a section x-x at a distance x from C. The intensity of load is $w = 1.5\,x$ KN/m

$S.F$ between C and A when $x < 2$

$$S.Fx\text{-}x = -1.5\,x \times \frac{x}{2} = -0.75\,x^2$$

$S.F_C = 0$

$S.F$ at $x = 1$ m, $= -0.75$ KN

$S.\ F.$ at $x = 2$ m, $= 0.75\,(2)^2 = -3$ KN

$S.\ F.$ between A and B When $x > 2$

$$S.F._{xx} = R_A - 0.75x^2$$
$$= 31.25 - 0.75\,x^2$$

$S.\ F.$ *at* $x = 2$ m, $= 31.25 - 0.75\,(2)^2 = 31.25 - 3 = 28.25$ KN

$S.\ F.$ *at* $x = 4$m, $= 31.25 - 0.75\,(4)^2 = 31.25 - 12 = 19.25$ KN

$S.\ F.$ will be Zero When $F_{xx} = 31.25 - 0.75\,(x)^2 = 0$

or $\quad x^2 = \dfrac{31.25}{0.75} = 41.66$ or $x = 6.45$ *from* C

$S.\ F.$ at $x = 10$ m or S. F. at B $= 31.25 - 0.75\,(10)^2 = 31.25 - 75$
$= -43.75$ KN

Bending moment

$$B.\ M_c = 0$$

Bending moment between C and A when $x < 2$

$$M_{xx} = 0.75x^2 \times \frac{x}{3} = 0.25\,x^3$$

$$B.M_x = 1\text{m},\ 0.25\,(1)^3 = .25 \text{ KN} - \text{m}$$
$$B.\ M_x = 2\text{m} = 0.25(2)^3 = 2 \text{ KN} - \text{m}$$

Bending moment between A and B When $x > 2$

$$M_{xx} = R_A\,(x-2) - w.\ \frac{x}{3}$$
$$= R_A\ (x-2) - (0.75x^2)\,\frac{x}{3}$$
$$= 31.25\,(x-2) - 0.25x^3$$

$B.M.$ at $x = 3$ m, $= 31.25\,(3-2) - 0.25\,(3)^3 = 31.25 - 6.75 = 24.50$KN–m

at $x = 6$m, $= 31.25\ (6-2) - 0.25\,(6)^3 = 125 - 54 = 71$KN – m

at $x = 8$m, $= 31.25\ (8-2)\ 0.25\,(8)^3 = 187.5 - 128 = 59.5$ KN –m at $x = 9\text{m}, = 31.25\,(9-2) - 0.25\,(9)^3 = 218.75 - 182.25 = 36.50$ KN–m

at $x = 10$ m, $= 31.25\,(10-2) - .75\,\dfrac{(10)^3}{3} = 250 - 250 = 0$

Bending moment will be maximum where shear force is Zero i.e. at $x = 6.45\text{m}$ from C

B. M. at $x = 6.45\text{m} = 31.25\,(6.45-2) - 0.25\,(6.45)^3$

$= 139.0625 - 67.0840 = 71.97$ KN m

S. F. and *B.M.* diagrams are show in fig. 5.51

Beams Subjected To Inclined Loading

Example 5.30

A beam AB 10 metres long is loaded as shown in fig. 5.52. Draw the S.F and B. M. diagrams.

→ +ve

← −ve

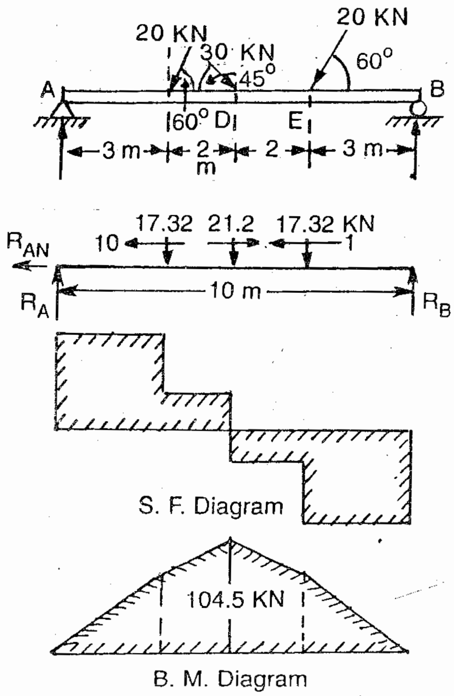


Fig. 5.52

Solution

Resolving the forces vertically and honizontally, the total horizontal force on the beam is

$$+\overleftarrow{20}\,\text{Cos}60° - \overrightarrow{30}\,\text{Cos}\,45° + \overleftarrow{20}\,\text{Cos}\,60°$$

$$20 \times \frac{1}{2} - 30 \times \frac{1}{\sqrt{2}} + 20 \times \frac{1}{2}$$

$$= + 10 - 21.2 + 10 = \overrightarrow{-1.2}\text{ KN}$$

$$\therefore\ R_{AH} = \overleftarrow{1.2}\text{ KN}$$

Taking moments about B

$$R_{AV} \times 10 = 20 \sin 60^\circ \times 7 + 30 \sin 45^\circ \times 5 + 20 \sin 60^\circ \times 3$$

$$= 20 \times \frac{\sqrt{3}}{2} \times 7 + 30 \times \frac{1}{\sqrt{2}} \times 5 + 20 \times \frac{\sqrt{3}}{2} \times 3 = 279.25$$

$$R_{AV} = 27.925\text{ KN}$$

$$R_B = (17.32 + 21.2 + 17.32) - 27.925 = 27.925\text{ KN.}$$

Shear Force.

$S.F_A = 27.925$ KN.

$S.\ Fc. = 27.925 - 17.32 = 10.605$ KN.

$S.F_D = 10.625 - 21.21 = -10.625$ KN

$S.F_E = 10.625 - 17.32 = -27.925$ KN.

$S.F_B = 27.925$

Bending moment

$B.\ M_A = 0$

$B.M_C = 27.925 \times 3$

$= 83.77$ KN–m

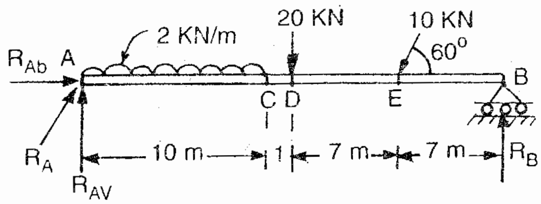
$B.M_D = 27.925 \times 5 - 17.32 \times 2 = 104.5$ KN

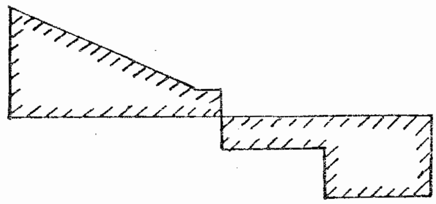
$B.M_E = 27.925 \times 7 - 17.32 \times 4 - 21.2 \times 2 = 83.77$ KN

$B.M_B = 27.925 \times 10 - 17.32 \times 7 - 21.2 \times 5 - 17.32 \times 2 = 0$

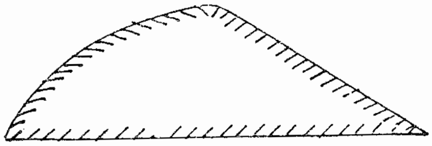
Example 5.31

A beam AB of span 25 metres is resting on fixed support at A and on rollers at B. It carries a.u.d. L.of 2KN/m over the portion AC and a Point load of 20KN at D. A load of 10KN inclined at 60° to the beam is applied at E. Calculate support reactions and draw the S. F. and B.M. diagrams.





S. F. Diagram



B. M. Diagram

Fig. 5.53

Solution

Calculation for support reactions R_{Ah}, R_{Av} and R_B

Since $\Sigma H = 0$

$$\overrightarrow{R_{Ah}} - \overleftarrow{10} \cos 60° = 0$$

$$R_{Ah} = 5 \text{ KN}$$

Since $\Sigma v = 0$

$$R_{Av} + R_B - 2 \times 10 - 20 - 10 \sin 0° = 0$$

$$\therefore R_{Av} + R_B = 40 + 10 \times 0.866 = 48.66 \text{ KN.}$$

Taking moments about A

$$-R_B \times 25 + 10 \sin 60° \times 18 + 20 \times 11 + 2 \times 10 \times \frac{10}{5} = 0$$

$$25 \ R_B = 476 \quad \text{or} \quad R_B = 19.04$$

$$R_{AV} + R_B = 48.66$$

$$R_{AV} + 19.04 = 48.66 \quad \text{or} \quad R_{AV} = 29.62 \text{ KN}$$

Shear Force

S.F. at A = 29.62 KN.

S.F. at $C = 29.62 - 2 \times 10 = 9.62$KN

S.F. at $D = 29.62 - 2 \times 10 - 20 = -10.38$KN.

S.F. at $E = 29.62 - 2 \times 10 - 20 - 10 \sin 60$

$= 29.62 - 20 - 20 - 10 \times 0.866$

$= -19.04$ KN.

Bending moments –

B.M. at A = zero

B.M. at C = RA $\times 10 - wx.x/2$

$= 29.62 \times 10 - 2 \times 10 \times 10/2$

$= 296.2 - 100 = 196.2$ KN.m

B.M. at $D = 29.62 \times 11 - 2 \times 10 \ (10/2 + 1)$

$= 205.82$ KN–m

B.M. at $E = 29.62\ (18) - 2 \times 10 \ (10/2 + 1 + 7) - 20 \times 7$

$= 133.28$ KN–m

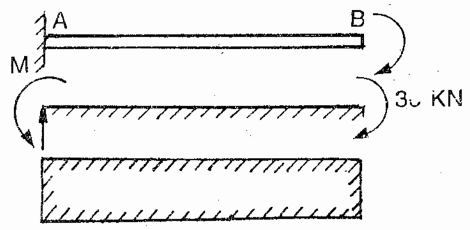
B.M. at *B.* $= 29.62 - 2 \times 10 \ (10/2 + 1 + 7 + 7 - 20\ (14) - 10 \times 0.866 \times 7$

= Zero

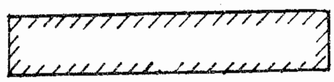
Cantilevers Subjected to couples

Example 5.32

A moments of 30 KN -mm is applied at the free end of a cantilever of span 3 meters. Draw the shear force and bending moments Diagrams.



S. F.D.



B.M.D.

Fig. 5.54

Reactions at the fixed end of the cantilever are shown. It consists of an anticlockwise moments of 30 KN.m and an upward reaction $R_A = 0$, Hence shear force will be a straight line and B.M.between B and A will be 30 KN–m.

Example 5.33

Draw B.M. diagram for the cantilever shown in fig. 5.55

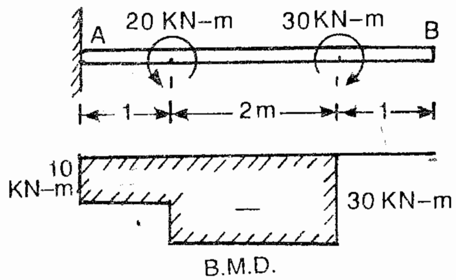


Fig. 5.55

Bending moments between B and D = 0

$B.M.$ between C and D $= -30$ KN – m

$B.M.$ between C and A $= -30 + 20$

$= -10$ KN – m

Example 5.34

A cantilever AB of span 6 meters is fixed at A and loaded as shown in the figure. Determine the reaction at A and draw the shear force and bending moments diagrams.

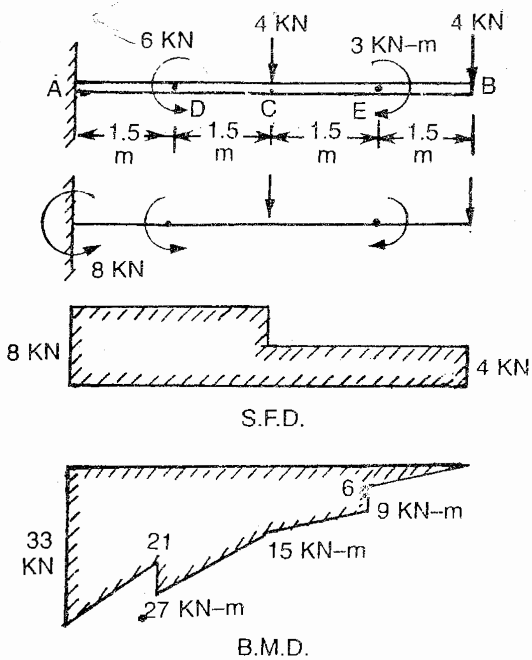


Fig. 5.56

Solution

Total load on the cantilever, $= 4 + 4 = 8$ KN

Taking moments about A,

$$- 4 \times 6 - 3 - 4 \times 3 + 6$$

$$= -24 - 3 - 12 + 6 = 33 \text{ KN} - \text{m} \quad \text{(clockwise)}$$

Balancing reacting moments at $A = +33$ KN-m anticlockwise

Hence the reaction at A will consist of an upward force of 8KN and an anticlockwise reacting moment of 33 KN – m

Shear Force from A to $C = 8$ KN

Shear Force from C to $B = 4$ KN

Bending moments at B = Zero

Bending moment just on the right hand side of E

$$= -4 \times 1.5 = -6 \text{ KN-m}$$

Bending moment just on the left hand side of E

$$= -6 - 3 = -9 \text{ KN} - \text{m}$$

Bending moment at C $= -4 \times 3 - 3 = -15$ KN – m

Bending moment just on the right hand side of D,

$$= -4 \times 4.5 - 3 - 4 \times 1.5 = -27 \text{ KN} - \text{m}$$

Bending moment just on the left hand side of D.

$$= -27 + 6 = -21 \text{ KN} - \text{m}$$

Bending moments at $A = -4 \times 6 - 3 - 4 \times 3 + 6 = -33$ KN-m

Beam with a couple at the centre :

The figure shows a beam AB of span L hinged at A and B and subjected to a couple M $KN - m$ at mid span.

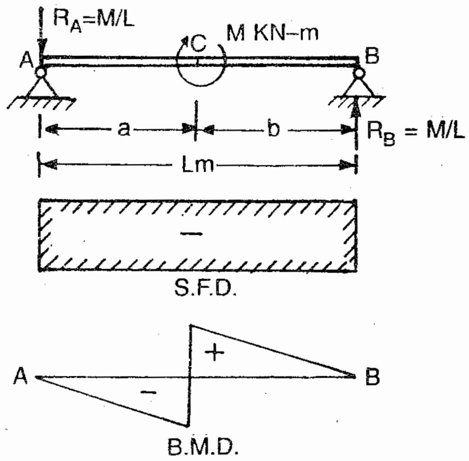


Fig. 5.57

Taking moments of all the forces about A

$$-R_B \times L + M = 0$$

$$R_B = \frac{M}{L} \text{ KN} \quad \text{(upwards)}$$

Taking moments about B

$$R_A \times L + M = 0$$

or $$R_A = -\frac{M}{L}. \quad \text{(downwards)}$$

Shear Force at A $= \dfrac{-M}{L}$. As there is no load between A & B, the $S.F.$ between A & B is constant throughout and is equal to $-M/L$

$B.M.$ at $A = 0$

$B.M.$ just on the left hand side of $C = -\dfrac{M}{L}.a$

$B.M.$ just on the right hand side of $C = +\dfrac{M}{L}.b$

$B.M.$ at $B = 0$

Shear Force and Bending moment diagrams are shown in the figure.

Simply supported beam susbjected to a couple at one end.

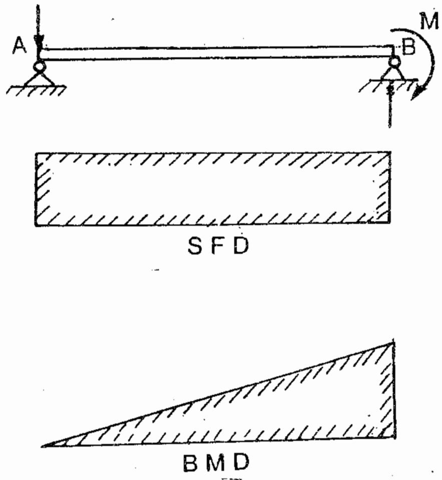


Fig. 5.58

Taking moments of all the forces about A,

$$M - R_B \times L = 0$$

$$R_B = \frac{M}{L} \qquad \text{(upwards)}$$

Taking moments of all forces about B,

$$M + R_A \times L = 0$$

$$R_A = -\frac{M}{L} \qquad \text{(downward)}$$

Shear force in the beam is constant throughout $= \frac{M}{L}$

Bending moments at any section x-x from A,

$$= -\frac{M}{L} . x$$

Shear force and $B.M.$ diagrams are shown in the figure.

Example 5.35

Draw the shear force and bending moment diagrams for a beam of span 8 meters simply supported at A and B. The beam carries a uniformly distributed load 10 KN/m from A to C. An anticlockwise couple of 120 KN–m is also acting at 2 meters from end B.

Total load couple is not accounted. also also in S.F.

Solution

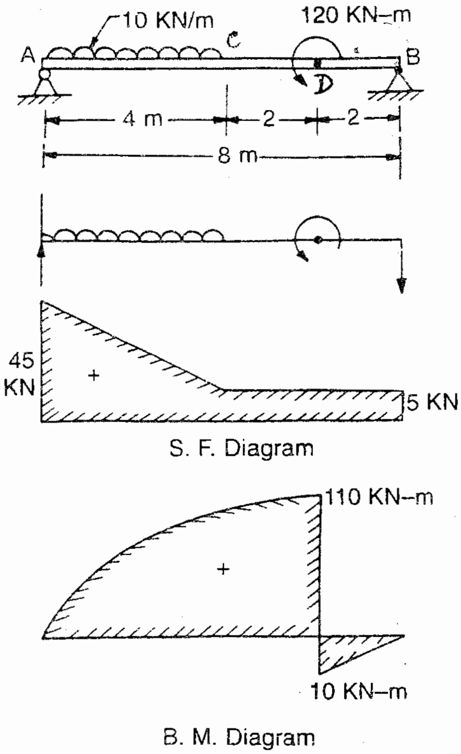


Fig. 5.59

Taking moments of all Forces about A

$$R_B \times 8 = 120 - 10 \times 4 \times \frac{4}{2}$$

$$R_B = \frac{120-80}{8} = 5 \text{ KN} \downarrow \quad \text{(downwards)}$$

$$R_A = 10 \times 4 + 5 = 45 \quad \text{(upwards)}$$

Shear force between C and $B = +5$ KN

Shear force at $A = 45$ KN

From A to C shear force will change from 45 KN to 5 KN

$B.M.$ at A = Zero

$$B.M. \text{ at } C = 45 \times 4 - 10 \times 4 \times \frac{4}{2}$$

$$= 180 - 80 = +100 \text{ KN–m}$$

$B.M.$ just on the right side of $D = -5 \times 2 = -10$ KN–m

$B.M.$ just on the left side of $D = -10 + 120 = +110$ KN–m

Example 5.36

Draw the shear force and bending moment diagrams for the beam shown in figure 5.60

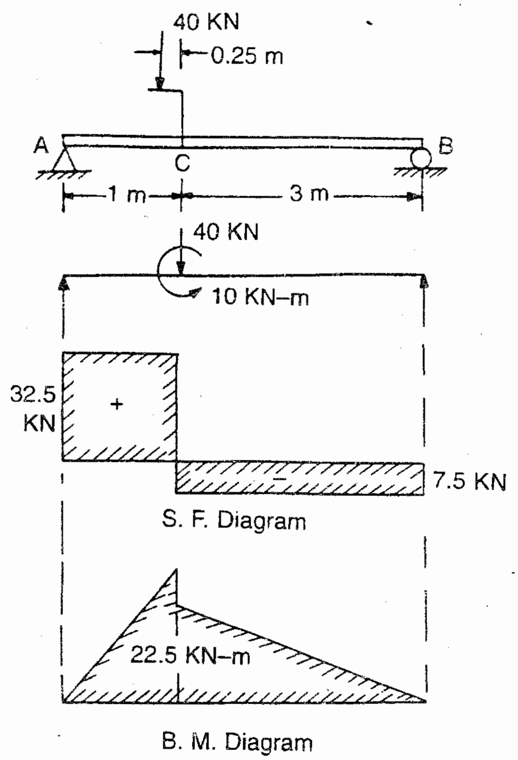


Fig. 5.60 B.M.Diagram

Solution

The load on the bracket will produce an anticlockwise moment of $40 \times 0.25 = 10$ KN at C and a vertical load of 40 KN at C.

Taking moments about B.

$R_A \times 4 = 40 \times 3 + 10 = 130$

$R_A \quad = 325$ KN

$R_B \quad = 40 - 32.5 = 7.5$ KN

Shear force at A = 32.5 KN

Shear force between A and C = 32.5KN

Shear force between C and B will be 7.5 KN

Bending moment at A = Zero

Bending moment at $C = 32.5 \times 1 = 32.5$ KN–m

Bending moment at C will drop from 32.5 KN–m to (32.5 - 10 KN-m due to the couple) = 22.5 KN–m

Bending moment at B $= 32.5 \times 4 - 40 \times 3 - 40 \times 0.25$
$= 130.0 - 120 - 10$
$=$ Zero

Example 5.37

Draw the shear force and bending moment diagrams for the beam shown in figure 5.61

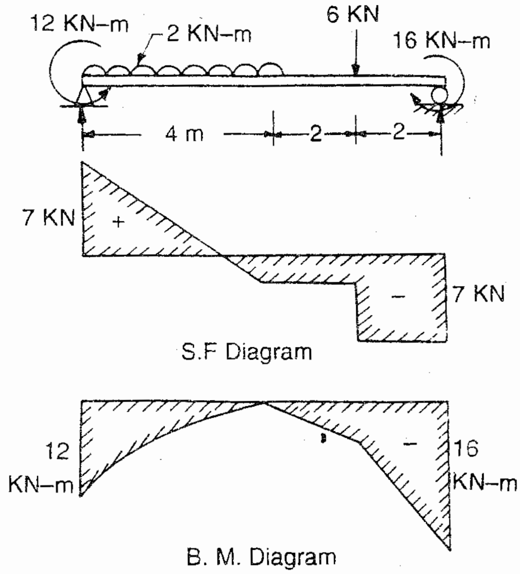


Fig. 5.61

Solution

Taking moments of all forces about B

$$R_A \times 8 - 12 - 2 \times 4 \left(\frac{4}{2} + 4\right) - 6 \times 2 + 16 = 0$$

$$R_A = \frac{12 + 48 + 12 - 16}{8} = \frac{56}{8} = 7\text{KN}.$$

$$R_B = 6 + 2 \times 4 - 7 = 7 \text{ KN}$$

Shear force -

$S.\ F.$ at $A = +7$ KN.
$S.\ F.$ at $x\text{-}x = 7 - 2 \times x$
$S.\ F.$ is zero at $x = 3.5$ m from A.
$S.\ F.$ at $C = 7 - 4 \times 2 = 1$ KN
$S.\ F.$ between C and D
$= -1$ KN
$S.\ F.$ at $D = -1 - 6 = 7$ KN
$S.\ F.$ at $B = 7$ KN.

Bending moment : -

$B.M.$ at A = − 12 KN − m

$B.M.$ at $C = +7\times 4 - 12 - 2\times 4\times\frac{4}{2} = 0$

$B.M.$ at $D = 7\times 6 - 12 - 2\times 4\ (4/2 + 2)$
$= 42 - 12 - 32 = -2$ KN − m.

B. M. at $B = 7\times 8 - 2\times 4\ (4/2 + 4) - 12 - 6\times 2$
$= 56 - 48 - 12 - 12 = 56 - 72$
$= -16$ KN-m.

SUMMARY

1. A beam remains in stable equilibrium under the following conditions.
   (a) Algebraic sum of all forces in any direction is Zero
   (b) Algebraic sum of the moments of all forces about any point is zero.
   $\Sigma H = 0$, $\Sigma V = 0$ and $\Sigma M = 0$
2. Shear force at a section of a beam is the algebraic sum of all vertical forces to any one side of the section.
3. Bending moment at a section of a beam is the algebraic sum of the moments of all forces to one side of the Section.
4. When external forces acting on the portion of a beam to the left of the section tend to push that part up, the shear forces is positive or when the external forces acting on the portion of a beam to the right of the sction tend to push that part down the shear forces is positive.
5. Moments producing compression in the top fibre and tension in the bottom fibre are positive.
6. Moments which try to bend the beam upwards and cause compression in the bottom and tension in the top fibres are taken negative
7. $B.M.$ at a section is positive if it is sagging and negative if it is hogging
8. $B.M.$ is maximum at the point where $S.F.$ is zero or where it changes direction from +Ve to −Ve or Vice -Versa.
9. The point of contra flexure or the point of inflexion is the point where $B.M.$ Change its sign from positive to negative or Vice - Versa $B.M.$ at the point of Contra flexure is zero.
10. $\frac{dF}{dx} = w$, and $\frac{dM}{dx} = F$

QUESTIONS

1. How are beams classified? show by drawing sketch various types of beams you know.
2. Show by sketches no. of reactions offered by.
   (a) Simple or Roller support
   (b) Hinged support
   (c) Fixed support

3. Define the following.
   (a) Shear force at a section of a beam
   (b) Bending moment at a section of a beam.
4. Establish the relationship between *S. F.* and *B. M.* at a section of a beam.
5. Define point of Contraflexure. What is the maximum value of *B. M.* at this point ?

EXERCISES

6. A cantilever of span 4 meters, supports concentrated loads of 5 KN at the free end and point loads of 4 KN and 3 KN at 1 metre and 2 metres from the fixed end. Draw the *S. F.* and *B. M.* diagrams. ($S. F._{max}$ = 12 KN, M_{max} = 30 KN-m)
7. A cantilever 3 metre long carries a uniformly distributed load of 10 KN over a length of 2 metres from the fixed end A and a point load of 10 KN at the free end. Draw the *S. F.* & *B. M.* diagrams. (SF_A = 30 KN, BM_A = 130 KN-m)
8. A simply supported beam of span 4 metres carries two point load of 4 KN each at 1 metre and 3 metres from the left end support. It also carries a *u.d.l.* of 2 KN/m over a central length of 2 metres. Calculate the maximum shear force and bending moment and draw the *S. F.* & *B. M.* diagrams.

   $S.F._{max}$ at A & B = 6 KN, ($B.M._{max}$ at mid span = 14 KN-m)
9. Draw the shear force and bending moment diagrams for the beam shown in figure 5.62. Calculate the Value of *S.F.* & *B. M.* at *C, D* & *E.*

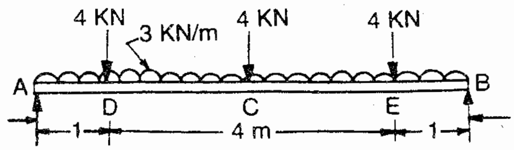


Fig. 5.62

$S.F._A$ = B = 15 KN
$S.F._D$ = E = 8 KN
$S.F._C$ = 2 KN
$B.M._C$ = 25.5 KN-m
$BM_D = BME$
= 13.5 KN-m

10 Draw the shear force and bending moment diagrams and calculate the maximum and minimum value of *S. F. and B. M.* for the beam shown in figure 5. 63.

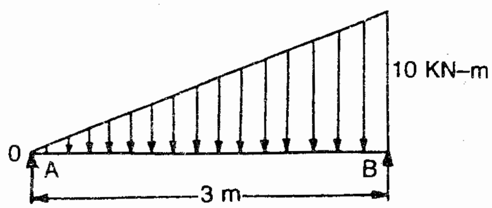


Fig. 5.62

($S.F._A$ = 5 KN , $S.F._B$ = 10 KN
BM_{max} = 5.77 KN-m at 1.268m from A)

11. A timber beam of span 10 metres 120 mm × 120 mm section floats horizontally in sea water. Two equal weights sufficient to immerse it are placed on the beam 2.8 m from each end. If the weight of timber per cubic metre is 7 KN and that of water 10 KN/. m^3, calculate the value of each load. Draw the *S. F.* and *B. M.* diagrams and state the value of max ^{m} *B. M.* (216 N and 169.3 N–m)
12. A beam 8 metres long is simply supported at *A* and *B* 5 metres apart overhangs 3 metres beyond support B. It carries a *u.d.l.* of 10 KN/m. over the entire length and a concentrated load of 30 KN at the free end. Draw the shear force and bending moment diagrams.
13. Draw the shear force and bending moment diagrams for the beam shown in figure-5.64

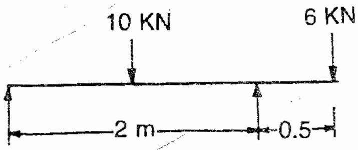


Fig. 5.64

14. A beam 9 metres long is supported at *A* and *B* 6 metres apart. It overhangs 3 metres beyond support *B* and carries a uniformly varying load of 3KN/m as shown in figure 5.65. Draw the shear force and bending moment diagrams and state the values of max *S.F.* and *B.M.*

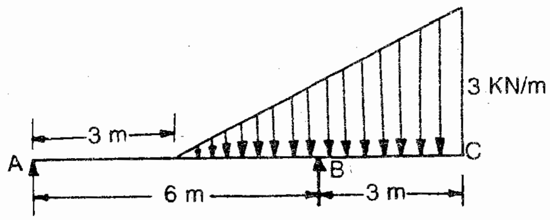


Fig. 5.65

Max ^{m} $SF._B$ = 6.75 KN

Max $BM._B$ = 11.25KN

15. A simply supported beam *AB* of span 6 metres carries a concentrated load of 4 KN at *C*, a distance of 1 metre from *A*. An anticlockwise couple of 8 KN is also acting at *C*. Draw the shear force and bending moment diagrams.
16. A beam of span 6 metres is loaded as shown in figure 5.66. Draw the B.M. and S.F diagrams.

S.F

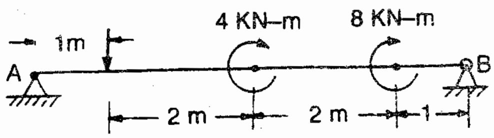


Fig. 5.66

17. A simply supported beam *AB* of span 5.5 metres carries a *udl* of 10 KN/m over a length of 4 metres from end *A*. It is subjected to a clockwise couple of 10 KN – m at a distance of 1 metre from end *B*. Construct the *B.M.* and *S.F.* diagrams.

18 Draw the *S.F.* and *B.M.* diagrams for the cantilever shown in figure 5.67

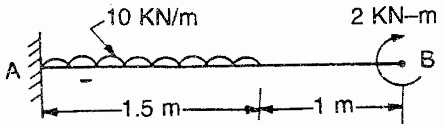


Fig. 5.67

19. A number of persons are standing in a queue on a narrow cantilever 4 metres long. Assuming that the average wf of a person is 600 N and he takes about 300 mm space while standing, calculate the maximum S.F. and B.M.
20. A shaft is fixed at one end on a lathe machine. While thread cutting was performed if the movement of the chisel put a load equivalent of 40 KN at the end. Calculate the S.F. and B.M. produced at the fixed end. The length of the shaft is 4 metres.
21. A chajja is loaded with triangular loading in such a manner that the intensity of loading is zero at the free end and 40 KN/m at the free fixed end. Construct the S.F. and B.M. if the span of the chajja is 2.5 m.
22. Calculate the reactions in the case of a beam shown in fig. and construct the *S.F.* and *B.M.* diagrams.

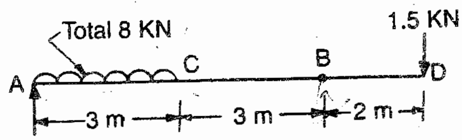


Fig. 5.68

$R_A = 5.5$ KN, $R_B = 4$ KN

□□□

6

Moment Of Inertia

First Moment Of An Elemental Area

The first moment of an elementary area about any axis in the plane of the area is the product of the area and the perpendicular distance between the elementary area and the axis.

If δa is the area of the small elemental area and x and y are the distances from OY and OX then.

Ist moment about $OX = \delta a.y$

Ist moment about $OY = \delta a.x$.

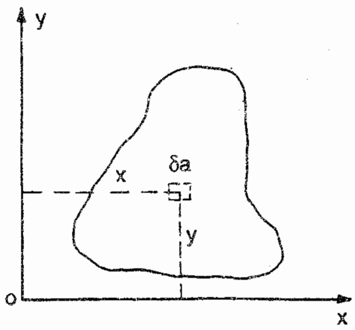


Fig. 6.1

Second Moment Of An Elemental Area

The second moment of an elementary area about any axis in the plane of the area is the product of the area and square of the perpendicular distance between the elementary area and the *axis*. This is also called ''MOMENT OF INERTIA''

Refering to figure 6.1 the moment of inertia about X - axis is $\delta a\, y^2$ and

Moment of inertia about y-axis = $\delta a.\, x^2$

If the whole body has an area A which consists of such elementary areas like δa, then the moment of inertia of the area A about any axis in the plane of the area is given by the summation of the second moment of area about the same axis of all the elements of areas contained in the total area A.

Moment of inertia about OX – *axis*

$$I_{xx} = \Sigma\, \delta a.y^2$$

Moment of inertia about OY – axis

$$I_{yy} = \Sigma\, \delta a.x^2$$

Units.

The units of moment of inertia are mm^4 and m^4.

Radius Of Gyration

It is defined as the distance at which the area may be supposed to be concentrated to produce the same moment of inertia about the given axis.

If the moment of inertia of area A about the x-axis is denoted by $I_{x\text{-}x}$, then the radius of gyration is defined by

$$K_{xx} = \sqrt{\frac{I_{x-x}}{A}}$$

Similarly the radius of Gyration with respect to Y - axis is given by

$$K_{yy} = \sqrt{\frac{I_{y-y}}{A}}$$

The Units of radius of gyration is mm.

Theorem Of Parallel Axes

The moment of inertia of an area about any axis is equal to the moment of inertia of the area about a parallel axis passing through the centroid of the area and the square of the perpendicular distance between the two axes.

Refering to figure 6.2 Let I_{XG} be the moment of inertia about an axis passing through the centroid G. Let y be its perpendicular distance from the axis OX which is parallel to GX, then

$$I_{x\text{-}x} = I_{xG} + Ay^2$$

and $I_{y\text{-}y} = I_{yG} + Ax^2$

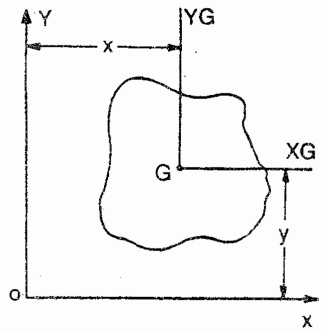


Fig. 6.2

Theorem Of Perpendicular Axes

Moment of inertia of a plane area about an axis perpendicular to the area and passing through its centroid is equal to the sum of the moment of inertia of the area about two mutually perpendicular axes passing through the centroid and in the plane of the area.

$$I_{zz} = I_{xx} + I_{y\text{-}y}$$

Polar Moment Of Inertia

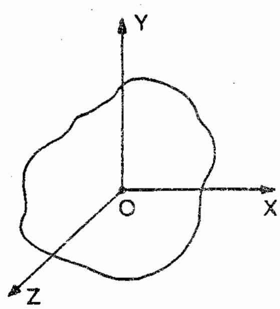


Fig. 6.3

The moment of inertia of a plane area with respect to an axis perpendicular to the plane of the area is called polar moment of inertia

$$I_{zz} = I_{xx} + I_{yy}$$

Section Modulus

It is the property of a section and is determined by dividing moment of inertia or the second moment of the area about an axis passing through the centroid of the section by the distance of extreme fibre of the section from the axis. It is denoted by the letter Z,

$$Z = \frac{\text{M.I about centroidal axis}}{\text{Distance of extreme fibre of Section from the axis through the centroid.}}$$

Unit of section modulus is mm^3

Moment Of Inertia Of Standard Sections

1. Rectangular section : A rectangular section of width b and depth d is shown in figure 6.4. Consider a strip of width b and thickness dy at a distance y from the x–x axis

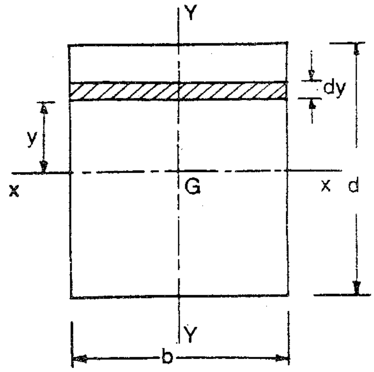


Fig. 6.4

Area of the strip = $b.dy$.

Second moment of area of this strip about x–x–axis = $b.dy.y^2$

Total moment of inertia of all the strips about x-x axis

$$I_{xx} = 2\int_0^{d/2} b.y^2 \,.\, dy = \frac{bd^3}{12}$$

Similarly moment of inertia about y–y axis

$$I_{yy} = \frac{db^3}{12}$$

2. Hollow rectangular section

$$I_{xx} = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$I_{yy} = \frac{DB^3}{12} - \frac{db^3}{12}$$

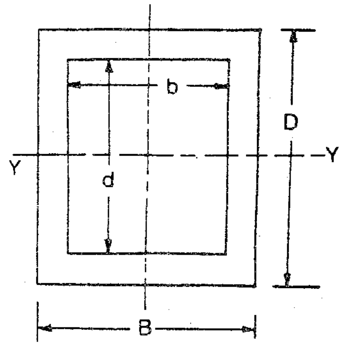


Fig. 6.5

3. Circular section of radius R.

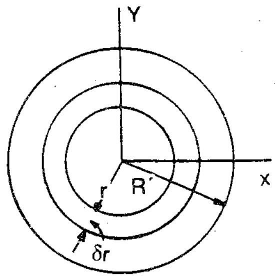


Fig. 6.6

Consider an elementary ring of radius r and thickness dr.

Area of the ring = $2\pi\, r.dr$

Polar moment of inertia of the ring about an axis passing through O

$$= 2\pi\, r \,.\, dr \,.\, r^2$$

Polar moment of inertia of the whole circle

$$= \int_0^R 2\pi\, r^3 \,.\, dr = \frac{\pi R^4}{2}$$

But $I_P = I_{xx} + I_{yy}$, for circle $I_{xx} = I_{yy}$

or $I_P = 2\, I_{xx}$

or $I_{xx} = \frac{I_P}{2} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}$

4. Hollow Circular Section.

$$I_{xx} = I_{yy} = \frac{\pi}{4}(R^4 - r^4)$$

$$= \frac{\pi}{64}(D^4 - d^4)$$

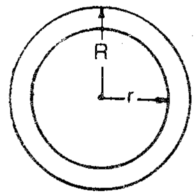


Fig. 6.7

Moment Of Inertia Of A Triangular Lamina

Let ABC be a triangle of base b and height h

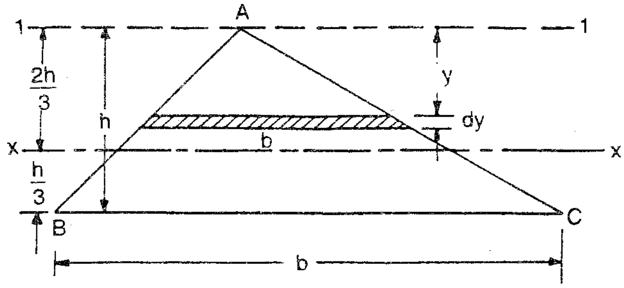


Fig. 6.8

(a) Moment of inertia about axis 1-1 through the vertex and parallel to the base

Consider an element of thickness dy at a distance y from the vertex A

Width of the element $b' = \frac{b.y}{h}$

Area of the Element $= b'.\,dy = \frac{b}{h}.\,y\,.\,dy$

Moment of inertia of the element about 1–1

$$= \frac{by}{h}.\,dy\,.\,y^2 = \frac{b}{h}.\,y^3\,.\,dy$$

∴ Moment of inertia of the whole lamina about axis 1–1

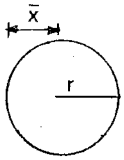
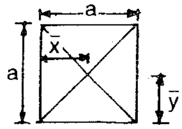
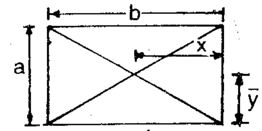
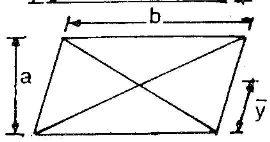
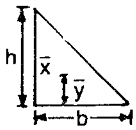
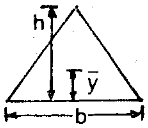
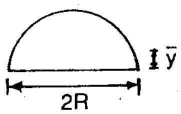
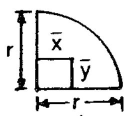
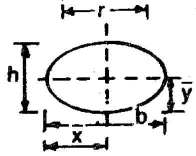
$$I_{1-1} = \frac{b}{h}\int_o^h y^3\,.\,dy = \frac{bh^3}{4}$$

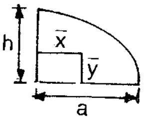
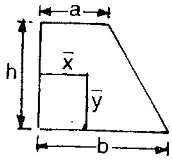
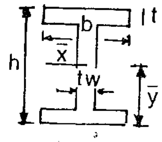
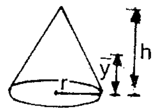
(b) Moment of inertia of the lamina about the centroidal axis parallel to the base. The centroial axis passes at a distance of $\frac{2}{3}h$ from the vertex.

$$I_{xx} = I_{1-1} + A\left(\frac{2}{3}h\right)^2$$

$$= \frac{b\,h^3}{4} + \frac{1}{2}b.h\frac{4}{9}h^2 = \frac{b\,h^3}{36}$$

Centres Of Gravity Of Important Figures
TABLE 6.1

S. No.	Figure	Area/Volume	Distance of C.G. or Centroid
1.		$A = \pi r^2$	$\bar{x} = r$
2.		$A = a^2$	$\bar{x} = a/2$ $\bar{y} = a/2$
3.		$A = ab$	$\bar{x} = b/2$ $\bar{y} = a/2$
4.		$A = ab$	$\bar{y} = a/2$
5.		$A = \frac{b\,h}{2}$	$\bar{x} = b/3$ $\bar{y} = h/3$
6.		$A = \frac{b\,h}{2}$	$\bar{y} = \frac{h}{3}$
7.		$A = \frac{\pi}{8}(2\,R)^2 = \frac{\pi R^2}{2}$	$\bar{y} = \frac{4R}{3\pi}$
8.		$A. = \frac{\pi r^2}{4}$	$\bar{x} = \bar{y} = \frac{4r}{3\pi}$
9.		$A = \frac{2}{3}b\,h$	$\bar{x} = b/2$ $\bar{y} = h/2$

S. No.	Figure	Area/Volume	Distance of C.G. or Centroid
10.		$A = \frac{1}{2} bh$	$\bar{x} = \frac{3}{8} b$ $\bar{y} = \frac{3}{8} h$
11.		$A = \frac{1}{2}(a+b)\,h$	$\bar{x} = \frac{a^2 + ab + b^2}{3\,(a+b)}$ $\bar{y} = \frac{h}{3} \times \frac{(2a+b)}{(a+b)}$
12.		$A = 2bt + (h - 2t)\,t_w$	$\bar{x} = \frac{b}{2}$ $\bar{y} = \frac{h}{2}$
13.		$V = \frac{1}{3}\pi r^2 h$	$\bar{y} = \frac{h}{4}$
14.		$V = \frac{2}{3}\pi r^3$	$\bar{y} = \frac{3r}{8}$

Example 6.1

Determine, for the plane area shown in fig 6.9

(a) Moment of inertia about x -x axis and about the base AB

(b) Moment of inertia about Y–Y axis and about side AD

(c) Least radius of gyration

(d) Section modulus

Solution

(a) Moment of inertia about

x–x axis

$$I_{xx} = \frac{b\,d^3}{12}$$

$$= \frac{60(100)^3}{12} = 5 \times 10^6 \text{ mm}^4$$

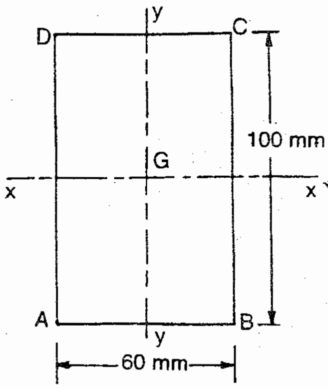


Fig. 6.9

Moment of inertia about the base AB

$$I_{AB} = I_{xx} + A_y^2$$
$$= 5 \times 10^6 + (60)(100)(50)^2$$
$$= 20 \times 10^6 \text{ mm}^4$$

(b) Moment of inertia about Y–Y axis

$$I_{yy} = \frac{d\,b^3}{12} = \frac{100(60)^3}{12}$$
$$= 18 \times 10^5 \text{mm}^4$$

Moment of inertia about side AD

$$I_{AD} = I_{yy} + Ax^2$$
$$= 18 \times 10^5 + (60)(100)(30)^2$$
$$= 72 \times 10^5 \text{ mm}^4$$

(c) Radius of gyration

$$K_{xx} = \sqrt{I_{xx}/A} = \sqrt{\frac{5 \times 10^6}{60 \times 100}} = 28.8 \text{ mm}$$

$$K_{yy} = \sqrt{I_{yy}/A} = \sqrt{\frac{18 \times 10^5}{(60)(100)}} = 17.32 \text{ mm}$$

Least radius of gyration = 17.32

(d) Section Modulus

$$Z_{xx} = \frac{I_{xx}}{y} = \frac{5 \times 10^6}{50} = 10^5 \text{ mm}^3$$

$$Z_{yy} = \frac{I_{yy}}{x} = \frac{18 \times 10^5 \text{mm}^3}{30} = 60 \times 10^3 \text{mm}^3$$ **Answer.**

Example 6.2

Determine the moment of inertia of the rectangular hollow section shown in fig 6.10 about its centroidal axes.

Solution

Moment of inertia about x– axis

$$I_{xx} = \frac{B\,D^3}{12} - \frac{b\,d^3}{12}$$
$$= \frac{(80)(120)^3}{12} - \frac{40\,(40)^3}{12}$$
$$= 1152 \times 10^4 - 21.33 \times 10^4$$
$$= (1152 - 21.33) \times 10^4 \text{ mm}^4$$
$$= 1130.67 \text{ mm}^4$$

$$I_{yy} = \frac{D\,B^3}{12} - \frac{d\,b^3}{12}$$

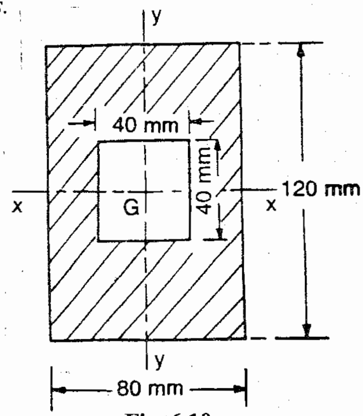


Fig. 6.10

$$= \frac{120\,(80)^3}{12} - \frac{40\,(40)^3}{12}$$
$$= 512 \times 10^4 - 21.33 \times 10^4$$
$$= 490.67 \times 10^4 \text{ mm}^4$$ **Answer.**

Example 6.3

Determine the moment of inertia of the rectangular section in which a circular hole of 20 mm dia has been drilled as shown in figure 6 .11

Solution

M.I of the given section will be

M.I of the rectangular section − *M.I* of the circular hole

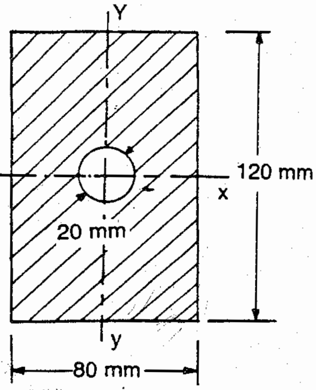


Fig. 6.11

$$I_{xx} = \frac{b\,d^3}{12} - \frac{\pi}{64}(D)^4$$
$$= \frac{80\,(120)^3}{12} - \frac{\pi}{64}(20)^4$$
$$= 1152 \times 10^4 - 785 \times 10^4$$
$$= 1151.215 \times 10^4 \text{ mm}^4$$
$$I_{yy} = \frac{d\,b^3}{12} - \frac{\pi}{64}(D)^4$$
$$= \frac{120\,(80)^3}{12} - \frac{\pi}{64}(20)^4$$
$$= 512 \times 10^4 0 - .785 \times 10^4$$
$$= 511.215 \times 10^4 \text{ mm}^4$$

Example 6.4

Determine the moment of inertia of the I-Section shown in fig 6.12

Solution

Moment of inertia of the given section will be the sum of the *M.I.* of the rectangular section (1), (2) and (3) as shown in the figure.

For rectangular sections (1) and (3)

$$I_{xx1} = I_{xG} + Ay_1^2$$
$$= \frac{60\,(10)^3}{12} + 60 \times 10\,(105)^2$$
$$= 5000 + 6615000$$
$$= 6.620000 \text{ mm}^4 = 6.62 \times 10^6$$
$$\therefore I_{xx1} = I_{xx3} = 6.62 \times 10^6 \text{ mm}^4$$

For rectangular section (2)

$$I_{xx2} = I_{XG}$$

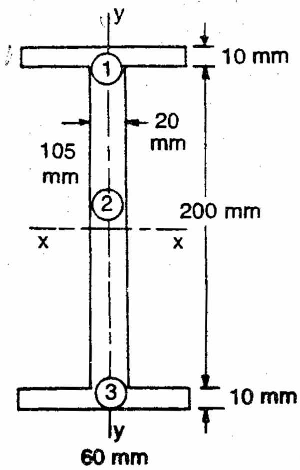


Fig. 6.12

$$= \frac{20\,(200)^3}{12} = 13.3\times\ 10^6\ \text{mm}^4$$

Moment of inertia of the given section about x–axis

$$I_{xx} = I_{xx_1} + I_{xx_3} + I_{xx_2}$$

$$I_{xx} = 6.62\times 10^6 + 6.62\times 10^6 + 13.3\times 10^6\ \ 26.54\times 10^6\ \text{mm}^4$$

Moment of inertia about Y – axis

Section (1) and (3)

$$I_{yy_1} = I_{yy_3} = I_G + A_x 2$$

$$= \frac{d\,b^3}{12} = \frac{10\,(60)^3}{12} + 0$$

$$= 18\times 10^4\ \text{mm}^4$$

M.I of Section (2) about Y–axis

$$= \frac{d\,b^3}{12} = \frac{200\,(20)^3}{12} + 0$$

$$= \frac{4}{3}\times 10^5\times 13.3\times 10^4\ \text{mm}^4$$

Moment of inertia of the given section about *Y–axis*

$$I_{yy} = I_{yy_1} + I_{yy_2} + I_{yy_3}$$

$$= 18\times 10^4 + 18\times 10^4 + 13.3\times 10^4$$

$$= 49.3\times 10^4\ \text{mm}^4$$ **Answer.**

Example 6.5

1. *Determine the moment of inertia of the Unsymmetrical I section shown in fig. 6.13 about its centroidal axes.*

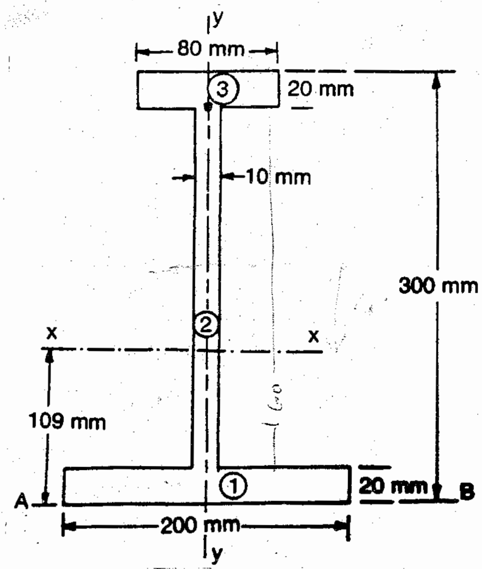


Fig. 6.13

Solution :-

Let $\bar{y}$ be the distance of X–axis from the axis of reference AB,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{(a_1 + a_2 + a_3)}$$

$$= \frac{(200)\,(20)\,(10) + (260)\,(10)\,(150) + (80)\,(20)\,(290)}{4000 + 2600 + 1600}$$

$= 109$ mm from AB.

Moment of inertia of the given section will be the sum of the M.I of the rectangular sections (1), (2) and (3) as shown in fig. 6.13

For rectangular section (1)

$$I_{xx} = I_{xG} + A_{y_1}{}^2$$

$$= \frac{200\,(20)^2}{12} + (200)\,(20)\,(109 - 10\,)^2$$

$$= 13.3 \times 10^4 + 3920.4 \times 10^4 = 3933.7 \times 10^4 \text{ mm}^4$$

For the 2nd rectangular section

$$I_{xx_2} = I_{xG} + A\,y_2{}^2$$

$$= \frac{10\,(260)^3}{12} + (10)\,(260)\,(41)^2$$

$$= 1464.6 \times 10^4 = 437.06 \times 10^4 = 1901 \times 10^4 \text{ mm}^4$$

For the third section.

$$I_{xx_3} = I_{xG} + A.y_3{}^2$$

$$= \frac{80\,(20)^3}{12} + (80)\,(20)\,(101)^2$$

$$= 5.33 \times 10^4 + 5241.76 \times 10^4 = 5247 \times 10^4 \text{ mm}^4$$

Moment of inertia of the whole section about X–axis

$$I_{xx} = I_{xx_1} + I_{xx_2} + I_{xx_3}$$

$$= (3938.7 + 1901 + 5247) \times 10^4 = 11082.52 \times 10^4 \text{mm}^4$$

The section is symmetrical about Y–axis

Hence $I_{yy} = I_{yy_1} + I_{yy_2} + I_{yy_3}$

$$= \frac{20\,(200)^3}{12} + \frac{260\,(10)^3}{12} + \frac{(20)\,(80)^3}{12}$$

$$= 1333.3 \times 10^4 + 2.16 \times 10^4 + 85.5 \times 10^4$$

$$I_{yy} = 1420.96 \times 10^4 \text{ mm}^4$$

Example 6.6

Determine the moment of inertia of an equal angle section 10 mm × 100mm × 12m about both the principal axes.

$$\bar{y} = \frac{(100)\,(12)\,6 + (88 \times 12)\,(56)}{(100 \times 12) + (88 \times 12)}$$

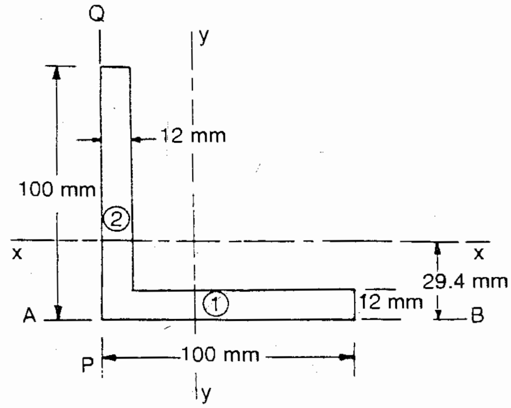


Fig. 6.14

$$= \frac{7200 + 59136}{1200 + 1056} = \frac{66336}{2256} = 29.40 \text{ mm from } AB$$

$$\bar{x} = \frac{(100)(12)(50) + (88)(12)(6)}{(100 \times 12) + (88 \times 12)} = \frac{60000 + 66336}{2256} = 29.40 \text{ from } PQ$$

Moment of inertia of the section about x – axis

$$I_{xx} = I_{xx_1} + I_{xx_2}$$

$$I_{xx_1} = \frac{100\,(12)^3}{12} + (100)\,(12)\,(29.4 - 6\,)^2$$

$$= 14400 + 657072 = 671472 \text{ mm}^4$$

$$I_{xx_2} = \frac{12(88)^3}{12} + (88)\,(12)\,(56 - 29.4)^2$$

$$= 681472 + 747183.36 = 1428655.36 \text{ mm}^4$$

$$I_{xx} = 671472 + 1428655.36 = 2100127.36 \text{ mm}^4$$

Moment of inertia about Y – *axis*

$$I_{yy} = I_{yy_1} + I_{yy_2}$$

$$I_{yy_1} = \frac{12\,(100)^3}{12} + (100)\,(12)\,(50 - 29.\,4)^2 =$$

$$= 10^6 + 509232 = 1509232 \text{mm}^4$$

$$I_{yy_2} = \frac{88\,(12)^3}{12} + (88 \times 12)\,(29.4 - 6)^2 = 12672$$

$$= 12672 + 578223.36 = 590895.36$$

$$I_{yy} = 1509232 + 590895.36 = 11100127.36 \text{ mm}^4$$ **Answer.**

Example 6.7

An Unequal angle section 100 mm × 80 mm × 10 mm stands with 100 mm side vertical Fig. 6.15 Determine the moment of inertia about horizontal and vertical axis passing through the centroid of the section.

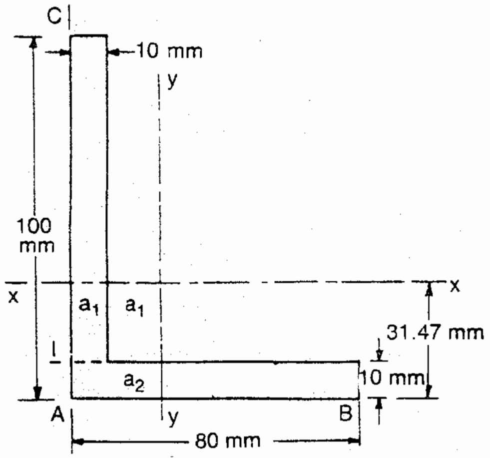


Fig. 6.15

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$$= \frac{900\,(55) + 800 \times 5}{900 + 800}$$

$$= \frac{49500 + 4000}{1700} \qquad = 31.47 \text{ mm from } AB$$

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} = \frac{900 \times 5 + 800 \times 40}{1700}$$

$$= 21.47 \text{ mm from } AC$$

$$I_{xx} = I_{xx_1} + I_{xx_2}$$

$$= \frac{10\,(90)^3}{12} + 900\,(55 - 31.41)^2 + \frac{80\,(10)^3}{12} + 80 \times 10\,(31.47 - 5\,)^2$$

$$= 60.75 \times 10^4 + 49.82 \times 10^4 \quad + .66 \times 10^4 + 56.05 \times 10^4$$

$$= 167.28 \times 10^4 \text{ mm}^4$$

$$I_{yy} = \frac{90\,(10)^3}{12} + 900\,(21.47 - 5)^2 + \frac{10\,(80)^3}{12} + 800\,(40 - 21.47\,)^2$$

$$= 75 \times 10^4 + 24.41 \times 10^4 + 42.66 \times 10^4 + 27.46 \times 10^4$$

$$= 195.28 \times 10^4 \text{ mm}^4 \qquad \textbf{Answer.}$$

Example 6.8

Determine the momet of inertia of the T - Section shown in figure 6.16

Solution

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(120 \times 16)\,(128) + (120 \times 16)\,(60)}{(120 \times 16) + (120 \times 16)}$$

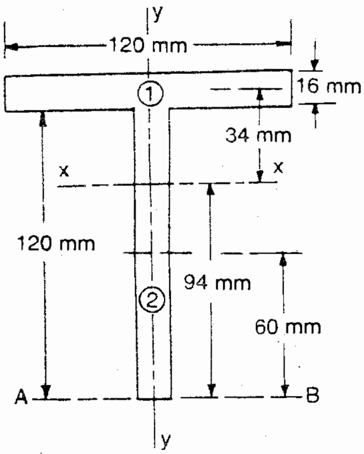


Fig. 6.16

$$= \frac{245760 + 115200}{3840}$$

$= 94$ mm from AB

$$I_{xx_1} = \frac{120\,(16)^3}{12} + (120)\,(16)\,(34)^2$$

$$= 40960 + 2219520$$

$$= 2260480 \text{ mm}^4$$

$$I_{xx_2} = \frac{16\,(120)^3}{12} + (120)\,(16)\,(34)^2$$

$$= 2304000 + 2219520$$

$$= 4523520$$

$$I_{xx} = I_{xx_1} + I_{xx_2}$$

$$= 2260480 + 4523520$$

$$= 6784000$$

$$= 678.4 \times 10^4 \text{ mm}^4$$

$$I_{yy} = I_{yy_1} + I_{yy_2}$$

$$\frac{16\,(120)^3}{12} + \frac{120\,(16)^3}{12}$$

$$= 2304000 + 40960 = 2344960 \; mm^4$$

$= 234.496 \times 10^4$ mm^4 **Answer.**

Example 6.9

Locate the position of centroidal axis and calculate the ***moment of*** *inertia of the section shown in figure 6.17*

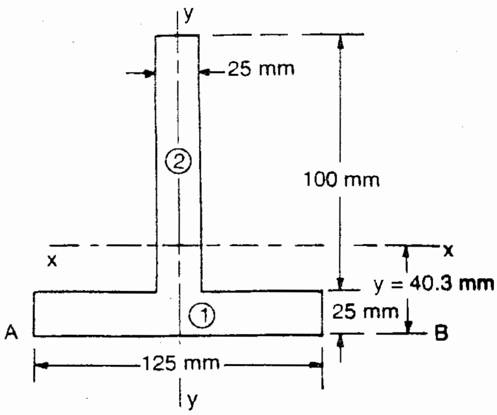


Fig. 6.17

Solution

Let y be the vertical distance of x – axis from the axis of reference AB

$$\bar{y} = \frac{(125)\,(25)\,(125) + (100)\,(25)\,(75)}{(125)\,(25) + (100)\,(25)} = 40.3\text{mm}$$

$\bar{y}$ = 40.3 mm from AB.

Moment of inertia of the section will be the summation of M.I of sections (1) and (2)

$$I_{xx} = I_{xx_1} + I_{xx_2}$$

$$\text{Now } I_{xx_1} = I_{xG} + Ay^2$$

$$= \frac{1}{12}(125)(25)^3 + (125)(25)(40.3 - 12.5)^2$$

$$= 162760.42 + 2415125$$

$$= 2.58 \times 10^6 \text{ mm}^4$$

$$I_{xx_2} = \frac{(25)(100)^3}{12} + (25)(100)(75 - 40.3)^2$$

$$= 2.08 \times 10^6 + 3.01 \times 10^6$$

$$= 5.09 \times 10^6 \text{ mm}^4$$

$$I_{xx} = I_{xx_1} + I_{xx_2} = (2.58 + 5.09) \times 10^6 = 7.67\ 10^6 \text{ mm}^4$$

$$I_{yy} = I_{yy_1} + I_{yy_2}$$

$$= \frac{(100)\ (25)^3}{12} + \frac{(25)\ (125)^3}{12}$$

$$= 4.19 \times 10^6 \text{ mm}^4$$

Example 6.10.

Determine the moment of inertia about the centroidal axes of the section shown in fig. 6.18

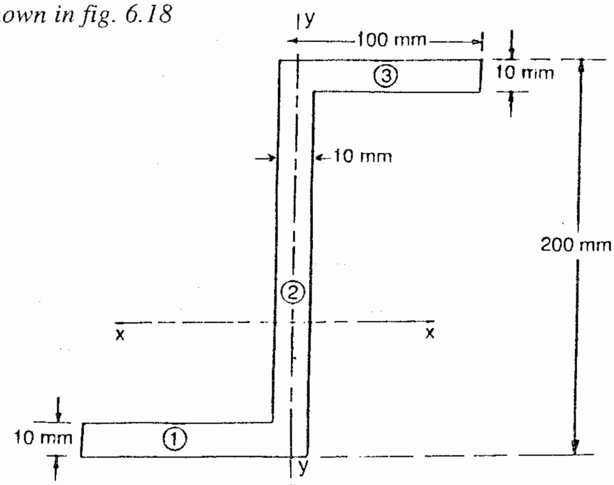


Fig. 6.18

Moment of inertia of the sectional will be the sum of the M. I. of the rectangular sections 1, 2 and 3.

$I_{xx} = I_{xx_1} + I_{xx_2} + I_{xx_3}$

For rectangular section (1) applying theorem of parallel axis.

$$I_{xx_1} = I_{xG} + Ay2$$

$$= \frac{1}{12}(90)\,(10)^3 + (90\times 10)\,(95)^2$$

$$I_{xx_2} = \frac{1}{12}(10)\,(200\ ^3$$

$$I_{xx_3} = I_{xG} + Ay2 = \frac{1}{12}(90)\,(10)3 + (90 \times 10)\,(95)^2$$

$$I_{xx} = 2\left[\frac{1}{12}(90)\,(10)^3 + (90)\,(10)\,(95)^2\right] + \left[\frac{1}{12}(10)\,(200)^3\right]$$

$$I_{xx} = 22.92 \times 10^6 \text{mm}^4$$

$$I_{yy} = 2\left[\frac{(10)(90)^3}{12} + (90)\,(10)\,(50)^2\right] + \left[\frac{1}{12}(200)\,(10)^3\right]$$

$$= 5.72 \times 10^6 \text{ mm}^4$$ **Answer**

Example 6.11

Determine the moment of inertia of the section shown in fig 6.19

Solution

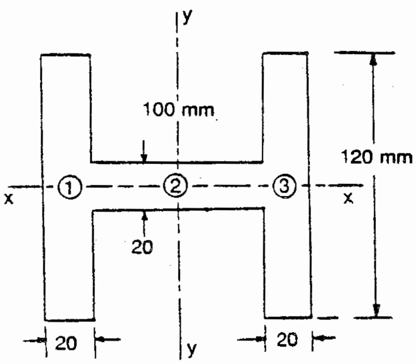


Fig. 6.19

Moment of inertia about X–X axis

$$I_{xx} = I_{xx_1} + I_{xx_2} + I_{xx_3}$$

$$= \frac{20(120)^3}{12} + \frac{100(20)^3}{12} + \frac{20(120)^3}{12}$$

$$= 288 \times 10^4 + 6.66 \times 10^4 + 288 \times 10^4$$

$$I_{xx} = 582.66 \times 10^4 \text{ mm}^4$$

$$I_{yy_1} = I_{yy_3} = \frac{120\,(20)^3}{12} + 120\,(20)\,(60)^2$$

$$= 8 \times 10^4 + 864 \times 10^4 = 872 \times 10^4$$

$$I_{yy_2} = \frac{20(100)^3}{12} = 166.6 \times 10^4$$

$$I_{yy} = I_{yy_1} + I_{yy_2} + I_{yy_3}$$
$$= 872 \times 10^4 + 166.6 \times 10^4 + 872 \times 10^4$$
$$= 1910.6 \times 10^4 \text{ mm}^4$$

Example 6.12

Locate the centroidal axes of the channel section shown in figure 6.20 and calculate the moment of inertia about the axis of X and axis of Y of the section

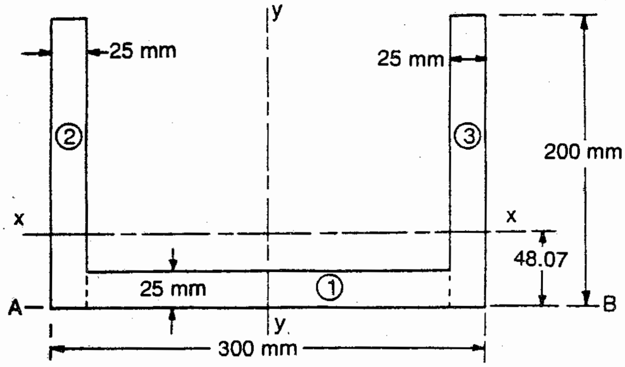


Fig. 6.20

Let $\bar{y}$ *be* the distance of x – axis from the axis of reference AB

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{a_1 + a_3 + a_3}$$

$$= \frac{[(250)(25)(12.5)] + [(200)(25)(100)] + [(200)(25)(100)]}{[(250)(25) + 2(200)(25)]}$$

$\bar{y}$ = **48.07** mm from AB

Total ***M. I.*** of the section will be summation of M. I. of rectangular **sections (1)** + (2) + (3) as shown in figure 6.20

For rectangular section (1)

Applying theorem of parallel axis

$$I_{xx} = I_{xG} + A.y^2$$

$$= \frac{(250)(25)^3}{12} + (250)(25)(48.07 - 12.5)^2$$

$$= 023.3 \times 10^4 \text{ mm}^4$$

For rectangular section 2 and 3

$$I_{xx} = 2 [I_{xG} + A_{y2}]$$

$$= 2\left[\frac{(25)(200)^3}{12} + (25)(200)(100 - 48.07)^2\right]$$

$$= 2 (3001.46 \times 10^4)$$

Total I_{xx} of the channel section

$$I_{xx} = 823.3 \times 10^4 + 2 (3001.46 \times 10^4)$$
$$= 6825.52 \times 10^4 \text{ mm}^4$$

The section is symmetrical about Y–axis

Hence $I_{yy} = I_{yy}$ of (1) $+ I_{yy}$ of (2) + I_{yy} of (3)

For rectangular Section (1)

$$I_{yy} = I_y G + A_x 2$$
$$= \frac{(25)(250)^3}{12} + 0 = 3255.2 \times 10^4 \text{ mm}^4$$

For rectangular sectio (2) and (3)

$$I_{yy} = 2\,[\,I_y G + A_x 2]$$
$$= 2\left[\frac{(200)\,(25)^3}{12} + 200\,(25)\,(250 - 125)^2\right]$$
$$= 2\,[26.04 \times 10^4 + 9453.1 \times 10^4]$$
$$= 18958.2 \times 10^4 \text{ mm}^4$$

Total $I_{yy} = 3255.2 \times 10^4 + 18958.2 \times 10^4$
$$= 22213.4 \times 10^4 \text{ mm}^4$$

Example 6.13

Determine the polar moment of inertia of a hollow circular section shown in fig 6.21

Solution

Moment of inertis of the hollow section about x–axis and y-axis

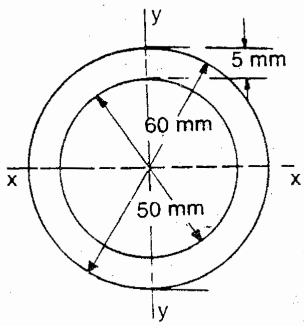


Fig. 6.21

$$I_{xx} = I_{yy} = \frac{\pi}{64}(D^4 - d^4)$$
$$= \frac{\pi}{64}(60^4 - 50^4)$$
$$= \frac{\pi}{64}(1296 \times 10^4 - 625 \times 10^4)$$
$$= \frac{\pi}{64}(671 \times 10^4) = 32.93 \times 10^4 \text{ mm}^4$$

Polar moment of inertia of the given section

$$I_{zz} = I_{xx} + I_{yy}$$
$$= 32.93 \times 10^4 + 32.93 \times 10^4$$
$$= 65.86 \times 10^4 \text{ mm}^4$$ **Answer**

Example 6.14

Determine the moment of inertia of the section shown in fig 6.22 about the edge AB

Solution

Moment of inertia of the square about AB

$$I_{AB} = I_{xG} + A_{y2} = \frac{bd^3}{12} + A_{y2}$$
$$I_{AB} = 675 \times 10^6 + 2025 \times 10^6 = 2700 \times 10^6 \text{ mm}^4$$
$$= \frac{300(300)^3}{12} + (300)(300)(150)^2$$

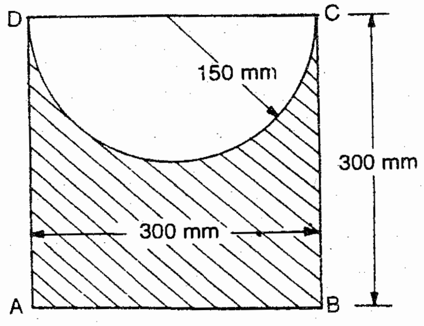


Fig. 6.22

Moment of inertia of the semicircular portion which has been removed

$I_{AB} = I_{xG} + Ay^2$

$$= \frac{1}{2} \cdot \frac{\pi}{64} (150)^4 + \frac{\pi}{4} (150)^2 \cdot \left[300 - \frac{2 \times 150}{3\pi}\right]^2$$

$$= 1242.52 \times 10^4 + 176.625 \times 10^2 (300 - 31.83)^2$$

$$= 1242.52 \times 10^4 + 127020.13 \times 10^4$$

$$= 128262.65 \times 10^4 = 1282.65 \times 10^6$$

Moment of inertia of the given section about AB

$= 2700 \times 10^6 - 1282.65 \times 10^6$

$= 1417.35 \times 10^6 \text{ mm}^4$ **Answer.**

Example 6.15

Determine the moment of inertia of a square section 120 mm × 120 mm about its diagonal from which a hole of 50 mm has been punched out.

Solution

The square is made up of two triangles, so the moment of inertia of

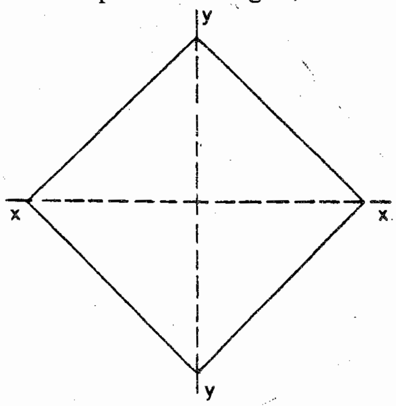


Fig. 6.23

the square is the sum of the M.I. of the triangles about the base

$$I_{xx} = 2 \cdot \frac{bh^3}{12} \quad \text{where } b = \sqrt{(120)^2 + (120)^2} = 169.7 \text{ mm}$$

$$\text{and } h = \frac{120}{\sqrt{2}} = 84.86 \text{ mm}$$

$$I_{xx} = \frac{2(169.7)(84.86)^3}{12}$$

$$I_{xx} = 1728.61 \times 10^4 \text{ mm}^4$$

Moment of inertia of the circular hole

$$I_{xx} = \frac{\pi}{64}(50)^4 = 30.67 \times 10^4 \text{ mm}^4$$

Moment of inertia of the given section about its diagonal is

$$I_{xx} = I_{yy} = 1728.61 \times 10^4 - 30.67 \times 10^4$$
$$= 1697.94 \times 10^4 \text{ mm}^4$$ **Answer.**

Example 6.16

Determine the moment of inertia of the shaded portion about AB axis as shown in figure. 6.24

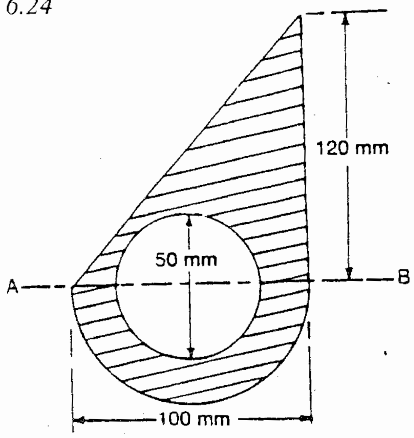


Fig. 6.24

Solution

Moment of inertia of the triangle about AB

$$I_1 = \frac{b h^3}{12} = \frac{100(120)^3}{12}$$
$$= 1440 \times 10^4 \text{ mm}^4$$

Moment of inertia of the semicircle of 100 mm diameter about AB

$$I_2 = \frac{1}{2}\left[\frac{\pi d^4}{64}\right] = \frac{\pi}{128} \times (100)^4$$
$$= 245.43 \times 10^4 \text{ mm}^4$$

Moment of inertia of the circular hole of 50 mm diameter

$$I_3 = \frac{\pi}{64}(50)^4 = 30.67 \times 10^4 \text{ mm}^4$$

Moment of inertia of the composite section

$$I = I_1 + I_2 - I_3$$
$$= 1440 \times 10^4 + 245.43 \times 10^4 - 30.67 \times 10^4$$
$$= 1654.76 \times 10^4 \text{ mm}^4$$

Example 6.17

Locate the centroid of the shaded area shown in figure 6.25 and calculate the moment of inertia of the section about x–x and y–y axes.

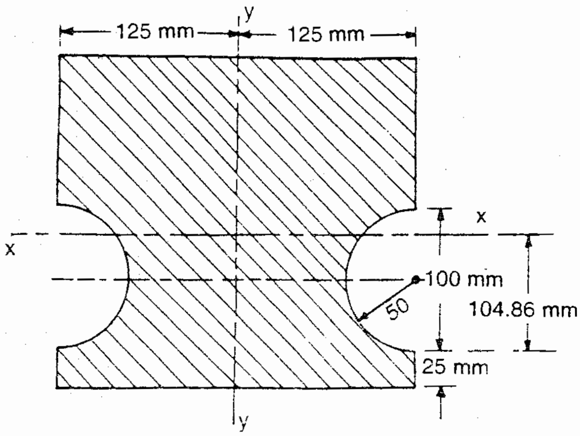


Fig. 6.25

Solution

Let $\bar{y}$ be the distance of x-axis from the axis of reference AB.

$$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{(a_1 - a_2)}$$

$$= \frac{(250)\,(200)\,(100) - \frac{\pi}{4}(100)^2 \times (75)}{(250)\,(200) - \frac{\pi}{4}(100)^2}$$

$$= \frac{5 \times 10^6 - 5^8 \times 10^6}{5 \times 10^4 - .785 \times 10^4} = \frac{4.42 \times 10^6}{4.215 \times 10^4} = 104.86 \text{ mm}$$

$$\bar{y} = 104.86 \text{ mm from } AB$$

Moment of inertia of the given section will be M.I. of rectangular section (1) – M.I. of the circle for rectangular portion (1) shown in the figure

$$I_{xx_1} = I xG + A_{y2}$$

$$= \frac{(250)\,(200)^3}{12} + (250)\,(200)\,(104.86 - 100)^2$$

$$I_{xx_1} = 16677 \times 10^4 \text{ mm}^4$$

I_{xx} of the circular portion

$$I_{xx_2} = I_{xG} + A_{y2}$$

$$I_{xx} = \frac{\pi}{64}(100)^4 + \frac{\pi}{4}(100)^2\ (104.86 - 75)^2$$

$$= 1195 \times 10^4$$

Net I_{xx} of the given section

$$I_{xx} = I_{xx_1} - I_{xx_2}$$

$$= 16677 \times 10^4 - 1195 \times 10^4 = 15481 \times 10^4 \text{ mm}^4$$

Moment of inertia about Y – axis

$$I_{yy} = I_{yy} \text{ of rectangle} - I_{yy2} \text{ of circle}$$

$$I_{yy_1} = (I_{YG} + A.x^2) = \frac{(200)\,(250)^3}{12}$$

$$= 26041 \times 10^4$$

$$I_{yy2} = \frac{\pi}{64}(100)^4 = 490.8 \times 10^4$$

$$I_{yy} = I_{yy_1} - I_{yy_2}$$

$$= 26041 \times 10^4 - 490.8 \times 10^4$$

$$= 25540 \times 10^4 \text{ mm}^4$$

Example 6.18

Determine the moment of inertia of the compound section shown in fig. 6.26

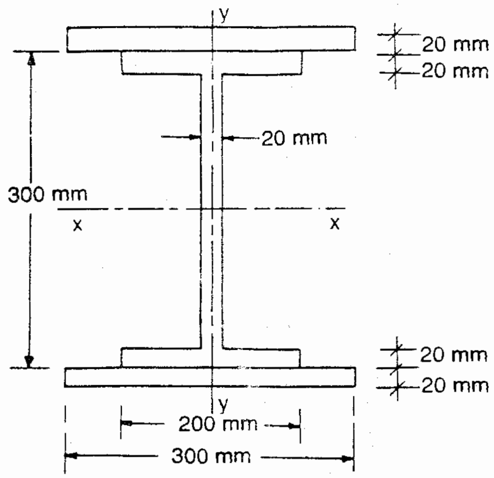


Fig. 6.26

Solution

$$I_{xx} \text{ for the joist} = \frac{200\,(300)^3}{12} - \frac{180\,(260)^3}{12} = 186.36 \times 10^6 \text{ mm}^4$$

I_{xx} for the plates

$$= 2\left[\frac{300\,(20)^3}{12}\right]$$

$$= 0.4 \times 10^6 \text{ mm}^4$$

I_{xx} for the compound section

$= 186.36 \times 10^6 + 0.4 \times 10^6$

$= 186.76 \times 10^6 \text{ mm}^4$

I_{yy} for the joist

$$= \frac{300\,(200)^3}{12} - \frac{260\,(180)^3}{12} = 126.23 \times 10^6 \text{ mm}^4$$

I_{yy} for the plates

$$= 2\left[\frac{20\,(300)^3}{12}\right]$$

$= 90 \times 10^6 \text{ mm}^4$

I_{yy} for the compound section $= (126.23 + 90) \times 10^6$

$I_{yy} = 216.23 \times 10^6$ **Answer**

Example 6.19

Determine the I_{xx} and I_{yy} of the compound section shown in figure 6.27. Also calculate the least radius of gyration.

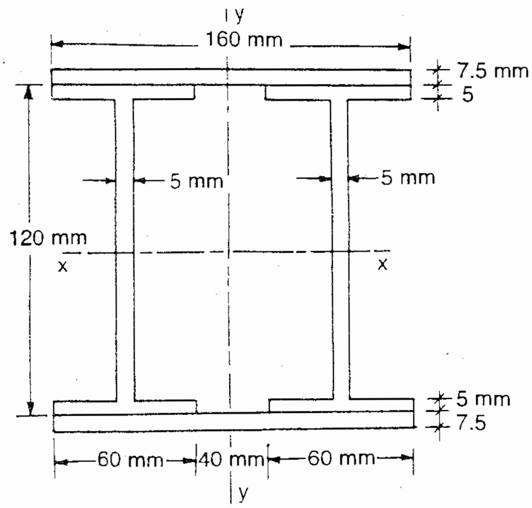


Fig. 6.27

Solution.

$$I_{xx} \text{ of joists} = 2\left[\frac{60 \times 120^3}{12} - \frac{55 \times 110^3}{12}\right] = 508 \times 10^4 \text{ mm}^4$$

$$I_{xx} \text{ of plates} = \left[\frac{160 \times 135^3}{12} - \frac{160 \times 120^3}{12}\right] = 976 \times 10^4 \text{ mm}^4$$

I_{xx} for the compound section $= 1484 \times 10^4 \text{ mm}^4$

I_{yy} of plates = $\dfrac{15 \times 160^3}{12} = 512 \times 10^4 \text{ mm}^4$

I_{yy} for Joists = $2\left[\dfrac{10 \times 60^3}{12} - \dfrac{110 \times 5^3}{12} + 1150 \times 50^2\right] = 612 \times 10^4$ mm

I_{yy} of the section = $(512 + 612) \times 10^4 = 1124 \times 10^4 \text{ mm}^4$

Area of the compound section = (1200 + 1200 + 1150 + 1150)

= 4700 mm^2

Least radius of *gyration* $K = \sqrt{\dfrac{112 \times 10^4}{4700}} = \sqrt{2400}$

$K = 48.98$ mm. Answer.

SUMMARY

1. Moment of inertia of a body about an axis is the sum of the product of the areas of all the elements constituting the body and the square of their respective distances of centre of gravity from the axis of reference

   . $I_{x-x} = \Sigma\, \delta_a . y^2$ and $I_{y-y} = \Sigma\, \delta_a . x^2$

2. Radius of gyration is the distance from the axis of rotation where the total mass or area of the body is supposed to be concentrated , so that its moment of inertia about the axis is the same as that with the actual distribution of mass.

   $K_{xx} = \sqrt{\dfrac{Ix-x}{A}}$ or $K_{yy} = \sqrt{\dfrac{Iy-y}{A}}$.

3. Theorem of parallel axes states that the moment of inertia of a plane figure about an axis is equal to its M.I. about a parallel axis through its C.G. plus the product of its area and the square of the perpendicular distance between the two axis

   $I_{x-x} = I_{xG} + A_y^2$

   $I_{y-y} = I_{yG} + A_{x2}$

4. Theorem of perpendicular axes states that the moment of inertia of a plane figure is equal to the sum of the M.I. of figure about the axes at right angles to each other in its plane and intersecting each other at the point where the perpendicular axis passes through it

   $I_{zz} = I_{xx} + I_{yy}$

5. Section modulus is defined as the moment of inertia divided by the distance of the extreme fibre of the section from the axis through the centroid of the section.

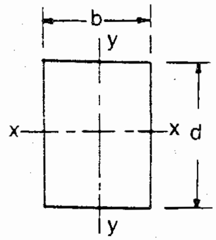
   $Z = \dfrac{I}{y}$

6. Polar moment of inertia of a plane area with respect to an axis perpendicular to the plane of the area is called polar moment of inertia

   $I_{zz} = I_{xx} + I_{yy}$

Moment of inertia of standard sections.

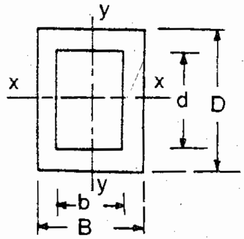
7. Rectangular section.



$$I_{xx} = \frac{b\,d^3}{12} \text{ about } x - \text{axis}$$

$$I_{yy} = \frac{d\,b^3}{12} \text{ about } y - \text{axis}$$

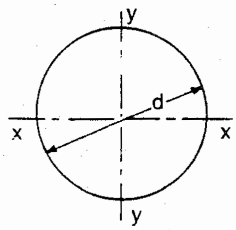
8. Hollow rectangular section



$$I_{xx} = \frac{B\,D^3}{12} - \frac{b\,d^3}{12}$$

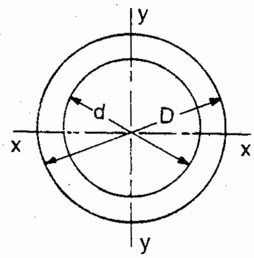
$$I_{yy} = \frac{D\,B^3}{12} - \frac{d\,b^3}{12}$$

9. Circular section



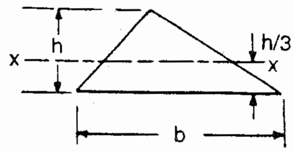
$$I_{xx} = I_{yy} = \frac{\pi}{64} D^4$$

10. Hollow circular section



$$I_{xx} = I_{yy} = \frac{\pi}{64} (D^4 - d^4)$$

11. Triangle



$$I_{xx} = \frac{b\,h^3}{36}$$

QUESTIONS

1. Explain what do you understand by the terms moment of inertia and radius of gyration ?

2. State the theorem of parallel axes. Derive an expression for the moment of inertia of a tringle about an axis passing through the C.G and parallel to the base.
3. State the theorem of perpendicular axes. Derive an expression for the polar moment of inertia of a solid circular plate about an axis perpendicular to both the axis of *X* and *Y*.
4. What is section modulus ? Derive expressions for the section modulus in the following cases

   (a) A square Section (b) Rectangular section (c) Circular section

EXERCISES

5. Locate the centroidal axes and determine the moment of inertia about horizontal axis passing through the centroid of the section. fig 6.28

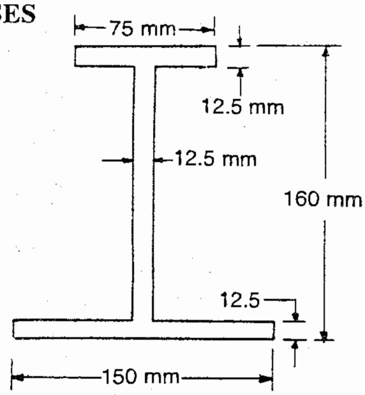


Fig. 6.28

6. Determine the moment of inertia of an equal angle section $100^{mm} \times 100^{mm} \times 20^{mm}$ about both the horizontal and vertical axis passing through the centroid. fig 6.29

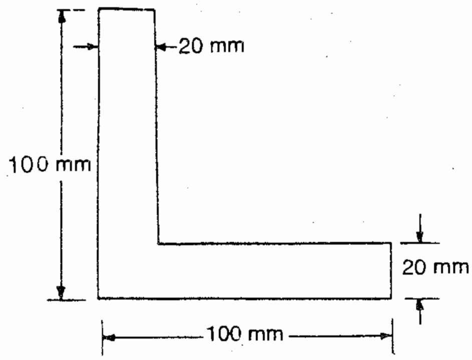


Fig. 6.29

7. Calculate the moment of inertia of a T–section about both the vertical and horizontal axis. fig 6.30

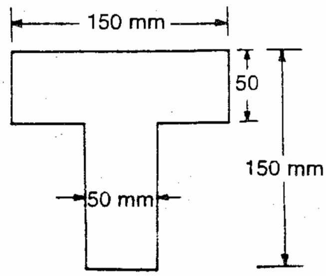


Fig. 6.30

8. Determine the moment of inertia of the channel section about a horizontal axis passing through the centroid. fig 6.31

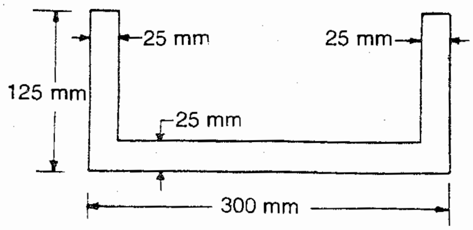


Fig. 6.31

9. Determine the I_{xx} and I_{yy} of the compound section shown in figure. 6.32
$I_{xx} = 13686.66 \times 104\ mm^4$ and
$I_{yy} = 12836.66 \times 104\ mm^4$

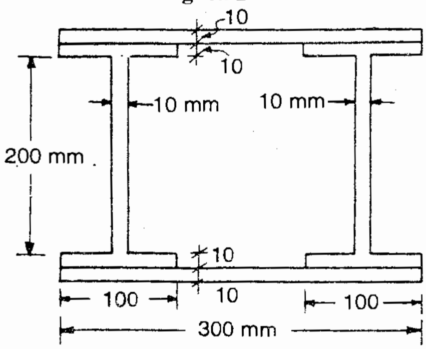


Fig. 6.32

10. Determine the moement of inertia of the compound section along the axis of X only passing through the centroid also find the radius of gyration. Fig 6.33

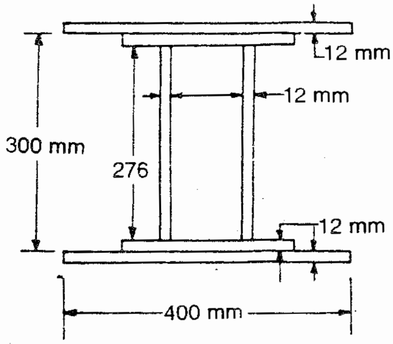


Fig. 6.33

11. A composite section is made of 300 mm × 100 mm channel and two plates 300 mm× 18.75mm. Determine the Ixx and Iyy through the centroidal axes given that Ixx for channel section = $2775 \times 10^4\ mm^4$, $I_{yy} = 473.43 \times 10^4\ mm^4$. Area of channel section = $5756\ mm^2$ position of C.G from back of channel = 25.1mm. fig 6.34

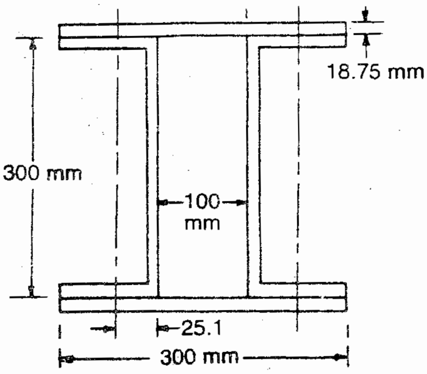


Fig. 6.34

□□□

7

Stresses In Beams

I. Bending Stresses

When a freely supported beam is subjected to forces acting at right angles to its horizontal axis, the beam bends as shown in figure. 7.1 (b)

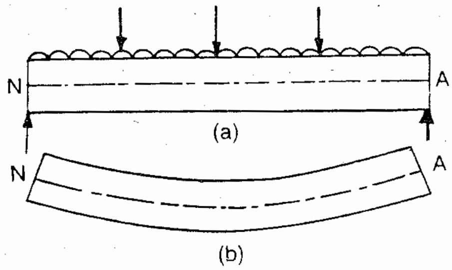


Fig. 7.1

These forces acting on the beam produce the following effects

(i) At any cross section of the beam perpendicular to the longitudinal axis bending stresses as well as shearing stresses are induced.

(ii) The beam undergoes deflection perpendicular to its longitudinal axis.

Pure Bending

When a couple is applied to the ends of a beam the bending produced is known as pure bending. Only bending stresses are set up and no shearing stresses are induced.

Ordinary Bending Or Simple Bending

When a number of vertical forces act on a beam not forming a couple, the bending action is called simple bending. Both bending stresses and shearing stresses are set up at any cross–section perpendicular to the longitudinal axis of the beam.

Bending Stresses

When a beam bends the upper layers are shortened and lower layers are elongated. Since the upper layers are compressed, therefore compressive stresses are induced in these fibres. In the lower layers which are elongated tensile stresses are set up. Because bending action produces these tensile and compressive stresses, therefore these stresses are called **bending stresses.**

Neutral Surface

In between the upper and lower layers there exists a layer in the beam containing fibres which do not undergo any elongation or shortening. This surface is not subjected to either tension or commpression and remains un affected. It remains neutral. Hence this surface is known as neutral surface of the beam.

Neutral axis

The inter-section of the neutral surface with any cross-section of the beam perpendicular to its longitudinal axis is called neutral axis. All fibres above the neutral axis are in compression and all fibres below the neutral axis are in a state of tension.

Assumptions in theory of simple bending

1. The material of the beam is uniform throughout
2. Each cross-section of the beam is symmetrical about the plane of bending.
3. The radius of curvature of the beam before bending is very large in comparision to the transverse dimensions of the beam.
4. The loads are applied to the beam in the plane of bending.
5. Transverse cross-sections of the beam remain plane before and after bending.
6. Young's modulus has the same value in compression and tension.
7. Hook's law applies to each longitudinal layer.
8. The resultant pullor thrust across a transverse section of the beam is zero.

Bending Equation.

When a beam is loaded it bends and bending stresses are induced. The relation ship between the bending moment, bending stress, radius of curvature in which the beam bends, modulus of elasticity and moment of inertia of the cross-section of the beam is given by the following equation, known as bending equation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

Proof of Bending Equation

Consider a portion of a uniform beam subjected to simple bending as shown in figure 7.2 (a).

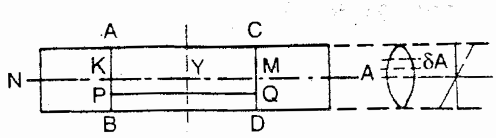


Fig. 7.2 (a)

Consider a portion of the beam between paralled sections *A B* and *C D*. Let *y* be the distance of the fibre *P* Q From the neutral surface. After bending the planes assume the position A_1 B_1 and C_1 D_1 and the fibre *P* Q elongates to P_1 Q_1 as shown in figure 7.2(b). Let θ be the angle subtended at the intersection of A_1B_1 and C_1 D_1. Let *R* be the radius of curvature of neutral surface. Then the radius of curvature of the fibre P_1 Q_1 will be $(R + y)$

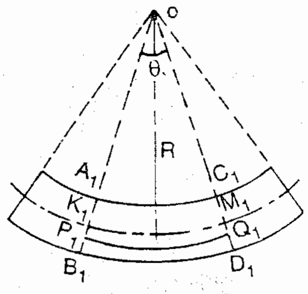


Fig. 7.2 (b)

$$\text{Strain} = \frac{P_1Q_1 - PQ}{PQ} = \frac{P_1Q_1 - KM}{KM} = \frac{P_1Q_1}{KM} - 1$$

$$\text{Strain} = \frac{(R+y)\,\theta}{R\,\theta} - 1 = \frac{R+y}{R} - 1 = \frac{y}{R}$$

$$\text{Strain} = \frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \text{ or } \sigma = \frac{E}{R}\,.\,y = k.y$$

Since E and R are constants therefore σ is directly proportional to y. Hence we can conclude that bending stresses at any layer varies directly with its distance from the neutral axis. It is zero at the neutral axis and maximum at the top most and bottom most fibres of the beam. The maximum stresses in the outermost fibres of the beam are called SKIN STRESSES.

Position of Neutral Axis

The position of neutral axis and radius of curvature can be determined from the condition.

That the forces distributed over any given cross-section of the beam must give rise to a resisting couple which balances the external couple M.

Consider a small elemental area dA at a distance y from the neutral axis.

$$\text{Force on the elemental area} = \sigma \times dA = \frac{E}{R}\,.\,y.dA$$

$$\therefore \text{ Sum of the forces acting on the section of the beam} = \int \frac{E}{R}\,.\,y.\,dA$$

Now all such forces which are distributed over the cross-section, represent a system equivalent to a couple. Therefore, the resultant of these forces must be equal to Zero.

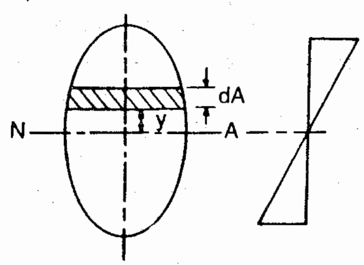


Fig. 7.2 (c)

$$\text{or} \quad \frac{E}{R}\int y.dA = 0$$

Which means that moment of the area of the cross-section about the neutral axis is Zero. Hence neutral axis passes through $C.\ G.$ of the section.

Moment of Resistance

Moment of the force acting on the elemental area about $N.A = \frac{E}{R}.y.$

$$dA.y \qquad = \frac{E}{R}\, y^2.\ dA$$

Adding all such moments over the cross-section and equating the resultant moment to the applied moment

$$M = \int \frac{E}{R}.\, y^2.\ dA$$

$$\text{or} \quad M = \frac{E}{R}\int y^2.\ dA$$

Now $\int y^2.\ dA$ = Moment of inertia of the cross section about the $N.\ A$

$$\text{Hence} \quad M = \frac{E}{R}.\ I$$

$$\text{or} \quad \frac{M}{I} = \frac{E}{R}$$

We have already established that

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$\therefore \quad \frac{\boldsymbol{M}}{\boldsymbol{I}} = \frac{\sigma}{\boldsymbol{y}} = \frac{\boldsymbol{E}}{\boldsymbol{R}}$$

This equation is known as bending equation where

M = Bending moment or Moment of resistance in N–mm

I = Moment of inertia in mm^4.

σ = Bending stress in MPa

y = Maximum distance of the fibre from the $N.\ A.$ in mm

E = Modulus of elasticity in N/mm^2

R = Radius of curvature in mm.

Example 7.1

A Cantilever 4 metres long is subjected to a uniformly distributed load of 1 KN per metre run over the entire span. The section of the Cantilever is 40 mm wide and 60 mm deep. Determine the bending stresses produced. what point load may be placed at the free end to produce the same bending stress.

Solution

Moment of inertia of the section $I = \dfrac{bd^3}{12}$

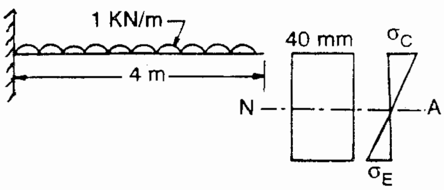


Fig. 7.3

$$I_{xx} = \frac{(40)\,(60)^3}{12}$$

$$= 72 \times 10^4\ mm^4$$

Max. B.M will occur at the fixed end of the cantilever $M = \dfrac{wl^2}{2}$

$$M = \frac{1 \times 1000\,(4)^2}{2} = 8 \times 10^3 \text{ N–m} = 8 \times 10^6 \text{ N–mm}$$

Appling bending equation

$$\frac{M}{I} = \frac{\sigma}{y} \quad \text{or} \quad \sigma = \frac{M}{I} \cdot y$$

Maximum stress will be induced at the extreme fibre from the neutral axis i.e. at $y = \dfrac{60}{2} = 30$ mm

$$\sigma = \frac{8 \times 10^6}{72 \times 10^4} \times 30 = 333.3 \text{ MPa}$$

When the *u.d.l.* is replaced by a point load *W* at the free end then stress produced is 333.3 MPa

Hence $M = \dfrac{\sigma}{y} \cdot I = \dfrac{333.3 \times 72 \times 10^4}{30} = 8 \times 10^6$ N–mm

Max^{m}. bending moment at the fixed end = $W.l$

$W.l = 8 \times 10^6$ or $W = \dfrac{8 \times 10^6}{4 \times 1000} = 2 \times 10^3 \text{ N} = 2$ KN **Answer**

Example 7.2

A cantilever of rectangular section is 4 metres long and subjected to a uniformly distributed load of 20 KN per metre run over the entire span. If the allowable bending stress is limited to 160 MPa determine the dimensions of the beam taking depth equal to twice the width.

Solution

Maximum *B.M.* will occur at the fixed end

$$M_{max} = \frac{wl^2}{2} = \frac{20\,(4)^2}{2} = 160 \text{ KN–m} = 160 \times 10^6 \text{ N–mm}$$

Moment of inertia of the section

$$I = \frac{bd^3}{12} \text{ and } y_{max} = \frac{d}{2}$$

$$\therefore \text{ Section modulus } Z = \frac{I}{y} = \frac{bd^2}{6}$$

$$M_r = \sigma \times Z$$

or $$Z = \frac{Mr}{\sigma} = \frac{160 \times 10^6}{160} = 10^6 \text{ mm}^3$$

$$\frac{bd^2}{6} = 10^6 \text{ Now } d = 2b$$

or $$\frac{b\,(2b)^2}{6} = 10^6 \text{ or } b^3 = \frac{6}{4} \times 10^6$$

or $b = 114.47$ mm $= 114.5$ mm

$d = 229$ mm **Answer**

Example 7.3

A mild steel cantilever 100mm wide and 40 mm deep is fixed at one end in a wall. The over hang length is 1.25 metres. If a clockwise turning moment 3000 N-m is applied at the free end, determine the radius to which the cantilever will be bent. Also Calculate the vertical displacement of the free end. Take E = 200 KN/mm^2.

Solution

Moment of inertia of the Section

$$I = \frac{bd^3}{12}$$

$$I = \frac{(100)\,(40)^3}{12} = \frac{160}{3} \times 10^4 \text{ mm}^4$$

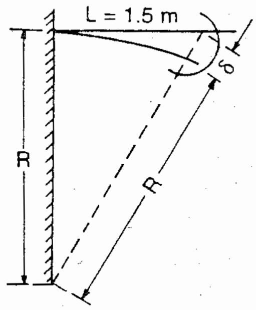


Fig. 7.4

Applying bending equation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\text{or} \quad R = \frac{EI}{M} = \frac{200 \times 10^3 \times 160}{3000 \times 10^3 \times 3} \times 10^4 \text{ mm}$$

$$= 35.5 \times 10^3 \text{ mm}$$

$$\text{or} \quad R = 35.5 \text{ metres}$$

For displacement

From the property of a circle we know that the tangent from a point is equal to the product of segments of any secant from that point.

$$l^2 = \delta \;\; (2R + \delta)\text{ , Neglecting } \delta^2$$

we have $\delta = \dfrac{l^2}{2R} = \dfrac{(1.25)^2}{2 \times 35.5}$ or $\delta = 21.8$ mm **Answer**

Example 7.4

In a Cantilever two strain gauges are placed at a distance 65 mm and the stresses observed had a difference of 27 MPa when a concentrated load W acts to the right of the strain gauges. If the section modulus is 150 × 10^3 mm^3, determine the value of load W. *(ENGG. Services)*

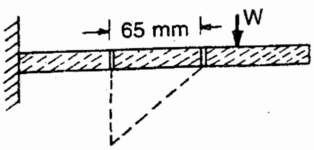


Fig. 7.5

Moment of resistance of the section

$$M_r = \sigma \times Z$$

$$= 27 \times 150 \times 10^3 \text{ N} - \text{mm} \quad \text{... (i)}$$

Bending moment due to the applied load W

$$M = W \times 65 \text{ N} - \text{mm} \quad \text{... (ii)}$$

Equating (i) and (ii) we get

$$W \times 65 = 27 \times 150 \times 10^3$$

$$\text{or} \quad W = \frac{27 \times 150 \times 10^3}{65} \text{ Newtons}$$

$$= 62.30 \quad \text{KN}$$

Example 7.5

A beam of circular section 7 metres long is supported at C and attached to the foundation at A as shown in figure 7.6. The beam supports a u.d.l. of 6 KN/m over the portion B C. If the permissible bending stress is 250 MPa find the diameter of the beam.

Solution

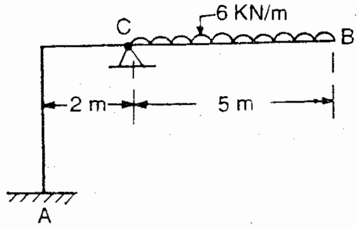


Fig. 7.6

Max. B. M. at $C, M = \dfrac{wl^2}{2}$

$$M = \frac{6000\,(5)^2}{2} = 75000 \text{ N–m}$$

Section modulus

$$Z = \frac{I}{y} = \frac{\frac{\pi}{64} d^4}{d/2}$$

$$Z = \frac{\pi}{32} d^3$$

Bending stress allowed = 250 MPa

Now $M_r = \sigma . Z$

or $75000 \times 10^3 = 250 \times \dfrac{\pi}{32} (d)^3$

or $d^3 = \dfrac{75000 \times 10^3 \times 32}{250 \times \pi}$

or $d = 145.1$ mm **Answer**

Example 7.6

A beam is loaded by a couple of magnitude 1.5 KN-m at each end as shown in figure 7.7 . The beam is 30 mm wide and 60 mm deep. Determine the maximum compressive and tensile stresses produced and draw the stress diagram.

Solution

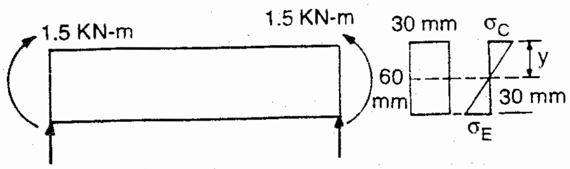


Fig. 7.7

Maximum bending stress will occur at the extreme fibre of the section. Moment of inertia of the section about x – axis

$$I_{xx} = \frac{bd^3}{12} = \frac{(30)\,(60)^3}{12} = 54 \times 10^4 \text{ mm}^4$$

$M_r = 1.5 \times 10^6$ N – mm.

Applying bending equation

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\text{or} \qquad \sigma = \frac{M}{I} \cdot y = \frac{1.5 \times 10^6 \times 30}{54 \times 10^4} = 83.3 \text{ MPa}$$

Maximum Compressive stress $\sigma_c = 83.3$ MPa

Maximum tensile stress $\sigma_t = 83.3$ MPa **Answer**

Example 7.7

A cantilever 3 metres long carries a uniformly distributed load of 1 KN per metre run over the whole span. The cross-section of the beam is rectangular 60 mm wide and 100 mm deep with a hole of 20 mm diameter at the centre. Determine the maximum bending stress induced in the beam.

Solution

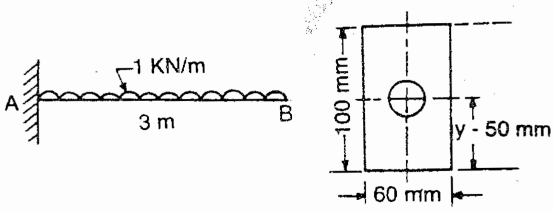


Fig. 7.8

Maximum bending moment $= \dfrac{wl^2}{2}$

$$M = \frac{1 \times 1000 \times (3)^2}{2} \times 1000 = 4.5 \times 10^6 \text{ N-mm}$$

Moment of inertia of the section

$$I = \frac{bd^3}{12} - \frac{\pi}{64}(d)^4$$

$$= \frac{60\,(100)^3}{12} - \frac{\pi}{64}(20)^4 = 499.21 \times 10^4 \text{ mm}^4$$

$$\bar{y} = \frac{100}{2} = 50 \text{ mm}$$

Applying bending equation

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\text{or} \quad \sigma = \frac{M.y}{I} = \frac{4.5 \times 10^6 \times 50}{499.2 \times 10^4} = 45.07 \text{N/mm}^2$$

Maximum bending stress will occur at the extreme fibre

$\sigma = 45.07$ MPa **Answer**

Example 7.8

A simply supported beam 5 metres long of rolled steel section carries two point loads 120 KN each at 300 mm from ends as shown in figure 7.9 (a). Determine the maximum bending stresses in tension and compression.

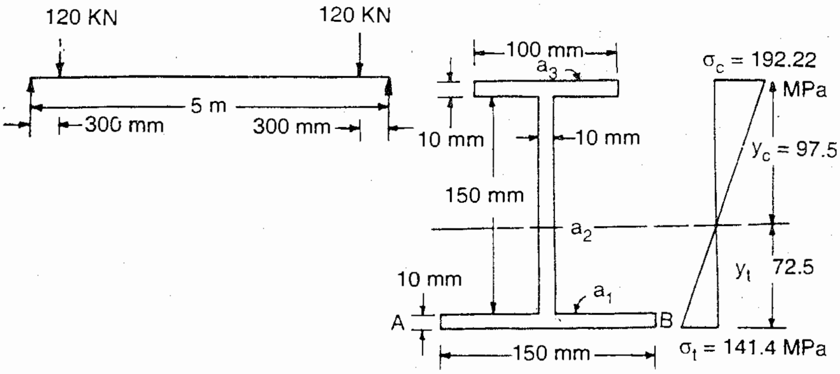


Fig. 7.9

Solution

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$= \frac{(150)(10) \times 5 + (150 \times 10)(85) + (100 \times 10)(165)}{(150 \times 10) + (150 \times 10) + (100 \times 10)} = 72.5 \text{ mm}$$

from AB

$$I_{xx} = \frac{(150)(10)^3}{12} + (150)(10)(67.5)^2 + \frac{(10)(150)^3}{12} + (10)(150)(12.5)^2$$

$$+ \frac{(100)(10)^3}{12} + (100)(10)(92.5)^2$$

$$I_{xx} = 1.25 \times 10^4 + 683.43 \times 10^4 + 281.25 \times 10^4 + 23.43 \times 10^4 + 0.83 \times 10^4 + 855.62 \times 10^4$$

$$= (1845.81) \times 10^4 \text{ mm}^4$$

Maximum bending moment will occur under each load

$$M_{max} = 120 \times 10^3 \times 300 = 36 \times 10^6 \text{ N-mm}$$

$$\text{Max}^m \text{ tensile stress } \sigma_t = \frac{M}{I} \times y_t = \frac{36 \times 10^6}{1845.81 \times 10^4} \times 72.5$$

$$= 141.4 \text{ MPa}$$

$$\text{Max}^m \text{ Compressible strees } \sigma_c = \frac{M}{I} \cdot y_c = \frac{36 \times 10^6}{1845.81 \times 10^4} \times 97.5$$

$$= 192.22 \text{ MPa}$$ **Answer.**

Example 7.9

A beam of T – section is subjected to a Couple of 6000 N – m at each end. Determine the maximum tensile and Compressive stresses induced in the beam. The flange is 120 mm × 20 mm and the web in 100 mm × 20 mm as shown in figure 7.10

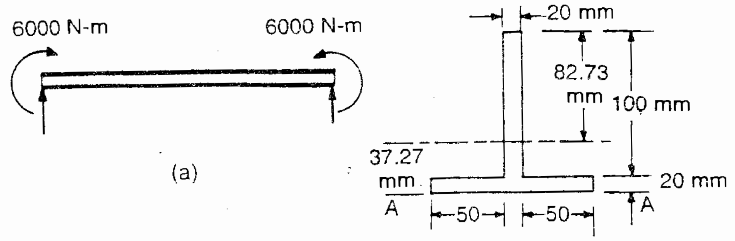


Fig. 7.10

Let $\bar{y}$ be the distance of the centroid of the section from the reference axis $A-A$ as shown

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$$\bar{y} = \frac{120 \times 20 \times 10 + (100)(20)(70)}{(120)(20) + (100)(10)}$$

$$= \frac{24000 + 140000}{2400 + 2000} = \frac{16400}{4400} = 37.27 \text{ mm from } A - A$$

or $y_t = 37.27$ mm and $y_c = (120 - 37.27) = 82.73$ mm

Now the moment of inertia of the section will be

$$I_{xx} = I_{gg} + A.{}_y{}^2 = \frac{bd^3}{12} + A_y{}^2$$

$$= \frac{1}{12}(120)(20)^3 + (120)(20)(37.27 - 10)^2$$

$$+ \frac{1}{12}(20)(100)^3 + (100)(20)(70 - 37.27)^2$$

$$I_{xx} = [8 \times 10^4 + 12995.83 \times 10^2 + .166 \times 10^7 + 2142.52 \times 10^3]$$

$$= (8 \times 10^4 + 129.96 \times 10^4 + 166.6 \times 10^4 + 214.25 \times 10^4)$$

$$= 518.87 \times 10^4 \text{ mm}^4$$

Now Appling bending equation

$$\frac{M_r}{I} = \frac{\sigma_t}{y_t} \quad \text{Where } M_r = M = 6000 \times 10^3 \text{ N-mm}$$

$$\text{or } \sigma_t = \frac{M.y_t}{I} = \frac{6000 \times 10^3 \times 37.27}{518.87 \times 10^4} = 1.15 \times 37.27$$

$$= 43.09 \text{ MPa}$$

$$\sigma_c = \frac{6000 \times 10^3 \times 82.73}{518.87 \times 10^4}$$

$$= 1.15 \times 82.73 = 95.13 \text{ MPa}$$

Hence bending stress intension = 43.09 MPa

Bending stress in Compression = 95.13 MPa

Example 7.10

A simply supported horizontal beam of span 4 metres has a section as shown in figure . 7.11 (a) Calculate the maximum uniformly distributed load the beam can carry if the maximum permissible stress in tension is 50 MPa and 70 MPa in compressions.

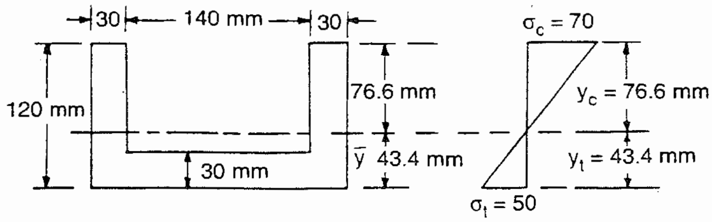


Fig. 7.11

$$\bar{y} = \frac{2\,(120 \times 30)\,60 + (140 \times 30)15}{2\,(120 \times 30) + 140 \times 30} = 43.4 \text{ mm}$$

$$I = 2\left[\frac{1}{12}(30)\,(120)^3 + 120 \times 30\,(16.6)^2\right] + \left[\frac{1}{12}(140)\,(30)^3 + 140 \times 30\,(28.4)^2\right]$$

$$= 1434.5 \times 1\,0^4 \text{ mm}^4.$$

Moment of resistance when a tensile stress of 50 MPa develops in the bottom most fibre

$$M_r = \frac{\sigma . I}{y_t} = \frac{50 \times 1434.5 \times 10^4}{43.4} = 165.26 \times 10^5 \text{ N-mm}$$

Moment of resistance when compressive stress of 70 MPa develops in the top most fibre at

$y_c = 76.6$ from the neutral axis

$$\text{Mr} = \sigma \times \frac{I}{y_c} = \frac{70 \times 1434.5 \times 10^4}{76.6} = 131 \times 10^5 \text{ N-mm}$$

Hence if both conditions are to be satisfied then bending moment must not exceed 131×10^5 N – mm

$$\text{or} \quad \frac{wl^2}{8} = 131 \times 10^5 \text{ N-mm} = 131 \times 10^2 \text{ N-m}$$

$$\text{or} \quad w = \frac{131 \times 10^2 \times 8}{(4)^2} = 65.5 \times 10^2 \text{ N/m}$$

$$= 6.55 \text{ KN/m}$$ **Answer**

Example 7.11

A rectangular beam 100 mm wide 200 mm deep and 4 metres long is simply suppord at ends. It Carries a u.d.L. of 5 KN per metre run over the entire span. If this load is removed and two loads W KN each are placed at one metre from each end, calculate the greatest value which may be assigned to W So that the maximum bending stress remains same as before.

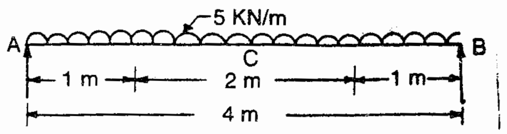


Fig. 7.12 (a)

Solution

Section modulus $Z = \frac{bd^2}{6} = \frac{(100)(200)^2}{6} \text{ mm}^3 = \frac{4}{6} \times 10^6 \text{ mm}^3$

$Max.\ B.M.$ will occur at mid span $= \frac{wl^2}{8} = \frac{5(4)^2}{8}$

$= 10 \text{ KN}-\text{m} = 10 \times 10^6 \text{ N}-\text{mm}$

From bending equation $Mr = \sigma \times Z$

or $\sigma = \frac{Mr}{Z} = \frac{10 \times 10^6}{\frac{4}{6} \times 10^6} = \frac{60}{4} = 15 \text{ MPa}$

Maximum stress produced is 15 MPa. When the *u.d.l* is replaced by two point loads W KN each, then Maximum B. M will be $= W \times 1$ KN –m

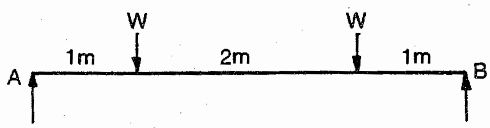


Fig. 7.12 (b)

$M_{max} = W \times 10^6$ N –m, $\sigma = 15$ MPa, $Z = \frac{4}{6} \times 10^6 \text{ mm}^3$

or $M = \sigma \times Z = 15 \times \frac{4}{6} \times 10^6 \text{ N}-\text{mm} = 10 \text{ KN}-\text{m}$

Equating Mr to Maximum B. M. we get $W \times 1$ KN–m = 10 KN-m
or W = 10 KN

Example 7.12

A floor has to carry a load of 300 KN/sq. m. If the span of each joist which is 120 mm wide and 300 mm deep is 4 metres, calculate their spacing centre to centre. The maximum permissible bending stress is not to exceed 120 MPa.(JMI)

Solution

Moment of inertia of each joist

$I = \frac{bd^3}{12} = \frac{(120)(300)^3}{12}$

$= 270 \times 10^6 \text{ mm}^4$

Section modulus $Z = \frac{I}{y}$

$= \frac{270 \times 10^6}{150} = 1.8 \times 10^6 \text{ mm}^3$

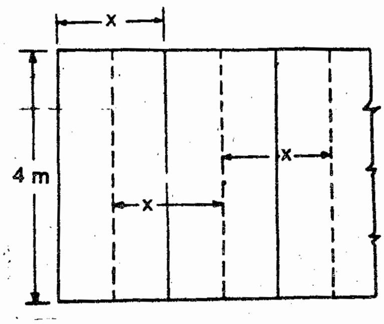


Fig. 7.13

Moment of resistance $M_r = \sigma \times z$

$$= 120 \times 1.8 \times 10^6$$
$$= 216 \times 10^6 \text{ N–mm}$$
$$= 216 \times 10^3 \text{ N–m}$$

Let the spacing of the joists be x metres then rate of loading on the joist

$$w = (300 \times x \times 1) \text{ KN/metre.}$$

$$\text{Maximum } B.M = \frac{wl^2}{8} = \frac{(300x)(4)^2}{8} = 600x \text{ KN–m} = 600 \times 10^3 x \text{ N-m}$$

Equating M_r to maximum bending moment.

$$600 \times 10^3 x = 216 \times 10^3$$

$$x = \frac{216}{600} \times \frac{10^3}{10^3} = .360 \text{ metres}$$

Spacing of joists = 0.36 metre c/c = 360 mm **Answer**

Example 7.13

A rolled steel joist with simply supported endsspans 10 metres. It is required to carry a load of 16 KN at its mid span. If the maximum fibre stress due to bending is not to exceed 120 MPa and the central deflection is not to exceed $\frac{1}{320}$ of the span, find a suitable depth of the joist. Take E = 200 KN/mm^2

Solution

$$\text{Central deflection} = \frac{1}{320} \times l$$

or $$\delta = \frac{wl^3}{48EI} = \frac{l}{320}$$

or $$I = \frac{wl^2 \times 320}{48 \times 200 \times 10^3} = \frac{16 \times 10^3 \times (10 \times 10^3)^2 \times 320}{48 \times 200 \times 10^3}$$

$$I = 5.33 \times 10^7 \text{ mm}^4$$

Applying bending equation

$$\frac{M}{I} = \frac{\sigma}{y}, \text{ B.M. at mid span} = \frac{WL}{4} = \frac{16 \times 10}{4} = 40 \text{ KN–m}$$

or $$y = \frac{\sigma \times I}{M} = \frac{120 \times 5.33 \times 10^7}{40 \times 10^6}$$

$$y = (30 \times 5.33) = 160 \text{ mm}$$

depth of the joist = 320 mm **Answer**

Example 7.14

A cast iron pipe 540 mm internal diameter and 30 mm wall thickness is running full of water and supported over a length of 8 metres. Determine the maximum stress intensity in the metal if the density of cast iron is 72 KN/m^3 and that of water 10 KN/m^3. (Patna Univ.)

Solution

Internal area of the pipe $= \dfrac{\pi}{4}(540)^2 = 229022.\text{mm}^2$

Cross-sectional area of the metal $= \dfrac{\pi}{4}(600^2 - 540^2)$

$= 53721.23 \text{ mm}^2$

Moment of inertia of the section $= \dfrac{\pi}{64}(600^4 - 540^4)$

$= 21.387 \times 10^8 \text{ mm}^4$

Section modules $Z = \dfrac{I}{y} = \dfrac{21.387 \times 10^8}{300} = 7.129 \times 10^6 \text{ mm}^3$

Weight of pipe per meter length

$$= \frac{53721.23}{(1000)^2} \times 1 \times 72 \times 10^3 = 3867.92 \text{ Newton}$$

Weight of water in the pipe of one meter length

$$= \frac{229022.1}{(1000)^2} \times 1 \times 10 \times 10^3 = 2290.22 \text{ Newton}$$

Total weight of pipe when full of water

$= (3867.92 + 2290.22) = 6058.14$ N/m

Maximum bending moment $= \dfrac{wl^2}{8}$

$$B.M. = \frac{6058.14\,(8)^2}{8} = 484665.12 \text{ N–m.}$$

From bending equation we know

$$M_r = \sigma.Z$$

$$\sigma = \frac{Mr}{Z} = \frac{48465.12 \times 10^3}{7.129 \times 10^6} = 6.798 \text{N/mm}^2$$

Maximum stress intensity $= 6.798$ MPa **Answer**

Flexural strength of a section

The moment of resistance offered by a beam is called its flexural strength or the strength of the beam. The strength of a beam section depends upon its section modulus $Z = \dfrac{I}{y}$. It is therefore necessary to know the value of section modulus for various sections.

Section modulus

$$Z = \frac{\text{M.O.I about the axis passing through C. G. of the section}}{\text{Maximum distance of the layer of the cross-section of the beam from the neutral axis}}$$

$$Z = \frac{I}{y}$$

Now from the bending equation

$$\frac{M}{I} = \frac{\sigma}{y}$$

or $$M = \sigma \times \frac{I}{y} = \sigma \times Z$$

From the above equation it is obvious that the moment of resistance of a beam is proportional to the section modulus since the stress will be same for a homogenous material of the beam section

Section modulus for various sections

(i) Rectangular section

Let b and d the width and depth of the section

(a) When N. A. is parallel to the width of the section

$$Z = \frac{I_{x-x}}{y}$$

$$= \frac{1}{12} bd^3 \times \frac{1}{d/_2}$$

$$= \frac{bd^2}{6}$$

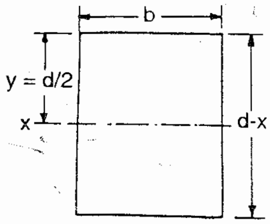


Fig. 7.14

(b) when N.A. is parallel to the depth of the section

$$Z = \frac{I_{y-y}}{x}$$

$$= \frac{1}{12} db^3 \times \frac{1}{b/2}$$

$$= \frac{db^2}{6}$$

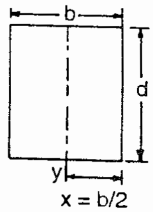


Fig. 7.15

(ii) Square section

Let 'b 'be the side of the square. The section modulus will be same

$$Z = \frac{I_{x-x}}{y} = \frac{I_{y-y}}{x}$$

$$= \frac{1}{12} b . b^3 \times \frac{1}{b/2} = \frac{1}{6} b^3$$

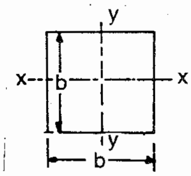


Fig. 7.16

(iii) Circular section

Let 'd' be the diameter of a solid circular section

$$Z = \frac{I_{x-x}}{y} = \frac{I_{y-y}}{x}$$

$$= \frac{\pi}{64} d^4 \times \frac{1}{d/2} = \frac{\pi}{32} d^3$$

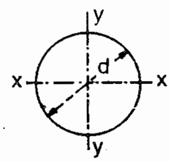


Fig. 7.17

Example 7.15

Three beams each of length L, same allowable bending stress σ are subjected to equal bending moment M.

If the cross-sections of the beams are a square, a rectangle with depth twice the width and a circle, determine the ratio of the weights of circular and rectangular beams with respect to the square beam. *(Oxford univ.)*

Solution

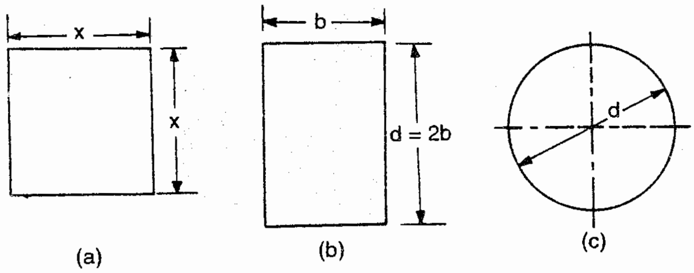


Fig. 7.18

Since all the beams have the same allowable stress σ and bending moment M hence the section modulus Z of all the beams must be equal

(a) Square section – Let the side of the square be x then

$$Z_1 = \frac{bd^2}{6} = \frac{x^3}{6}$$

(b) Rectangular section – Let the breadth of the beam be 'b' and depth = $2b$

$$\text{Section modulus} \quad Z_2 = \frac{bd^2}{6} = \frac{b(2b)^2}{6} = \frac{2}{3}b^3$$

(c) Circular section -

Let 'd' be the diameter of the circular section

Section modulus

$$Z_3 = \frac{I}{y} = \frac{\pi}{64}\frac{d^4}{d/2} = \frac{\pi}{32}d^3$$

Now $\quad Z = Z_1 = Z_2 = Z_3$

$$Z = \frac{x^3}{6} = \frac{2}{3}b^3 = \frac{\pi}{32}d^3$$

or $\quad d = 1.193\,x$ and $b = 0.6299\,x$

The weights of the beams are proportional to their sectional areas

$$\therefore \frac{\text{Wt of rectangular beam}}{\text{Wt of square beam}} = \frac{\text{Area of rectangular beam}}{\text{Area of square beam}}$$

$$= \frac{2b^2}{x^2} = \frac{2\,(0.6299x)^2}{x^2} = 0.7936$$

$$\frac{\text{Wt of circular beam}}{\text{Wt of square beam}} = \frac{\text{Area of circular beam}}{\text{Area of square beam}}$$

$$= \frac{\frac{\pi}{4}(d)^2}{x^2} = \frac{\frac{\pi}{4}(1.193x)^2}{x^2} = 1.118 \qquad \textbf{Answer}$$

Example 7.16

Calculate the dimensions of the strongest section that can be cut out of a circular log of wood 240 mm in diameter

Solution

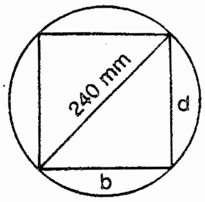


Fig. 7.19

The beam which offers maximum moment of resistance is considered as the strongest beam

$M_r = \sigma \times Z$

So for beams of same material σ, being common Z should be maximum for maximum strength

Let b = breadth and d = depth of the section

For maximum utility of the log of wood the corners of the section must lie on the circumference. Hence the diagonal of the section must be equal to the diameter of the log of wood for least wastage

$$b^2 + d^2 = (\text{diameter})^2 = (240)^2$$

$$\text{or} \quad d^2 = (240^2 - b^2)$$

$$Z = \frac{I}{y} = \frac{bd^2}{6} = \frac{b}{6}(240^2 - b^2)$$

For Z to be maximum $\frac{dz}{bd}$ should be equal to zero $\quad \frac{dz}{db} = 0$

$$\frac{d}{db}\frac{(57600\,b - b^3)}{6} \quad \text{or} \quad 3b^2 = 57600$$

$$\text{or} \quad b^2 = \frac{57600}{3} = 19200 \quad \text{or } b = 138.5 \text{ mm}$$

$$d^2 = \sqrt{57600 - 19200} = \sqrt{38400}$$

$$d = 195.5 \text{ mm} \qquad \textbf{Answer}$$

Example 7.17

A beam of I – section is shown in figure 7.20 compare its flexural strength with

(a) A rectangular section of the same area and same $\frac{b}{d}$ ratio

(b) A solid circular section of the same area

Solution

The moment of inertia of the I – section

$$I_{xx} = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$= \frac{80\,(100)^3}{12} - \frac{70\,(80)^3}{12}$$

$$= 368.0 \times 10^4 \text{ mm}^4$$

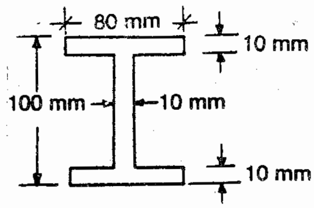


Fig. 7.20

Section modulus $Z_I = \frac{I}{y} = \frac{368 \times 10^4}{50}$

$= 73.6 \times 10^3 \text{ mm}^3$

Area of the I – section

$A = 80 \times 10 + 80 \times 10 + 80 \times 10$

$= 2400 \text{ mm}^2$

(ii) For rectangular section

$\frac{b}{d} = \frac{80}{100}$ or $b = .8d$

Area $= b \times d = .8d \times d = 2400$

or $.8d^2 = 2400$ or $d = 54.77$ m

and $b = .8 \times 54,77 = 43.81$ mm

$\therefore \quad Z_2 = \frac{bd^2}{6} = \frac{43.81 \times (54.77)^2}{6}$

$= 21.90 \times 10^3 \text{ mm}^3$

Circular section

Area $= \frac{\pi}{4} d^2 = 2400$

or $d = \sqrt{\frac{2400 \times 4}{\pi}} = 55.27$ mm

$Z_3 = \frac{\pi}{32} d^3 = \frac{\pi}{32} (55.27)^3 = 16.58 \times 10^3 \text{ mm}^3$

The ratio of flexural strength of the

I, Rectangular and circular section $Z_1 : Z_2 : Z_3$

is $73.6 \times 10^3 : 21.9 \times 10^3 : 16.58 \times 10^3$

$4.43 : 1.32 : 1$

Hence I – section is 4.43 times stronger than the circular section and rectangular is 1.32 times stronger than the circular section

Example 7.18

Compare the weights of two equally strong beams of circular section made of the same material, one being of solid section and the other of hollow section with internal diameter being 40% of the external diameter.

(Calcutta Univ.)

Solution

Since the beams are equally strong therefore moment of resistance of both the beams must be equal

$M_{r\,(\text{solid})} = M_{r\,(\text{hollow})} \therefore \sigma_s \times Z_s = \sigma_h \times Z_h$

or $Z_s = Z_h$

Section modulus of solid section

$$Z_s = \frac{I}{y} = \frac{\frac{\pi}{64} d_s^4}{d/2} = \frac{\pi}{32} d_s^3$$

Section modulus of hollow section

$$Z_h = \frac{I}{y} = \frac{\frac{\pi}{64}\left(D_h^4 - d_h^4\right)}{D_h/2} = \frac{\frac{\pi}{32}\left[\left(D_h^4\right)-\left(.4D_h\right)^4\right]}{D_h}$$

$$Z_h = \frac{\pi}{32}\frac{\left(D_h^4 - 0.0256 D_h^4\right)}{D_h} = \frac{\pi}{32} \times 0.9744\ D_h^3$$

$$\therefore \quad \frac{Z_h}{Z_s} = \frac{\frac{\pi}{32}\times 0.9744\, D_h^3}{\frac{\pi}{32}\cdot d_s^3} = \frac{0.9744\, D_h^3}{d_s^3}$$

$$\text{or} \quad \frac{D_h^3}{d_s^3} = \frac{1}{0.9744} = 1.0262$$

As the material is same, the ratio of their weights

$$\frac{W_s}{W_h} = \frac{\text{Volume}\times \text{ density of solid section}}{\text{Volume}\times \text{ density of hollow section}}$$

$$= \frac{\text{Area}\times \text{ length}\times \text{ density of solid section}}{\text{Area}\times \text{ length}\times \text{ density of hollow section}}$$

Area of solid section = $\frac{\pi}{4}\, d_s^2$

Area of hollow section = $\frac{\pi}{4}\left[D_h^2 - (.4D_h)^2\right] = \frac{\pi}{4}\times 0.84\ D_h^2$

$$\therefore \quad \frac{W_s}{W_h} = \frac{\frac{\pi}{4}\, d_s^2}{\frac{\pi}{4}\times 0.84\, D_h^2} = \frac{d_s^2}{0.84\, D_h^2}$$

Since $D_h = 1.0086\, d_s$

$$\therefore \quad \frac{W_s}{W_h} = \frac{d_s^2}{0.84\ (1.0086\, d_s)^2} = 1.17$$

Hence the weight of the solid beam is 1.17 times the weight of the hollow beam

Flitched beams (Transformed section method)

In order to increase the strength of timber beams steel plates are sand witched between two timber cross-sections. Such beams are called **Flitched beams.** Steel plates act as reinforcement for timber. The timber and steel sections are bolted together very tightly so that there is no slip between the two materials. The stresses induced in the two materials are in the ratio of their modulii of elasticity. The moment of resistance of flitched beams can be calculated by converting the area of one material into an equivalent area in terms of the other material. This method is known as **transformed section method**. The important point to be kept in mind is that the distance of various sections of the transformed material with respect to the neutral axis should remain the same as in the original beam. Following examples will explain the transformed section method.

Example 7.19

A flitched beam consists of a timber joist 150 mm × 250 mm strengthened by steel plates 10 mm × 200 mm on either side of the joist. If the stresses in steel and timber are not to exceed 120 MPa and 7 MPa, then find the moment of resistance of the flitched beam. Take m = 20

Solution

The equivalent moment of inertia of the cross-section as if the entire beam is made of timber

$$I_{xx} = (I_t + mI_s)$$

$$I_{xx} = \frac{150\,(250)^3}{12} + 20 \times 2\,\frac{(10)\,(200)^3}{12}$$

$$= \frac{1}{12}(234375 + 320000) \times 10^4 \text{ mm}^4$$

$$= 46197.9 \times 10^4 \text{ mm}^4$$

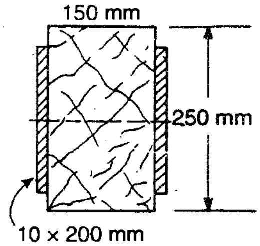


Fig. 7.21

Moment of resistance of the flitched beam

$$M_r = \sigma_t \times \frac{I}{y}$$

$$= 7 \times \frac{46197.9 \times 10^4}{125} \qquad = 2587.08 \times 10^4 \text{ N–mm}$$

$M_r = 25.87$ KN–m **Answer.**

Example 7.20

A flitched beam consists of two wooden joists 100 mm × 200 mm with a steel plate 10 mm × 140 mm placed symmetrically between them. If σ_w = 7 MPa and E_s = 200 KN/mm² and E_w = 10 KN/mm². Determine corre - sponding stress in steel plate and the moment of resistance of the flitched beam.

Solution

From the symmetry of the cross–section it can be said that the neutral axis lies at 100 mm from the top fibre of the wooden joist.

Stress in top fibre of the joist at A = 7 MPa

$$\text{Stress in timber at } B = \frac{70}{100} \times 7$$

$$= 4.9 \text{ MPa}$$

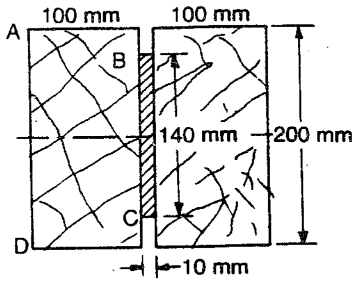


Fig. 7.22

$$\text{Modular ratio } \frac{E_s}{E_w} = \frac{200 \times 10^3}{10 \times 10^3} = 20$$

Moment of inertia of the transformed cross-section about $x - x$

$$I_{xx} = (I_w + mI_s)$$

$$I_{xx} = \left[2\,\frac{(100)\,(200)^3}{12} + 20\,\frac{(10)\,(140)^3}{12}\right]$$

$$I = (13333. + 4573) \times 10^4 = 17906 \times 10^4\ \text{mm}^4$$

$\bar{y}$ = 100 mm and σ = 7MPa

Applying bending equation

$$M_r = \sigma_W \times \frac{I}{y} = \frac{7 \times 17906 \times 10^4}{100} = 125346\ \text{N–mm}$$

$$= 125.346\ \text{N–m}$$ **Answer.**

Example 7.21

A wooden beam 150 mm × 200 mm is reinforced at the bottom by a steel plate 10 mm × 150 mm. If the allowable stress in timber is 8 MPa, calculate the moment of resistance of the beam.

Take m = 15.

Solution

$$\bar{y} = \frac{(150 \times 200)\,(110) + m\,(10)\,(150)\,(5)}{(150)\,(200) + m\,(10)\,(150)}$$

Where m = 15

$\bar{y}$ = 65 mm from *AB*

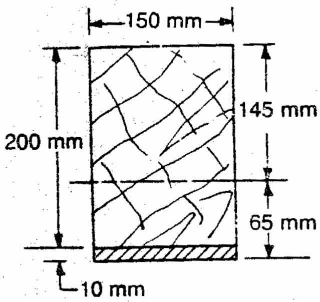


Fig. 7.23

Equivalent moment of inertia of the section

$$I_{xx} = (I_t + m\,I_s)$$

$$I_{xx} = \frac{(150)\,(200)^3}{12} + (150)\,(200)\,(45)^2 + \frac{15\,(150)\,(10)^3}{12} + (150)(10)\,(60)^2$$

$$= 24193.73 \times 10^4\ \text{mm}^4$$

y maximum = (210 – 65) = 145 **mm**

Stress in timber = σ_w = 8 *MPa*

Moment of resistance = $\sigma \times \dfrac{I}{y}$

$$M_r = \frac{8 \times 24193.73 \times 10^4}{145} = 13.348\ \text{KN–m}$$ **Answer**

Example 7.22

A wooden beam 150 mm × 250 mm is to be reinforced with two steel flitches 10 mm × 150 mm in section. Compare the strengths of the beams for the following cases

(i) Flitches are attached to top and bottom

(ii) Flitches are attached symmetrically on the sides.

Take m = 20

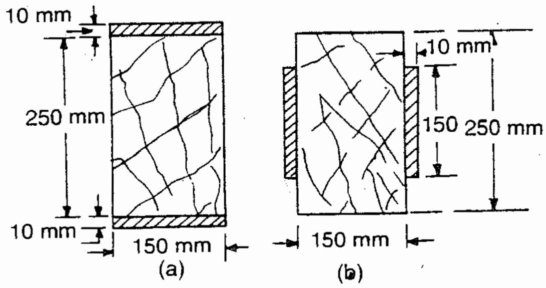


Fig. 7.24

Solution

Case (i) The equivalent moment of inertia as if the entire beam is made of timber

$$I_1 = I_w + m\, I_s$$

$$I_1 = \frac{(150)\,(250)^3}{12} + 20 \times 2\left[\frac{150\,(10)^3}{12} + (150)\,(10)\,(130)^2\right]$$

$$= 120981.25 \times 10^4 \text{ mm}^4$$

Let σ_w be the stress in timber

Moment of resistance

$$M_1 = \sigma_w \cdot \frac{I}{y} = \frac{\sigma_w \times 120981.25 \times 10^4}{125} = 967.85 \times 10^4\, \sigma_w$$

$$M_1 = 967.85 \times 10^4\, \sigma_w$$

Case (ii) The equivalent moment of inertia of the entire beam in terms of timber

$$I_2 = (I_w + m\, I_s)$$

$$I_2 = \frac{150\,(250)^3}{12} + 20 \times 2\left\{\frac{1}{12}(10)\,(150)^3\right\}$$

$$= (19531.25 + 11250) \times 10^4 = 30781.25 \times 10^4 \text{ mm}^4$$

Moment of resistance

$$M_2 = \sigma_w \cdot \frac{I_2}{y} = \frac{(30781.25 \times 10^4)}{125}\, \sigma_\omega = 246.25 \times 10^4\, \sigma_\omega$$

$$= 246.25 \times 10^4\, \sigma_\omega$$

Thus the ratio of moment of resistance

$$\frac{M_2}{M_1} = \frac{246.25 \times 10^4 \sigma_\omega}{967.85 \times 10^4 \sigma_\omega} = 0.254 \qquad \textbf{Answer.}$$

II. Shearing Stresses In Beams

The stress caused by the shearing force at a section of a beam is called shear stress. When a beam is loaded not only bending stresses are induced but shearing stresses are also induced. The effect of vertical shearing stress on a beam is to cause sliding on a vertical section. The vertical shear stress is always accompanied by a horizontal shear stress. Shear stress varies along the depth of the section shearing stress is maximum at the neutral axis and diminishes to zero at the outermost fibre on either side of the neutral axis. These stresses cause diagonal tension and compression inclined at 45 degrees to the horizontal. The variation in intensity of vertical shearing force may be analysed as follows.

Distribution of Shear Stress

Consider an element of length dx cut from a beam as shown in figure 7.25

Let M be the bending moment at the left side of the element and

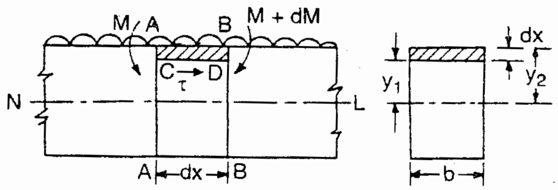


Fig. 7.25

$M + dM$ be the bending moment at the right side of the element. If y is measured upwards from the neutral axis, then the bending stress at the left section A A

$$\sigma = \frac{M}{I} \cdot y$$

Where I denotes the moment of inertia of the entire cross-section about the neutral axis. Similarly the bending stress at the right section B B is

$$\sigma' = \frac{(M + dM) \cdot y}{I}$$

Now consider the equilibrium of the shaded element $ACDB$. The force acting on an area dA of the face AC is the product of area and the stress.

$$\therefore \qquad \sigma .\, dA = \frac{M}{I} . y . dA.$$

The sum of all such forces over the left face AC is found by integration

$$\int_{y_1}^{y_2} \frac{M . y}{I} . dA.$$

Similarly the sum of all normal forces over the right face *BD* is given by

$$\int_{y_1}^{y_2} \frac{(M+dM)y}{I} \, . \, dA$$

Since these two integrals are unequal, some horizontal force must act on the shaded element to keep it in equilibrium. This horizontal shearing force acts on the lower face *CD*.

Let τ be the shearing stress and b, be the width of the beam at the position where τ acts then horizontal shearing force along the face $C\,D = \tau . b . \, dx$. For equilibrium of the element *ABCD,* we have

$$\Sigma F_h = \int_{y_1}^{y_2} \frac{M.y}{I} \, . \, dA - \int_{y_1}^{y_2} . \frac{(M+dM)}{I} . \, y \, . \, dA \; + \tau b \, . \, dx = 0$$

Solving we get

$$\tau = \frac{1}{Ib} \frac{dM}{dx} \int_{y_1}^{y_2} y \, .dA$$

The term $\frac{dM}{dx}$ represent the shear force V at the section A–A

$$\tau = \frac{V}{Ib} \int_{y_1}^{y_2} y . \, dA$$

The term $\int_{y_1}^{y_2} y . \, dA$ is the first moment of the shaded area about N–A.

Let it be equal to $A\bar{y}$.

$$\tau = \frac{V.A.\bar{y}}{Ib}$$

Where $\quad \tau$ = Shear stress at any section

$A \, . \, \bar{y}$ = *Moment of area (between the section and extreme end on the same side of the neutral axis) about the N – A..*

I = Moment of inertia about C.G.

b = Width of the section.

V = Total shear force at the section.

Variation Of Shear Stress

(1) Rectangular Section

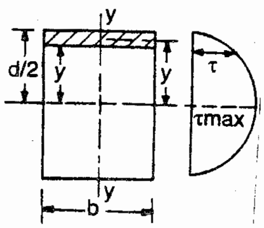


Fig. 7.26

Consider a rectangular section of width b and depth d

Area of the shaded portion= b . $\left(\frac{d}{2} - y\right)$

Distance of C.G. of this area from N-A.

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Moment of this area about $N\text{–}A = b\left(\frac{d}{2}-y\right)\times\frac{1}{2}\left(\frac{d}{2}+y\right)$

$$A.\bar{y} = \frac{b}{2}\left(\frac{d^2}{4}-y^2\right)$$

Intensity of Shear Stress $\tau = \frac{V.A\bar{y}}{I.b}$

$$\tau = \frac{V.}{Ib}\cdot\frac{b}{2}\left(\frac{d^2}{4}-y^2\right) = \boxed{\frac{V}{2I}\left(\frac{d^2}{4}-y^2\right)}$$

The intensity of shear stress depends upon the variable y. It decreases with increase in the value of y and Vice - Versa

τ at the top when $y = \frac{d}{2}$

$$\tau_{min} = \frac{V}{2I}\left(\frac{d^2}{4}-\frac{d^2}{4}\right) = \text{Zero}$$

τ at the neutral axis, when $y = 0$

$$\tau_{\max} = \frac{V}{2I}\left(\frac{d^2}{4}-0\right) = \frac{Vd^2}{8I}$$

$$\tau_{\max} = \frac{Vd^2}{8I} = \frac{Vd^2}{8}\times\frac{1}{\frac{1}{12}bd^3} = \frac{Vd^2}{8}\times\frac{12}{bd^3}$$

$$= \frac{3}{2}\frac{V}{bd} = 1.5\,.\frac{V}{bd}$$

$$\tau_{\max} = 1.5\,.\frac{V}{bd} = 1.5\,.\text{ Average shear stress}$$

The above equation shows that the variation in shear stress is parabolic and that in a rectangular section, maximum shear stress at mid depth is 1.5 times the average shear stress.

Circular Section

A beam of solid circular section is shown in figure. 7.27 Consider an elementary strip at a distance y from the *N.A.*

Let b = breadth of the strip

dy = thickness of the strip

Area of the strip = $b.dy$

Moment of area of the strip about N.A = $b.dy.y$

Moment of the shaded area about the N.A = $A.\bar{y}$

$$A.\bar{y} = \int_{y=y}^{y=R} b.y\,.\,dy \quad --- \quad --- \quad \text{(i)}$$

Now referring to the figure, width of the section

$$b = 2\sqrt{R^2-y^2}$$

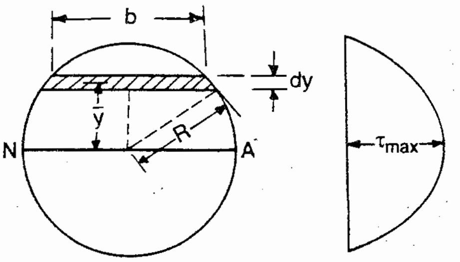


Fig. 7.27

or $\quad b^2 = 4\,(R^2 - y^2)$

Differentiating we get

$$2b.db = 4\,(-2y)\,dy, \text{ Since } R \text{ is constant}$$
$$= -8y.dy$$

or $\quad y.dy = -\dfrac{b}{4}\,.\,db$

The value of b will be zero at the top and maximum at Neutral axis

$\therefore$ When $\quad y = R \quad , \quad b = o$

and when $\quad y = Y \quad , \quad b = b$

Therefore by substituting these values in equation (i) we get

$$A.\bar{y} = \frac{1}{4}\int_b^o -b^2.\,db$$

$$= \frac{1}{4}\left[\frac{-b^3}{3}\right]_b^o = \frac{1}{4}\left[0 - \left(-\frac{b^3}{3}\right)\right] = \frac{b^3}{12}$$

Now $\tau = \dfrac{V}{I.b}\,.\,A.\bar{y}$

$$= \frac{V}{Ib}\cdot\frac{b^3}{12} = \frac{Vb^2}{12I} \quad \text{put } b^2 = 4(R^2 - Y^2)$$

or $\quad \tau = \dfrac{V\,4}{12I}\,(R^2 - y^2) = \dfrac{V\,(R^2 - y^2)}{3I}$

Shear stress has a parabolic variation and will be maximum when y will be zero.

$$\tau_{max} = \frac{VR^2}{3I}$$

$$= \frac{VR^2}{3 \times \frac{\pi}{4} R^4} = \frac{4}{3}\,\frac{V}{\pi R^2}$$

or $\quad \tau_{max} = 1.33\ \tau$ average

Example 7.23

A laminated timber beam 120 mm × 150 mm is made of three 50 mm × 120 mm wide planks glued to-gether as shown in figure 7.28, to resist longitudinal shear. The beam is simply supported over a span of 3 metres. If the allowable shear stress in the glued joint is 5 MPa, determine the safe point load the beam can carry at the centre.

Solution.

Let W be the point load at the centre.

Then the maximum shear force $= \dfrac{W}{2}$

$$I_{N.A} = \frac{1}{12} bd^3 = \frac{1}{12} \times 120\,(150)^3 = 3375 \times 10^4\ \text{mm}^4$$

Shear stress at the glued joint where the permissible shear stress is 5 MPa

A = Area of $ABCD$, the area above the glued joint C–D

$= 120 \times 50$

$= 6000\ \text{mm}^2$

$\bar{y} = \left(\dfrac{50}{2} + \dfrac{50}{2}\right) = 50\ \text{mm}$

$A\bar{y} = 6000 \times 50$

$= 300 \times 10^3$

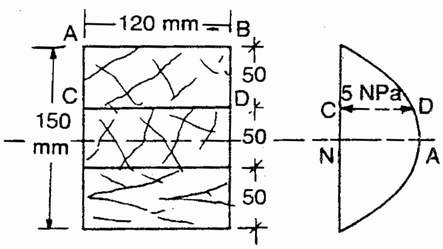


Fig. 7.28

$$\tau_{N\text{-}A} = \frac{V.A\bar{y}}{I.b} = \frac{W}{2} \times \frac{300 \times 10^3}{3375 \times 10^4 \times 120}$$

$$5 = \frac{W}{2} \times \frac{300 \times 10^3}{3375 \times 10^4 \times 120}$$

$$\text{or} \quad W = \frac{5 \times 2 \times 3375 \times 10^4 \times 120}{300 \times 10^3} = \frac{3375 \times 120 \times 10}{300}$$

$= 3375 \times 4 = 13500$ Newton

$= 135$ KN

Example. 7.24

A simply supported steel beam of I-section 120mm× 50mm with 5mm thick flanges and web carries a uniformly distributed load of 2KN/m on a span of 16 metres. Determine the maximum intensity of shear stress on a vertical section 5 metres from one end. What is the ratio to the average shear intensity at the section ?

Moment of inertia of the section

$$I_{xx} = \frac{1}{12} BD^3 - \frac{1}{12} bd^3$$

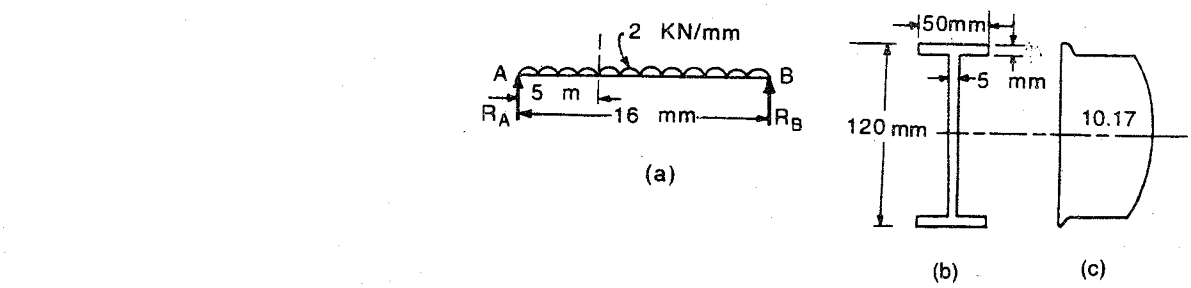


Fig. 7.29

$$= \frac{1}{12}(50)(120)^3 - \frac{1}{12}(45)(110)^3$$
$$= 7200000 - 499125$$
$$= 690.08 \times 10^4 \text{ mm}^4$$

Area of the section $= 2(50)(5) + (110) \times (5)$

$$= 500 + 550 = 1050 \text{ mm}^2$$

Reactions $R_A = R_B = \frac{wl}{2} = \frac{2 \times 16}{2} = 16$ KN

Shear force at 5 metres from A = $R_A - w.x$

$$= 16 - 2 \times 5 = 6 \text{ KN} = 6000\text{N}$$

Average shear stress $\tau_{av} = \frac{\text{Shear force}}{\text{Area of Cross–section}}$

$$\tau_{av} = \frac{6 \times 10^3}{1050} = 5.71 \text{ MPa}$$

$$\tau_{max} = \frac{V.A.\bar{y}}{Ib} \text{ and will be at } N.A$$

A $\bar{y}$ = Moment of the area above $N.A$ about $N.A$

$$= (50 \times 5)(55 + 2.5) + (5 \times 55)(55/2)$$
$$= (250 \times 57.5) + (275 \times 27.5)$$
$$= 14375 + 7562.5 = 21937.5 \text{ mm}^2$$

b is the width of the section at which shear stress is to be determined.

$$\tau_{max} = \frac{16 \times 21937.5 \times 1000}{690.08 \times 10^4 \times 5} = 10.71 \text{ MPa}$$

Ratio of $\frac{\tau_{max}}{\tau_{av}} = \frac{10.17}{5.71} = 1.78$

Example 7.25

A T-Section 200mm20mm is used as beam with 200mm side horizontal. The beam has to resist a shear force of 15 KN. Find the maximum intensity of shear stress across the section and sketch the distribution of shear stress across the section.

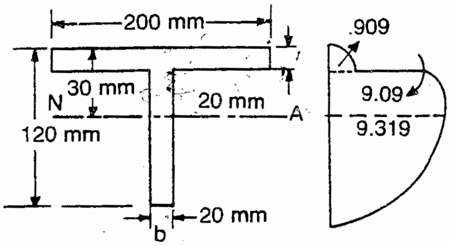


Fig. 7.30

Solution

$$\bar{y} = \frac{(200 \times 20)\,(10) + (100 \times 20)\,(50 + 20)}{(200 \times 20) + (100 \times 20)}$$

$$= \frac{4000 + 140{,}000}{4000 + 2000} = \frac{180{,}000}{6000}$$

$$= 30 \text{ mm.}$$

$$I_{xx} = \left[\frac{200(20)^3}{12} + (200 \times 20)\,(30 - 10)^2\right] + \left[\frac{(20)\,(100)^3}{12} + (100 \times 20)\,(40)^2\right]$$

$$= 660{,}0000 \text{ mm}^4 = 660 \times 10^4 \text{mm}^4$$

Area of the flange = 200 × 20 = 4000 mm^2

Distance of C.G from N.A = (30 – 10) = 20 mm, b = 200 mm

Intensity of shear stress $\tau = \dfrac{V.A\,\bar{y}}{I.b}$

$$\tau_{aa} = \frac{15 \times 10^3 \times (4000)\,(20)}{6600000 \times 200} = \frac{60}{66} = 909 \text{ MPa}$$

Shear stress at the junction of flange and Web

$$= .909 \times \frac{200}{20} = 9.09 \text{ MPa}$$

Maximum shear stress will occur at the neutral axis

Area of the section above the neutral axis

= (200 × 20) + (20) (30 – 10)

= 4000 + 400

$A\,\bar{y}$ = 4000 × 20 + 400 × 5 = 80,000 +2,000 = 82000 mm^3

$$\tau = \frac{V.A\,\bar{y}}{I \times b} = \frac{15 \times 10^3 \times 82{,}000}{660 \times 10^4 \times 20}$$

$$= \frac{15 \times 82}{66 \times 2} = 9.318 \text{ MPa} \qquad \textbf{Answer.}$$

Example. 7.26

A channel section as shown in the figure is used as a beam with 200 mm base vertical. At a certain cross-section it has to resist a shear force of 120 KN. calculate the maximum intensity of shear stress induced in the section and sketch the distribution of stress across the section.

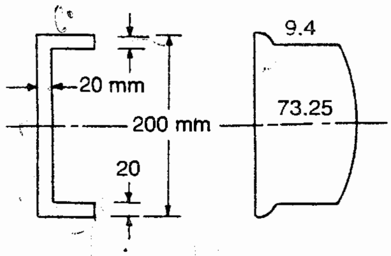


Fig. 7.31

Solution

$$I_{xx} = \frac{1}{12}(60)(200)^3 - \frac{1}{12}(60 - 10)(200 - 40)^3$$

$$= 40 \times 10^6 - 17.06 \times 10^6 = 22.94 \times 10^6 \text{ mm}^4$$

Intensity of shear stress at the top is Zero

(ii) For intensity of shear stress in the flange at the junction of flange and Web

$$A = 60 \times 20 = 1200 \text{ mm}^2$$

$$\bar{y} = 90 \text{ mm}$$

$$A\bar{y} = 1200 \times 90 = 108 \times 10^3 \text{ mm}^3$$

$$\tau = \frac{V \cdot A\bar{y}}{I \cdot b} = \frac{120000 \times 108 \times 10^3}{22.93 \times 10^6 \times 60}$$

$$= 9.4 \text{ MPa}$$

(iii) Intensity of shear stress in the web at the junction of web and the flange.

Shear stress will increase by $\frac{B}{b}$ $\frac{\text{width of flange}}{\text{thickness of web}}$

$$\tau_{aa} \text{ for Web} = \frac{9.4 \times 60}{10} = 56.4 \text{ MPa}$$

(iv) Intensity of shear stress at the neutral axis

Area of the portion above the N.A

$$A = (60 \times 20) + 10 \times 80$$

$$A\bar{y} = 1200 \times 90 + 800 \times 40$$

$$= 108000 + 32000 = 140 \times 10^3 \text{ mm}^3$$

$$b = 10 \text{ mm (thickness of web)}$$

$$\tau_{xx} = \frac{V.A\bar{y}}{I.b} = \frac{120 \times 10^3 \times 140 \times 10^3}{22.93 \times 10^6 \times 10}$$

$= 73.25$ MPa **Answer.**

Example 7.27

The section of a beam is a triangle with base b and height h, the base being placed horizontally. At a certain cross-section the shear force is V. Prove that the maximum intensity of shear stress occurs at $\frac{h}{2}$ *and its* *magnitude is* $\frac{3V}{b.h}$ *and that the shear stress intensity at the neutral axis is* $\frac{8V}{3b\,h}$.

Solution

Let the intensity of shear stress be maximum at a distance x from the top.

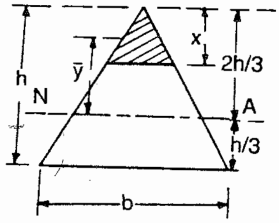


Fig. 7.32

$$\tau_x = \frac{V.A\,\bar{y}}{I.b}$$

$$\tau_x = \frac{V \times \frac{1}{2}\left(b.\frac{x}{h}\right).x\left(\frac{2}{3}h - \frac{2x}{3}\right)}{\frac{bh^3}{36}.\frac{b}{h}x}$$

$$= \frac{12V}{bh^2}\left(x - \frac{x^2}{h}\right)$$

For maximum shear stress its derivative must be equal to zero

$$\frac{d(\tau_x)}{dx} = \frac{12V}{bh^2}\left(1 - \frac{2x}{h}\right) = 0 \quad \text{or} \quad x = \frac{h}{2}$$

$$\therefore\ \tau_{max} = \frac{12V}{bh^2}\left(\frac{h}{2} - \frac{1}{h}.\frac{h^2}{4}\right) = \frac{3V}{bh}$$

$$\tau_{n.a} = \frac{V\left(\frac{1}{2}.\frac{2}{3}.b.\frac{2}{3}h\right)\left(\frac{1}{3}.\frac{2}{3}.h\right)}{\frac{bh^3}{36} \times \frac{2}{3}b} = \frac{\dfrac{V \times \frac{4}{81}bh^2}{2b^2h}}{108}$$

$$= \frac{8V}{3bh}$$ **Answer.**

Example 7.28

Find the ratio of the maximum shear stress to the mean shear stress of the beam section shown in figure 7.33

Solution

For analysing the shear stress distribution let us consider the two semi circular grooves as a hole of 60 mm diameter.

Moment of inertia of the section about $x - x$ axis

$$I_{xx} = \left[\frac{1}{12}(90)(90)^3 - \frac{\pi}{64}(60)^4\right]$$

$$= 483.14 \times 10^4\ \text{mm}^4$$

Shear stress at the edges will be Zero

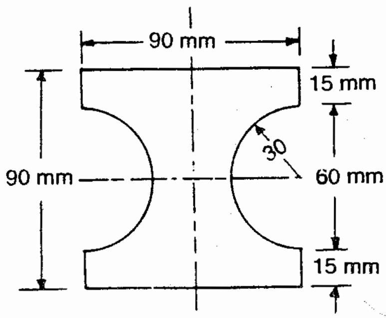


Fig. 7.33

Shear stress at $B-B$

A = Area of the section above $B-B = 90\times 15 = 1350\ mm^2$

$\bar{y}$ = Distance of its $C.G$ from $N.A = 30 + 7.5 = 37.5$ mm and width b = 90 mm.

$$\text{Hence } \tau_{B-B} = \frac{V.A\bar{y}}{I\times b} = \frac{V\times 1350\times 37.5}{483.14\times 10^4\times 90} = 1.16\times 10^4\ V\ \ MPa.$$

Intensity of shear stress at $N.A$

$$A\bar{y} = \left[(90\times 45)\left(\frac{45}{2}\right) - \frac{\pi}{2}(30)^2 \cdot \frac{4}{3\pi}\times 30\right] = 73125$$

$$b = (90-60) = 30\ mm$$

$$\tau_{max} = \frac{V\times A\bar{y}}{I.b}$$

$$= \frac{V\times 73125}{483.14\times 10^4\times 30} = 5.045\times 10^{-4}\ V\ \ MPa$$

$$\tau_{mean} = \frac{\text{Shear Force}}{\text{Area of Cross-section}}$$

$$= \frac{V}{[90\times 90 - \frac{\pi}{4}(60)^2]} = \frac{V}{8100 - 2827.43}$$

$$= \frac{V}{5272.57}\ \text{MPa}$$

$$\frac{\tau_{max}}{\tau_{mean}} = \frac{5.045\times 10^{-4}\ V}{\frac{V}{5272.57}} = 5.045\times 5272.57\times 10^{-4}$$

$$= 2.66$$ **Answer.**

Example 7.29

A beam section as shown in fig.7.34 is subjected to a shear force V. Find the ratio of the shear stresses at the section and at the neutral axis. The section is at a distance $\frac{h}{8}$ from the neutral axis. *(Roorkee Univ.)*

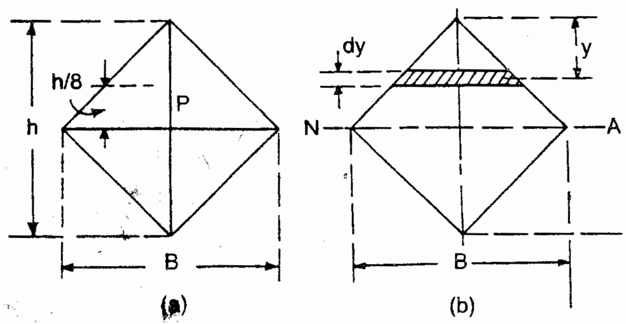


Fig. 7.34

Solution :

The horizontal diagonal is the $N.\ A$ of the Section. Now consider a horizontal strip of thickness dy at a distance y from the top.

Width of the strip $= \dfrac{2yB}{h}$

Distance of the strip from $N.\ A = \left(\dfrac{h}{2} - y\right)$

Moment of the strip about $N.\ A$

$$= \left(\frac{2yB}{h} \cdot dy\right)\left(\frac{h}{2} - y\right)$$

$$= \left(By - \frac{2By^2}{h}\right) dy$$

$$A\bar{y} = \int_0^{h/2} \left(By - \frac{2By^2}{h}\right) dy = \left[\frac{By^2}{2} - \frac{2By^3}{3h}\right]_0^{h/2}$$

$$= \left(\frac{Bh^2}{8} - \frac{Bh^2}{12}\right) = \frac{Bh^2}{24}$$

Now $= I = 2\,\dfrac{B\,(h/2)^3}{12} = \dfrac{Bh^3}{48}$

Shear stress at the section is

$$\tau = \frac{V}{IB} \cdot A\,\bar{y}$$

$$= \frac{V \times 48}{Bh^3 \times b} \times \frac{Bh^2}{24} = \frac{2V}{hb}$$

$$= \frac{2V \cdot h}{h \cdot 2yB} = \frac{V}{yB}$$

At the given Section

$$y = \frac{h}{2} - \frac{h}{8} = \frac{3h}{8}$$

$$\tau_{PQ} = \frac{V}{By} = \frac{V \times 8}{B \times 3h} = \frac{8V}{3Bh}$$

At neutral axis $\tau_{N.A} = \dfrac{V}{By} = \dfrac{V}{B.h/2} = \dfrac{2V}{Bh}$

$$\therefore \qquad \frac{\tau_{NA}}{\tau_{PQ}} = \frac{2V}{B.h} \Big/ \frac{8V}{3B.h} = \frac{3}{4}$$

And $\dfrac{\tau_{PQ}}{\tau_{NA}} = 1.33$ **Answer.**

SUMMARY

1. Bending equation is

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

Where M = Moment of resistance which is equal to the applied bending moment on the beam

I = Moment of inertia of the beam section about $N.A.$

σ = Bending stress at a distance y from the neutral axis.

E = Modulus of elasticity of beam material

R = Radius of curvature of the beam

2. Neutral axis is the axis accross a section which divides the section into tension Zone and compression Zone.
3. Neutral axis passes through the centroid of the section under simple bending.
4. Neutral axis remains unaffected and the stress at $N.A.$ is Zero.
5. Maximum bending stresses are induced in the extreme fibres of the section. This stress is called skin stress.
6. Moment of resistance is the sum of moments due to internal stresses and is numerically equal to the applied moment.
7. Section modulus $Z = \dfrac{I}{Y}$
8. The flexural strength of a section means the moment of resistance offered by it.
9. In Composite section, the total moment of resistance is the sum of the *moments of resistance of individual sections.*

   $M_r = M_1 + M_2$
10. Shear stress is the stress caused by the shear force at a section of a beam.
11. $\tau = \dfrac{V}{Ib} \cdot A\bar{y}$
12. In case of rectangular section.

$$\tau = \frac{V}{2I}\left(\frac{d^2}{4} - y^2\right)$$

$\tau_{max} = 1.5\ \tau$ Awarage..

13. For circular section

$$\tau = \frac{V}{3I}(R^2 - y^2)$$

$\tau_{max} = 1.33\ \tau$ Average

14. In T – section, maximum shear stress will occur at neutral axis
15. Maximum shear stress at the top of a rectangular section is zero.
16. Transverse shear is always accompanied by a complementary shear.

QUESTIONS

(1) How bending stresses are induced in a beam? Explain. What is the nature of stress in top most fibre and bottom most fibre of a beam subjected to simple bending?

(2) State the assumptions made in the theory of simply bending.

(3) What is the difference between neutral axis and neutral surface of a beam? Why should the bending stress be zero at the neutral axis

(4) Prove the bending equation for beams subjected to pure bending.

(5) What is section modulus ? How it is related to the flexural strength of a section.

(6) What is shear stress ? Explain.

(7) Derive an expression for shear stress τ at any point in the transverse section of a beam subjected to a shear force V.

(8) Prove that in case of rectangular section, the maximum shear stress is 1.5 times the average shear stress.

EXERCISES

(9) The moment of inertia of a beam section 300 mm deep is 60×10^6 mm^4. Determine the largest simply supported span over which a beam of this section can be used for carrying a *u.d.l.* of 5 KN per metre run. The maximum fibre stress is limited to 80 MPa. Also calculate the value of a concentrated load W that the beam can carry at its centre on a span of 8 metres .

(l = 7.1 metres, W = 16 KN)

(10) A simply supported beam of rolled steel section carries two point loads 100 KN each at 250 mm from the supports. Determine the maximum bending stresses. ($\sigma_c = \sigma_E =$ 161.6 MPa)

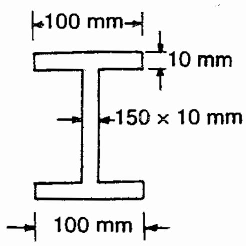


Fig. 7.35

(11) A T – section beam having flangel 100 mm × 20 mm and web 20 × 100 mm is simply supported over a span of 6 metres. It carries a *u.d.l* of 300 N/m run including its own weight over the entire span together with a load of 250 N at mid span. Calculate the maximum tensile and Compression stresses induced in the beam. (σ_t = 2.5.87 MPa and σ_c = 12.93 MPa)

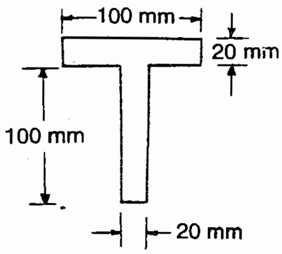


Fig. 7.36

(12) A horizontal girder 10 metres long rests on supports at ends. Form one of its ends. A up to the centre it carries a load of 15 N/m run and from the centre to the end B a load of 30 N/m. Determine the maximum bending moment acting on the beam. If the depth of the beam is 400 mm, find the moment of inertia of the beam so that the maximum stress produced may not exceed 140 MPa.

(M_{max} = 227.41 N –m, $I = 32.488 \times 10^4$ mm^4)

(13) A timber beam 160 mm wide and 300 mm deep is simply supported on a span of 5 metres. It carries a *u.d.l.* of 3000 N per metre run over the whole span and three equal loads W Newton each placed at mid span and quarter span points. If the stress in timber is not to exceed 8 MPa, determine the maximum value of W. (3970 Newtons)

(14) A simply supported timber beam of 4 metres span carries a *u.d.l.* of 200 N/m run over its entire span and a point load of 500 N at its mid span. Calculate the dimensions of beam if depth is 2 times the breadth working bending stress in timber is not to exceed 15 MPa. (b = 121.6 mm, d = 243.2 mm

(15) A beam is of square section of side 100 mm. If the permissible bending stress is 60 MPa find the moment of resistance when the beam is placed such that (a) two sides are horizontal (b) one diagonal is vertical. Also determine the ratio of the flexural strengths of the section in the two positions.

(M_1 = 10KN–m, M_2 = 7.07*KN–m*, $\frac{M_1}{M_2}$ = 1.414)

(16) A flitched beam consists of two timber sections 100 mm× 150 mm each strengthened by a steel plate 30 mm× 100 mm as shown in the figure. It the beam is simply supported over a span of 8 metres, determine the uniformly distributed load the beam can carry if the stress in timber is not to exceed 7.5 MPa. Obtain the corresponding stress in steel also. Take modular ratio m = 20.
(w = 7.344 KN/m, σ_s = 100 MPa)

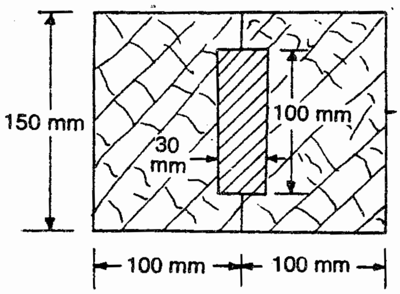


Fig. 7.37

(17) A wooden beam 160 mm wide and 300 mm deep is reinforced with two steel plates 160 mm wide and 10 mm deep one each at the top and bottom of the section. Calculate the moment of resistance of composite section if the working bending stress in timber is not to exceed 10 MPa. (Take E_s = 200 KN/mm^2, E_w = 10 KN/mm^2 (*M .R* 126.542 KN –m)

(18) A steel beam of rectangular section 120 mm wide and 200 mm deep is subjected to a shear force of 240 KN. Determine the maximum shear stress at the neutral axis and sketch the shear distribution diagram. (τ_{max} = 9 MPa **Answer**)

(19) A rectangular beam of span 8 metres is simply supported at ends. The beam has a section 60 mm × 120 mm deep. Determine the uniformly distributed load per metre run the beam can support if the maximum permissible shear stress is not to exceed 4 MPa. (w = 4.8 KN/m **Answer**)

(20) A simply supported beam of span 4 metres carries a uniformly distributed load of 9 KN/m over its entire span. The cross section of the beam is a T – section with flange and web both 100 mm × 20 mm. Determine the average shear stress and the maximum shear stress. Also calculate the intensity of shear stress at sections A –A and B – B as shown in the figure. (τ_{av} = 4.50 MPa, τ_{max} = 10.8 MPa τ_{A-A} = 4.725 MPa, τ_{B-B} = 10.41 MPa) Fig. 7.38

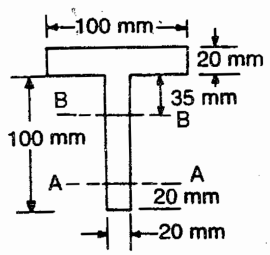


Fig. 7.38

(21) A beam of span 3 metres supports a uniformly distributed load w N/m. The beam has an I – section 80 mm deep and 60 mm wide. The flanges are 5 mm thick and the web is 3.5 mm thick. If the shear stress is limited to 5 MPa. determine the value of w. **Ans.** $w =$ 8.22 KN/m.

(22) A beam of square section is placed horizontally with one diagonal placed horizontally. If the shear force at a section of the beam is V, determine the maximum shear stress and draw the stress distribution diagram for the section.

$$(\tau_{max} = \frac{9}{8} \tau_{av.})$$

(23) A beam of circular section has 160 mm diameter. If the beam is subjected to a maximum shear force of 150 KN, determine the maximum shearing stress (31.25 MPa)

□ □ □

8

Elastic Deflection Of Beams

When a beam is laterally loaded not only bending and shear stresses are induced but the beam also deflects at right angles to its longitudinal axis.

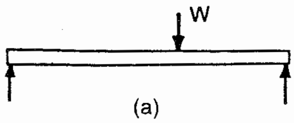


Fig. 8.1 (a)

Definition

Deflection at any point in a loaded beam is the amount of deviation of its neutral surface from its original position before loading. It is represented by the letter 'y' Downward deflection is negative.

Elastic Curve

When a beam is laterally loaded every point on the neutral surface is subjected to some vertical displacement or deflection. The line joining these deflected positions at various points is called elastic curve as shown in figure 8.1 (b)

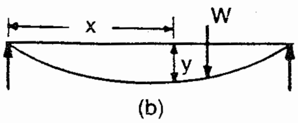


Fig. 8.1 (b)

In the design of beams care should be taken to see that the beam does not deflect more than the permissible values under a given loading condition. The indianstandard specification for steel beams and plate girders restricts the maximum deflection to $\frac{1}{325}$ of span.

Slope

The slope of a point on abeam is the angle which the tangent on it, in its deflected position makes with the x – axis. It is also called inclination and represented by the letter "i" or "θ"

Methods of Determining beam deflections :–

The following are the common methods for the determination of slope and deflection.

(i) Double Integration method.

(ii) Moment – area method.

(iii) Macaulay's method.

DOUBLE – INTEGRATION METHOD

The differential equation of the elastic curve of a bent up beam is given by.

$$EI\,\frac{d^2y}{dx^2} = M \qquad \text{---} \qquad \text{---} \qquad \text{(i)}$$

Where x and y are the coordinates as shown in fig. 8.2, y represents the deflection of the beam, E is the modulus of elasticity and I is the moment of inertia of the beam section. M represents the bending moment at a distance x from one end of the beam. The bending moment M is a function of x and if the above equation (i) is integrated twice we obtain the deflection y as a function of x..

An expression for the curvature at any point along the deflection curve of the beam is

$$\frac{I}{R} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}$$

Generally the slope of the neutral surface of the beam is very small ie the term $\left(\frac{dy}{dx}\right)$ is very small hence $\left(\frac{dy}{dx}\right)^2$ is still smaller and therefore can be neglected. Hence we may write.

$$\frac{I}{R} = \frac{d^2y}{dx^2}$$

Now from bending equation we have

$$\frac{M}{I} = \frac{E}{R} \qquad \text{or} \qquad \frac{1}{R} = \frac{M}{EI}$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{M}{EI} \qquad \text{or} \qquad EI\,\frac{d^2y}{dx^2} = M$$

Hence

$$\text{Slope} \quad \frac{dy}{dx} = \frac{1}{EI}\int M\,dx \quad \text{and Deflection } y = \frac{1}{EI}\iint M\,dx$$

Relation between slope, deflection and radius of curvature :–

The elastic curve of a loaded beam is shown in figure 8.2. Consider a short length δs on the elastic curve. let (x, y) be the cordinates of A and $(x + \delta x, y + \delta y)$ be the co-ordinates of point B on the curve. Let the tangents at A and B make angles of θ and $(\theta + \delta\theta)$ with the x–axis. The angles between the normal Ac and Bc at A and B respectively will be $\delta\theta$. Let R be the radius of curvature.

Now $\delta s = R\,\delta\,\theta$ and in the limiting case

$$\frac{\delta s}{\delta\theta} = \frac{ds}{d\theta} = .R$$

$$\text{or,} \qquad \frac{1}{R} = \frac{d\theta}{ds}$$

Again in the approx triangle ABD

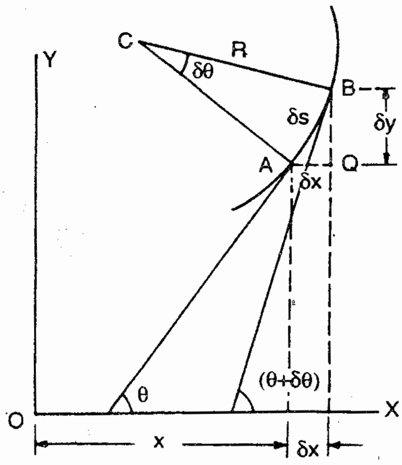


Fig. 8.2

$$\frac{dx}{ds} = \cos\theta \quad \text{and} \quad \frac{dy}{dx} = \tan\theta$$

Differentiating $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(\tan\theta) = \sec^2\theta\,\dfrac{d\theta}{dx}$

$$\frac{d^2y}{dx^2} = \sec^2\theta\,\frac{d\theta}{ds}\cdot\frac{ds}{dx}$$

$$= \sec^2\theta\,\frac{1}{R}\,.\,\sec\theta = \sec^3\theta\,.\,\frac{1}{R}$$

$$\frac{d^2y}{dx^2} = (sec^2\theta)^{3/2} \times \frac{1}{R}$$

$$\frac{d^2y}{dx^2} = (1 + \tan^2\theta)^{3/2} \times \frac{1}{R}$$

$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \times \frac{1}{R}$$

or, $$\frac{I}{R} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}$$

Generally the slope of the neutral axis of the beam is very small ie the term $\dfrac{dy}{dx}$ is small hence $\left(\dfrac{dy}{dx}\right)^2$ is still smaller and therefore negligible. Hence we may write that $\dfrac{1}{R} = \dfrac{d^2y}{dx^2}$

Again from the bending equation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

or, $\frac{1}{R} = \frac{M}{EI}$

or, $\frac{M}{EI} = \frac{d^2y}{dx^2}$ or, $M = EI \cdot \frac{d^2y}{dx^2}$

Hence slope $= \frac{dy}{dx} = \int \frac{M}{EI} \cdot dx$

and Deflection $y = \int\int \frac{M}{EI} .dx$

DEFLECTION OF CANTILEVERS

Standard Cases

Cantilever with a concentrated load at the free end :–

A cantilever AB of span L is fixed at end A and a point load W acts at the free end B. consider a section $x - x$ at distance x from the free end. Let M_x be the bending moment at the section $x - x$. Let y_B be the deflection under the load. Let i_B be the angle of slope.

$$EI \cdot \frac{d^2y}{dx^2} = M = - W.x$$

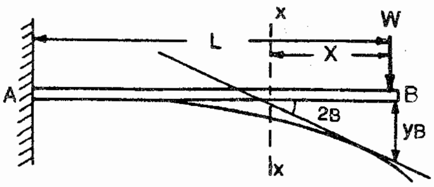


Fig. 8.3

Integrating

$$EI \cdot \frac{dy}{dx} = - W\frac{x^2}{2} + C_1$$

Since $\frac{dy}{dx} = 0$

when $x = L$

$\therefore$ $0 = -\frac{WL^2}{2} + C_1$ or, $C_1 = +\frac{WL^2}{2}$

Hence $EI \cdot \frac{dy}{dx} = -\frac{Wx^2}{2} + \frac{WL^2}{2}$ --- --- (i)

The maximum slope will be at the free end when $x = 0$.

Therefore slope at B.

$EI \cdot i_B = \frac{WL^2}{2}$ or, $i_B = \frac{WL^2}{2EI}$ radians

Integrating further , $EI .y = -\frac{Wx^3}{6} + \frac{WL^2x}{2} + C_2$

Since the deflection is zero at the fixed end when $x = l$

$$0 = -\frac{WL^3}{6} + \frac{WL^3}{2} + C_2$$

or, $C_2 = -\frac{WL^3}{3}$

or, $EIy = -\frac{Wx^3}{6} + \frac{WL^2x}{2} - \frac{WL^3}{3}$ --- --- (ii)

In order to determine deflection at the free end B put $x = 0$ in the above equation.

$$EI \cdot y_B = \frac{-WL^3}{3}$$

Therefore maximum deflection will occur at the free end

$$y_B = \frac{-WL^3}{3EI}$$

Negative sign shows that the deflection is downward

$$y_{max} = \frac{-WL^3}{3EI}$$

Example 8.1

A Cantilever 4 m long supports a load of 50 KN at its free end. If the moment of inertia of the section is 300 × 10⁶ mm⁴. Determine the maximum deflection. Take E = 200 GN/m². Also calculate the slope at the fre end.

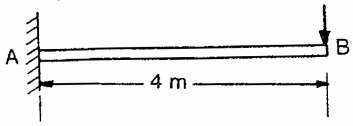


Fig. 8.4

$$y_B = \frac{WL^3}{3EI}$$

$$= \frac{50 \times 10^3 \times (4 \times 1000)^3}{3 \times 200 \times \frac{10^9}{10^6} \times 300 \times 10^6}$$

$$= 17.77 \text{ mm}$$

$$i_B = \frac{WL^2}{2EI} = \frac{50 \times 10^3 \times (4 \times 1000)^2}{2 \times 200 \times 10^3 \times 300 \times 10^6} = 0.0066 \text{ radans}$$

Cantilever with a Concentrated Load not at the free end :–

A cantilever AB of span L is fixed at A and a point load W acts at C at distance L_1, from the fixed end A. Since the portion CB is unloaded it will remain straight.

The slope at C will be $i_C = \dfrac{WL_1^2}{2EI}$

Deflection at C $\quad y_C = \dfrac{WL_1^3}{3EI}$

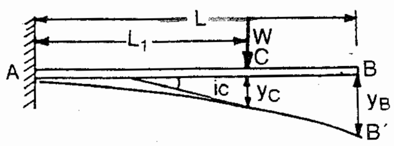


Fig. 8.5

Slope at B will be same as slope at C.

Therefore $i_B = i_C = \dfrac{WL_1{}^2}{2EI}$

And deflection at B

$$y_B = y_c + i_C (L - L_1)$$

$$y_B = \frac{WL_1{}^3}{3EI} + \frac{WL_1{}^2}{2EI}(L - L_1)$$

Cantilever with several point Loads

When several point loads are acting simultaneously the deflection at any point will be the algebraic sum of the deflections at the point due to the point loads acting individually.

Example 8.2

A cantilever beam is loaded as shown below. Determine the deflection at the free end.

Take $I = 1500 \times 10^4$ mm^4 and $E = 200$ KN/mm^2.

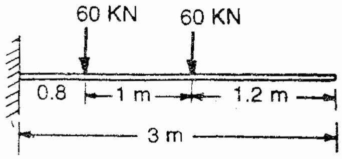


Fig. 8.6

Solution

The Maximum deflection will be the sum of the deflections due to W_1 and W_2

$$y_{m1} = \frac{W_1 L_1^3}{3EI} + \frac{W_1 L_1^2}{2EI}(L - L_1)$$

$$= \frac{60 \times 10^3 \times (.8 \times 1000)^3}{3EI} + \frac{60 \times 10^3}{2EI}(.8 \times 1000)^2 (3 - 0.8) \times 1000$$

$$y_{m1} = \frac{60 \times 10^3 \times (.8)^2 \times (1000)^3}{1500 \times 10^4 \times 200 \times 10^3}\left[\frac{1}{3} \times .8 + \frac{2.2}{2}\right]$$

$$= \frac{60 \times 10^3 \times (.8)^2 \times (2.466) \times 10^9}{1500 \times 10^4 \times 200 \times 10^3} = 17.49 \text{ mm}$$

$$y_{m2} = \frac{W_2 l_2^3}{3EI} + \frac{W_2 l_2^2}{2EI}(l - l_2)$$

$$= \frac{W_2 l_2^{\ 2}}{EI}\left[\frac{l_2}{3} + \frac{(l - l_2)}{2}\right]$$

$$= \frac{60 \times 10^3 \times (1.8)^2 \times (1000)^2}{1500 \times 10^4 \times 200 \times 10^3}\left[\frac{1.8 \times 1000}{3} + \frac{(3 - 1.8) \times 1000}{2}\right]$$

$$= \cancel{7.776} \quad 77.76$$

$y_{max} = y_{m1} + y_{m2} = 20.2 + 7.77 = 29.97$ mm **Answer**

95.25 mm

Example : 8.3

A cantilever has a span of 3 metres and carries two point loads W_1 and W_2 at a distance of a and 2 a from the fixed end. obtain an expression for the maximum deflection at the free end.

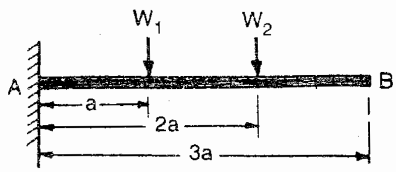


Fig. 8.7 (a)

Consider this as two cantilevers

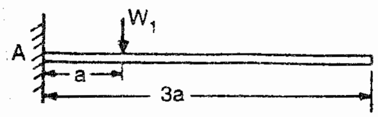


Fig. 8.7 (b)

Maximum deflection due to W_1 at B

$$y_{B1} = \frac{W_1 a^3}{3EI} + \frac{W_1 a^2}{2EI}(3a - a)$$

$$= \frac{W_1 a^3}{3EI} + \frac{W_1 a^3}{EI} = \frac{W_1 a^3}{EI}\left(1 + \frac{1}{3}\right) = \frac{4W_1 a^3}{3EI}$$

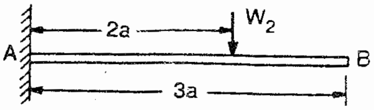


Fig. 8.7 (c)

Maximum deflection due to W_2 at B

$$y_{B_2} = \frac{W_2(2a)^3}{3EI} + \frac{W_2(2a)^2}{2EI}(3a - 2a)$$

$$= \frac{8W_2 a^3}{3EI} + \frac{4W_2 a^3}{2EI} = \frac{8W_2 a^3}{3EI} + \frac{2W_2 a^3}{EI}$$

$$= \frac{W_2 a^3}{EI}\left(\frac{8}{3} + 2\right) = \frac{14W_2 a^3}{3EI}$$

Total deflection $y_{B1} + y_{B2}$

$$y_{B_1} + y_{B_2} = \frac{4W_1 a^3}{3EI} + \frac{14}{3}\frac{W_2 a^3}{EI} \quad = \frac{2a^3}{3EI}\{2\,W_1 + 7\,W_2\}$$

$$y_{max} = \frac{2\,a^3}{3EI}(2\,W_1 + 7\,W_2)$$ **Answer**

Example 8.4

A horizontal cantilever of uniform section and length l carries two vertical point loads W_1 and W_2 as shown below find the deflection at the free end in terms of E and I

Solution

Consider the cantilever without the load W_2, then the deflection at free end

$$y_1 = \frac{W_1 l^3}{3EI}$$

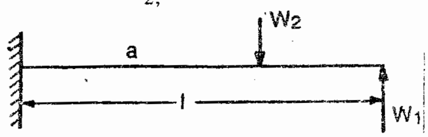


Fig. 8.8

If now W_1 is removed then deflection due to W_2 alone

$$y_2 = \frac{W_2 a^3}{3EI} + \frac{W_2 a^2}{2EI}(l - a)$$

$\therefore$ Net deflection of the free end

$$y = y_1 - y_2$$

$$= \frac{W_1 l_1^3}{3EI} - \frac{W_2 a^3}{3EI} - \frac{W_2 a^2}{2EI}(l - a)$$

$$= \frac{1}{6EI}\left\{2\,W_1 L_1^3 - 2W_2 a^3 - 3w_2 a^2(l - a)\right\}$$

$$= \frac{1}{6\,EI}\left\{2\,W_1 l_1^3 - W_2 a^2(3l - a)\right\}$$ **Answer**

Example : 8.5

An electric pole stands 4 metres above ground level. A force W_1 acting at 2 metres above the ground level pulls it towards left where as a force W_2 at 3 metres above the ground level pulls it towards right as shown in figure 8.9.. Calculate the forces W_1 and W_2 and find its ratio so that the pole remains vertical and deflection at the top of the pole is zero.

Solution :–

The pole will remain vertical with no deflection at the top only when deflection due to W_1 towards left and deflection due to W_2 towards right are equal.

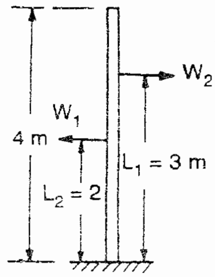


Fig. 8.9

Hence $y_1 = y_2$

$$y_1 = \frac{W_1 L_1^{\ 3}}{3EI} + \frac{W_1 L_1^{\ 2}}{2EI}(L - L_1)$$

$$= \frac{W_1 (2)^3}{3EI} + \frac{W_1 (2)^2 (4-2)}{2EI}$$

$$= \frac{W_1}{EI}\left(\frac{8}{3} + \frac{8}{2}\right) = \frac{20}{3}\frac{W_1}{EI}$$

$$y_2 = \frac{W_2 L_2^{\ 3}}{3EI} + \frac{W_2 L_2^{\ 2}}{2EI}(L - L_2)$$

$$= \frac{W_2}{EI}\left[\frac{(3)^3}{3} + \frac{9}{2}(4-3)\right] = \frac{27 W_2}{2EI}$$

Equating y_1 and y_2

$$\frac{20}{3}\frac{W_1}{EI} = \frac{27}{2}\frac{W_2}{EI} \qquad \therefore \frac{W_1}{W_2} = \frac{27}{2} \times \frac{3}{20} = \frac{81}{40}$$

$$\therefore \qquad \frac{W_1}{W_2} = \frac{81}{40} = 2.025$$ **Answer.**

Cantilever with uniformly distributed Load w over the whole span

A cantilever AB of span L is fixed at end A and $u.d.l.$ of w per unit length acts from A to B. consider a section $X - X$ at distance x from the free end.

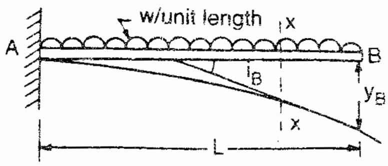


Fig. 8.10

$$EI.\frac{d^2 y}{dx^2} = M = -\frac{wx^2}{2}.$$

Integrating $EI\,\dfrac{dy}{dx} = -\dfrac{wx^3}{6} + C_1$

Since $\dfrac{dy}{dx} = 0$ When $x = L$

$$\therefore \quad 0 = -\frac{wL^3}{6} + C_1 \quad \text{or} \quad C_1 = \frac{wL^3}{6}$$

$$\text{or} \quad EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wL^3}{6} \qquad \text{--- --- (i)}$$

The maximum slope will occur at the free end when $x = 0$ therefore slope at B.

$$EI\,.\,i_B = \frac{wL^3}{6}$$

$$i_B = \frac{wL^3}{6EI} \text{ radians}$$

Integrating again we get

$$EI\,.\,y = -\frac{wx^4}{24} + \frac{wL^3x}{6} + C_2$$

Since the deflection is zero at the fixed end when $x = L$

$$\therefore \quad 0 = -\frac{wL^4}{24} + \frac{wL^3}{6}L + C_2$$

$$\text{or,} \quad C_2 = -\frac{1}{8}wL^4$$

$$\text{or,} \quad EIy = -\frac{wx^4}{24} + \frac{wL^3}{6}.x - \frac{wL^4}{8} \qquad \text{--- --- (ii)}$$

In order to determine deflection at the free end put $x = 0$ in the above equation.

$$EI\,.\,y_B = \frac{-wL^4}{8}$$

$$y_B = \frac{-wL^4}{8EI}$$

Therefore maximum deflection will occur at the free end.

$$Y_{\max} = -\frac{wL^4}{8EI}$$

Example : 8.6

A cantilever AB of span 3 metres is 300 mm deep. A uniformly distributed load of wN/metre induces a maximum bending stress of 90 MPa. Determine the maximusm slope and deflection produced if the moment of inertia of the section is 700×10^5 mm^4 and the modulus of elasticity of the material is 200 GN/m^2

Solution :

Using bending equation

$$\frac{M}{I} = \frac{\sigma}{y}$$

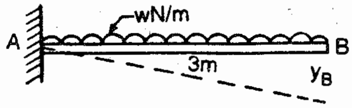


Fig. 8.11

or, $M = \frac{\sigma I}{y}$

$$= \frac{90 \times 700 \times 10^5}{150} \text{ N-mm} = 42 \times 10^6 \text{ N-mm.}$$

$$= 42 \times 10^3 \text{ N-m}$$

Maximum bending moment $M = \frac{wL^2}{2}$

or, $\frac{wL^2}{2} = 42 \times 10^3$

or, $w = \frac{42 \times 10^3 \times 2}{(3)^2} = 9.33 \text{ N/m}$

Maximum deflection due to w N/m at B.

$$y_B = \frac{wL^4}{8EI} = \frac{9.33 \times (3 \times 1000)^4}{8 \times 200 \times 10^3 \times 700 \times 10^5} = 6.74 \text{ mm}$$

Slope at the free end

$$i_B = \frac{wL^3}{6EI} = \frac{9.33 \times (3 \times 1000)^3}{6 \times 200 \times 10^3 \times 700 \times 10^5}$$

$= .0029$ radian **Answer**

Example 8.7

A uniform cantilever 4 metres long is subjected to a uniformly distributed load 10 N/m over its entire span. Determine the dimensions of the beam if the maximum deflection at the free end is 12 mm.

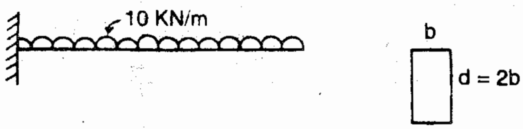


Fig. 8.12

Take $E = 200$ KN/mm^2 and width to depth ratio as 1:2

Solution

Deflection at the free end

$$y = \frac{wl^4}{8EI}$$

$$12 = \frac{10 \times (4)^4 (1000)^4}{8 \times 200 \times 10^3 \times I}$$

or $I = \frac{10 \times 16 \times 16 \times 10^{12}}{8 \times 200 \times 10^3} = \frac{4}{3} \times 10^8 \text{ mm}^4$

Now $d = 2b$

$$\therefore \quad I = \frac{b(2b)^3}{12} = \frac{2}{3} b^4$$

$$\text{or} \quad \frac{2}{3}b^4 = \frac{4}{3} 10^8 \text{ mm}^4$$

$$\text{or} \quad b^4 = \frac{4}{3} \times \frac{3}{2} \times 10^8 = 2 \times 10^8 \text{ mm}$$

$$b = 118.92 \text{ mm}$$

$$\therefore \quad d = 2 \times 118.92 = 237.84 \text{ mm} \qquad \textbf{Answer.}$$

Cantilever partially Loaded with *u.d.l.* over a length L_1 *from the fixed end :*

A cantilever AB of span L is fixed at A and a *u.d.l.* of wper unit length acts over a length AC. The portion AC may be treated as a cantilever with *u.d. l.*. Hence slope and deflection at C may be written as.

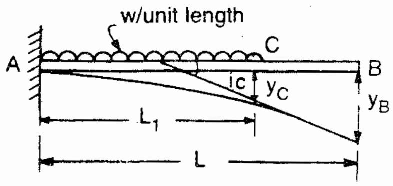


Fig. 8.13

$$i_C = \frac{w L_1^3}{6EI} \quad \text{and} \quad y_C = \frac{w L_1^4}{8EI}$$

Since there is no load on portion CB it remains straight.

$$\text{Hence } i_B = i_C = \frac{wL_1^3}{6EI}$$

$$y_B = y_C + i_C (L - L_1)$$

$$y_B = \frac{wL_1^4}{8EI} + \frac{wL_1^3}{6EI}(L - L_1)$$

Example : 8.8

A cantilever AB is 5 metres long. A u.d l of 12 KN/m acts over a portion 3 metres from the fixed end and a concentrated load of 30 KN acts at the free end B. Determine the value of the maximum vertical displacement of the elastic curve in terms of the flexural rigidity EI.

Solution :

Maximum deflection due to point load at B

$$y_1 = \frac{wL^3}{3EI} = \frac{30\,(5)^3}{3EI} = \frac{1250}{EI}$$

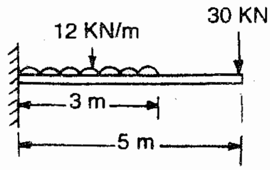


Fig. 8.14

Deflection at B due to *u. d. l.* on portion AC.

$$y_2 = \frac{wL_1^4}{8EI} + \frac{wL_1^3}{6EI}(L - L_1)$$

$$y_2 = \frac{12(3)^4}{8EI} + \frac{12(3)^3}{6EI}(5 - 3)$$

$$= \frac{121.5}{EI} + \frac{108}{EI} = \frac{229.5}{EI}$$

Therefore total deflection

$$y_{max} = \frac{1250}{EI} + \frac{229.5}{EI} = \frac{1479.5}{EI} \qquad \textbf{Answer.}$$

Cantilever partially loaded with *u.d. L* from the free end

A cantilever *AB* of span *L* is fixed at *A* and a u. d. l of *w* per unit length acts over the portion *CB*. This case may be treated as the difference of a cantilever with u.d.l. over the entire span and a cantilever with u.d.l. acting over the portion *AC*.

$$\text{Slop } i_B = \left[\frac{wL^3}{6EI} - \frac{wL_1^3}{6EI}\right]$$

$$\text{Deflection } y_B = \frac{wL^4}{8EI} - \left[\frac{wL_1^4}{8EI} + \frac{wL_1^3}{6EI}(L - L_1)\right]$$

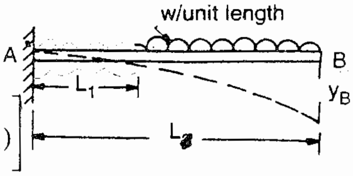


Fig. 8.15

Example 8.9

A Cantilever beam is subjected to a uniformly distributed load extending from the mid point of the beam to the free end. Determine the slope and deflection of the free end.

Slope at the free end

$$i_B = \left[\frac{wl^3}{6EI} - \frac{w(l/2)^3}{6EI}\right] = \frac{7wl^3}{48EI}$$

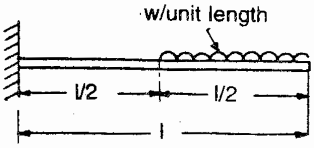


Fig. 8.16

Deflection of the free end

$$y_B = \frac{wl^4}{8EI} - \left\{\frac{wl_1^3}{8EI} + \frac{wl_1^3}{6EI}(l - l_1)\right\}$$

Put $l_1 = \frac{l}{2}$

$$\therefore \quad y_B = \frac{wl^4}{8EI} - \left\{\frac{w\left(\frac{l}{2}\right)^3}{8EI} + \frac{w\left(\frac{l}{2}\right)^3}{6EI}\left(l - \frac{l}{2}\right)\right\} = \frac{41wl^4}{384EI} \quad \textbf{Answer}$$

Example : 8.10

A hollow circular cantilever has internal diameter 125 mm, and thickness of metal 30 mm, and is loaded with 200 KN/m run for a distance of one metre from the free end. Determine the deflection at the free end if the length of the cantilever is 3 metres. take E = 120 KN/mm².

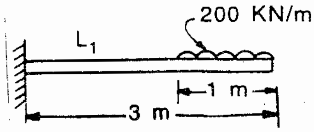


Fig. 8.17

Solution :

Moment of inertia of the section

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(185^4 - 125^4)$$

$$= 4451.43 \times 10^4 \text{ mm}^4$$

Deflection at the free end

$$Y_B = \frac{w}{EI}\left[\frac{L^4}{8} - \left\{\frac{(L_1)^4}{8} + \frac{(L_1^3)}{6}(L - L_1)\right\}\right]$$

$$= \frac{w}{EI}\left[\frac{(3000)^4}{8} - \left\{\frac{(2000)^4}{8} + \frac{(2000)^3}{6}(3000 - 2000)\right\}\right]$$

$$= \frac{w}{EI}\left[\frac{81}{8}\times 10^{12} - \left\{\frac{16}{8}\times 10^{12} + \frac{8}{6}\times 10^{9}\times 1000\right\}\right]$$

$$y_B = \frac{w}{EI}\times 10^{12}\left[\frac{81}{8} - \left\{\frac{16}{8} + \frac{8}{6}\right\}\right] = \frac{w}{EI}\times 10^{12}\times \frac{163}{24}$$

$$= \frac{200\times 10^{12}\times 163}{120\times 10^{3}\times 4451.43\times 10^{4}\times 24} = \frac{163\times 2\times 10^{6}}{12\times 24\times 4451.43}$$

$y_B = 25.4$ mm **Answer.**

Example 8.11

A Cantilever 2 metre long is loaded as shown. Calculate the deflection at the free end if the section is 120 mm × 200 mm. Take E = 100 KN/mm²

Solution

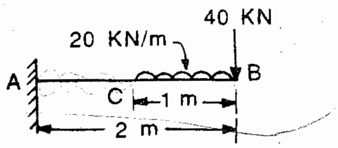


Fig. 8.18

Moment of inertia of the section

$$I = \frac{120\,(200)^3}{12} = 8\times 10^{6}\ \text{mm}^4$$

y_{B_1} due to point load

$$y_{B_1} = \frac{Wl^3}{3EI}$$

$$= \frac{40\times 10^{3}\times (2)^3\times (1000)^3}{3\times 80\times 10^{6}\times 100\times 10^{3}} = 13.3\ \text{mm}$$

y_{B_2} due to *u.d.L*

$$y_{B_2} = \frac{wl^4}{8EI} - \left\{\frac{wl_1^4}{8EI} + \frac{wl_1^3}{6EI}(l - l_1)\right\}$$

$$y_{B_2} = \frac{20\times 10^{3}}{EI}\left[\frac{16}{8} - \frac{1}{8} - \frac{1}{6}\right]\times (1000)^3$$

$$= \frac{20\times 10^{3}\times 41(1000)^3}{100\times 10^{3}\times 80\times 10^{6}\times 24} = 4.27\ mm$$

$$y = y_{B_1} + y_{B_2}$$
$$= 13.3 + 4.27 = 17.57\ mm$$ **Answer.**

Example 8.12

A Cantilever beam of length L carries a point load W at its free end. The beam for the first half of its length (from fixed end to mid point) is made of diameter D and for remaining length is $\frac{D}{2}$. Show that the deflection at the free end is.

$$y = \frac{23\,WL^3}{384\,EI_2}$$

Where I_2 is the moment of inertia of the smaller section (AMIE)

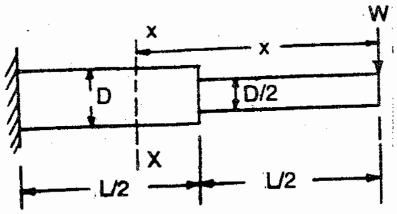


Fig. 8.19

Solution

$$I_1 = \frac{\pi}{64} D^4 \quad \text{and} \quad I_2 = \frac{\pi}{64} (D/2)^4 = \frac{\pi D^4}{64 \times 16}$$

$$\text{or} \qquad I_1 = 16\, I_2$$

Consider a section at a distance x from the free end.

$$M_x = -W.x, \qquad EI \frac{d^2y}{dx^2} = -W.x$$

Integrating twice we get

$$y = \int_o^{L/2} \int_o^{L/2} \frac{Wx}{EI_2} dx\, dx + \int_{L/2}^{L} \int_{L/2}^{L} \frac{Wx}{EI_1} dx\, dx$$

$$y = \int_o^{L/2} \frac{Wx}{EI_2} .x\, d\, x + \int_{L/2}^{L} \frac{Wx}{EI_1} .\, x\, d\, x$$

$$= \left[\frac{Wx^3}{3EI_2}\right]_o^{L/2} + \left[\frac{Wx^3}{3EI_1}\right]_{L/2}^{L}$$

$$= \frac{WL^3}{24EI_2} + \frac{WL^3}{3EI_1} - \frac{WL^3}{24\, EI_1}$$

Put $I_1 = 16\, I_2$, then

$$y = \frac{WL^3}{24\, EI_2} + \frac{WL^3}{48\, EI_2} - \frac{WL^3}{384\, EI_2}$$

$$= \frac{WL^3}{EI_2}\left[\frac{1}{24} + \frac{1}{48} - \frac{1}{384}\right]$$

$$= \frac{WL^3}{EI_2}\left[\frac{16+8-1}{384}\right] = \frac{WL^3}{EI_2}\left(\frac{23}{384}\right) \qquad \text{or} \quad y = \frac{23\, WL^3}{384\, EI_2}$$

Cantilever with a gradually varying load

A cantilever of span *L* carrying a uniformly varying load whose intensity varies from zero at *B* to *w* per unit run at the fixed end *A* is shown in figure 8.20

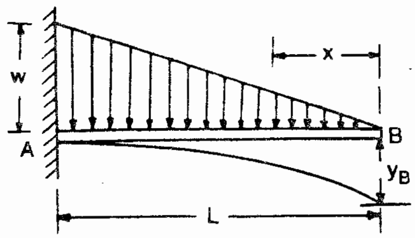


Fig. 8.20

Consider a section *x–x* at a distance *x* from the free end ***B***.

∴ Intensity of loading at distance $x = w\left(\frac{x}{L}\right)$

$$M_{xx} = \frac{1}{2} w\left(\frac{x}{L}\right) x \cdot \frac{x}{3} = -\frac{w x^3}{6 L}$$

$$EI \cdot \frac{d^2y}{dx^2} = -\frac{wx^3}{6 L}$$

Integrating we get

$$EI \cdot \frac{dy}{dx} = -\frac{wx^4}{24 L} + C_1 \qquad --- \qquad --- \qquad \text{(i)}$$

Slope $\frac{dy}{dx}$ is zero at $x = L$

$$\therefore \quad 0 = -\frac{wx^4}{24L} + C_1 \qquad \text{or,} \qquad C_1 = \frac{w L^3}{24}$$

Putting the value of a C_1 in equation

$$EI \cdot \frac{dy}{dx} = -\frac{wx^4}{24L} + \frac{wL^3}{24} \qquad --- \qquad --- \qquad \text{(ii)}$$

Slope is maximum when x = 0

$$\therefore \quad EI \cdot i_B = \frac{w L^3}{24} \qquad \text{or,} \qquad i_B = \frac{w L^3}{24EI} \text{ radian}$$

Integrating equation no (ii) we get

$$EI \cdot y = \frac{-wx^5}{120L} + \frac{w L^3 x}{24} + C_2$$

Deflection y is zero when $x = L$

$$\therefore \quad 0 = \frac{-wL^4}{120} + \frac{w L^4}{24} + C_2 \quad \text{or,} \quad C_2 = \frac{-w L^4}{30}$$

$$\therefore \quad EI \cdot y = \frac{-w x^5}{120 L} + \frac{w L^3 x}{24} - \frac{w L^4}{30}$$

Maximum deflection occurs at the free end when $x = 0$

$$\therefore \quad EI \cdot y_B = \frac{-w L^4}{30} \qquad \text{or,} \qquad y_B = \frac{-w L^4}{30E I}$$

Example 8.13.

A contilever 3 metres long carries a uniformly varying load whose intensity varies from zero at the free end to 6 KN/m at the fixed end. Determine the slope and deflection at the free end. Take $I = 400 \times 10^4\ mm^4$ *and* $E = 120\ KN/mm^2$

Solution :

Slope at the free end

$$i_B = \frac{w L^3}{24E I} = \frac{6 \times (3000)^3}{24 \times 120 \times 10^3 \times 400 \times 10^4}$$

$$= .014 \text{ radian}$$

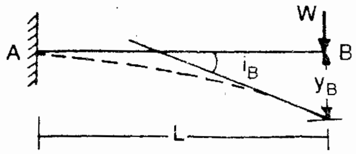
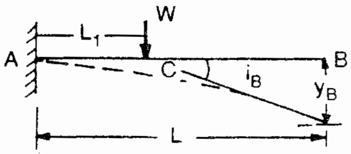
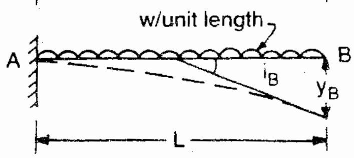
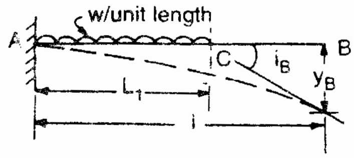
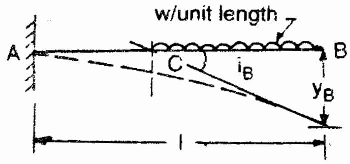
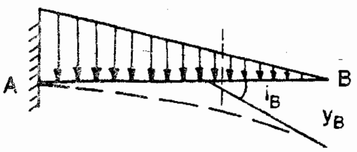
Maximum deflection

$$\dot{y}_B = \frac{w L^3}{30 E I} = \frac{6 \times (3000)^3}{30 \times 120 \times 10^3 \times 400 \times 10^4}$$

$$y_B = 33.75 \text{ mm}$$ **Answer.**

TABLE No. – 8.1

Standard Cases Of Slope And Deflection For Cantilevers

S. No.	Type of Loading	Max. Slope	Max. Deflection.
(1)		$i_B = \frac{WL^2}{2EI}$	$y_B = \frac{WL^3}{3EI}$
(2)		$i_B = i_C = \frac{WL_1^2}{2EI}$	$y_B = \frac{WL_1^3}{3EI} + \frac{WL_1^2}{2EI}(L-L_1)$ $y_C = \frac{WL_1^3}{3EI}$
(3)		$i_B = \frac{wL^3}{6EI}$	$y_B = \frac{wL^4}{8EI}$
(4)		$i_B = i_C = \frac{wL_1^3}{6EI}$	$y_C = \frac{wL_1^4}{8EI}$ $y_B = \frac{wL_1^4}{8EI} + \frac{wL_1^3}{6EI}(L-L_1)$
(5)		$i_B = \frac{wL^3}{6EI} - \frac{wL_1^3}{6EI}$	$y_B = \frac{wL^4}{8EI} - \left\{\frac{wL_1^4}{8EI} + \frac{wL_1^3}{6EI}(L-L_1)\right\}$
(6)		$i_B = \frac{wL^3}{24EI}$	$y_B = \frac{wL^4}{30EI}$

DEFLECTION OF BEAMS

Simply supported beam with a concentrated load at mid span

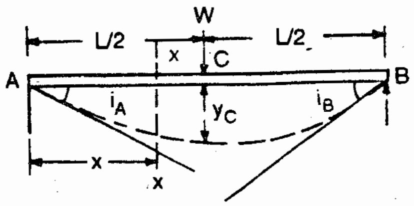


Fig. 8.21

AB is a simply supported beam of span L and carries a point load W at the centre. Consider a section x–x in the portion AC at a distance x from A.

Support reactions $R_A = R_B$

$$= \frac{W}{2}$$

Bending moment at x–x

$$M_x = +\frac{W}{2}.x$$

$$EI.\frac{d^2y}{dx^2} = +\frac{W}{2}.x$$

Integrating we get

$$EI.\frac{dy}{dx} = +\frac{W}{2}.\frac{x^2}{2} + C_1$$

At the centre the slope is zero *i.e.* $\frac{dy}{dx} = 0$ when $x = \frac{L}{2}$

$$0 = \frac{W}{2}.\frac{1}{2}\left(\frac{L}{2}\right)^2 + C_1 \qquad \text{or,} \qquad C_1 = -\frac{WL^2}{16}$$

or, $$EI.\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL^2}{16} \qquad \text{---} \qquad \text{---} \qquad \text{(i)}$$

Slope will be maximum at the supports therefore, slope at A when $x = 0$

$$EI.i_A = -\frac{WL^2}{16} \text{ radians}$$

$$i_A = i_B = -\frac{WL^2}{16EI} \text{ radians}$$

Integrating again, we get

$$EIy = +\frac{Wx^3}{12} - \frac{WL^2}{16}.x + C_2$$

Since deflection is zero at ends

i.e. $y = 0$ at $x = 0$ $\therefore$ $C_2 = 0$

or, $$EIy = \frac{Wx^3}{12} - \frac{WL^2x}{16} \qquad \text{---} \qquad \text{---} \qquad \text{(ii)}$$

Deflection at C when $x = \dfrac{L}{2}$

$$EI\, y_c = \frac{W}{12}\left(\frac{L}{2}\right)^3 - \frac{WL^2}{16}\ (L/2)$$

$$= \frac{W}{12}\frac{L^3}{8} - \frac{WL^2}{16} \cdot \frac{L}{2} = -\frac{WL^3}{48}$$

$$EI\,.\,y_C = -\frac{WL^3}{48} \qquad \text{or,} \quad y_C = -\frac{WL^3}{48EI}$$

Negative value shows that the deflection is downward.

$$Y_{max} = \frac{WL^3}{48EI}$$

Example : 8.14

A rolled steel joist rests freely on supports 8 metres apart with a point load of 1200 Newton acting at its mid span. If the maximum permissible bending stress is not to exceed 120 MPa and the central deflection not to exceed 1/320 of span, determine the depth of the joist. Take E = 200 KN/mm^2.

Solution :

$$\text{Maximum permissible deflection} = \frac{1}{320} \times 8 \times 1000 = 25 \text{ mm}$$

$$\text{Deflection at mid span } y_{\max} = \frac{WL^3}{48EI} = 25 \text{ mm}$$

$$\therefore \quad I = \frac{WL^3}{48 \times 25E} = \frac{WL^3}{1200E}$$

Permissible bending stress $\sigma = 120$ MPa

Applying bending equation

$$\frac{M}{I} = \frac{\sigma}{y} \qquad \text{or,} \quad y = \frac{\sigma I}{M} = \frac{120\, WL^3}{1200E} \times \frac{1}{\frac{WL}{4}} = \frac{120 L^2 \times 4}{1200E}$$

$$y = \frac{120 (8 \times 1000)^2 \times 4}{1200 \times 200 \times 10^3} = 128 \text{ mm}$$

$\therefore$ Depth of the Joist = 128 × 2 = 256 mm. **Answer**

Example 8.15

A simply supported steel beam 5 metres long is circular in cross–section of 120 mm diameter. What heaviest central point load can be placed on it so that the maximum deflcetion of the beam does not exceed 13.245 mm. Calculate the slope at supports then. Take E = 200 × 10^3 N/mm^2

Solution

Moment of inertia of the beam

$$I = \frac{\pi}{64}(120)^4$$

$$= 1017.87 \times 10^4 \text{ mm}^4$$

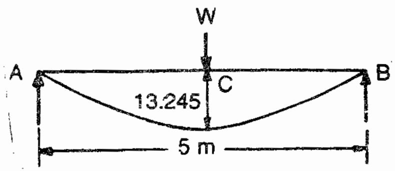


Fig. 8.22

$$y_{max} = \frac{W l^3}{48EI}$$

$$13.245 = \frac{W(5)^3 (1000)^3}{48 \times 200 \times 10^3 \times 1017.87 \times 10^4}$$

$$W = \frac{13.245 \times 48 \times 200 \times 10^3 \times 1017.27 \times 10^4}{(5)^3 \times (1000)^3}$$

$$= 10.347 \text{ KN}$$

$$\text{Slope } i_A = i_B = \frac{W l^2}{16EI}$$

$$= \frac{10.347 (5)^2 \times (1000)^2}{16 \times 200 \times 1017.27 \times 10^4}$$

$= .00746$ radians **Answer.**

Example 8.16

A simply supported beam AB of span 6 metres crosses an other beam CD of 9 metres span simply supported at ends as shown figure 8.23. The two beams are of same material and have equal cross-sectional area. If a concentrated load of 8 KN is applied at the Junction of the two beams, determine the support reactions of the two beams. (J. M. I)

Solution

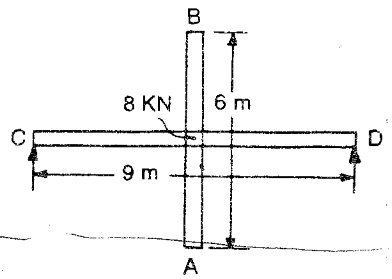


Fig. 8.23

Since both the beams are of same material and equal cross-section, therefore EI for both the beams will be same. Deflection at the centre in both directions will be equal.

Let W_{AB} be the load taken by beam AB and W_{CD} be the load taken by beam CD.

$\therefore\ W_{AB} + W_{CD} = W = 8\text{KN}$

Deflection at the centre

$$\frac{W_{AB}(6)^3}{48EI} = \frac{W_{CD}(9)^3}{48EI}$$

$$\frac{W_{AB} \times 216}{48EI} = \frac{W_{CD} \times 729}{48EI}$$

$$W_{AB} = \frac{729}{216} W_{CD}$$

Now $W_{AB} + W_{CD} = 8$ KN

$$\frac{729}{216} W_{CD} + W_{CD} = 8 \text{ KN}$$

$$W_{CD} = 8 \times \frac{216}{945} = 1.82 \text{ KN}$$

Hence $W_{AB} = (8 - 1.82) = 6.18$ KN

Since the load is placed at the centre the support reactions will be equal. Hence the reaction in case of the beam AB

$$= \frac{W_{AB}}{2} = \frac{6.18}{2} = 3.09 \text{ KN}$$

Reaction in case of beam CD will be

$$\frac{W_{CD}}{2} = \frac{1.82}{2} = .91 \text{ KN}$$ **Answer.**

Simply supported beam with a uniformly distributed load w per unit length.

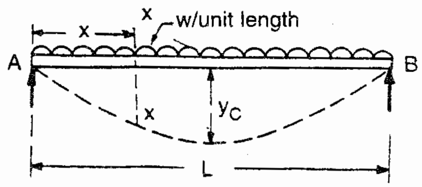


Fig. 8.24

AB is a simply supported beam of span L and carries a uniformly distributed load w per unit length over the whole span. Consider a section x–x at a distance x from A.

Support reaction $R_A = R_B$

$$= \frac{wL}{2}$$

Bending moment at x–x,

$$M_x = \frac{wL}{2} . x - \frac{wx^2}{2}$$

$$EI . \frac{d^2y}{dx^2} = \frac{wL}{2} . x - \frac{wx^2}{2}$$

Integrating we get,

$$EI . \frac{dy}{dx} = \frac{wL}{2} . \frac{x^2}{2} - \frac{wx^3}{6} + C_1$$

At mid span the slope is zero *ie* $\frac{dy}{dx} = 0$ at $x = \frac{L}{2}$

$$0 = \frac{wL}{2} . \frac{1}{2} . \left(\frac{L}{2}\right)^2 - \frac{w}{6} (L/2)^3 + C_1$$

or, $C_1 = -\frac{wL^3}{24}$

$$\therefore\ EI\,.\frac{dy}{dx} = \frac{wL}{4}x^2 - \frac{w}{6}x^3 - \frac{wL^3}{24} \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

Slope will be maximum at the supports therefore slope at A, when $x=0$

$$EI\,.\,i_A = -\frac{wL^3}{24} \qquad \text{or,} \qquad i_A = -\frac{wL^3}{24EI} = i_B$$

Integrating again,

$$EI\,y = \frac{wL}{12}x^3 - \frac{wx^4}{24} - \frac{wL^3}{24}x + C_2$$

Since the deflection is zero at A, we have

$$y = 0 \text{ at } x = 0 \ \therefore\ C_2 = 0$$

$$\text{or,} \quad EI\,.\,y = \frac{wL}{12}x^3 - \frac{w}{24}x^4 - \frac{wL^3}{24}x \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

Maximum deflection will occur at mid span when $x = L/2$

$$EI\,.\,y_c = \frac{wL}{12}\left(\frac{L}{2}\right)^3 - \frac{w}{24}\left(\frac{L}{2}\right)^4 - \frac{wL^3}{24}\,(L/2)$$

$$= \frac{-5wL^4}{384}$$

$$\text{and} \quad Y_{max} = \frac{5wL^4}{384EI}$$

Negative value shows that the deflection is downward.

Example 8.17.

A simply supported beam of span 2.5 metres and rectangular section 25 × 75mm carries a uniformly distributed load of 3 KN/metre. Determine the maximum slope and deflection of the beam. E = 100 GN/m^2

Solution :

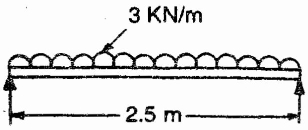


Fig. 8.25

Moment of inertia of the section

$$I_{xx} = \frac{1}{12}bd^3 = \frac{1}{12}(25)\,(75)^3 \text{ mm}^4$$

$$= 878906.25 \text{ mm}^4$$

$$\text{Maximum slope } i_A = i_B = \frac{1}{24}\frac{wL^3}{EI}$$

$$i_A = \frac{1\times 3\times (2.5\times 1000)^3}{24\times 100\times \dfrac{10^9}{10^6}\times 878906.25} = 0.022 \text{ radian}$$

Maximum deflection

$$y_c = \frac{5wL^4}{384EI} = \frac{5\times 3\times (2.5\times 1000)^4}{384\times 100\times \dfrac{10^9}{10^6}\times 878906.25}$$

$= 17.36$ mm. **Answer.**

Example 8.18

A beam of uniform rectangular section is supported at ends and carries a uniformly distributed load over the entire span. calculate the minimum depth of the section if the maximum permissible stress in the material is 10 N/mm² and the central deflection is not to exceed 12.5 mm in a span of 5 metres. Take E = 12 KN/mm²

Solution

$$\frac{M}{I} = \frac{\sigma}{y}$$

or $$M = \sigma \cdot \frac{I}{y} = \frac{wl^2}{8}$$

$$y_{max} = \frac{5wl^4}{384EI} = \frac{5\,l^2}{48EI} \cdot \frac{wl^2}{8}$$

or $$y_{max} = \frac{5\,l^2}{48EI} \cdot \sigma \cdot \frac{I}{y} = \frac{5\,l^2}{48EI} \cdot \sigma \cdot \frac{I \cdot 2}{d} \quad (\because y = \frac{d}{2})$$

or $$12.5 = \frac{5 \times (5)^2\,(1000)^2 \times 10 \times 2}{48 \times 12 \times 10^3 \times d}$$

or $$d = \frac{5 \times 25 \times 10^6 \times 10 \times 2}{48 \times 12 \times 10^3 \times 12.5} = 347.2 \text{ mm}$$

depth of plank = 347.2 mm **Answer.**

Example. 8.19

A rectangular beam 30 mm wide and 60 mm deep is freely supported at ends. If the beam is 4 metres long and carries a u.d.l of 4 KN/m over the whole span, determine the magnitude of a concentrated load that may be placed at the mid span so that the deflection at the centre may be doubled. Take E = 200 KN/mm²

Solution

Maximum deflection at the centre due to u.d.l. $y_{max1} = \dfrac{5wl^4}{384EI}$

Maximum deflection at the centre due to point load W, $y_{max2} = \dfrac{Wl^3}{48EI}$

Since max^m. deflection due to point load should be double the maximum deflection due to u.d.l

$$\therefore \quad y_{m2} = 2y_{m1}$$

or $$\frac{Wl^3}{48EI} = 2 \times \frac{5wl^4}{384EI}$$

or $$W = \frac{2 \times 5wl}{8} = \frac{2 \times 5 \times 4 \times 4}{8}$$

$$= 20 \text{ KN}$$ **Answer.**

Example 8.20

A simply supported beam of span 4 metres carries a u.d.l. of 2 KN/m on the whole span in addition to a conceutrated load of 10 KN at its mid span. Calculate the maximum deflection at the centre and the slope at the ends. Take I = 400 × 10⁴ mm⁴ and E = 200 KN/mm².

Solution.

Max^m. slope = slope due to point load + slope due to *u.d.l*

Maximum slope $\theta_{max} = \theta_1 + \theta_2$

$$\theta_{max} = \frac{W l^2}{16EI} + \frac{wl^3}{24EI}$$

$$= \frac{l^2}{8EI}\left[\frac{W}{2} + \frac{w.l}{3}\right] = \frac{16\times(1000)^2}{8\times200\times10^3\times400\times10^4} + \left[\frac{10\times10^3}{2} + \frac{2\times4\times10^3}{3}\right]$$

$$\theta_{max} = \frac{16\times(1000)^2\times10^3\,(7.66)}{8\times200\times10^3\times400\times10^4} = .01915 \text{ radian.}$$

Maximum deflection = Deflection due to point load + Deflection due to *u.d.l.*

$$y_{max} = \frac{Wl^3}{48EI} + \frac{5wl^4}{384EI}$$

$$= \frac{l^3}{48EI}\left[W + \frac{5}{8}w.l\right] = \frac{(4)^3\,(1000)^3\,(10+\frac{5}{8}\times 8)\times10^3}{48\times200\times10^3\times400\times10^4}$$

$$= \frac{64\times10^9\times15\times10^3}{48\times200\times10^3\times400\times10^4} = 25 \text{ mm}$$ **Answer.**

Examle 8.21

A simply supported beam of span l with uniforly varying load. is shown in fig 8.26.Determine slope at A and B and the max^m. defleetion.

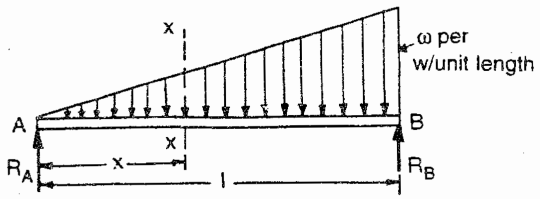


Fig. 8.26

Solution

Taking moments about B

$$R_A\,.l = \frac{wl}{2}\cdot\frac{l}{3} \qquad \text{or} \qquad R_A = \frac{wl}{6}$$

and $$R_B = \frac{wl}{3}$$

Consider a section x-x at a distance x from A

Rate of loading at x-$x = \frac{w.x}{l}$

$$M_{xx} = R_A.x. - \frac{1}{2}\frac{wx}{l}\,.\,x\,.\,\frac{x}{3} = \frac{wl}{6}x - \frac{wx^3}{6l}$$

But $$EI\,\frac{d^2y}{dx^2} = M = \frac{wlx}{6} - \frac{wx^3}{6l} \qquad \text{---} \qquad \text{---} \qquad \text{(i)}$$

Integrating, we get

$$EI\,\frac{dy}{dx} = \frac{w\,lx^2}{12} - \frac{W\,x^4}{24l} + C_1 \qquad \text{---} \qquad \text{---} \qquad \text{(ii)}$$

Where C_1 is the first constant of integration

Integrating again

$$EI\,.\,y\ \frac{w\,lx^3}{36} - \frac{w\,x^5}{120l} + C_1x + C_2 \qquad \text{---} \qquad \text{---} \qquad \text{(iii)}$$

Where C_2 is a constant of integration

Appling conditions of zero deflection at ends

ie $y = 0$, When $x = 0$ and $x = l$, we have $C_2 = 0$ and When $x = l$, $y = 0$ substituting these values in (iii)

$$0 = \frac{w\,l^4}{36} - \frac{w\,l^5}{120l} + C_1\,l \qquad \text{or} \qquad C_1 = -\frac{7w\,l^3}{360}$$

Substituting this value of C_1 in equation (ii)

$$EI\,\frac{dy}{dx} = \frac{wlx^2}{12} - \frac{wx^4}{24l} - \frac{7wl^3}{360} \qquad \text{---} \qquad \text{---} \qquad \text{(iv)}$$

Slope will be maximum at $-A$ or B

Putting $x = l$, we get slope at B

$$EI\,\frac{dy}{dx} = \frac{wl}{12}\,.\,l^2 - \frac{w.l^4}{12l} - \frac{7w\,l^3}{360} = \frac{wl^3}{45}$$

or $$i_B = \frac{wl^3}{45EI}$$

and by putting $x = 0$ in equation (iv) we get slope at A

$$i_A = \frac{7wl^3}{360EI} \qquad = \frac{7wl^3}{360EI} \quad \text{radians}$$

Now substituting the value of C_1 in equation (iii)

$$EI\,y = \frac{wlx^3}{36} - \frac{wx^5}{120l} - \frac{7wl^2x}{360}$$

or $$y = \frac{1}{EI}\left[\frac{wlx^3}{36} - \frac{wx^5}{120l} - \frac{7wl^3x}{360}\right] \qquad \text{(v)}$$

Maximum deflection will occur, where the slope is zero

$$\therefore \text{ put } \left[\frac{wlx^2}{12} - \frac{wx^4}{24l} - \frac{7wl^3}{360}\right] = 0$$

or $x = 0.519l$

Substituting this value of x in equation (v) we get

$$y_{max} = \frac{1}{EI}\left[\frac{wl}{36}(0.519l)^3 - \frac{w}{120l}(0.591l)^5 - \frac{7wl^3}{360}(0.591l)\right]$$

$$y_{max} = \frac{0.0065wl^4}{EI}$$

Example 8.22

A simply supported beam of span l with a uniformly distributed triangular load, is shown in fig 8.27. Determine max^m slope and deflection.

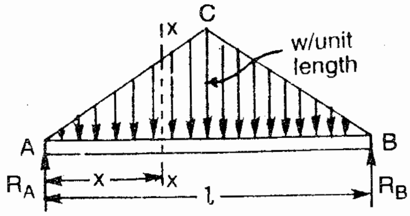


Fig. 8.27

Solution

Since the beam is symmetrically loaded

$\therefore R_A = R_B = \frac{1}{2}\left(\frac{1}{2}wl\right) = \frac{wl}{4}$

Consider a section x-x at a distance x from A

Rate of loading y at the section $= \frac{2wx}{l}$

Bending moment $M_{xx} = R_A.x - \frac{x.y}{2} \cdot \frac{x}{3}$

$$M_{xx} = \frac{wl}{4} \cdot x - \frac{x^2}{6} \cdot \frac{2w.x}{l} = \frac{wlx}{4} - \frac{wx^3}{3l}$$

Now $\quad EI \frac{d^2y}{dx^2} = M = \frac{wlx}{4} - \frac{wx^3}{3l}$

Integrating we get

$$EI \frac{dy}{dx} = \frac{wlx^2}{8} - \frac{wx^4}{12l} + C_1 \quad \text{---} \quad \text{---} \quad \text{(i)}$$

$$EI\,y = \frac{wlx^3}{24} + \frac{wx^5}{60l} + C_1\,l + C_2 \quad \text{---} \quad \text{---} \quad \text{(ii)}$$

Where C_1 and C_2 are constants of integration. Deflection is zero at A, i_e $y = 0$, when $x = 0$ and $\frac{dy}{dx} = 0$ When $x = \frac{l}{2}$, on applying the first condition we get $C_2 = 0$ and on applying the second condition, we have

$$\frac{wl}{8}\left(\frac{l}{2}\right)^2 - \frac{w}{12l}\left(\frac{l}{2}\right)^4 + C_1 = 0 \quad \text{or} \quad C_1 = -\frac{5wlx^3}{192}$$

Therefore equatins (i) and (ii) can be written as

$$EI \frac{dy}{dx} = \frac{wlx^2}{8} - \frac{wx^4}{12l} - \frac{5wl^3}{192}$$

and $\quad EI\,y = \frac{wlx^3}{24} - \frac{wx^5}{60l} - \frac{5wl^3}{192} \cdot x$

By symmetry the above equations are equally valid for portion BC of the beam as for AC.

Deflection will be maximum at the mid span, put $x = \frac{l}{2}$

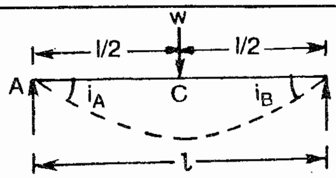
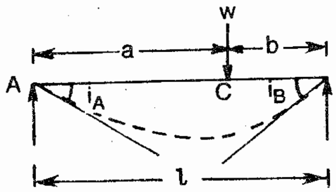
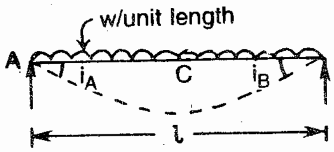
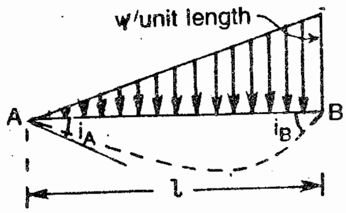
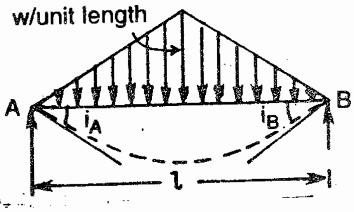
$$EI\,y_{max} = \frac{wl}{24}\left(\frac{l}{2}\right)^3 - \frac{w\left(\frac{l}{2}\right)^5}{6.l} - \frac{5wl^3}{192}\cdot\left(\frac{l}{2}\right)$$

$$y_{max} = \frac{wl^4}{120EI}$$

Slope at A When $x = 0$ $\qquad i_A = \dfrac{5wl^3}{192EI} = i_B$

Table No. – 8.2

Standard Cases Of Slope And Deflections For Beams

<table>
<tr><th>S. No.</th><th>Type of Loading</th><th>$M_{ax.}$ Slope</th><th>M_{ax}. Deflection</th></tr>
<tr><td>(1)</td><td>
</td><td>$i_A = i_B = \dfrac{Wl^2}{16EI}$</td><td>$y_c = y_{max} = \dfrac{Wl^3}{48EI}$</td></tr>
<tr><td>(2)</td><td>
</td><td>$i_A = \dfrac{Wb(l^2-b^2)}{6EIl}$
$i_B = \dfrac{Wb(l^2-a^2)}{6EIl}$</td><td>$y_{max} = \dfrac{Wb\,(l^2-b^2)^{3/2}}{9\sqrt{3}\,EIl}$
at $x = \sqrt{\dfrac{l^2-b^2}{3}}$
$y_c = \dfrac{Wa^2b^2}{3EIl}$</td></tr>
<tr><td>(3)</td><td>
</td><td>$i_A = i_B = \dfrac{wl^3}{24EI}$</td><td>$y_c = y_{max} = \dfrac{5wl^4}{384\,EI}$</td></tr>
<tr><td>(4)</td><td>
</td><td>$i_A = \dfrac{7\,wl^3}{360\,EI}$
$i_B = \dfrac{wl^3}{45\,EI}$</td><td>$y_{max} = \dfrac{2.5wl^4}{384\,EI}$.
at $x = 0.591$ from A</td></tr>
<tr><td>(5)</td><td>
</td><td>$i_A = i_B =$
$\dfrac{5wl^3}{192\,EI}$</td><td>$y_c = y_{max} = \dfrac{wl^4}{120\,EI}$</td></tr>
</table>

Example 8.23

A simply supported beam 240 mm deep supports a load of w KN/m over the entire span. If the allowable central deflection is $\frac{1}{320}$ of the span and the maximum fibre stress is not to exceed 120 MPa, determine the span of the beam and the intensity of loading per metre run. Take E = 200 KN/mm^2 and $I = 600 \times 10^4$ mm^4.

Solution

Maximum B. M. due to u.d.l $= \dfrac{wl^2}{8}$

Using bending equation $\dfrac{M}{I} = \dfrac{\sigma}{y}$

$$\sigma = \frac{M}{I} \cdot y = \frac{wl^2}{8I} \cdot \frac{d}{2} = \frac{wl^2}{16I} \cdot d \qquad \text{---} \qquad \text{---} \qquad \text{(i)}$$

Maximum central deflection $y_{max} = \dfrac{5\,wl^4}{384\,EI} \qquad \text{---} \qquad \text{---} \qquad \text{(ii)}$

$$\frac{\sigma}{y_{max}} = \frac{wl^2 d}{16\,I} \Big/ \frac{5\,wl^4}{384\,EI} = \frac{24\,d.E}{5\,l^2}$$

$$\frac{120}{\dfrac{l}{320}} = \frac{24\,d.E}{5\,l^2} \quad \text{or} \quad \frac{d}{l} = \frac{5 \times 320 \times 120}{24 \times 200 \times 10^3} = \frac{4}{100}$$

$\therefore\ l = \dfrac{100}{4} \times d$ and depth of the beam is 240 mm

$\therefore\ l = \dfrac{100}{4} \times 240$ mm $= 6000$ mm

Hence length of the beam = 6000 mm = 6 metres

From equation (i) we have

$$\sigma = \frac{wl^2.d}{16I}$$

$$w = \frac{16I \times \sigma}{d.l^2} = \frac{16 \times 600 \times 10^4 \times 120}{240 \times (6000)^2} = 1.33 \text{ N/mm}$$

$w = 1.33$ KN/metre **Answer.**

Example 8.24

A wooden plank 400 mm wide and 100 mm deep in section rests freely on two supports at the same horizontal level, which are 4 m apart. A man weighting 660 N stands in the middle of the plank carrying on his shoulders a load of bricks weighing 240 N. Find

(a) Maximum bending stress developed in the plank

(b) Maximum deflection of plank

Take weight of timber 8 KN/m^3 and E = 10 KN/mm^2

(PUNJAB)

Solution

The self weight of the plank will act as a u.d.l. over the whole span of the plank

$$\text{Self weight of the plank} = \frac{400}{1000} \times \frac{100}{1000} \times 4000 \times 8 = 1280 \text{ N}$$

$$\text{Moment of inertia of the plank} = \frac{400\,(100)^3}{12} \text{ mm}^4$$

$$= 33.33 \times 10^6 \text{ mm}^4$$

Maximum bending moment

$$M = \frac{wl^2}{8} + \frac{Wl}{4}$$

$$= \frac{1280 \times 4^{2}}{8} + (660 + 240) \times \frac{4}{4} = 1540 \text{ N-m} = 3460 \text{ N-m}$$

$$= 1540 \times 10^3 \text{ N-mm}$$

Appling bending equation

$$\sigma = \frac{M}{I} \cdot y$$

or $$\sigma = \frac{1540 \times 10^3}{33.33 \times 10^6} \times \frac{100}{2} = 2.31 \text{ N/mm}^2$$

$$\sigma = 2.31 \text{ MPa}$$

Maximum deflection y_m will be the sum of $y_{m_1} + y_{m_2}$

$$y_m = y_{m_1} + y_{m_2}$$

$$= \frac{5}{384} \frac{wl^4}{EI} + \frac{Wl^3}{48\,EI}$$

$$= \frac{l^3}{48EI}\left(\frac{5}{8}wl + W\right)$$

$$= \frac{l^3}{48EI}\left(\frac{5}{8} \times 1280 + 900\right) = \frac{l^3}{48EI}(800 + 900)$$

$$= \frac{(4)^3 \times (1000)^3 \times 1700}{48 \times 10 \times 10^3 \times 33.33 \times 10^6} = 6.88 \text{ mm}$$

$y_{max} = 6.88$ mm **Answer.**

Example 8.25

A beam AB of span 4 metres is simply supported at A and B. A cantilever PQ of length 2.5 metres which is fixed at P meets the beam AB at mid point Q, there by forming a rigid joint at Q. A vertical load 15 KN is applied vertically at common joint Q, find the reactions at ends of the simply supported beam. *(AMIE)*

Solution

The deflection at Q of the cantilever and the beam will be equal since the Joint Q is rigid.

Let W = load carried by the beam

$\therefore$ Load carried by the cantitever $= (15 - W)$

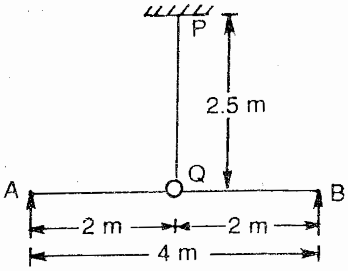


Fig. 8.28

Deflcetion of the beam at mid point Q

$$y = \frac{Wl^3}{48EI} = \frac{W(4)^3}{48EI} = \frac{64}{48}\frac{W}{EI}$$

$$= \frac{4}{3}\frac{W}{EI} \qquad \text{---} \qquad \text{---} \qquad \text{(i)}$$

Deflection of the cantilever at Q

$$y = \frac{(15-W)\,l^3}{3\,EI} = \frac{(15-W)\,(2.5)^3}{3\,EI}$$

$$= \frac{15.625}{3\,EI}(15-\text{W}) \qquad \text{---} \qquad \text{---} \qquad \text{(ii)}$$

$\therefore$ Equating (i) and (ii)

$$\frac{4W}{3EI} = \frac{15.625}{3EI}(15-w)$$

$$\text{or } 4W = 15.625 \times 15 - 15.625\,W$$

or $\quad 19.625\,W = 15.625 \times 15$

$$\text{or } W = \frac{15.625 \times 15}{19.625} = 11.94 \text{ KN.}$$

Load Carried by the cantilever = 15 – 11.94 = 3.057 KN

$$R_A = R_B = \frac{11.94}{2} = 5.97\text{KN}$$ **Answer**

Example 8.26

A cast iron water pipe 250 mm external diameter and 25 mm thick rests on two supports 8 metres apart. Calculte the maximum stress in the outer fibre of the material when empty and when full of water. Also determine the corresponding maximum deflection. Density of cast iron is 72 KN/m^3 and E = 210 GN/m^2 JMI

Solution

Moment of inertia of the pipe section

$$I = \frac{\pi}{64}(250^4 - 200^4) = 11320.7 \times 10^4 \text{ mm}^4$$

$$y = 250/2 = 125 \text{ mm}$$

$$\text{Section modulus } Z = \frac{I}{y} = \frac{11320.7 \times 10^4}{125} = 90.50 \times 10^4 \text{ mm}^3$$

$$\text{Volume of the pipe} = \frac{\pi}{4}(250^2 - 200^2) \times 8 \times 1000 \text{ mm}^3$$

$$= 1413.71 \times 10^5 \text{ mm}^3$$

$$Wt.\text{ of pipe} = \frac{1413.71 \times 10^5 \times 72 \times 10^3}{(1000)^3} = 10.178 \text{ KN}$$

$$\text{Volume of water} = \frac{\pi}{4}(200)^2 \times 8 \times 1000 = 25.132 \times 10^7 \text{ mm}^3$$

$$\textbf{Weight of water} = \frac{25.132 \times 10^7 \times 10 \times 10^3}{(1000)^3} = 2.513 \textbf{ KN}$$

$$\text{Weight of water} = \frac{25.132 \times 10^7 \times 10 \times 10^3}{(1000)^3} = 2.513 \text{ KN}$$

Total weight = 10.178 + 2.513 = 12.791 KN

This load will be the total u.d.l. acting on the pipe.

Stress in the outer fibre when the pipe is empty

$$M = \frac{wl^2}{8} = \frac{Wl}{8} = \frac{10.178 \times 10^3 \times 8 \times 1000}{8} \quad [\text{W=w} \times l]$$

$$\sigma = \frac{M}{Z} = \frac{10.178 \times 10^6}{90.56 \times 10^4} = 11.23 \text{ MPa}$$

Stress in the outer fibre when the pipe is full

and $$M = \frac{wl^2}{8} = \frac{WL}{8} = \frac{12.791 \times 10^3 \times 8 \times 1000}{8}$$

$$\sigma = \frac{12.791 \times 10^6}{90.56 \times 10^4} = 14.32 \text{ MPa}$$

Deflection when the pipe is empty

$$y_1 = \frac{5wl^4}{384\,EI} = \frac{5Wl^3}{384\,EI}$$

$$= \frac{5 \times 10.178 \times 10^3 \times (8)^3 \times (1000)^3}{384 \times 210 \times 10^3 \times 11320.7 \times 10^4}$$

$$= 2.85 \text{ mm}$$

Max^{m} Deflection of the pipe when full

$$= \frac{5}{384} \times \frac{12.791 \times 10^3 \times 8^3 \times (1000)^3}{210 \times 10^3 \times 11320.7 \times 10^4}$$

$= 3.586$ mm. **Answer**

MOMENT AREA METHOD

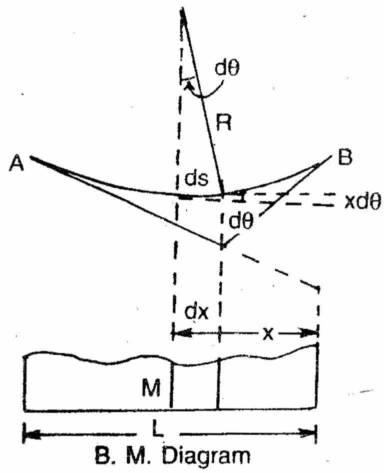


Fig. 8.29

Moment area method was developed as an alternative to double integration method. The slope and deflection at any single point on a beam can be more easily determined by this method, with the help of Mohr's Moment area theorems stated below.

Theorem I.

The angle between the tangents at any two chosen points A and B on the deflection curve of a beam is given by the area of the $B.\ M.$ diagram between these points divided by the product of E and I. Where E is the modulus of elasticity and I the moment of inertia of the section about the neutral axis

$$\theta = \int_A^B \frac{M\,dx}{EI}$$

The elastic curve between points A and B of a loaded beam is shown in figure 8.29 Let us consider an element of this curve of length ds. Let R be the radius of curvature of the beam. From bending equation we know that

$$\frac{M}{I} = \frac{E}{R} \quad \text{or} \quad \frac{1}{R} = \frac{M}{EI} \qquad \text{---} \qquad \text{---} \qquad (i)$$

The element of length ds subtends an angle $d\theta$ measured with respect to the centre of curvature of the element ds,

$$\therefore\ ds = R d\theta \quad \text{or} \quad \frac{1}{R} = \frac{d\theta}{ds}$$

Substituting $\frac{1}{R} = \frac{d\theta}{ds}$ in equation (i) we get

$$\frac{d\theta}{ds} = \frac{M}{EI} \quad \text{or} \quad d\theta = \frac{M}{EI}\ .\ ds$$

Since the elemental length ds is very small, it may be represented by its horizontal projection dx. We may thus write

$$d\theta = \frac{M}{EI}\ .\ dx$$

Let L be the length of the beam between points A and B. The angle θ between the tangents at A and B may be found by summing up $d\theta$ between the limits θ and L. Hence we get

$$\theta = \int_o^L d\theta = \int_o^L \frac{M\,dx}{EI}$$

$$\text{or} \quad \theta = \frac{\text{A}}{\text{EI}} = \frac{\text{Area of B.M . diagram over AB}}{\text{E.I}}$$

Theorem II

If A and B are two points on the deflection curve of a loaded beam, the vertical distance of B from the tangent drawn to the curve at A is given by the moment of the area of $B.\ M.$ diagram between A and B taken about A divided by the product of E and I.

Referring to the same figure, we have already established that

$$d\theta = \frac{M}{EI}\ .\ dx$$

The vertical distance between the tangents at A and B from B is Bb as shown in the figure. The length Bb made by the bending of the element of length ds is the vertical element $x\ \ d\theta$. Hence

$$x\,d\theta = \frac{Mx}{EI}\ .\ dx$$

The right side of this equation represents the moment of the shaded area $M.dx$ about a vertical line passing through B, divided by EI. Integrating we get

$$Bb = \int_A^B \frac{Mxdx}{EI}$$

or
$$y = \frac{A.\bar{x}}{EI}$$

Standard Cases

Cantilever with a point load at the free end

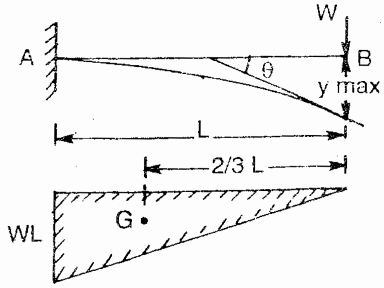


Fig. 8.30

A cantilever AB of span L with a point load W is shown in figure. 8.30 The bending moment diagram is a triangle with maximum $B.M$ at $A = WL$.

The $C.G$ of $B.M.$ diagram is at $\frac{2}{3}L$ from the reference line passing through B.

Slope.

Maximum slope

$$\theta_{max} = \frac{A}{EI}$$

$$= \frac{1}{2}\frac{L.WL}{EI} = \frac{WL^2}{2EI}$$

$$\theta_{max} = \frac{WL^2}{2EI}$$

Maximum deflection

$$y_{max} = \frac{A.\bar{x}}{EI} = \frac{1}{2}WL^2 \times \frac{2}{3}L \times \frac{1}{EI} = \frac{WL^3}{3EI}$$

$$y_{max} = \frac{WL^3}{3EI}$$

Example 8.27

A cantilever 4 metres long supports a point load of 50 KN at its free end. If the moment of inertia of the section is $300 \times 10^6\ mm^4$. Calculate the slope and deflection at the free end. Take $E = 200KN/mm^2$

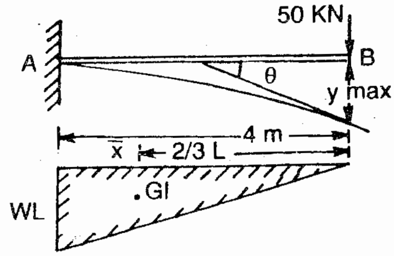


Fig. 8.31

Solution

Area of the moment diagram

$$= \frac{1}{2} \times L \times WL = \frac{1}{2} WL^2$$

C. G of the triangle $\bar{x} = \frac{2}{3} \times L$

Maximum slope

$$\theta_{max} = \frac{A}{EI}$$

$$= \frac{WL^2}{2EI}$$

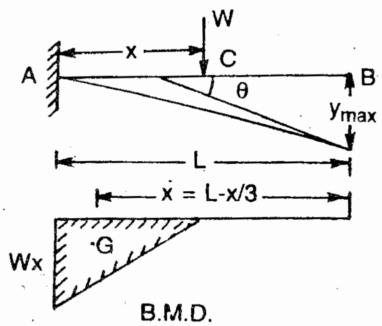
$$= \frac{50 \times 10^3 \times (4000)^2}{2 \times 200 \times 10^3 \times 300 \times 10^6} = .0066 \text{ radian}$$

Maximum deflection will occur at the free end

$$y_{max} = \frac{A.\bar{x}}{EI} = \frac{1}{2} \frac{WL^2}{EI} \times \frac{2}{3} L$$

$$= \frac{WL^3}{3EI} = \frac{50 \times 10^3 \times (4000)^2}{3 \times 200 \times 10^3 \times 300 \times 10^6} = 17.7 \text{ mm} \qquad \textbf{Answer}$$

Cantilever with a point load not at the free end.



B.M.D.

Fig. 8.32

A cantilever AB of span L with a point load W at C is shown in figure 8.32

Area of the $B.\ M.$ diagram $= \frac{1}{2} x \cdot W \cdot x = \frac{1}{2} Wx^2$

Max. slope $= \frac{A}{EI} = \frac{W \cdot x^2}{2 E_I}$

$\bar{x}$ = Distance of $C.\ G.$ of the $B.\ M$ diagram from the free end

$$= \left(L - x + \frac{2}{3} x\right) = \left(L - \frac{x}{3}\right)$$

Maximum deflection $= \frac{A.\bar{x}}{EI}$

$$y_{max} = \frac{Wx^2}{2EI}\left(L - \frac{x}{3}\right)$$

Deflection at C, $y_c = \frac{Wx^2}{2EI}\left(\frac{2}{3} x\right) = \frac{Wx^3}{3EI}$

Slope at C, $\theta_c = \frac{Wx^2}{2EI}$.

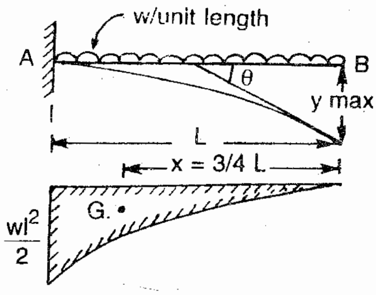


Fig. 8.33

Cantilever with U.d.L on the whole span

A cantilever AB of span L carrying a uniformly distributed load w per unit length is shown in figure 8.33

The maximum bending moment at the fixed end $= \frac{wL^2}{2}$

Distance of $C.\ G.$ of the $B.\ M.$ diagram from the free end $\bar{x} = \frac{3}{4} L$

Area of the $B.\ M.$ diagram $A = \frac{1}{3} L \times \frac{wL^2}{2} = \frac{wL^3}{6}$

Maximum slope $\theta_{max} = \frac{A}{EI} = \frac{wL^3}{6EI}$

Maximum deflection $y_{max} = \frac{A\bar{x}}{EI} = \frac{wL^3}{6EI} \times \frac{3}{4} L$

$$y_{max} = \frac{wL^4}{8EI}$$

Example 8.28

A cantilever A B of span 3 metres carries a u.d.l of 9.33 N/m over the entire span. Determine the maximum slope and deflection if the moment of inertia of the section is 7000 × 10^4 mm^4 and modulus of elasticity is200 GN/m^2.

Solution

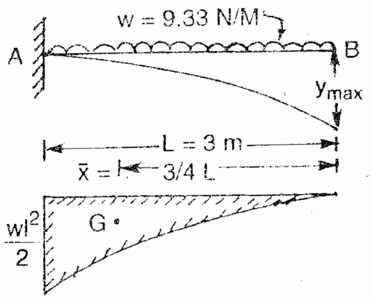


Fig. 8.34

Area of *B. M.* diagram

$$= \frac{1}{3} \times L \times \frac{wl^2}{2} = \frac{wl^3}{6}$$

Maximum slope θ

$$\theta_{max} = \frac{A}{EI} = \frac{wL^3}{6EI}$$

$$= \frac{9.33 \times (3000)^3}{6 \times 200 \times \frac{10^9}{10^6} \times 7000 \times 10^4}$$

$$= .0029 \text{ radian}$$

Maximum deflection

$$y_{max} = \frac{A\bar{x}}{EI} \quad \text{where} \quad \bar{x} = \frac{3}{4}L$$

$$= \frac{wl^3}{6EI} \times \frac{3}{4}l = \frac{wl^4}{8EI}$$

$$= \frac{9.33 \times (3 \times 1000)^4}{8 \times 200 \times 10^3 \times 7000 \times 10^4} = 6.74 \text{ mm} \quad \textbf{Answer}$$

Example 8.29

A cantilever of span L metres carries a U.d.l of w N/m over half of its length from the free end. Determine the slope and deflection at the free end.

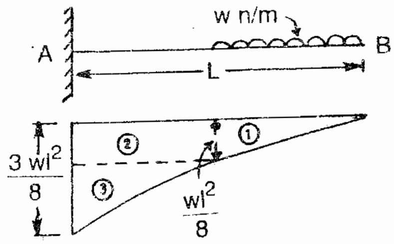


Fig. 8.35

Solution

Maximum slope and deflection will occur at the free end.

Total area of bending moment diagram $A = A_1 + A_2 + A_3$

$$A_1 = \frac{1}{3} \times \frac{L}{2} \times \frac{wL^2}{8} = \frac{wL^3}{48}$$

$$A_2 = \frac{L}{2} \times \frac{wL^2}{8} = \frac{wL^3}{16}$$

$$A_3 = \frac{1}{2} \times \frac{L}{2} \times \frac{2wL^2}{8} = \frac{wL^3}{16}$$

Maximum slope $\theta_{max} = \frac{A}{EI} = \frac{1}{EI}\left(\frac{wL^3}{48} + \frac{wL^3}{16} + \frac{wL^3}{16}\right) = \frac{7wL^3}{48EI}$

For maximum deflection find $(A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3) = A\bar{x}$

$$A_1\bar{x}_1 = \frac{wL^3}{48} \times \frac{3L}{8} = \frac{3wL^4}{384}$$

$$A_2\bar{x}_2 = \frac{wl^3}{16} \times \frac{3L}{4} = \frac{3wL^4}{64}$$

$$A_3\bar{x}_3 = \frac{wL^3}{16} \times \frac{5L}{6} = \frac{5wL^4}{96}$$

$$\therefore A\bar{x} = \left(\frac{3\,wL^4}{384} + \frac{3\,wL^4}{64} + \frac{5\,wL^4}{96}\right) = \frac{41wL^4}{384}$$

$$y_{max} = \frac{A\bar{x}}{EI} = \frac{41wL^4}{384EI}$$

Simply supported beam with a point load at mid span.

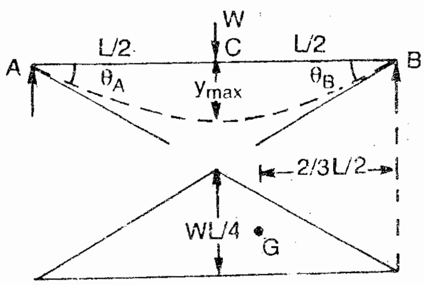


Fig. 8.36

A simply supported beam AB of span L with a point load W at the centre is shown in figure 8.36. Maximum bending moment will occur at the centre $M_{max} = \frac{WL}{4}$

Considering the portion BC only, as the result will be same for the portion AC

Area of $B.\,M.$ diagram between B and $C = \frac{1}{2} \cdot \frac{L}{2} \cdot \frac{WL}{4}$

$$A = \frac{WL^2}{16}$$

Maximum slope at A or $B = \frac{\text{Area of B. M. diagram}}{\text{EI}}$

$$\theta_{max} = \frac{WL^2}{16EI}$$

$\bar{x}$ = distance of $C.\,G$ of the $B\,M.$ diagram between B and C from support B

$$= \frac{2}{3} \times \frac{L}{2} = \frac{L}{3}$$

$$y_{max} = \frac{A\bar{x}}{EI} = \frac{WL^2}{16} \cdot \frac{L}{3EI} = \frac{WL^3}{48EI}$$

Simply supported beam with U.d.L. on whole span

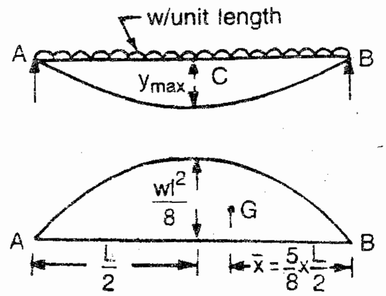


Fig. 8.37

A simply supported beam AB of span L and carrying a uniformly distributed load w per unit length is shown in figure 8.37

Maximum slope θ_{max} = Angle between the tangents at B and C

$$\text{Area of } B.\ M.\ \text{diagram over portion } BC = \frac{2}{3} \times \frac{L}{2} \times \frac{wL^2}{8}$$

$$A = \frac{wL^3}{24}$$

$$\text{Maximum slope } \theta_A = \theta_B = \frac{A}{EI} = \frac{wL^3}{24EI}$$

$$\bar{x} = \frac{5}{8} \times \frac{L}{2} = \frac{5L}{16}$$

Maximum deflection at C

$$y_{max} = \frac{A\bar{x}}{EI} = \frac{wL^3 \times 5L}{24EI \times 16} = \frac{5wL^4}{384\ EI} = \frac{5wL^4}{384EI}$$

A simply supported beam with a point load not at the centre

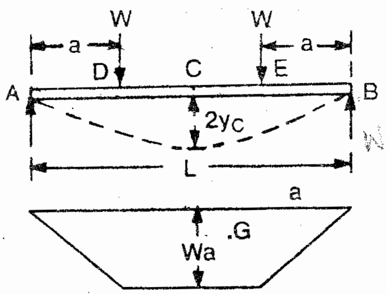


Fig. 8.38

A simply supported beam AB of span L with a point load W at a distance a from left support A is shown in figure 8.38

Add a load W at a distance a from the right hand support B to produce symmetry of loading on the beam. This will double the deflection at the centre. Hence from the Mohr's theorem w can state that $2y_c = \frac{A\bar{x}}{EI}$

or $$y_c = \frac{A\bar{x}}{2EI}$$

Area of $B.M.$ diagram between B and C

$$= \left(\frac{1}{2} a \times Wa\right) + \left(\frac{L}{2} - a\right)(Wa)$$

and $$A\bar{x} = \left(\frac{1}{2}Wa^2\right) \times \left(\frac{2}{3}a\right) + \left(\frac{L}{2} - a\right)(\text{Wa})\left[a + \frac{1}{2}\left(\frac{L}{2} - a\right)\right]$$

$$= \frac{W a^3}{3} + \text{Wa}\left(\frac{L}{2} - a\right)\left(\frac{L}{4} + \frac{a}{2}\right)$$

$$= \frac{W a^3}{3} + \frac{W a}{8}(L - 2a)(L + 2a)$$

$$y_c = \frac{A\bar{x}}{2EI} = \frac{1}{2EI} \times \frac{Wa}{24}(3L^2 - 4a^2)$$

$$y_c = \frac{Wa\,(3L^2 - 4a^2)}{48EI}$$

MACAULAY'S METHOD

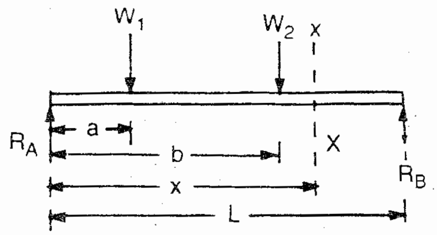


Fig. 8.39

When several point loads act on a beam Macaulay's method provides a much easier solution for determing the slope and deflection at any section on the beam.

A simply supported beam AB of span L is shown in figre. 8.39 Let W_1 and W_2 act at distances a and b from the support A. Consider a section x-x at a distance x from A then

$$EI\frac{d^2y}{dx^2} = M_x = R_{A.x} - W_1(x - a) - W_2(x - \text{b}) \qquad \text{---} \qquad \text{---} \qquad \text{(i)}$$

Integrating we get

$$EI\frac{dy}{dx} = R_A\frac{x^2}{2} + C_1 \,\vdots - W_1\frac{(x-a)^2}{2} \,\vdots - \text{W}_2\frac{(x-b)^2}{2} \qquad \text{---} \qquad \text{---} \qquad \text{(ii)}$$

$$EI_y = R_A\frac{x^3}{6} + C_1x + C_2 - \,\vdots\, \frac{W_1(x-a)^3}{6} \,\vdots - \frac{W_2(x-b)^3}{6} \qquad \text{---} \qquad \text{(iii)}$$

The following important points must be kept in mind

(i) The constant of integration C_1 should be written after the first term. The constant C_1 is valid for all values of x

(ii) The quantity (x-a) should be integrated as $\frac{(x-a)^2}{2}$ and not as $\left(\frac{x^2}{2} - ax\right)$ similarly (x-b) as $\frac{(x-b)^2}{2}$

(iii) The quantity $\frac{(x-a)^2}{2}$ should be integrated as a whole *ie* as $\frac{(x-a)^3}{6}$ and $\frac{(x-b)^2}{2}$ as $\frac{(x-b)^3}{6}$. The constant C_2 is written after $C_1 x$. The constant C_2 is valid for all values of x.

The constant C_1 and C_2 can be evaluated if the end conditions are known.

When a beam is simply supported the deflection is zero at ends *i.e.* $y = 0$, at $x = 0$, and $x = L$. Puttingthese values in deflection equation we get $C_2 = 0$ and putting $x = L$ and $y = 0$ in the deflection equation C_1 can be evaluated. Once the constants C_1 and C_2 are known, slope and deflection can be easily determined.

NOTE :- If for any value of x, the quantity within brackets in any term is negative and is raised to power higher than 1, the term is to be neglected.

A simply supported beam with a concentrated load not at the centre

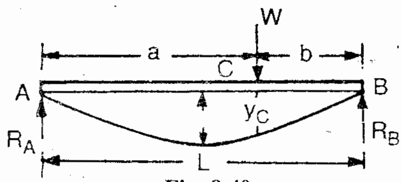


Fig. 8.40

A simply supported beam AB of span L carries a point load W at C as shown in figure 8.40 Let $AC>CB$. consider a section x-x at a distance x from A, then

$$M_x = R_A . x \;\vdots\; - W(x-a)$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{L} x \;\vdots\; - W(x-a)$$

Integrating we get

$$EI\frac{dy}{dx} = \frac{Wbx^2}{2L} + C_1 \;\vdots\; \frac{-W(x-a)^2}{2}$$

Integrating again

$$EI\,y = \frac{Wbx^3}{6L} + C_1 x + C_2 \;\vdots\; \frac{-W(x-a)^3}{6}$$

At A the deflection is zero ie at $x = 0, y = 0 \therefore C_2 = 0$

At B the deflection is zero $\therefore$ at $x = L, y = 0$

$$\therefore \qquad 0 = \frac{WbL^2}{6} + C_1 L - \frac{W(L-a)^3}{6}$$

or $$C_1 L = \frac{-W(L-a)^3}{6} - \frac{WbL^2}{6}$$

or $$C_1 L = \frac{Wb^3}{6} - \frac{WbL^2}{6} \quad \text{or} \quad C_1 = \frac{-Wb}{6L}(L^2 - b^2)$$

Hence slope and deflection at any section can be found from the following equations

$$EI\frac{dy}{dx} = \frac{Wbx^2}{2L} - \frac{Wb}{6L}(L^2 - b^2)\frac{-W(x-a)^2}{2} \quad \text{(Slope equation)}$$

$$EI\,y = \frac{Wbx^3}{6L} - \frac{Wb}{6L}(L^2 - b^2)\,x\frac{-W(x-a)^3}{6} \quad \text{(Deflection equations)}$$

Deflection under the load

Put $x = a$ in the deflection equation

$$EI\,y_C = \frac{Wba^3}{6L} - \frac{Wb}{6L}(L^2 - b^2)\,a\frac{-W(a-a)^3}{6}$$

$$= \frac{Wba^3}{6L} - \frac{Wb(L^2 - b^2)a}{6L}$$

$$= \frac{-Wba}{6L}(L^2 - b^2 - a^2) \text{ But } L = (a+b)$$

$$EI\,y_c = \frac{-Wba}{6(a+b)}[(a+b)^2 - b^2 - a^2]$$

$$= \frac{-Wba}{6(a+b)}[a^2 + 2ab + b^2 - b^2 - a^2]$$

$$= \frac{-Wa^2b^2}{3(a+b)}$$

or $$y_c = \frac{-Wa^2b^2}{3EIL}$$

To find maximum deflection

Maximum deflection will occur in the larger portion AC and at the point of maximum deflection the slope will be zero

Hence equating the slope at a section in AC to zero, we have

$$0 = \frac{Wbx^2}{2L} - \frac{Wb}{6L}(L^2 - b^2)$$

$$\text{or} \quad x^2 = \frac{L^2 - b^2}{3} \quad \text{or} \quad x = \sqrt{\frac{l^2 - b^2}{3}} = \sqrt{\frac{a^2 + 2ab}{3}}$$

The maximum deflection can now be determined from the deflection equation.

$$EI\, y_{max} = \frac{Wb}{6L}\left(\frac{L^2 - b^2}{3}\right)^{3/2} - \frac{Wb}{6L}(L^2 - b^2)\left(\frac{L^2 - b^2}{3}\right)^{1/2}$$

$$= \frac{Wb\ (L^2 - b^2)^{3/2}}{9\sqrt{3}\ EI.L}$$

$$y_{max} = \frac{Wb\ (a^2 + 2ab)^{3/2}}{9\sqrt{3}EIL} = \frac{Wb\ (a^2 + 2ab)^{3/2}}{9\sqrt{3}\ EI\,(a+b)}$$

Example. 8.30

A steel beam of rectangular section 4 meters long carries a concentrated load of 40 KN at 1 metre from the right end support. Determine the deflection of the beam under the load and the maximum deflection. Take $E = 200\ KN/mm^2$ and $I = 600 \times 10^4\ mm^4$.

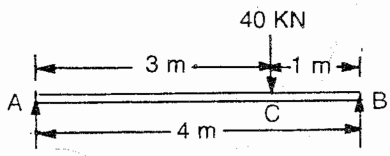


Fig. 8.41

$$y_c = \frac{W a^2 b^2}{3\ EIL} = \frac{40 \times 10^3 \times (3000)^2\ (1000)^2}{3 \times 200 \times 10^3 \times 600 \times 10^4 \times 4000} = 25 \text{ mm}$$

$$y_{max} = \frac{W\, b\, (L^2 - b^2)^{3/2}}{9\sqrt{3}\ EIL}$$

$$= \frac{40 \times 10^3 \times 1000[\ (3000)^2 - (1000)^2\ ]^{3/2}}{9\sqrt{3} \times 2000 \times 10^3 \times 600 \times 10^4 \times 4000}$$

$= 31$ mm **Answer**

$$y_{max} \text{ will occur at } \sqrt{\frac{L^2 - b^2}{3}} = \sqrt{\frac{16 - 1}{3}} = \sqrt{5}$$

$\doteq 2.33$ metres from A

Example 8.31

A horizontal beam AB having unifrom section is 5 meters long and is simply supported at ends. It carries two points loads of 5 KN and 7.5 KN placed at 1 meter and 3 meters from support A If the moment of inertia of the section is 400 × 10^4 mm^4 and modulus of elasticity is 200 GN/m^2, determine the deflection of the beam under the two loads

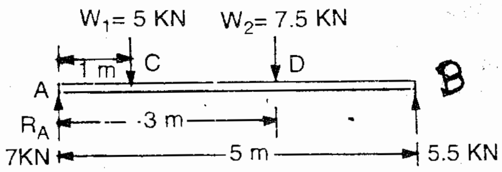


Fig. 8.42

Solution

Support reactions R_A = 7 KN and R_B = 5.5 KN

Consider a Section x -x at distance x from A

$$M_x = R_A . x - W_1 (x - a) - W_2 (x - b)$$

$$EI \frac{d^2y}{dx^2} = 7x \;\vdots\; - 5(x-1) \;\vdots\; - 7.5(x-3)$$

$$EI \frac{dy}{dx} = \frac{7x^2}{2} + C_1 \;\vdots\; \frac{-5(x-1)^2}{2} \;\vdots\; \frac{-7.5(x-3)^2}{2}$$

$$EI\,y = \frac{7x^3}{6} + C_1 x + C_2 \;\vdots\; \frac{-5(x-1)^3}{6} \;\vdots\; \frac{-7.5(x-3)^3}{6}$$

At $x = 0$, $y = 0$ $\therefore$ $C_2 = 0$

And at $x = 5$, $y = 0$

$$\therefore \quad 0 = \frac{7(5)^3}{6} + 5C_1 - \frac{5}{6}(5-1)^3 \; \frac{-7.5}{6}(5-3)^3$$

or $C_1 = -16.5$

Deflection equation is given by

$$EI\,y = \frac{7x^3}{6} - 16.5x \;\vdots\; \frac{-5}{6}(x-1)^3 \;\vdots\; \frac{-7.5}{6}(x-3)^3$$

To determine deflection under W_1 put $x = 1$ in the deflection equation

$$EIy_c = \frac{7(1)^3}{6} - 16.5 \times 1 \;\vdots\; \frac{-5}{6}(1-1)^3 \;\vdots\; \frac{-7.5}{6}(1-3)^3$$

The third and fourth terms are to be neglected

$$\therefore EI.\, y_C = \frac{7(1)^3}{6} - 16.5 \times 1$$

$$= 1.16 - 16.5 = -15.34$$

$$\therefore EI.\, y_C = 15.34 \times 10^{12}$$

$$y_C = \frac{-15.34 \times 10^{12}}{200 \times \frac{10^9}{10^6} 400 \times 10^4} = \frac{-15.34 \times 10}{8}$$

$$= 19.17 \text{ mm}$$

Deflection under D can be found by putting $x = 3$ in the deflection equation.

$$EI\, y_D = \frac{7(3)^3}{6} - 16.5 \times 3 \;\Big|\; \frac{-5}{6}(3-1)^3 \;\Big|\; \frac{-7.5}{6}(3-3)^3$$

$$= \frac{7 \times 27}{6} - 49.5 \frac{-5}{6}(2)^3$$

$$= 31.5 - 49.5 - 6.66 = 24.66$$

$$y_D = \frac{-24.66 \times 10^{12}}{200 \times \frac{10^9}{10^6} \times 400 \times 10^4}$$

$$= \frac{24.66 \times 10}{8}$$

$$= 30.82 \text{ mm} \qquad \textbf{Answer}$$

Example 8.32

A simply supported beam AB of unifrom section and span L meters supports two concentrated loads W each at L/4 and 3L/4 from support A. Determine the deflection of the beam by Macaulay's method in terms of flexural rigidity EI

(i) Deflection under the two loads

(ii) Deflection at mid span.

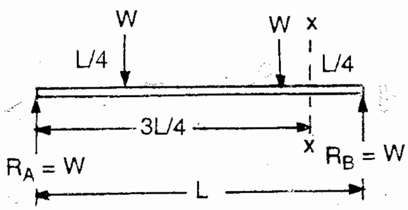


Fig. 8.43

Solution

Support reactions $R_A = R_B = W$

Consider a section x-x at a distance x from A

$$M_x = R_A \,.\, x - W(x - L/4) - W(x - 3L/4)$$

$$E\,\frac{d^2y}{dx^2} = W.x \;\Big|\; - W(x - L/4) \;\Big|\; - W(x - 3L/4) \qquad \text{---} \qquad \text{---} \qquad \text{(i)}$$

Integrating we get

$$EI.\frac{dy}{dx} = \frac{Wx^2}{2} + C_1 \;\Big|\; \frac{-W}{2}\left(x - \frac{L}{4}\right)^2 \;\Big|\; -\frac{W}{2}\left(x - \frac{3L}{4}\right)^2 \qquad \text{---} \qquad \text{---} \qquad \text{(ii)}$$

Integrating again

$$EI.y = \frac{Wx^3}{6} + C_1 x + C_2 \;\Big|\; -\frac{W}{6}(x - L/4)^3 \;\Big|\; -\frac{W}{6}\left(x - \frac{3L}{4}\right)^3 \qquad \text{---} \qquad \text{---} \qquad \text{(iii)}$$

At $x = 0,\ y = 0 \qquad \therefore\ C_2 = 0$

Again at $x = L,\ y = 0\ \therefore$ From equation (iii)

$$EI\,y = \frac{WL^3}{6} + C_1 L + 0 \;\Big|\; -\frac{W}{6}(L - L/4)^3 \;\Big|\; -\frac{W}{6}\left(L - \frac{3L}{4}\right)^3$$

$$\therefore\ C_1 = \frac{-3}{32}WL^2$$

Hence the deflection equation becomes

$$EI\,y = \frac{Wx^3}{6} - \frac{3}{32}WL^2.x \;\Big|\; -\frac{W}{6}\left(x - \frac{L}{4}\right)^3 \;\Big|\; -\frac{W}{6}\left(x - \frac{3L}{4}\right)^3$$

(i) For deflection under the first load put $x = L/4$

$$EI\,y_c = \frac{W}{6}(L/4)^3 - \frac{3}{32}WL^2(L/4) \;\Big|\; -\frac{W}{6}\left(\frac{L}{4} - \frac{L}{4}\right)^3 \;\Big|\; -\frac{W}{6}\left(\frac{L}{4} - \frac{3L}{4}\right)^3$$

$$= \frac{WL^3}{6 \times 64} - \frac{3\,WL^3}{32 \times 4}$$

$$y_c = \frac{WL^3}{EI}\left(\frac{1}{6 \times 64} - \frac{3}{32 \times 4}\right) = \frac{WL^3}{32\,EI}\left(\frac{1}{12} - \frac{3}{4}\right)$$

$$y_c = -\left(\frac{WL^3}{32EI} \times \frac{8}{12}\right) = \frac{-WL^3}{48EI}$$

(ii) Deflection under the 2nd load, put $x = \frac{3L}{4}$ in the deflection equation

$$EI\,y_D = \frac{W}{6}\left(\frac{3L}{4}\right)^3 - \frac{3}{32}WL^2\left(\frac{3L}{4}\right) \;\Big|\; -\frac{W}{6}\left(\frac{3L}{4} - \frac{L}{4}\right)^3 \;\Big|\; -\frac{W}{6}\left(\frac{3L}{4} - \frac{3L}{4}\right)^3$$

$$= \frac{27WL^3}{6 \times 64} - \frac{9WL^3}{32 \times 4} - \frac{WL^3}{6 \times 8} = \frac{WL^3}{48}$$

$$y_D = -\frac{WL^3}{48\,EI}$$

(iii) Deflection at mid span, put $x = L/2$

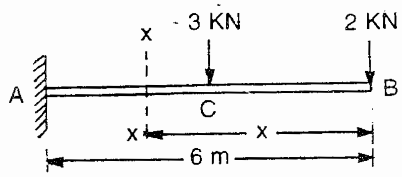
$$EI_{\text{at } x = \frac{L}{2}} = \frac{W}{6}\left(\frac{L}{2}\right)^3 - \frac{3}{32}WL^2(L/2) \;\Big|\; -\frac{W}{6}\left(\frac{L}{2} - \frac{L}{4}\right)^3 \;\Big|\; \frac{-W}{6}\left(\frac{L}{2} - \frac{3L}{4}\right)^3$$

$$= \frac{WL^3}{48} - \frac{3WL^3}{32 \times 2} - \frac{WL^3}{6 \times 64} = \frac{-11WL^3}{384}$$

$$y_{x=\frac{L}{2}} = \frac{-11WL^3}{384EI}$$

Example 8.33

A cantilever of uniform section 6 metres long carries a load of 2KN at the free end and 3 KN at 3 meters from the fixed end. Determime the maximum deflection of the cantilever at the free end by Macaulay's method. Take E = 200 GN/m² and I = 300 × 10⁵ mm⁴



Solution

Consider a section x-x at a distance x from B

Fig. 8.44

$$EI\frac{d^2y}{dx^2} = M_x = -2x \;\vdots\; -3(x-3) \qquad \text{---} \qquad \text{---} \qquad \text{(i)}$$

$$EI\frac{dy}{dx} = \frac{-2x^2}{2} + C_1 \;\vdots\; \frac{-3(x-3)^2}{2} \qquad \text{---} \qquad \text{---} \qquad \text{(ii)}$$

$$EI\,y = \frac{-2x^3}{6} + C_1 x + C_2 \;\vdots\; \frac{-3(x-3)^3}{6} \qquad \text{---} \qquad \text{---} \qquad \text{(iii)}$$

At $x = L, \frac{dy}{dx} = 0$ then from equation (ii) we have

$$0 = \frac{-2(6)^2}{2} + C_1 \quad \frac{-3(6-3)^2}{2} \qquad \text{or} \qquad C_1 = 49.5$$

At $x = L, y = 0$ then from equation (iii) we have

$$0 = \frac{-2(6)^3}{6} - 49.5x \times 6 + C_2 \quad \frac{-3(6-3)^3}{6}$$

$$= -72 - 49.5 \times 6 + C_2 - 13.5$$

$$= -72 + 297 + C_2 - 13.5 \qquad \text{or} \quad C_2 = -211.5$$

Putting the values of C_1 and C_2 in equation (iii)

$$EI\,y = -\frac{2x^3}{6} - 49.5x - 211.5 - \frac{-3(x-3)^3}{6}$$

For Deflection at the free end B, put $x = 0$

$$EI y_B = -211.5 \quad \text{or} \quad y_B = \frac{211.5 \times 10^{12}}{200 \times \frac{10^9}{10^6} \times 300 \times 10^5}$$

$$= \frac{211.5 \times 10}{60} = 35.25 \text{ mm} \qquad \textbf{Answer}$$

Example 8.34

A pull of 120 KN is applied to a pole AB at point A on the top as shown in fig. 8.45. If the diameter of the pole is 40 mm determine the value of the pull P to be applied at point C so that the deflection at the top of pole is Zero.

Solution

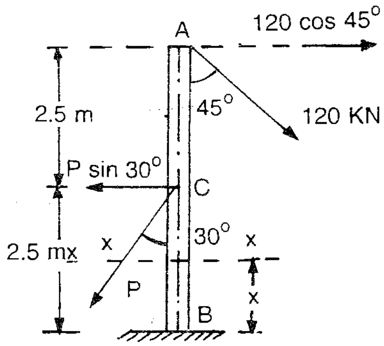


Fig. 8.45

Consider a section *x-x* at a distance *x* from *B*

$$M_{xx} = 120 \text{ Cos}45° (5 - x) - P \text{ Sin}30° (2.5 - x)$$

$$\frac{EId^2y}{dx^2} = -84.86\,(5-x) + 0.5\,(2.5 - x)$$

Integrating we get

$$\frac{EIdy}{dx} = -42.43\,(5-x)^2 + 0.25\,P\,(2.5-x)^2 + C_1 \quad \text{---} \quad \text{(i)}$$

$$EI_y = -14.14\,(5-x)^3 + 0.083\,P\,(2.5-x)^3 + C_1 x + C_2 \quad \text{---} \quad \text{---} \quad \text{(ii)}$$

where C_1 and C_2 are constants of integration

At $x = 0, \frac{dy}{dx} = 0$

$\therefore \quad 0 = -1060.75 + 1.5625\,P + C_1 = 0$

or $\quad C_1 = (678.88 - P)$

And $y = 0$, at $x = 0 \qquad \therefore 0 = -1767.5 + 1.296\,P + C_2$

or $\quad C_2 = (1767.5 - 1.296\,P)$

$\therefore$ Putting the values of C_1 and C_2 in (ii)

$$EI_y = 14.14\,(5-x)^3 + 0.83\,P\,(2.5-x)^3 + (678.88 - P\,x + (176.5 - 1.296\,\text{P})$$

But the deflection is Zero at *A* *i.e.* $y = 0$ when $x = 5$

$\therefore \qquad (678.88 - P)\,5 + (1767.5 - 1.296\,P) = 0$

(Neglecting terms in bracket that become – Ve)

or $\quad 678.88 \times 5 - 5P + 1767.5 - 1.296\,P = 0$

or $\quad 6.296\,P = 5161.9$

or $\qquad P = 819.86$ KN **Answer**

SUMMARY

1. Slope and deflection of a cantilever *AB* of length *L* and flexural rigidity *EI*

   (a) Point load *W* acting at the free end $i_B = \frac{Wl^2}{2\,EI}$ and $y_B = \frac{Wl^3}{3\,EI}$

2. Point load *W* acting at a distance l_1 from the fixed end

$$i_B = i_C = \frac{Wl_1^2}{2\,EI} \quad \text{and} \quad y_B = \frac{W\,l_1^3}{3\,EI} + \frac{Wl_1^2}{2\,EI}\,(l - l_1)$$

3. Uniformly distributed load w per unit length acting on the entire span.

$$i_B = \frac{w\,l^3}{6\,E_I} \text{ and } y_B \frac{wl^4}{8\,EI}$$

4. Gradually varying load from Zero at the free and to w per unit length at the fixed end.

$$i_B = \frac{wl^3}{24\,EI} \quad \text{and} \quad y_B = \frac{wl^4}{30\,EI}$$

Slope and deflection of a simply supported beam AB of span l and flexural rigidity EI

5. Point load W acting at mid span

$$i_A = i_B = \frac{Wl^2}{16\,EI} \quad \text{and} \quad y_c = \frac{Wl^3}{48\,EI}$$

6. Point load W acting at a distance a from A and b from B

$$i_A = \frac{Wb}{6\,EI}\,(l^2 - b^2) \text{ and } i_B = \frac{Wa}{6\,EI}\,(l^2 - a^2)$$

$$y_C = \frac{Wab}{6\,EIl}\,(l^2 - a^2 - b^2) \text{ and } y_{max} = \frac{Wa}{9\sqrt{3}\,EIl}$$

7. Uniformly distributed load w per unit length over the entire span

$$i_A = i_B = \frac{wl^3}{24\,EI} \quad \text{and} \quad y_B = \frac{5wl^4}{384\,EI}$$

8. Gradually varying load from Zero at A to w per unit llength at B

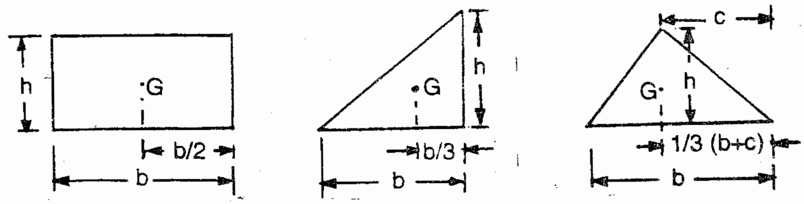
$$i_A = \frac{7wl^3}{360\,EI} \quad \text{and} \quad y_{max} = \frac{2.5wl^4}{384\,EI}$$

$$i_B = \frac{wl^3}{45\,EI} \quad \text{at} \quad x = 0.591 \text{ from } A..$$

9. Gradually varying triangular load from Zero at ends to w per unit length at mid span.

$$i_A = i_B = \frac{5wl^3}{192\,EI} \qquad y_{max} = \frac{wl^4}{120\,EI}$$

10. For Calculating $\bar{x}$ in respect of moment area method, some of the familiar *B. M.* diagrams are shown.

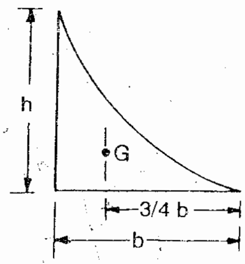


$A = bh.; \bar{x} = b/2$ (*i*) Rectangle

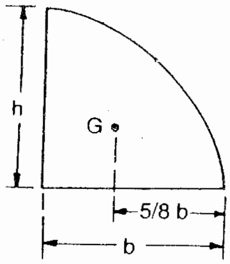
$A = \frac{1}{2}bh.\ ;\ \bar{x} = b/3$ (*ii*) Triangle

$A = \frac{1}{2}bh.; \bar{x} = 1/3\,(b+c)$ (*iii*) Triangle

Fig. 8.46



$A = \frac{bh}{3} \quad \bar{x} = \frac{3}{4}b.$



$A = \frac{2}{3}b; \quad \bar{x} = \frac{5}{8}b.h.$

EXERCISES

(1) A cantilever AB 2 metres long carries a load of 4 KN at the free end and 3 KN at 1 metre from the fixed end. Determine the maximum deflection of cantilever at the free end. Take $E = 200$ KN/mm^2 and $I = 1500 \times 10^4$ mm^4 ($y_B = 3.37$ mm)

(2) Calculate the maximum slope and deflection at the free end of a cantilever 3 metres long carrying a u.d.l. of 2 KN/m over the whole span and a point load of 2.5 KN at the free end. Take $E = 210$ KN/mm^2 and $I = 400 \times 10^6$ mm^4 ($i_B = .00241$ radian $y_B = 5.08$ mm)

(3) A uniformly loaded cantilever of span L has a deflection at the free end equal to .015 L. Find the slope of the deflection curve at the free end. (0.02 radian)

(4) A cantilever of length 2 metres carries a u. d. l. of 2.5 KN/m for a length of 1.25 metres from the fixed end and a point load of 1 KN at the free end. The beam is 120 mm wide and 240 mm deep, determine the deflection at the free end. Take $E = 200 \times$ KN/mm^2. ($y_{max} = 50.62$ mm)

(5) Calculate the minimum depth of rectangular beam 5 metres long and carrying a u.d.l. of w N/m over the whole span. The permissible deflection at the centre is 13 mm and a maximum fibre stress of 96 MPa. Take $E = 120$ KN/mm^2. ($d = 320.50$ mm)

(6) Compare the magnitudes of the slopes which occur at each end of a simply supported beam A B placed across a span of L metres when a load W Newtons is placed at a point $\frac{1}{3}$ rd of the span from the end B. Assume the beam to be horizontal. when W is removed. $\left(\frac{i_A}{i_B} = \frac{4}{5}\right)$

(7) A uniformly loaded steel beam supported at ends has a deflection at the mid span = 3.125 mm while the slope at the end is .01 radian. If the maximum permissible bending stress is limited to 90 MPa, determine the depth of the beam. Take $E = 200$ KN/mm^2 ($d = 30$ mm)

(8) A cantilever of length L Carries a uniformly distributed load of w per unit length for a distance $\frac{3}{4}L$ from the fixed end. calculate the slope and deflection at the free end.

$$\left(\theta = \frac{9\,wL^3}{128\,EI},\ y_B\ \frac{117\,wL^4}{2048\,EI}\right)$$

(9) A rectangular wooden beam 120 mm × 180 mm deep is simply supported at ends on a span 4 metres and carries a *u.d.l* of 6 KN/metre on the whole span. What point load at the centre should be placed so that the maximum deflection is doubled (W = 15 KN)

(10) A beam of span 6 metres carries a load of 5 KN at a distance of 4.8 metres from the left hand support. Calculate the maximum deflection and the deflection at the mid span. Take $E = 200$ KN/mm^2 and $I = 300 \times 10^4$ mm^4 (y_{max} = 21.7 mm, y_c = 21.45 m m)

□ □ □

9

Statically Indeterminate Beams

So far only determinate beams have been discussed where the number of unknown reactions was not more than three.

Statically indeterminate beams

When a system of forces acting on a plane of symmetry of a beam, Keeps it in static equilibrium and the number of unknown external reactions is more than three, the beam is called a statically indeterminate beam.

The three well known equations of static equilibrium $\Sigma H = 0$, $\Sigma V = 0$ and $\Sigma M = 0$, can not provide solution to more than three unknown quantities.

Hence more equations are formed with the help of deformation curve. Slope and deflection provide additional equations required in such cases to determine all unknown reactions.

Degree of indeterminancy

The degree of indeterminancy is given by the number of extra or redundant reactions. It is defined as the number of extra equations required for analysis in addition to the general equations of static equilibrium.

Types of statically indeterminate beams

Although several types of indeterminate structures exist but only the three main types of beams are given here.

Fig. 9.1

(i) Propped cantilevers

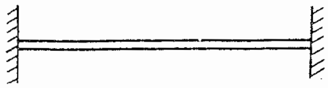


Fig. 9.2

(ii) Fixed beams

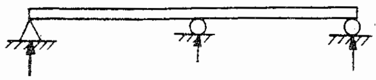


Fig. 9.3

(iii) Continuous beams

PROPPED CANTILEVERS

Props are supports provided to a cantilever to neutralise the effect of deflection that the cantilever undergoes due to applied loads. In other wards props are provided to produce an equal and opposite amount of deflection in the cantilever so that it is brought back to its original horizontal position.

Since up ward deflection produced by the prop is equal to the down ward deflection due to applied loads on the cantilever, by equating the two deflections, the reaction at the prop can be easily determined.

Sinking Of Prop

If the prop sinks by an amount δ, the algebraic sum of the deflections due to load on the cantilever and the deflections due to prop must be equal to δ.

The introduction of prop renders the cantilever indeterminate. Therefore such structures can not be analysed by the three equations of static equilibrium alone and therefore additional equations are obtained from consideration of slope or deflection while solving such problems. Following examples will help in understanding the procedure for calculating prop reactions for various types of loading.

Example 9.1

A cantilever of length l carries a concentrated load W at its mid span. If the free end be supported on a rigid prop find the reaction at the prop. Draw the S. F. and B. M. diagram for the cantilever.

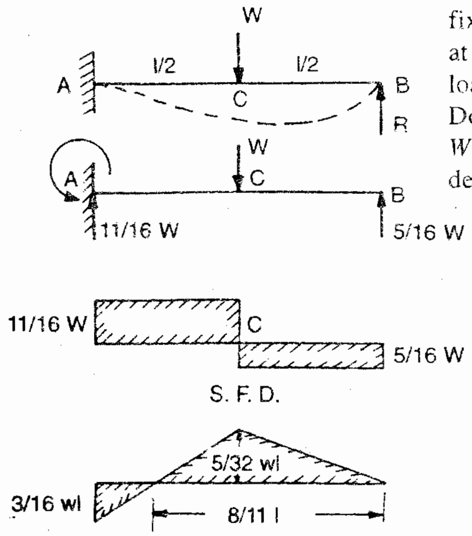


Fig. 9.4

A cantilever AB of span l is fixed at A and a prop is provided at B as shown in figure 9.4. A load W is acting at mid span. Down ward deflection due to load W will be equal to the upward deflection due to the prop.

$$\therefore \frac{W(l/2)^3}{3EI} + \frac{W(l/2)^2}{2EI} \cdot \frac{l}{2}$$

$$= \frac{1}{3}\frac{Rl^3}{EI}$$

$$\text{or} \quad R = \frac{5}{16}W$$

Therefore reaction at A

$$= W - \frac{5}{16}W = \frac{11}{16}W$$

Shear Force.

$S.\ F$ at any section between A and C will be equal to $\frac{11}{16}\ W$

$S.\ F.$ at any section between C and B will be $\frac{-5}{16}W$.

The shear force diagram can now be draw as shown in the figure.

B.M.

Bending moment at $B = 0$

$$B.\,M \text{ at } C \;=\; \frac{5}{16}\,W \cdot \frac{l}{2} = \frac{5}{32}\,Wl$$

$$B.\,M.\text{ at } A \;=\frac{5}{16}\,Wl - \frac{Wl}{2} = \frac{3}{16}\,Wl$$

Point of Contraflexure.

$$B.\,Mat_{xx} = \frac{5}{16}\;Wx - W\left(x - \frac{l}{2}\right)$$

By equating this equation to Zero, Point of contraflexure can be determined.

$$\frac{5}{16}x - x + \frac{l}{2} = 0$$

or $\quad x = \dfrac{8}{11}\,l$ from the propped end B.

Example 9.2

A Cantilever of length l carries a u.d.l w per unit run is propped at the free end. Find the reaction of the prop if it holds the free end to the level of the fixed end. Draw the B. M and S. F. diagrams.

A Cantilever AB of span l with a $u.\,d.\,l$ w/unit length is shown in fig. 9.5 End A is fixed and B is propped. Let R be the reaction of the prop.

Downward deflection

$$= \frac{w\,l^4}{8\,EI}$$

Upward deflection due to prop R

$$= \frac{R\,l^3}{3\,EI}$$

Since the cantilever remains horizontal, deflection at B is Zero.

$$\therefore\; \frac{R\,l^3}{3\,EI} = \frac{w\,l^4}{8\,EI}$$

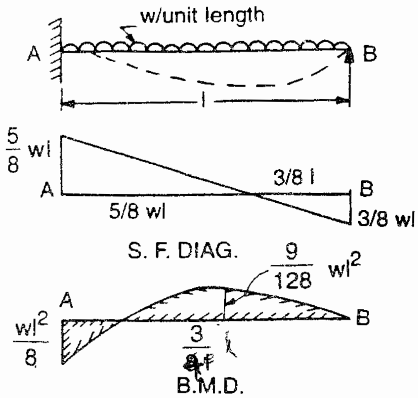


Fig. 9.5

or $R = \dfrac{3}{8}\,wl$ and Reaction at $A = \dfrac{5}{8}\,wl$

Shear Force.

S. F at any section xx will be zero, when

$$S.\ F_{xx} = wx - R = wx - \frac{3}{8} wl = 0 \quad \text{or } x = \frac{3}{8} l$$

$$S.\ F_B = -\frac{3}{8} wl,\ S.\ F.\ A = -\frac{3}{8} wl + wl = +\frac{5}{8} wl$$

Shear Force diagram can now be drawn as shown in the figure.

$$BM \text{ at } xx = M_{xx} = \frac{3}{8} wlx - \frac{wx^2}{2}$$

$$B.\ M \text{ at } A = \frac{3}{8} wl^2 - \frac{wl^2}{2} = -\frac{wl^2}{8}$$

Max m B. M will occur at $x = \frac{3l}{8}$, $M_{max} = \frac{9}{128} wl^2$

Point of contra flexure

Equating M_{xx} to Zero

$$M_{xx} = \frac{3}{8} wl - \frac{wx^2}{2} = 0$$

$$\therefore \quad x = 0 \text{ and } x = \frac{3}{4} l$$

B. M. diagram can now be drawn as shown in fig. 9.5

Deflection

At any section *xx* from *B*.

$$EI \frac{d^2y}{dx^2} = \frac{3}{8} wl\ .x - \frac{wx^2}{2}$$

$$EI \frac{dy}{dx} = \frac{3}{16} wl\ .\ x^2 - \frac{wx^3}{6} + C_1$$

At *A* the slope is Zero i.e. $\frac{dy}{dx} = 0$ at $x = l$

$$\therefore \quad 0 = \frac{3}{16} wl.^3 - \frac{wl^3}{6} + C_1 = 0 \text{ or } C_1 = \frac{-wl^3}{48}$$

$$\therefore EI \frac{dy}{dx} = \frac{3}{16} wl.x^2 - \frac{wx^3}{6} \frac{-wl^3}{48} \text{ (slope equation)}$$

Integrating again

$$EIy = \frac{wl\ .x^3}{16} - \frac{wx^4}{24} - \frac{wl^3}{48} x + C_2$$

Deflcetion is Zero at *A* , When $x = l$

$$0 = \frac{wl^4}{16} - \frac{wl^4}{24} - \frac{wl^4}{48} + C_2 \text{ or } C_2 = 0$$

$$\text{Hence } EIy = \frac{wl\ .x^3}{16} - \frac{wx^4}{24} - \frac{wl^3}{48} x$$

Maximum deflection will occur where the slope is Zero

Equating the slope equation to Zero

$$\frac{3}{16} wl.x^2 - \frac{wx^3}{6} - \frac{wl^3}{48} = 0$$

Sloving the above equation

Maximum deflection will occur at 0.422 l from the propped end B.

$$y_{max} = \frac{0.005415\, W\, l^4}{E\, I}$$

Example 9.3

A cantilever of span 4 metres is supported at the free end to the level of the fixed end. it carries a concentrated load of 40 KN at the centre of the span calculate the reaction of the prop and find the position and amount of maximum deflection. Take $E = 210$ KN/ mm^2 and $I = 1300 \times 10^4$ mm^4.

Deflection due to the load at the free end

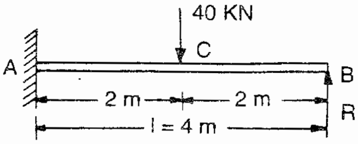


Fig. 9.6

$$y_B = \frac{5\, W\, l^3}{48\, E\, I}$$

The upward deflection due to reaction R must neutralize this down ward deflection.

$$\frac{R\, l^3}{3\, E\, I} = \frac{5}{48} \frac{W\, l^3}{E\, I}$$

$$\therefore \ R = \frac{15}{48} W = \frac{5}{16} W$$

$$= \frac{5}{16} \times 40 = 12.5 \text{ KN}$$ **Answer**

Example 9.4

A cantilever carries a concentrated load W at $\frac{3}{4}$ of its length from the fixed end and is proped at the free end to the level of the fixed end, find what proportion of the load is carried on the prop?

Solution

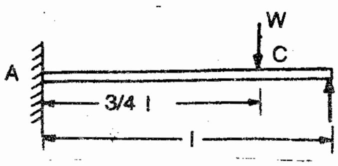


Fig. 9.7

Down ward deflection due to concentrated load W at $\frac{3}{4}\, l$ = upward deflection due to R

$$= \frac{W\, l_1{}^3}{3\, E\, I} + \frac{W\, l_1{}^2}{2\, E\, I} (l - l_1) = \frac{R l^3}{3\, E\, I}$$

$$\frac{R\, l^3}{3\, E\, I} = \frac{W}{3EI} (3/4)^3\, l^3 + \frac{W}{2EI} (3/4)^2\, l^2 \left(l - \frac{3}{4} l\right)$$

$$\frac{R\,l^3}{3\,E\,I} = \frac{W \times 27\,l^3}{3\,E\,I \times 64} + \frac{W}{2\,E\,I} \times \frac{9}{16}\,l^2 \times \frac{1}{4}\,l$$

$$\frac{R\,l^3}{3} = \frac{9}{64}\,W\,l^3 + \frac{9}{128}\,W\,l^3$$

$$R = 3\left[\frac{9}{64}\,W + \frac{9}{128}\,W\right]$$

$$= 3\left[\frac{(18+9)}{128}\,W\right] = \frac{81}{128}\,W \qquad \textbf{Answer}$$

Example 9.5

A uniform cantilever of span 5 metres is propped at the free end to the level of the fixed end. Calculate reaction on the prop. when the cantilever carries a uniformly distributed load of 20 KN per metre run over its whole length. Also determine the maximum deflection.

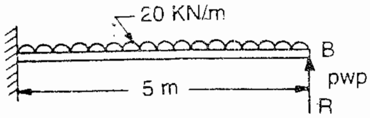


Fig. 9.8

Down ward deflection at B due to $u.d.l.$ on the

$$\text{Cantilever} = \frac{w\,l^4}{8\,E\,I}$$

Upward deflection due to the prop at the free end

$$= \frac{R\,l^3}{3\,E\,I}$$

In order that the cantilever may remain horizontal, deflection at B must be Zero.

$$\text{or} \qquad \frac{w\,l^4}{8EI} = \frac{R\,l^3}{3EI}$$

$$\text{or} \qquad R = \frac{3 \times w\,l^4}{8\,l^3} = \frac{3}{8}\,wl = \frac{3}{8} \times 20 \times 5$$

$$= \frac{300}{8} = 37.5 \text{ KN}$$

Example 9.6

A cantilever of span 4 metre carries a.u.d. l of 15 KN per metre run on the entire span and a point load of 20 KN at the free end which is supported to the same level as the fixed end. Calculate the reaction at the prop.

Solution

Down ward deflection at B due to $u.d.l.$ + point load = Upward deflection due to R at B

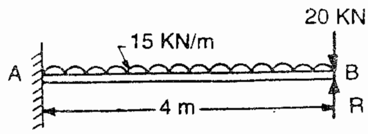


Fig. 9.9

$$\therefore \quad \frac{w\,l^4}{8\,E\,I} + \frac{W\,l^3}{3\,E\,I} = \frac{R\,l^3}{3\,E\,I}$$

$$\text{or} \qquad \frac{(w\,l)\,.l^3}{8} + \frac{W\,l^3}{3} = \frac{R\,l^3}{3}$$

$$\text{or} \qquad \frac{(15\times 4)}{8} + \frac{20}{3} = \frac{R}{3}$$

$$\text{or} \qquad R = 3\left[\frac{60}{8} + \frac{20}{3}\right] = 3\left[\frac{180 + 160}{24}\right]$$

$$= \frac{340}{8}\ \text{KN} = 42.5\ \text{KN}$$ **Answer.**

Example 9.7

A cantilever of effective length l carries a total load wl uniformly distributed throughout the length. If the cantilever is propped at a point $\frac{l}{4}$ *from the free end and the prop so adjusted that there is no deflection at the free end. Determine the reaction at the prop.*

Solution

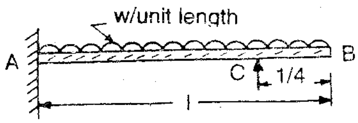


Fig. 9.10

Down ward deflection at B due to $u.\ d.\ l.$

$$= \frac{1}{8}\frac{w\,l^4}{E\,I} \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

Upward deflection at B due to prop at $\frac{l}{4}$ from the free end

$$= \frac{R\,l_1^{\,3}}{3\,E\,I} + (l - l_1)\frac{R\,l_1^{\,2}}{2\,E\,I}$$

$$= \frac{R\left(\frac{3}{4}l\right)^3}{3\,E\,I} + \left(l - \frac{3}{4}l\right)\frac{R\left(\frac{3}{4}l\right)^2}{2\,E\,I}$$

$$= \frac{R\times 27\,l^3}{3\times 64\,E\,I} + \frac{1}{4}\,\frac{l\times R\times 9\,l^2}{2EI\times 16}$$

$$= \frac{9}{64}\frac{Rl^3}{EI} + \frac{9\,R\,l^3}{128\,E\,I} = \frac{27\,R\,l^3}{128\,E\,I} \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

In order that the cantilever may remain horizontal deflection at B must be Zero

$$\therefore \qquad \frac{1}{8}\frac{w\,l^4}{E\,I} = \frac{27\,R\,l^3}{128\,E\,I}$$

$$R = \frac{128\times w\,l}{8\times 27} = \frac{16}{27}wl$$ **Answer**

Example 9.8

A cantilever A B 2 metres long rests on an other cantilever C D one metre long as shown in figure 9.11. If the cantilever A B is subjected to a u.d.l. of 1 KN/m over the whole length determine the reaction at C. If the *flexural rigidity of A B is twice that of C D, what will be the deflection at C. Take $EI_{CD} = 200$ KN–m²* *(Madras)*

Solution.

Let R be the reaction and y_c the deflection at C.

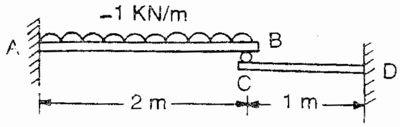


Fig. 9.11

Downward deflection of end B due to *u.d.l.*

$$= \frac{w\, l_{AB}^4}{8\, E\, I_{AB}}$$

Upward deflection of end *B* due to reaction *R* at *C*

$$= \frac{R\, l_{AB}^3}{3\, E\, I_{AB}}$$

Net down ward deflection of cantilever *A B* at *B*

$$= \frac{w\, l_{AB}^3}{8\, E\, I_{AB}} - \frac{R\, l_{AB}^3}{3\, E\, I_{AB}}$$

Reaction *R* at *C* causes downward deflection of *C*

$$= \frac{R\, l_{CD}^3}{3\, E\, I_{CD}}$$

As the deflections at *B* and *C* are same

$$\therefore \quad \frac{w\, l_{AB}^4}{8\, E\, I_{AB}} - \frac{R\, l_{AB}^3}{3\, E\, I_{AB}} = \frac{R\, l_{CD}^3}{3\, E\, I_{CD}}$$

Now $EI_{CD} = 200$ KN–m² $\therefore$ $EI_{AB} = 400$ KN–m²

or $$\frac{1\,(2)^4}{8 \times 400} - \frac{R\,(2)^3}{3 \times 400} = \frac{R \times (1)^3}{3 \times 200}$$

or $$0.5 - R \times \frac{2}{3} = \frac{R}{6} \quad \text{or} \quad 0.5 = R \times \frac{5}{6}$$

or $$R = \frac{0.5 \times 6}{5} = \frac{3}{5} \text{ KN}$$

Deflection at *C*

$$y_c = \frac{R\, l_{CD}^3}{3\, E\, I_{CD}}$$

$$= \frac{3 \times (1)^3}{5 \times 3 \times 200}$$

$= 1$ mm **Answer**

Example 9.9

A Cantilever 4 metre long carries a u.d.l of 20 KN per metre run over the entire span. A prop is provided at the free end which sinks 10 mm from the level of the fixed end. If E = 200 KN/mm² and I = 4000 mm⁴. Calculate the prop reaction.

Solution

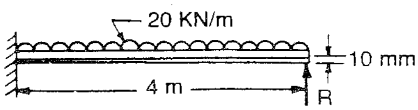


Fig. 9.12

Downward deflection due to *u.d.l* – upward deflection due to prop = y

$$\text{or } 10 \text{ mm} = \frac{w\,l^4}{8\,E\,I} - \frac{R\,l^3}{3\,E\,I}$$

$$\text{or } \frac{R\,l^3}{3\,E\,I} = \frac{w\,l^4}{8\,E\,I} - 10$$

$$R = \frac{(w\,.l)\,.l^3}{8\,E\,I} \times \frac{3\,E\,I}{l^3} - \frac{10 \times 3\,E\,I}{l^3}$$

$$= \frac{20 \times 4 \times 3}{8} - \frac{10 \times 3 \times 200 \times 10^3 \times 4000}{(4000)^3}$$

$$= 30 - \frac{3 \times 2 \times 4}{64} = 30 - \frac{3}{8}$$

$$= (30 - .375)\text{ KN} = 29.625\text{ KN}$$ **Answer.**

□ □ □

Statically Indeterminate Beams (Fixed Beams)

When the ends of a beam are firmly clamped so that the ends remain horizontal, the beam is then called a **fixed beam**. Such beams are also called **built-in beams** or **'Encastre beams.'**

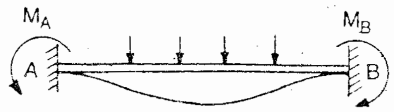


Fig. 9.13

When the ends are held firmly, the slopes at the ends of the beam are zero. Therefore a fixed beam may also be described as a beam to which certain couples are applied at the ends so that the ends remain horizontal and slopes at both the ends are zero. The moment induced at the ends due to fixed ends are called fixing moments or fixed end moments.

To determine the magnitude and nature of the fixing moments, the moment area method has been found to be quite easy.

Moment – Area theorems

Theorem I

If A and B are two points on a loaded beam, the angle between the tangents at A and B is given by the area of the bending moment diagram between A and B divided by EI.

Theorem II

If A and B are two points on a loaded beam the intercept AC on the vertical at A between the tangents at A and B is given by the moment of bending moment diagram taken about A. Similarly the intercept BD on the vertical at B is given by taking the moment of the bending moment diagram about B.

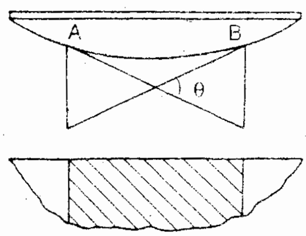


Fig. 9.14

Method Of Superposition

After determining the end moment M_A and M_B the bending moment diagram for the beam is constructed by super–imposing the fixing moment diagram on the free moment diagram. This is called method of super position.

First construct the free moment diagram for a simply supported beam. This is represented by the triangle ABC. The trapezium ABDE represents the fixing moment diagram. By plotting the negative ordinates due to M_A and M_B on the same side of the base line AB as the positive ordinates due to W, the overlapping portions cancell each other and the net bending

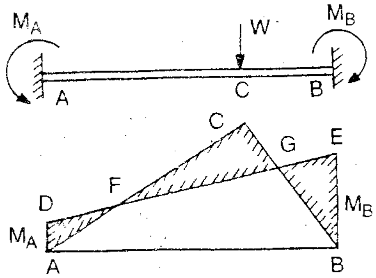


Fig. 9.15

moment is given by the ordinates of the shaded portion of the diagram. The following examples will be helpfull in understanding the method. Some standard cases of fixed beams are discussed below.

Fixed beam with a point load at the centre

A fixed beam AB of span L with a point load W at the mid span is shown in figure 9.16.

When the beam is freely supported maximum bending –moment will be at the centre and is equal to $\frac{WL}{4}$ as shown.

Since the load W is centrally placed the fixing moments M_A and M_B at the fixed ends will be equal. ∴ $M_A = M_B$

The fixing moment diagram will be rectangle.

Since the change of slope--between A and B is Zero;

Therefore according to theorem no. 1.

Area of free moment diagram + Area of fixing moment diagram = 0

$$\text{or} \quad \frac{1}{2} L \times \frac{WL}{4} + M_A \times L = 0$$

$$\text{or} \quad M_A = \frac{-WL}{8} = MB$$

Point of Contraflexure

Consider a section $x - x$ at a distance x from A, then

$$M_{xx} = R_A \cdot x - M_A = 0$$

$$\text{or} \quad \frac{W}{2} \cdot x - \frac{WL}{8} = 0$$

$$\text{or} \quad x = \frac{L}{4}$$

Similarly when section $x - x$ lies in the portion CB.

$$M_{xx} = R_A . x - M_A - W . (x - L/2)$$
$$= 0$$

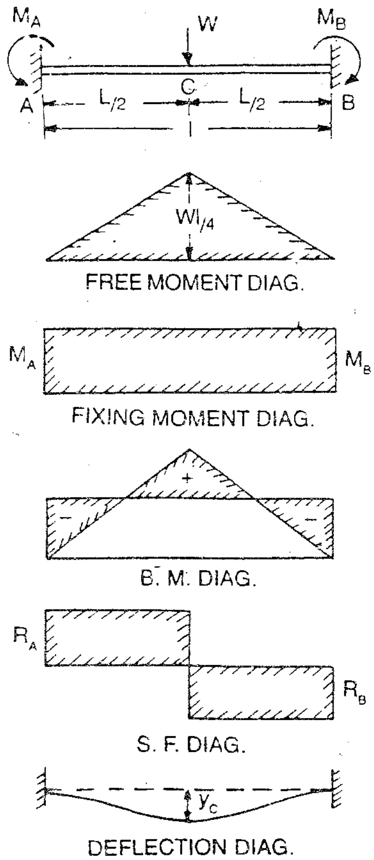


Fig. 9.16

or $\frac{W}{2} . x - \frac{WL}{8} - W.x + \frac{WL}{2} = 0$ or $x = \frac{3L}{4}$

So the point of contraflexure will be at a distance of

$$x = L/4 \text{ from } A \text{ and } \frac{3L}{4} \text{ from } A$$

Shear Force

Taking moments about B,

$$RA .L - M_A - W . \frac{L}{2} = - M_B$$

$$RA . L - \frac{WL}{8} - \frac{WL}{2} + M_B = 0$$

or $RA . L - \frac{WL}{8} - \frac{WL}{2} + \frac{WL}{8} = 0$ or $R_A = \frac{W}{2} = R_B$

Hence $S.F.$ diagram for a fixed beam is similar to the $S. F.$ diagram for a simply supported beam as shown in figure 9.16.

Deflection.

Consider a section $x - x$ in AC at a distance of x from A.

$$EI \frac{d^2 y}{d x^2} = + M_x = + \frac{W}{2} . x - M_A = + \frac{W}{2} . x - \frac{WL}{8}$$

$$EI \frac{dy}{dx} = + \frac{Wx^2}{4} - \frac{WL}{8} x + C_1$$

Now slope $\frac{dy}{dx}$ will be zero, when $x = 0$, hence $C_1 = 0$

Integrating again,

$$EI y = + \frac{Wx^3}{12} - \frac{WLx^2}{16} + C_1 x + C_2 = 0$$

At $x = 0$, deflection y at the support will be zero,

$$\therefore \; C_2 = 0$$

At $x = \frac{L}{2}$, y will be maximum

$$EI y_c = + \frac{W}{12}\left(\frac{L}{2}\right)^3 - \frac{WL}{16}\left(\frac{L}{2}\right)^2$$

$$= + \frac{WL^3}{96} - \frac{WL^3}{64} = - \frac{WL^3}{192}$$

$$y_c = - \frac{WL^3}{192EI} \text{ or } y_{max} = - \frac{WL^3}{192EI}$$

Negative value shows that the deflection is downward.

Fixed beam with a u.d.l. w/unit length over the entire span.

A fixed beam AB of span L with a uniformly distributed load w/unit length over the entire span is shown in figure 9.17

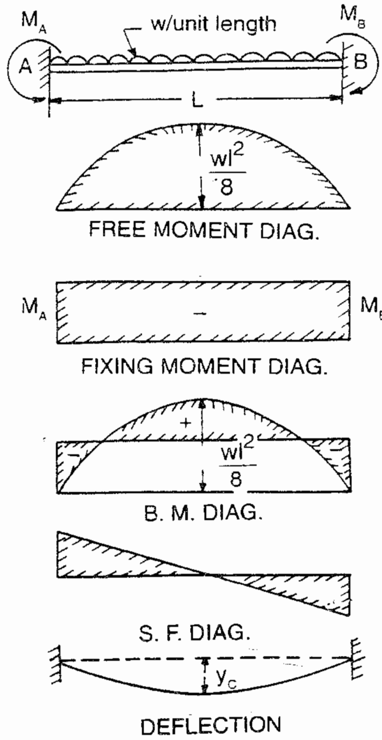


Fig. 9.17

Since the load is uniformly distributed over the entire span equal fixing moments are induced at A and B.

$$M_A = M_B$$

Since the change of slope between A and B is Zero.

Therefore,

Area of free moment diagram + Area of fixing moment diagram = 0

$$\frac{2}{3} L \times \frac{w L^2}{8} + M_A \times L = 0$$

$$\text{or } M_A = -\frac{w L^2}{12} = M_B$$

Point of contraflexure :

Consider a section $x - x$ at a distance of x from A

$$M_{xx} = R_A \,.\, x - M_A - w.x. \frac{x}{2} = 0$$

$$\text{or } \frac{w L}{2} . x - \frac{w L^2}{12} - \frac{w x^2}{2} = 0$$

$$\text{or } \frac{x^2}{2} - \frac{L x}{2} + \frac{L^2}{12} = 0,$$

$$\text{or } x^2 - L x + \frac{L^2}{6} = 0$$

$$\text{or } \quad x = \frac{L}{2} \pm 0.289 L \quad \text{or } x = 0.211 L \text{ and } 0.789 L$$

The points of contraflexure will be at a distance of 0.211 L and 0.789 L from A and B respectively.

Shear force

By symmetry the supports reactions R_A and R_B will be equal,

$$R_A = R_B = \frac{w l}{2}$$

The shear force diagram will be similar to the *S. F.* diagram for a freely supported beam.

Deflection

$$EI \frac{d^2y}{dx} = + M_x = + \frac{w L}{2} . x - w . \frac{x^2}{2} - \frac{w L^2}{12}$$

Integrating

$$EI \frac{dy}{dx} = + \frac{w L}{2} . \frac{x^2}{2} - \frac{w}{2} . \frac{x^3}{3} - \frac{w L^2}{12} . x + C_1$$

At $x = 0$, slope is zero ie. $C_1 = 0$

Integrating again

$$EI\,y = +\frac{wL}{2}\cdot\frac{x^3}{6} - \frac{w}{2}\cdot\frac{x^4}{12} - \frac{wL^2}{12}\cdot\frac{x^2}{2} + C_1 x + C_2$$

at $x = 0$, deflection y is zero $\therefore C_2 = 0$

and maximum deflection will occur at the centre, when $x = \frac{l}{2}$

$$EI\,y_c = +\frac{wL}{12}\left(\frac{L}{2}\right)^3 - \frac{w}{24}\left(\frac{L}{2}\right)^4 - \frac{wL^2}{24}\left(\frac{L}{2}\right)^2$$

$$EI\,y_c = \frac{+wL^4}{96} - \frac{wL^4}{96} - \frac{wL^4}{384}$$

$$EI\,y_c = -\frac{wL^4}{384}$$

$$y_c = \frac{-wL^4}{384EI} \quad \text{or} \quad y_{max} = \frac{wL^4}{384EI}$$

Negative sign means the deflection is downward.

Fixed beam with a point load not at the centre.

Since the load W is not centrally placed fixing moments M_A and M_B will be unequal.

Since change of slope between A and B is zero, therefore Area of free moment diagram + Area of fixing moment diagram = 0

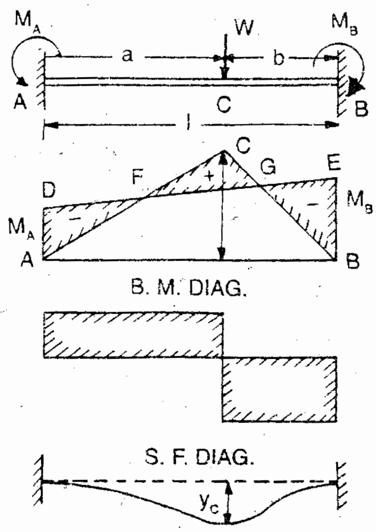


Fig. 9.18

or $$(M_A + M_B)\frac{L}{2} = \frac{-Wab}{L}\cdot\frac{L}{2}$$

or $$(M_A + M_B) = \frac{-Wab}{L} \qquad \text{.... (i)}$$

According to 2nd theorem, the moment of both the above areas about A must be equal.

$$\therefore (M_A + 2M_B)\frac{L^2}{6} =$$

$$\left\{\frac{Wab}{L}\times\frac{a}{2}\times\frac{2a}{3} + \frac{Wab}{L}\times\frac{b}{2}\left(a + \frac{b}{3}\right)\right\}$$

$$= \frac{Wab}{6}(2a + b)$$

or $$(M_A + 2M_B) = \frac{Wab(2a+b)}{L^2} \qquad \text{.... .... (ii)}$$

Solving (i) and (ii)

$$M_A = \frac{-W ab^2}{L^2} \quad \text{and} \quad M_B = \frac{-W a^2 b}{L^2}$$

Now consider a section $x - x$ at a distance x from A in the portion AC

$$M_{xx} = \frac{W b x}{L} - \left[M_A + (M_B - M_A) \cdot \frac{x}{L}\right]$$

$$= \frac{W b x}{L} - \frac{W a b^2}{L^2} - \frac{W a b (a-b) x}{L^3}$$

For point of contraflexure put $M_{xx} = 0$

$$\therefore \frac{Wbx}{L} - \frac{Wab^2}{L^2} - \frac{Wab (a- b)}{L^3} . \ x = 0$$

$$or \quad x = \frac{a.L}{(3a+b)} \quad \text{where } x \text{ is measured from } A$$

Similarly for point of contra-flexure in portion BC, We get, $x = \dfrac{b.L}{(a+3b)}$ where x is measured from B.

Shear Force –

Taking moments about B

$$R'_A \times L - M_A - Wb + M_B = 0$$

$$R'_A \times L \quad \frac{-W ab^2}{L^2} - Wb + \frac{W a^2 b}{L^2} = 0$$

$$R'_A = \frac{Wb^2 (3a+ b)}{L^3}$$

$$R'_B = -\frac{W a^2 (a+ 3b)}{L^3}$$

Shear Force diagram has been shown in fig. 9.18

Deflection under the load

$$y_c = \frac{W a^3 b^3}{3 L^3 E I}$$

Example 9.10

An encastre beam 5 m long carries a concentrated load of 16 KN at its centre. Determine the fixed end moments and the support reactions. Also calculate the maximum deflection under the load and draw the B. M. and shear force diagrams. Take E = 200 KN/mm² and I = 80 × 10⁶ mm⁴

Solution

For free moment diagram M_{ax} . $B.\ M = \dfrac{WL}{4} = \dfrac{16 \times 5}{4} = 20$ KN–m

Area of free moment diagram $= \dfrac{1}{2} \times 5 \times (20) = 50$

Area of fixing moment diagram

$M_A \times L = M_B \times L$

Since ends are fixed change of slope between A and B is Zero

$\therefore$ Area of free moment diagram + Area of fixing moment diagram =0

or $\frac{1}{2} \times 5 \times 20 + M_A \times L = 0$

or $M_A = -\frac{50}{5}$

$= -10$ KN– m $= M_B$.

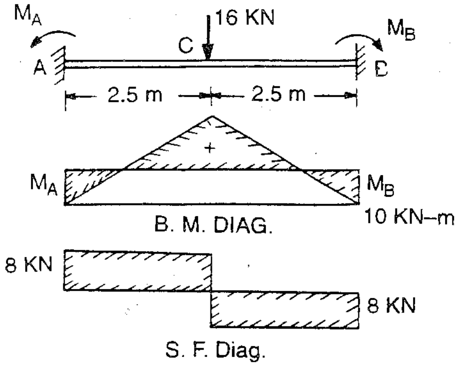


Fig. 9.19

Support reactions at A and $B = \frac{W}{2} = \frac{16}{2} = 8$ KN

Maximum deflection at $C = \frac{WL^3}{192EI}$

$$y_c = \frac{16 \times 10^3 \times (5 \times 1000)^3}{192 \times 200 \times 10^3 \times 80 \times 10^6} = 0.65 \text{ mm} \qquad \textbf{Answer.}$$

Example 9.11

A fixed beam AB of span 4 metres supports a load of 30 KN at a distance of 1 metre from support A. calculate the fixing moments at the ends and draw the B.M. and shear force diagrams.

Solution

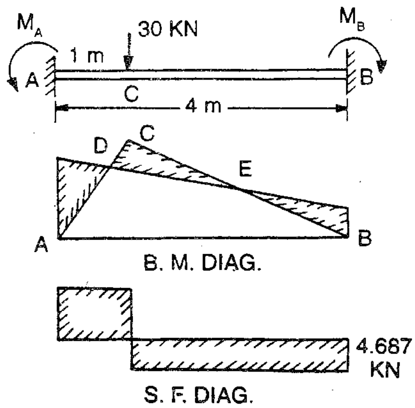


Fig. 9.20

For free moment diagram

$M_{max} = \frac{Wab}{L} = \frac{30 \times 1 \times 3}{4}$

$= 22.5$ KN–m

Fixing moment at A

$M_A = \frac{Wab^2}{L^2} = \frac{30\,(1)\,(3)^2}{(4)^2}$

$= 16.75$ KN–m

$M_B = \frac{Wa^2b}{L^2} = \frac{30\,(1)^2 \times 3}{(4)^2}$

$= 5.625$ KN–m

Shear Force

$R'_A = \frac{Wb^2\,(L + 2a)}{L^3}$

$$= \frac{30\,(3)^2\,(4 + 2 \times 1)}{(4)^3}$$

$= 25.31$ KN

$$R'_B = \frac{W a^2 (L + 2b)}{L^3}$$

$$= \frac{30\,(1)^2\,(4 + 2 \times 3)}{(4)^3} = 4.687 \text{ KN}$$

Point of Contra flexure, for point D when $x < a$

$$x = \frac{aL}{(3a + b)} = \frac{1 \times 4}{(3 \times 1 + 2)} = .66 \text{ m from } A$$

For point E *when* $x > a$

$$x = \frac{L(L + b)}{(L + 2b)} = \frac{4\,(4 + 3)}{(4 + 6)} = 2.8 \text{ m from } A$$

Example : 9.12

A built-in beam of span 6 metres carries two point loads 20 KN each at 1 metre and 5 metres from fixed end A. Find the moments at the supports. What is the central moment. Draw the S.F. & BM. diagrams. *(oxford)*

Solution :

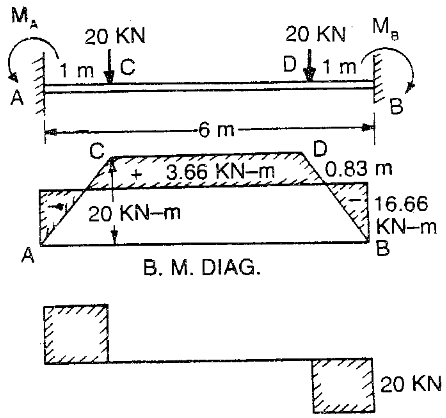


Fig. 9.21

Area of the free moment diagram

$$= \frac{1}{2} \times (6 + 4) \times 20 = 100$$

Area of the fixing moment diagram

$$= M_A \,.\, L = M_B \,.L$$

Since the change of slope between A and B is zero,

$$\therefore\ M_A \times L = -100$$

or $\quad M_A = \dfrac{-100}{6}$

$= -16.66$ KN–m

and $\quad M_B = -16.66$ KN–m

Central moment

$= (20 - 16.66)$

$= 3.36$ KN–m

For point of contraflexure,

$$M_{xx} = R_A \,.\, x - M_A = 0$$

or $\quad 20\,.x - 16.66 = 0 \quad$ or $\quad x = \dfrac{16.66}{20} = 0.833$ m

Point of contraflexure will occur at 0.83 m from either end. For shear force, Taking moments of all forces about B,

$$R_A \times 6 - M_A - 20 \times 5 - 20 \times 1 + MB = 0$$

$$R_A = \frac{20 \times 6}{6} = 20 \text{ KN} = R_B$$ **Answer.**

Example 9.13

AB is an encastre beam of 8 m effective span. It carries point loads of 10 KN each at quarter span points and at centre. Draw the B.M and S.F diagrams. What is the maximum B.M. and where the B.M. is Zero.

Solution

For free moment diagram

$R_A = R_B = 15$ KN

$M_C = 30$, $M_D = 40$,

$M_E = 30$KN–m

Area of free moment diagram =

$$2\left[\frac{1}{2} \times 2 \times 30 + \frac{1}{2}(30 + 40)\,2\right]$$

$= 200.$ .... (i)

Area of fixing moment diagram

$= M_A \times 8$

$= M_B \times 8$ .... (ii)

Since the change of slope between A and B is Zero.

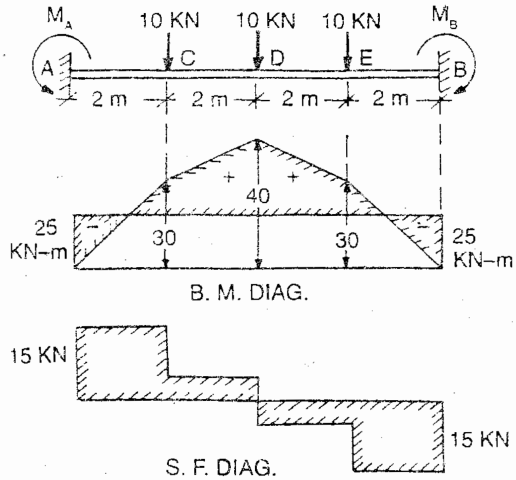


Fig. 9.22

$\therefore$ Area of free moment diagram

+ Area of fixing moment diagram = 0

or $\quad M_A \times 8 + 200 = 0$

or $\quad M_A = -\dfrac{200}{8} = -25$ KN–m $= M_B$

$M_A = M_B = -25$ KN–m

Maximum $B.M$ $(40 - 25) = +15$ KN-m and -25 KN-m (–ve)

For Point of contraflexure, equate M_{xx} to Zero

$$M_{xx} = R_A\,x - M_A = 0$$

or $\quad 15.x - 25 = 0 \quad$ or $\quad x = \dfrac{25}{15} = 1.66$ m from A

Zero Bending moment will occur 1.66 m from either end

Example 9.14

A fixed beam AB 6 metres long carries a uniformly distributed load of 2 KN/m over the whole span and a concentrated load of 10 KN at the centre. Draw the B.M. and S.F. diagrams and calculate the maximum deflection. Take E = 200 KN/mm² and I = 4 × 10⁴ mm⁴.

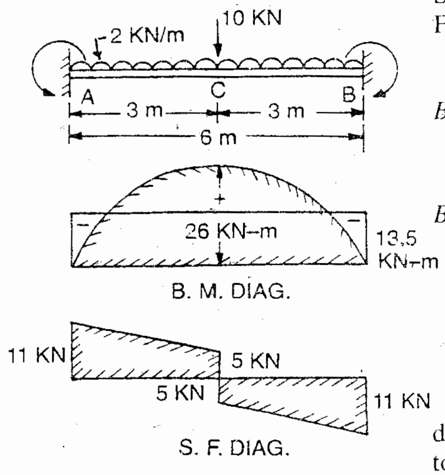


Fig. 9.23

Solution

For free moment diagram

$$R_A = R_B = 11 \text{ KN}$$

$$B.M_C = 11 \times 3 - 2 \times 3 \times \frac{3}{2}$$

$$= 33 - 9 = 26 \text{ KN–m}$$

$$B.M \text{ at } \frac{l}{4} = 11 \times \frac{6}{4} - 2 \times \frac{6}{4} \times \frac{6}{4} \times \frac{1}{2}$$

$$= 16.5 - 2.25 = 14.25 \text{ KN–m}$$

$$B.M_{\text{at}}\ 3\frac{l}{4} = 14.25 \text{ KN–m}$$

For fixing moments.

M_A = – [fixed end moment due to *u.d.l.* + fixed end moment due to point load]

$$= -\left[\frac{w\,l^2}{12} + \frac{W\,L}{8}\right]$$

$$= -\left[\frac{2 \times (6)^2}{12} - \frac{10 \times 6}{8}\right] = [6 + 7.5] = -13.5 \text{ KN–m}$$

$\therefore$ $M_A = M_B = -13.5$ KN–m

The combined diagram is shown in fig. 9.23

$S.F_A = 11$ KN

$S.\ F_c$ just to the left of $c = 11 - 2 \times\ 3 = 5$ KN

$S.\ F.$ just to the right of $c = 11 - 2 \times\ 3 + 10 = -5$ KN

$S.\ F_B = 1 - 2 \times\ 6 - 10 = -11$ KN

$$\text{Max}^{\text{m}}.\ \text{Deflection} = \frac{w\,L^4}{384E\,I} + \frac{W\,L^3}{192\,E\,I} = \frac{L3}{EI}\left[6\times\ \frac{2}{384} + \frac{10}{192}\right]$$

$$= \frac{0.0520 \times (6)^3 \times (1000)^3}{200 \times 10^3 \times 4 \times 10^4} = 1.404 \text{ mm} \quad \textbf{Answer.}$$

Example : 9.15

A fixed end beam AB has an effective span of 6 metres and loaded with 1 Kn/m on the whole span in addition a concentrated load of 12 KN at 2 m from A. Draw B. M. and S. F. diagrams. *(Rajasthan)*

Solution

For free moment diagram

$$R_A = \frac{w\,L}{2} + \frac{W\,b}{L}$$

$$= \frac{1 \times 6}{2} 3 + \frac{12 \times 4}{6} = 11 \text{ KN}$$

$$R_B = \frac{wL}{2} + \frac{Wa}{L}$$

$$= \frac{1 \times 6}{2} + \frac{12 \times 2}{6} = 7 \text{ KN}$$

$MA = 0$

$$\underset{x=2m}{M} = 11 \times 2 - 1 \times 2 \times \frac{2}{2}$$

$$= 20 \text{ KN–m}$$

$$\underset{x=3m}{M} = 11 \times 3 - 1 \times 3 \times \frac{3}{2} - 12 \times 1$$

$$= 16.5 \text{ KN–m}$$

$$\underset{x=4m}{M} = 11 \times 4 - 1 \times 4 \times \frac{4}{2} - 12 \times 2$$

$$= 12 \text{ KN–m}$$

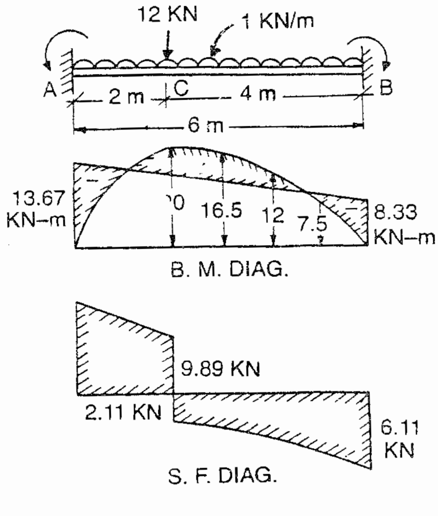


Fig. 9.24

$$\underset{x=5m}{M} = 11 \times 5 - 1 \times 5 \times \frac{5}{2} - 12 \times 3 = 7.5 \text{ KN–m}$$

$M_B = 0$

Fixing Moments

$$M_A = \frac{wL^2}{12} + \frac{Wab^2}{L^2} = \frac{1 \times (6)^2}{12} + \frac{12 \times 2\,(4)^2}{(6)^2} = 3 + 10.67 = 13.67 \text{K–m}$$

$$M_B = \frac{wL^2}{12} + \frac{Wa^2b}{L^2} = \frac{1 \times (6)^2}{12} + \frac{12\,(2)^2\,(4)}{(6)^2} = 3 + 5.33 = 8.33 \text{ KN–m}$$

Shear Force

Taking moments about B

$$R'_A \times 6 - 13.67 - 12 \times 4 - 1 \times 6 \times \frac{6}{2} + 8.33 = 0 \text{ or } R_A = 11.89 \text{ KN}$$

$R'_B = 6.11$ KN

$S.F_A = 11.89$ KN

$S.F.$ just to the left of C $= 11.89 - 1 \times 2 = 9.89$ KN

$S.F.$ just to the right of C $= 11.89 - 12 - 1 \times 2 = 2.11$ KN

$S.F_B = 11.89 - 12 - 1 \times 6 = -6.11$ KN

$B.M.$ and $S.F.$ diagrams are shown in fig. 9. 24

Example 9.16

An encastre beam of span 4 metres carries a u. d. l. of 1 KN/m over its entire length and two point loads of 2 KN and 4 KN at 1 metre and 2 metres from fixed end A. Draw the B.M. and S. F. diagram.

Solution :

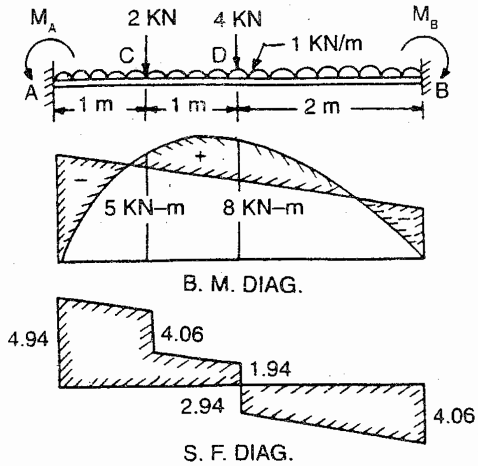


Fig. 9.25

For free moment diagram taking moment about B.

$R_A \times 4 - 2\times 3 - 4\times 2 - 1\times 4\times 4/2 = 0$

$4R_A = 6+8+8$

$R_A = 22/4 = 5.5$ KN

$R_B = 2+4+4-5.5$

$= 4.5$ KN.

$M_C = 5.5\times 1 - 1\times 1\times \frac{1}{2}$

$= 5.5 - 0.5 = 5$ KN–m

$M_D = 5.5\times 2 - 1\times 2\times 2/2 - 2\times 1$

$= 11-2-1 = 8$ KN–m

For fixing moments.

M_A = Due to $(udl + W_1 + W_2)$

$$M_A = -\left[\frac{w\,l^2}{12} + \frac{W\,a\,b^2}{L^2} + \frac{W\,a\,b^2}{L^2}\right]$$

$$M_A = -\left[\frac{1\times 4^2}{12} + \frac{2\times 1\times 3^2}{4^2} + \frac{4\times 2\times 2^2}{4^2}\right]$$

$$= -\left[\frac{+16}{12} + \frac{36}{16} + \frac{32}{16}\right] = -\{1.33 + 2.25 + 2\} = -558 \text{ KN–m}$$

$$M_B = -\left[\frac{w\,L^2}{12} + \frac{Wa^2b}{L^2} + \frac{W\,L}{8}\right] = -\left[\frac{1\times 4^2}{12} + \frac{2\times 1^2\times 3}{16} + \frac{4\times 4}{8}\right]$$

$$= -\left[1.33 + \frac{6}{16} + \frac{16}{8}\right] = -(1.33 + .375 + 2) = 3.705$$

$$= -3.705 \text{ KN–m}$$

Shear Force.

Taking moments about B.

$$R'_A\times 4 - M_A - w.\frac{L.L}{2} - 2\times 3 - 4\times 2 + M_B = 0$$

$$R'_A\times 4 = 5.58 + 1\times 4\times \frac{4}{2} + 6 + 8 - 3.705 = 23.775$$

$$R'_A = \frac{23.775}{4} = 5.94 \text{ KN}$$

$$R'_B = 1\times 4 + 2 + 4 - 5.94 = 4.06 \text{ KN}$$

$S.F.\ A\ = 5\ .\ 94$

$S.F.$ just left of $C = 5.94 - 1 \times\ 1 = 4\ .94$ KN

$S.\ F.$ Just to the right of $C = 5.94 - 1 - 2 \quad = 2.94$ KN

$S.F.$ Just to the left of $D = 5.94 - 1 \times\ 2 - 2 \ = 5.95 - 4 = 1.94$ KN

$S.F.$ Just to the right of $D = 5.95 - 1 \times\ 2 - 2 - = 5.\ 94 - 8 = -2.00$ KN

$S.\ F.$ at $B = 5.94 -1\times\ 4 - 2 - 4 = 5.\ 94 - 10 = \quad - 4.06$ KN

Eample : 9.17

A built-in beam of span 8 metres carries a uniformly distributed load of 1/2 KN per metre over the left half of the span. Calculate the support moments and draw B. M. and S. F. diagrams. *(J.M.I.)*

Solution

Supports reactions for a freely supported bear

$$R_A \times\ 8 = \frac{1}{2} \times\ 4\left(\frac{4}{2} + 4\right)$$

$R_A = 1.5$ KN

$R_B = (1/2\times\ 4 -1.5\ ) = 0.5$ KN

Since change of slop between A and B is zero.

Area of free moment diagram + Area of Fixing moment diagram = 0

Now area f free moment diagram = Area of parabolic figure

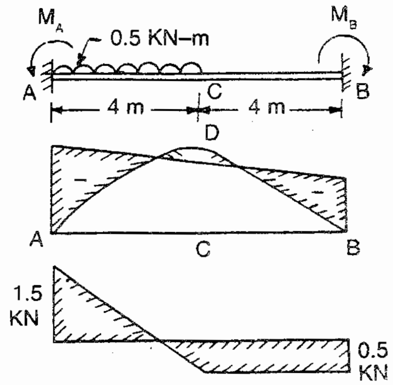


Fig.. 9.26 (a)

ADC+ Area of triangle DBC - Consider a strip dx at a distance x from A, then

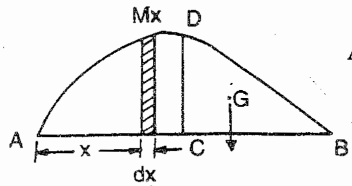


Fig . 9.26 (a)

$$M_x = \left(R_A\ .\ x - w\ .\ x\ .\ \frac{x}{2}\right)$$

Area for the strip $\quad = M_x.\ dx$

$$= \left(R_A\ .\ x - \frac{w\,x^2}{2}\right) dx$$

Total area of the figure $= \int_0^4 M_x\ .\ d_x$

$$= \int_0^4 \quad \left(R_A\ .\ x - \frac{w\,x^2}{2}\right) dx$$

Area of the triangle $\quad = \frac{1}{2} \times\ 4 \times\ 2$

Total area of free moment diagram

$$= \int_0^4 \quad \left(R_A\ .\ x - \frac{w\,x^2}{2}\right) dx\ +\ \frac{1}{2} \times\ 4 \times\ 2$$

Area of fixing moment diagram = $(M_A + M_B)\,\frac{L}{2}$

$$\therefore\ (M_A + M_B)\ \frac{L}{2} = \int_0^4 \left[1.5\,x - 0.5\frac{x^2}{2}\right] dx + 4$$

$$= \left[1.5\frac{x^2}{2} - 0.5\frac{x^3}{6}\right]_0^4 + 4$$

$$4\ (M_A + M_B) = (6.67 + 4) = 10.67$$

$$\text{or } (M_A + M_B) = \frac{10.67}{4} = 2.66 \quad \ldots\ldots \quad \ldots\ldots \quad \text{(i)}$$

According to 2nd theorem

Moment of fixing moment diagram about A = Moment of free moment diagram about A.

$$(M_A + 2M_B)\frac{L^2}{6} = \int_0^4 (Mx\,.dx)\,.\,x + \left(\frac{1}{2}\times 4 \times 2\right)(4 + 4/3)$$

$$= \int_0^4 \left[R_A\,.\,x - \frac{w\,x^2}{2}\right].\,x\,.\,dx\,. + 4\,\frac{(16)}{3}$$

$$= \int_0^4 \left[1.5\,x^2 - \frac{w\,x^3}{2}\right] dx + \frac{64}{3}$$

$$= \left[\frac{1.5\,x^3}{3} - \frac{0.5\,x^4}{8}\right]_0^4 + \frac{64}{3}$$

$$(M_A + 2\,M_B)\times\frac{8^2}{6} = 32 - 16 + \frac{64}{3}$$

$$\text{or} \quad M_A + 2\,M_B = 3.50 \quad \ldots\ldots \quad \ldots\ldots \quad \text{(ii)}$$

Solving (i) & (ii) $M_A + 2M_B = 2.66$

$M_{A\,+}\,2M_B = 3.50$

$M_A = 1.72$ KN–m and $M_B = 0.94$ KN–m **Answer.**

For Shear Force

Taking moments about B

$$R'_A \times 8 - 1.72 - 0.5\times 4\,(\frac{4}{2} + 4) + 0.94 = 0$$

$R'_A = 1.5975$

$R'_B = 2.15975 = 0.4025$

Example : 9.18

An encastre beam AB of span 8 metres carries a uniformly distributed load of 4 KN/metre over the left half of the span and a concentrated load of 8 KN at 2 metres from B. Calculate the fixing moments at the ends and draw the B.M and S.F. diagram. *(CalcuttaUniv)*

Solution –

For free moment diagram support reactions

$$R_A \times 8 = (4\times 4)(4/2 + 4)\ 8\times 2$$

$= 96 + 16 = 112$

$R_A = 14$ KN

$R_B = 10$ KN

Moment at C, M_C

$$M_C = R_A \times 4 - 4 \times 4\left(\frac{4}{2}\right)$$

$$= 14 \times 4 - 32 = 24 \text{ KN–m}$$

$$M_D = R_B \times 2 = 10 \times 2$$

$$= 20 \text{ KN–m}.$$

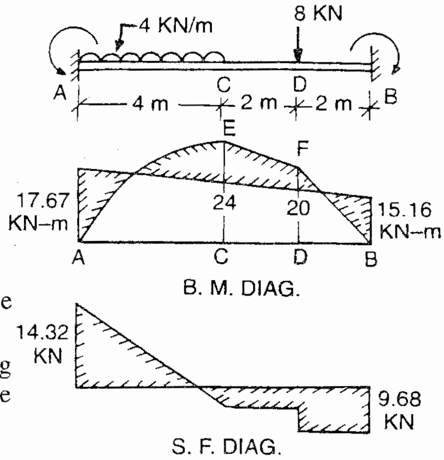


Fig . 9.27

Since the change of slope between A and B is Zero.

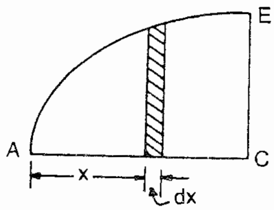
There fore Area of fixing moment diagram + Area of free moment diagram = 0

$$(M_A + M_B)\frac{L}{2} = \text{Area of ACE} +$$

Area of ECDE + Area of FDB

Area of parabolce figure ACE

Consider a strip dx at a distance x from A



$$M_x = (R_A . x - w x . x/2)$$

$$\text{Area } AEC = \int_0^4 (R_A . x - wx . x/2)\, dx$$

Total area of free moment diagram.

$$= \int_0^4 \left(R_A . - \frac{w x^2}{2}\right) dx + 2\left(\frac{24 + 20}{2}\right) + \frac{1}{2} \times 2 \times 20$$

$$= \int_0^4 \left(14 . x - \frac{w x^2}{2}\right) dx + 44 + 20$$

$$= \left[\frac{14 x^2}{2} - \frac{4 x . x^3}{2 \times 3}\right]_0^4 + 64 = \left[14 \times 8 - \frac{4 \times (4)^3}{2 \times 3}\right] + 64 = 133.33$$

$$\therefore\ (M_A + M_B)\frac{L}{2} = 133.33 \text{ or } (M_A + M_B)\frac{8}{2} = 133.33$$

$$M_A + M_B = 33.33 \quad \ldots. \quad \ldots. \quad \text{(i)}$$

Again according to 2nd theorem.

Moment of fixing moment diagram about A = Moment of free moment diagram about A

$$(M_A + 2 M_B)\frac{L^2}{6} = \int_0^4 \left(R_A . x - \frac{w x^2}{2}\right) x\, dx + 44\,(4 + 1) + 20\left(6 + \frac{2}{3}\right)$$

$$= \int_0^4 \left(14 x^2 - \frac{4 x^3}{2}\right) dx + 44 \times 5 + 20\,(6.66)$$

$$= \left[\frac{14x^3}{3} - \frac{4x^4}{8}\right]_0^4 + 220 + 133.2$$

$$= \left[\frac{14}{3}(4)^3 - \frac{4\,(4)^4}{8}\right] + 353.20$$

$$= (298.66 - 128) + 353.2 = 170.66 + 353.20$$

$$= 523.82$$

$$(M_A + 2\,M_B) \times \frac{8^2}{6} = 523.82 \quad \text{or} \quad (M_A + 2\,M_A) = 94.10 \quad \ldots\ldots \quad \text{(ii)}$$

Solving (i) + (ii)

$M_A = 17.67$ KN–m and $M_B = 15.6$ KN–m

For ShearForce.

Equating clockwise moments to anti clockwise moments

$$R'_B \times 8 + 17.67 = \left(4 \times 4 \times \frac{4}{2}\right) + 8 \times 6 + 15.16$$

$$8 \times R'_B = 32 + 48 - 2.57 = 77.43$$

$$R'_B = 9.68 \text{ KN.}$$

$$R'_A = (4 \times 4 + 8) - 9.68 = 14.32 \text{ KN}$$

The shear force diagram is shown in the figure 9.24

Example 9.19

A fixed beam of span L metres carries a uniformly varying load whose intensity varies from zero at one end to w at the other. Determine the fixing moments at the ends. *(Poona Univ.)*

Solution

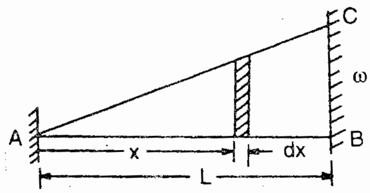


Fig. 9.28

Consider a strip dx at a distance x from A.

Load intensity at this section

$$= w.\frac{x}{L}$$

Total load on the strip

$$W = w.\frac{x}{L}.dx$$

This load W may now be treated as a point load acting at a distance x from A and $(L - x)$ from B.

Hence $M_A = \dfrac{wab^2}{L^2}$ and $M_B = \dfrac{wab^2}{L^2}$

Now $W = \left(\dfrac{wx}{L}\right)dx$, $a = x$, $b = (L - x)$

Therefore integrating between the limits, 0 and L we get the fixing moments.

$$M_A = \int_0^L \left(\frac{w \times x}{L}\right) \frac{dx.\, x\,(L-x)^2}{L^2}$$

or $$M_A = \frac{w}{L^3}\int_0^L x^2\,(L-x)^2\,dx = \frac{w}{L^3}\int_0^L x^2\,(L^2 - 2\,L\,x + x^2)\,dx$$

$$M_A = \frac{w\,l^2}{30}$$

and $M_B = \int_0^L \frac{w\,.\,x}{L}\,.\,dx\,.\,x^2\,(L-x)$ or $M_B = \dfrac{w\,L^2}{20}$ **Answer.**

Example 9.20

A built-in beam AB of span L metres carries a uniformly varying load which varies from zero at A to w at the mid span. Determine the fixing moments at A and B.

Solution -

Consider a strip dx at a distance x from A

Load intensity at this point $= \left(\dfrac{w.x}{L/2}\right)$

Total load at this point $= \left(\dfrac{w.x}{L/2}\right).\,dx$

$= \dfrac{2wx}{L}\,.\,dx$

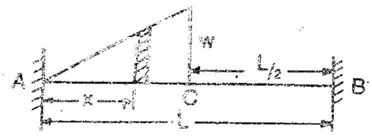


Fig. 9.29

Now consider this as a point load at x from A

Then $M_A = \dfrac{W.ab^2}{L^2}$

Putting $W = \dfrac{2wx.}{L}\,dx$, $a = x$, $b = (L-x)$

$$M_A = \int_o^{L/2} \frac{2wx.dx}{L}\,.\,\frac{x.(L-x)}{L^2} = \frac{2w}{L^3}\int_o^{L/2} x^2\,(L-x)^2\,dx$$

$$= \frac{2w}{L^3}\int_o^{L/2} x^2\,(L^2 - 2Lx + x^2 dx$$

$$= \frac{2w}{L^3}\left[\frac{L^2.x^3}{3} - \frac{2Lx^4}{4} + \frac{x^5}{5}\right]_o^{L/2} = \frac{wL^2}{30}$$

Now $$M_B = \frac{Wa^2b}{L^2} = \int_o^{L/2} \frac{(2w.x.dx)}{L}\,.\,\frac{x^2(L-x)}{L^2}$$

$$= \int_o^{L/2} \frac{2w.x^3}{L^3}\,(L-x)\,.\,dx = \frac{2w}{L^3}\int_o^{L/2} (x^3\,L - x^4)\,dx$$

$$= \frac{2w}{L^3}\left[\frac{Lx^4}{4} - \frac{x^5}{5}\right]_0^{L/2}$$

$$M_B = \frac{3wL^2}{160}$$

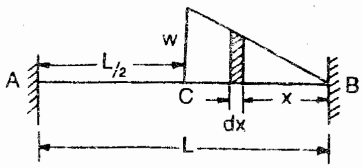


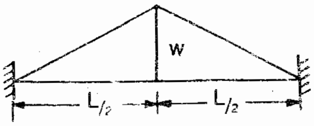
Fig. 9.30

Answer

When the load varies from zero at B to w at mid span, M_A will be $\frac{3}{160}wl^2$ and M_B will be $\frac{1}{30}wl^2$

When the load varies from zero at A to w at C and then decreases to zero at B.

This case may be treated as the combination of above two cases and M_A will be sum of (i) and (ii)



$$M_A = \frac{1}{30}wL^2 + \frac{3}{160}wL^2 = \frac{5}{96}wL^2$$

Fig. 9.31

and M_B will be the sum of the fixing moments at B in case (i) and (ii)

$$M_B = \frac{3}{160}wL^2 + \frac{1}{30}wL^2 = \frac{5}{96}wL^2$$

Sinking of a Support

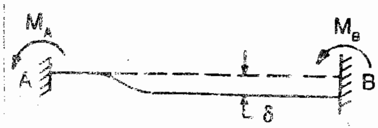


Fig. 9.32

If the prop B sinks by an amount δ below the level of A, it will result in the induction of a shear force equal to $\frac{12EI\delta}{L^3}$ throughout .

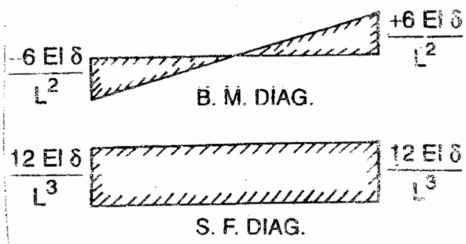
Bending moment at A,

$$M_A = \frac{-6EI\ \delta}{L^2}$$

Bending moment at B,

$$M_B = \frac{+6EI\ \delta}{L^2}$$

If the fixed beam carries a uniformly distributed load and one support is lower than the other then the support moments will be



$$M_A = \left[\frac{-wL^2}{12} - \frac{6EI\delta}{L^2}\right]$$ at the higher support.

Fig. 9.33

and $M_B = \left[\frac{-wL^2}{12} + \frac{6EI\delta}{L^2}\right]$ at the lower support.

Example 9.21

A fixed beam AB of span 5 metres carries a u.d.l. of 8 KN/metre. The support B sinks by 10 mm.

If $E = 2 \times 10^5$ N/mm^2 and $I = 66 \times 10^6$ mm^4. Draw the Bending moment diagram.

Solution

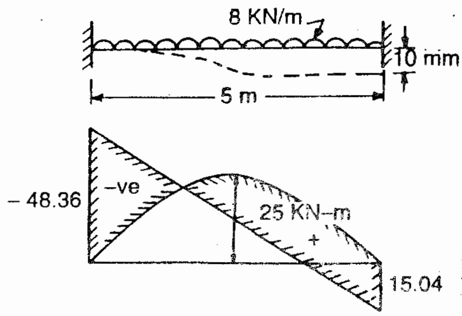


Fig. 9.34

Fixing moments due to *u.d.l.*,

$$M_A' = M_B' = -\frac{wL^2}{12}$$

$$= \frac{-8 \times 25}{12} = -16.66 \text{ KN-m.}$$

End moments caused by sinking of support

$$M_A'' = -M_B'' = \frac{-6EI\delta}{L^2}$$

$$= \frac{-6 \times 2 \times 10^5 \times 66 \times 10^6}{(5 \times 1000)^2} \times 10$$

$$= -31.7 \text{ KN-m}$$

$M_A'' = -31.7$ KN-m. , $M_B'' = +31.7$ KN-m

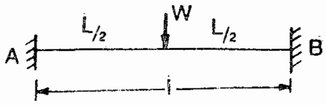
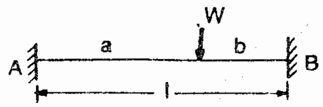
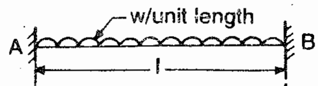
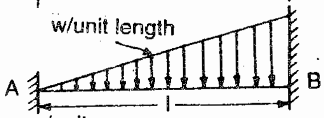
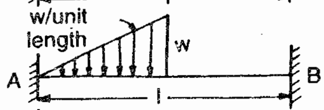
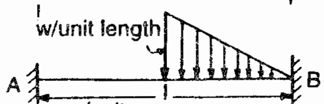
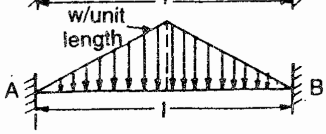
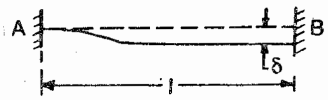
Net and moments due to *u.d.l.* and sinking of support *B*

$M_A = M_A' + M_A'' = -16.66 - 31.7 = -48.36$ KN-m.

$M_B = M_B' + M_B'' = -16.66 + 31.7 = +15.04$ KN-m **Answer**

Table 9.1

Standard Cases of Fixed End Beams

Type of Loading	Fixed End Moments.
	$M_A = M_B = -\dfrac{WL}{8}$
	$M_A = -\dfrac{Wab^2}{L^2}$ $\quad M_B = -\dfrac{Wa^2b}{L^2}$
	$M_A = M_B = \dfrac{wL^2}{12}$
	$M_A = -\dfrac{wL^2}{30}$ $\quad M_B = -\dfrac{wL^2}{20}$
	$M_A = -\dfrac{wL^2}{30}$, $\quad M_B = \dfrac{3}{160}wL^2$
	$M_A = \dfrac{3}{160}wL^2$ $\quad , M_B = -\dfrac{wL^2}{30}$
	$M_A = M_B = -\dfrac{5wL^2}{96}$
	$M_A = -\dfrac{6EI\delta}{L^2}$ $\quad M_B = +\dfrac{6EI\delta}{L^2}$

Statically Indeterminate Beams (Continuous Beams)

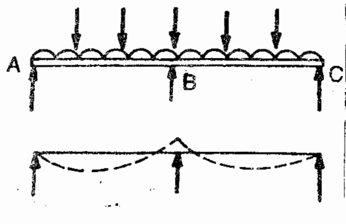


Fig. 9.35

A beam resting on more than two supports is called a continuous beam. The deflected form of a continuous beam under a loading system is shown in the figure. The elastic curve shows that the curvature at the supports is convex upwards. It means that the moments induced at the supports will be opposite in nature to the moments produced in the centre of different spans of the continuous beam. The moments induced at the supports are called support moments.

Clapeyron's Three Moments Theorem

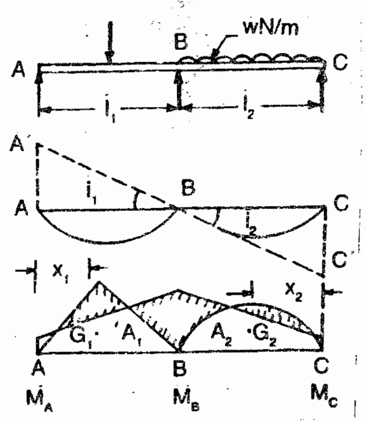


Fig. 9.36

Let AB and BC be two consecutive spans of length l_1 and l_2 of a continuous beam of any number of spans. The free moment diagrams for the loading on these spans is shown in the figure. Let x_1 be the distance of the $C.\ G$ of the moment diagram on span AB from. A. Similarly let x_2 be the distance of the $C.\ G$ of the moment diagram on span BC from C. let I_1 and I_2 be the moment of inertia of spans AB and BC respectively.

Let M_A, M_B and M_c be the support moments at A, B and C respectivly. Then clapeyron's theorm states that

$$M_A \cdot \frac{l_1}{I_1} + 2M_B\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right) + M_C\frac{l_2}{I_2} = \frac{-6A_1x_1}{l_1I_1} - \frac{6A_2x_2}{l_2I_2}$$

When both the spans AB and BC have similar sections then $I_1 = I_2$ and the equation can be written more simply as

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = \frac{-6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

Now, according to moment - Area method, the intercept AA' on the vertical at A between the tangent at A and B is given by the moment of the bending moment diagram between A and B divided by the flexural rigidity EI. The moment being taken about A.

Therefore

$$AA' = \frac{1}{EI_1}\left[A_1 x_1 + \frac{M_A l_1}{2} \times \frac{l_1}{3} + \frac{M_B l_1}{2} \times \frac{2l_1}{3}\right]$$

$$i_1 = \frac{AA'}{l_1} = \frac{A_1 x_1}{l_1 EI_1} + \frac{M_A l_1}{6EI_1} + \frac{M_B l_1}{3EI_1}$$

$$\text{and } i_2 \quad \frac{CC'}{l_2} = \frac{A_2 x_2}{l_2 EI_2} + \frac{M_C l_2}{6EI_2} + \frac{M_B l_2}{3EI_2}$$

Since the beam is Continuous $i_1 = -i_2$

$$\therefore \frac{A_1 x_1}{l_1 EI_1} + \frac{M_A l_1}{6EI_1} + \frac{M_B l_1}{3EI_1} = -\frac{A_2 x_2}{l_2 EI_2} - \frac{M_C l_2}{6EI_2} - \frac{M_B l_2}{3EI_2}$$

Transporting terms

$$M_A\left(\frac{l_1}{EI_1}\right) + 2M_B\left(\frac{l_1}{EI_1} + \frac{l_2}{EI_2}\right) + \frac{M_c l_2}{EI_2} = -\frac{6A_1 x_1}{l_1 EI_1} - \frac{6A_2 x_2}{l_2 EI_2}$$

And when $I_1 = I_2 = I$

Then the equation can be written as

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = \frac{-6A_1 x_1}{l_1} - \frac{6_2 x_2}{l_2}$$

Standard Cases

1. Continuous beam with point loads at mid spans and of constant I

Area of free moment diagram on span AB

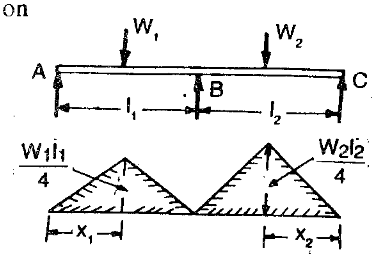


Fig. 9.37

$$A_1 = \frac{W_1 l_1}{4} \times \frac{l_1}{2}$$

$$x_1 = \frac{l_1}{2}$$

$$\frac{6A_1 x_1}{l_1} = 6\left[\frac{1}{2} \times l_1 \times \frac{W_1 l_1}{4} \times \frac{l_1}{2}\right]$$

$$= \frac{3}{8} W_1 l_1^2$$

Similarly Area of free moment diagram on span BC

$$A_2 = \frac{W_2 l_2}{4} \quad \text{and} \quad x_2 = \frac{l_2}{2}$$

$$\therefore \quad \frac{6A_2 x_2}{l_2} = \frac{3}{8} W_2 l_2^2$$

Now Applying three moments theorem on spans AB and BC

$$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = \frac{-6A_1\, x_1}{l_1} - \frac{6A_2\, x_2}{l_2}$$

$$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = \frac{-3}{8} W_1\, l_1^2 \frac{-3}{8} W_2\, l_2^2$$

2. Continuous beam with non - central point loads on each span

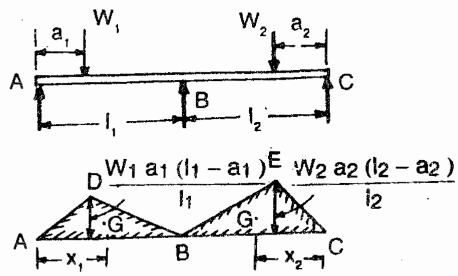


Fig. 9.38

Area of free moment diagram over span AB

$$A_1 = \frac{1}{2} l_1 \times \frac{W_1\, a_1 (l_1 - a_1)}{l_1}$$

$$x_1 = \frac{(l_1 + a_1)}{3}$$

$$\frac{6A_1 \bar{x}_1}{l_1} = \frac{6\, W_1 a_1 (l_1^2 - a_1^2)}{6\,(l_1)}$$

$$= \frac{W_1\, a_1 (l_1^2 - a_1^2)}{l_1}$$

Similarly

Area of moment diagram on span BC

$$A_2 = \frac{1}{2} \frac{l_2\, W_2\; a_2\,(l_2 - a_2)}{l_2}$$

$$x_2 = \frac{(l_2 + a_2)}{3}$$

$$\frac{6A_2\, x_2}{l_2} = \frac{W_2\, a_2 (l_2^2 - a_2^2)}{l_2}$$

Applying three moment theorem on spans AB and BC

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = -\frac{6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = \frac{-W_1 a_1}{l_1}(l_1^2 - a_1^2) \frac{-W_2 a_2}{l_2}(l_2^2 - a_2^2)$$

3. Continuous beam with *U.d.L* on each span

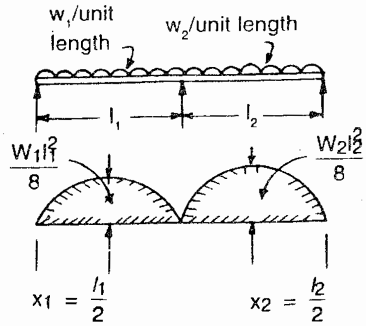


Fig. 9.39

Area of free moment diagam on span AB

$$A_1 = \frac{2}{3} \times l_1 \times \frac{w_1 l_1^2}{8} \quad \text{and} \quad x_1 = \frac{l_1}{2}$$

$$\therefore \quad \frac{6A_1 x_1}{l_1} = \frac{6}{l_1}\left[\frac{2}{3} l_1 \frac{w_1 l_1^2}{8}\right] \times \frac{l_1}{2}$$

$$= \frac{w_1 l_1^3}{4}$$

Similarly area of free moment diagram on span BC

$$A_2 = \frac{2}{3} \times l_1 \times \frac{w_2 l_2^2}{8} \quad \text{and} \quad x_2 = \frac{l_2}{2}$$

Hence $\dfrac{6A_2 x_2}{l_2} = \dfrac{w_2 l_2^3}{4}$

Now Applying three moments theorem on spans AB and BC

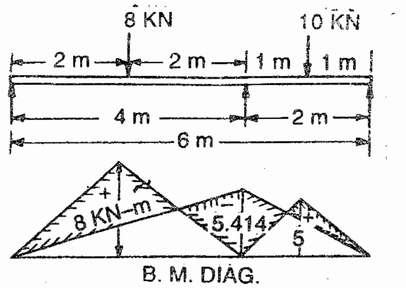
$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = -\frac{6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

or $$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = -\frac{w_1 l_1^3}{4} - \frac{w_2 l_2^3}{4}$$

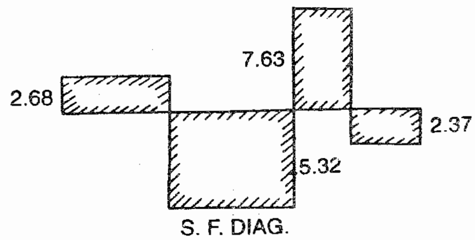
Example 9.23

A continuous beam ABC of span 6 metres is shown in figure 9.40 Draw the shear force and Bending moment diagrams.

Solution



B. M. Diaagram



S. F. Diagram
Fig. 9.40

Maximum free moment ordinate on span AB

$$M_{max} = \frac{W_1 l_1}{4} = \frac{8 \times 4}{4} = 8 \text{ KN-m}$$

Maximum free moment ordinate on span BC

$$M_{max} = \frac{W_2 l_2}{4} = \frac{10 \times 2}{4} = 5 \text{ KN-m}$$

Now applying three- moments theorem on spans AB and BC

$$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = \frac{-6A_1x_1}{l_1} - \frac{6A_2x_2}{l_2}$$

$$4M_A + 2M_B\,(4 + 2) + M_C \times 2 = \frac{-3}{8} W_1\, l_1^2 \frac{-3}{8} W_2\, l_2^2$$

$$4M_A + 12M_B + 2M_C = \frac{-3}{8} \times 8\,(4)^2 - \frac{3}{8} \times 10 \times (2)^2$$

$$4M_A + 12M_B + 2M_C = -48 - 15 = -63$$

Since end moments M_A and M_C are zero

$$\therefore\ 12\, M_B = -63$$

$$M_B = -5.25 \text{ KN-m}$$

Support Reactions

Taking moments about B of all forces to the left of B

$$R_A \times 4 - 8 \times 2 = -5.25$$

$$\text{or} \quad R_A = \frac{10.75}{4} = 2.68 \text{ KN}$$

Taking moments about B of all forces to the right of B

$$R_C \times 2 - 10 \times 1 = -5.25$$

$$R_C = \frac{4.75}{2} = 2.37 \text{KN}$$

Now $R_A + R_B + R_C = 10 + 8 = 18$

or $2.68 + R_B + 2.37 = 18$ or $R_B = 12.95\ KN$

B. M. and *S.F.* diagrams are shown in the figure.

Example 9.24

Draw the B.M and S.F. diagrams for the continuous beam shown in fig. 9.41

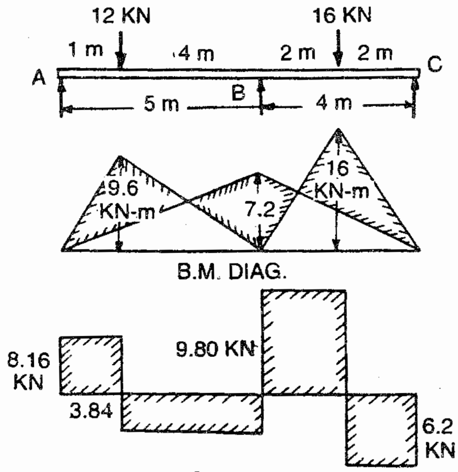


Fig. 9.41

Solution

Max. free moment ordinate on span AB

$$M_{max} = \frac{W_1 ab}{l_1} = \frac{12 \times 1 \times 4}{5}$$

$$= 9.6 \text{ KN-m}$$

Maximum free moment ordinate on span B_C

$$M_{max} = \frac{W_2 l_2}{4} = \frac{16 \times 4}{4}$$

$$= 16 \text{ KN-m}$$

Applying three moments theorem on span AB and BC

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = \frac{-6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

$$5M_A + 2M_B (5 + 4) + 4M_C = -\frac{W_1 a_1}{l_1}(l_1^2 - a_1^2) - \frac{3}{8} W_2 \, l_2^{\,2}$$

$$= \frac{12 \times 1}{5}(5^2 - 1^2) - \frac{3}{8} \, 12 \times 4^2$$

$5M_A + 18M_B + 4M_C = -57.6 - 72 = 129.6$

Since ends are simply supported $M_A = M_C = 0$

$\therefore \; 18M_B = 129.6$

$M_B = 7.2$ KN-m

Support Reactions

Taking moments about B of forces to the left of B

$R_A \times 5 - 12 \times 4 = -7.2$

or $R_A = \dfrac{48 - 7.2}{5} = 8.16$ KN

Taking moments about B of forces to the right of B.

$Rc \times 4 - 16 \times 2 = -7.2$

$R_C = \dfrac{32 - 7.2}{4} = 6.2$ KN

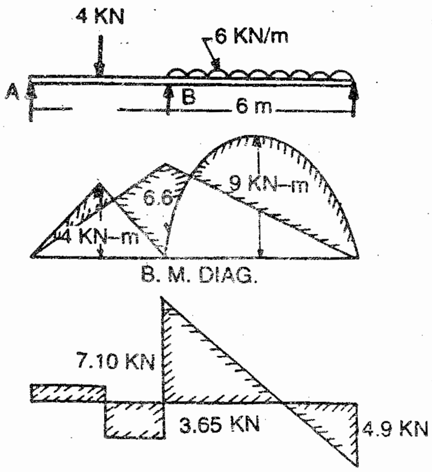
Now $R_A + R_B + R_C = 12 + 16 = 28$

$8.16 + R_B + 6.2 = 28$

$R_B = 28 - 8.16 - 6.2 = 13.64$ KN

Example 9.25

Determine the support moments and draw tne bending moment and shear force diagrms for the beam shown in figure 9.42 *(Bomboy Univ.)*



S. F. Diaagram Fig. 9.42

Solution

Maximum free moment ordinate on span AB

$$M_{max} = \frac{Wl}{4} = \frac{4 \times 4}{4} = 4 \text{ KN-m}$$

Maximum free moment ordinate on span BC

$$M_{max} = \frac{wl^2}{8} = \frac{2\,(6)^2}{8} = 9 \text{ KN-m}$$

Applying three moments theorem on spans AB and BC

$$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = \frac{-6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

$$4M_A + 2M_B\,(4+6) + 6M_C = -\frac{3W_1 l_1^2}{8} - \frac{w_2 l_2^3}{4}$$

Since ends are simply supported $M_A = M_C = 0$

$$\therefore\ 20\, M_B = \frac{-3}{8} 4\ (4)^2 - \frac{1}{4}\ (2)\,(6)^3 = -24 - 108$$

or $\quad M_B = -\dfrac{132}{20} = -6.6$ KN-m

$\therefore \quad M_A = 0, \quad M_B = -6.6 \quad$ and $\quad M_C = 0$

Bending moment diagram is shown in the figure

Support Reactions

Taking moments about B of forces to the left of B

$$R_A \times 4 - 4 \times 2 = -6.6$$

$$4R_A = -6.6 + 8 = 1.4$$

$$R_A = \frac{1.4}{4} = .35 \text{ KN}$$

Taking moments about B of forces to the right of B

$$R_C \times 6 - 2 \times 6 \times \frac{6}{2} = -6.6$$

$$6\, R_C = -6.6 + 36 = 29.4$$

$$R_C = 4.9 \text{ KN}$$

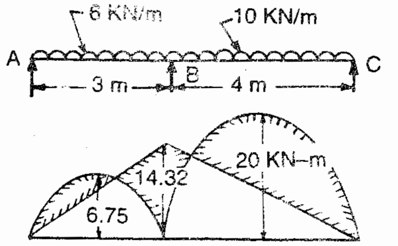
Now $R_A + R_B + R_C = 0.35 + R_B + 4.9 = 4 + 12 = 16$

or $\quad R_B = 16 - 0.35 - 4.9 = 10.75$

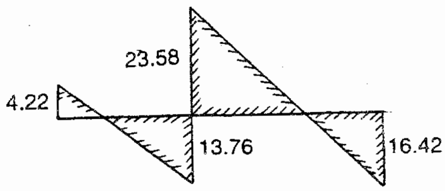
Shear force diagram can now be drawn as shown in the figure.

Example 9.26

Draw B.M. and S. F. diagram for the continuous beam ABC of span 7 metres. Span AB carries a u.d.l of 6 KN/m and span BC carries a u.d.l of 10 KN/m.



B. M. Diagram



S. F. Diagram
Fig. 9.43

Solution

For free moments Span AB

$$M_{max} = \frac{w_1 l_1^2}{8} = \frac{6\,(3)^2}{8} = 6.75 \text{ KN-m}$$

Span BC

$$M_{max} = \frac{w_2 l_2^2}{8} = \frac{10\,(4)^2}{8}$$

$$= 20 \text{ KN-m}$$

End moments $M_A = M_C = 0$

Applying three moments theorem on spans AB and BC

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = -\frac{6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

$$3\,M_A + 2M_B\,(3 + 4) + 4M_C = -\frac{w_1 l_1^3}{4} - \frac{w_2 l_2^3}{4}$$

$$= -\frac{6(3)^3}{4} - \frac{10(4)^3}{4}$$

$$14M_B = -40.5 - 160 = 200.5$$

$$M_B = -14.32 \text{ KN-m}$$

Support reactions

Taking Moments about B of all forces to the left of B

$$R_A \times 3 - 6 \times 3 \times \frac{3}{2} = -14.32 \quad \text{or} \quad R_A = 4.22 \text{ KN}$$

Taking moments about B of all forces to the right of B

$$R_C \times 4 - 10 \times 4 \times \frac{4}{2} = -M_B = -14.32 \quad \text{or} \quad R_C = 16.42$$

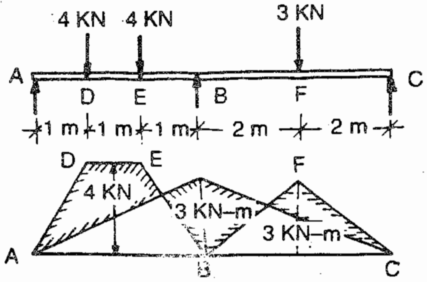
Now $R_A + R_B + R_C = 18 + 40 = 58$

$4.22 + R_B + 16.42 = 58$ or $R_B = 37.34$

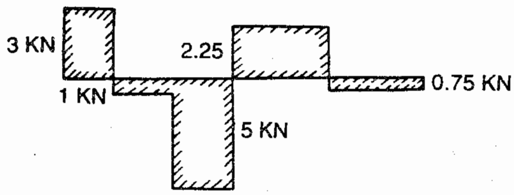
B. M and *S. F.* diagrams are shown in figure 9.43

Example 9.27

Determine the support moments for a continuous beam ABC as shown in figure 9.44 and draw the B.M and S.F. diagrams also locate the points of contraflexure.



B. M. Diagram



S. F. Diagram
Fig. 9.44

Solution

Since the ends A and C are simply supported

$$M_A = M_C = 0$$

Area of free moment diagram on AB

$$A_1 = (3 + 1) \times \frac{4}{2} = 8$$

$$x_1 = \frac{3}{2} = 1.5 \text{ m from } A$$

$$\frac{6 A_1 x_1}{l_1} = \frac{6 \times 8 \times 1.5}{3} = 24$$

Area of free moment diagram on span BC.

$$A_2 = \frac{1}{2} \times 4 \times 3 = 6$$

$$x_2 = 2\text{m from } C.$$

$$\therefore \quad \frac{6A_2x_2}{l_2} = \frac{6 \times 6 \times 2}{4} = 18$$

Now applying 3 - moments theorm on span AB and BC.

$$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = -\frac{6A_1x_1}{l_1} - \frac{6A_2x_2}{l_2}$$

Since M_A and M_C are zero

$$2M_B\,(3 + 4) = \frac{-6 \times 8 \times 1.5}{3} - \frac{6 \times 6 \times 2}{4}$$

$$14\,M_B = -42 \quad \text{or,} \quad M_B = \frac{-42}{14} = -3 \text{ KN-m}$$

Support reactions

Taking moments about B, of forces to the left of B.

$R_A \times 3 - 4 \times 2 - 4 \times 1 = -M_B = -3$

$3R_A = 12 - 3 = 9 \quad$ or, $\quad R_A = 3$ KN

Taking moments about B, of forces to the right of B.

$R_C \times 4 - 3 \times 2 = -3$

or, $\quad 4R_C = 3 \quad$ or, $\quad R_C = \frac{3}{4} = 0.75$KN

$R_A + R_B + R_C = 4 + 4 + 3$

or $\quad 3 + R_B + 0.75 = 11 \quad$ or, $\quad R_B = 11 - 3.75 = 7.25$ KN

B. M and *S. F* diagrams can now be drawn as usual.

Point of contraflexure.

Consider a section x-x at a distance x from A.

and equate M_{xx} to zero

$M_{xx} = R_A \times x - 4\,(x - 1) - 4\,(x - 2) = 0$

$3x - 4x + 4 - 4x + 8 = 0$

or, $\quad -5x = -12 \quad$ or, $\quad x = \frac{12}{5} = 2.4$ m from A

Similarly consider a section x_1-x_1 at x_1 from C in span BC and equate $M_{x_1x_1}$ to zero.

$M_{x_1x_1} = R_C \cdot x_1 - 3\,(x_1 - 2) = 0$

or, $\quad 0.75.x_1 - 3x_1 + 6 = 0$

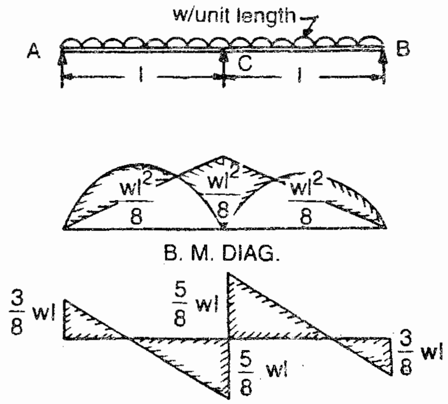
or, $\quad -2.25\,x_1 = -6$

or, $\quad x_1 = \frac{6}{2.25} = 2.66$ m from C.

Example 9.28

Use the three moments theorm to prove that in a beam uniformly loaded and supported at its two extremeties and continuous over the intermediate pier at its centre at the same level as the other two supports. The load taken by the pier is $\frac{5}{8}$ th of the total load on the beam.

(Oxford Univ.)



S. F. Diagram
Fig. 9.45

Solution -

Applying three moments theorm on span *AB-BC*

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = -\frac{6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

End moments $M_A = M_c = 0$

$$\therefore \; 2M_B (l + l) = -\frac{wl^3}{4} - \frac{wl^3}{4}$$

$$4l \times M_B = -\frac{wl^3}{2} \quad \text{or,} \quad M_B = -\frac{wl^2}{8}$$

Taking moments about *B*.

$$R_A \times l - w \times l \times \frac{l}{2} = -\frac{wl^2}{8}$$

$$R_A = \frac{\frac{-wl^2}{8} + \frac{wl^2}{2}}{l} \qquad \therefore \; R_A = R_C = \frac{3}{8} wl$$

Now $R_A + R_B + R_C = (wl + wl) = 2wl$

$$\frac{3}{8} wl + R_B + \frac{3}{8} wl = 2wl$$

$$R_B = 2wl - \frac{3}{4}wl = \frac{(8-3)}{4}\ wl\ = \frac{5}{4}wl$$

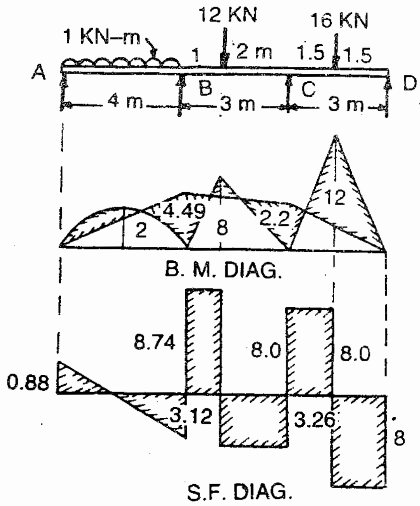
or, $\quad R_B = \dfrac{5}{4}wl \quad$ or $\quad \dfrac{5}{8}$ [Total load $2wl$]

$$= \frac{5}{8}(2wl)$$

B. M. and *S.F.* diagrams have been drawn above.

Example 9.29

A beam ABCD 10 metres long covers three spans of 4m, 3m and 3m, the supports being at the same level. On span AB there is a u.d.l of 1KN/m. On span BC a point of 12KN load acts at 1 m from B and a point load of 16KN at the mid span on span CD. Calculate the moments and reactions at the supports and draw the B.M and S.F. diagrams



S. F. Diagram. Fig. 9.46

Solution -

$M_A = M_D = 0$

Free moment ordinates for

$$\text{Span } AB,\ M_{max} = \frac{wl^2}{8} = \frac{1\times 4^2}{8} = 2\text{KN-m}$$

$$\text{Span } B_C,\ M_{max} = \frac{Wab}{l} = \frac{12\times 1\times 2}{3}\ \ 8 \text{ KN-m}$$

$$\text{Span } C_D,\ M_{max} = \frac{Wl}{4} = \frac{16\times 3}{4}$$

$$= 12 \text{ KN-m}$$

Now Applying three moments theorem on spans AB and BC

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C . l_2 = -\frac{6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

$$4M_A + 2M_B (4 + 3) + 3M_C = \frac{wl^3}{4} - \frac{W_2 a_2}{l_2} (l_2^2 - a_2^2)$$

$$14M_B + 3M_C = -1 \times \frac{(4)^3}{4} - \frac{12 \times 2}{3} (3^2 - 2^2)$$

$$14M_B + 3M_C = -16 - 40 = -56 \quad --- \quad \text{(i)}$$

Now applying three - moments theorem on spans BC and CD

$$M_B l_2 - 2M_C (l_2 + l_3) + M_D l_3 = -\frac{W_2 a_2}{l_2} (l_2^2 - a_2^2) - \frac{3}{8} \text{ W3 } l_3^2$$

$$3M_B + 2M_C (3 + 3) + 3M_D = -\frac{12 \times 1}{3} (3^2 - 1^2) - \frac{3}{8} .16. (3)^2$$

$$3M_B + 12M_C = -32 - 54 = -86 \quad --- \quad \text{(ii)}$$

Solving (i) and (ii) we get

$$M_B = -4.49 \text{ and } M_C = -2.286$$

Support reactions

Taking moments about B of all forces to the left of B

$$R_A \times 4 - 1 \times 4 \times \frac{4}{2} = M_B = -4.49$$

$$R_A = .88$$

Taking moments about C of all forces to the left of C

$$R_A (4 + 3) + R_B (3) - 1 \times 4 \left(\frac{4}{2} + 3\right) - 12 \times 2 = M_C = -2.28$$

or $R_B = 11.86$

Taking moments about C of all forces to the right of C

$$R_D \times 8 = 16 \times 1.5$$

$$R_D = 8$$

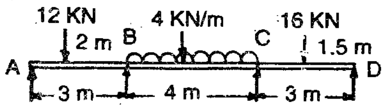
Now $R_A + R_B + R_C + R_D = 4 + 12 + 16$

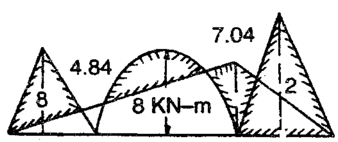
$.88 + 11.86 + R_C + 8 = 32$

$$R_C = 11.26$$

Example 9.30

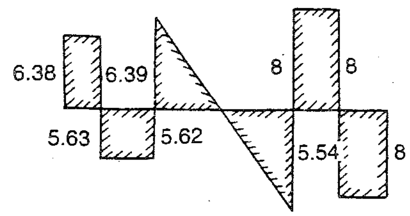
A continuous beam ABC is shown in fig.9.47 Draw the B.M. and S.F. diagrams





B.M. Diagram

B. M. DIAG.



S.F. Diagram Fig. 9.47

Solution

End moments $M_A = M_D = 0$

Free moment ordinates for

Span AB, $M_{max} = \dfrac{Wab}{l} = \dfrac{12 \times 1 \times 2}{3}$ 8 KN -m

Span BC, $M_{max} = \dfrac{Wl^2}{8} = \dfrac{4 \times (4)^2}{8} =$ 8 K N -m

Span CD, $M_{max} = \dfrac{Wl}{4} = \dfrac{16 \times 3}{4}$ = 12 KN -m

Applying three moments theorem on spans AB and BC

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C . l_2 = -\frac{6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

$$3M_A + 2M_B (3 + 4) + 4M_C = -\frac{W_1 a_1}{l_1} - (l_1{}^2 - a_1{}^2) - \frac{w_2 l_2^3}{4}$$

$$3M_A + 14M_B + 4M_C = -\frac{12 \times 1}{3} - (3^2 - 1^2) - 4 \times \frac{(4)^3}{4}$$

$$= -32 - 64 = -96$$

or $14M_B + 4M_C = -96$ --- --- --- (i)

Now applying three moments theorem on spans BC and CD

$$M_B l_2 + 2M_C (l_2 + l_3) + M_D . l_3 = -\frac{6A_2 x_2}{l_2} - \frac{6A_3 x_3}{l_3}$$

$$4M_B + 2M_C (4 + 3) + 3M_D = -\frac{w_2 l_2^3}{4} - \frac{3}{8} W_3 l_3^2$$

or $4M_B + 14M_C = \dfrac{-4\,(4)^3}{4} - \dfrac{3}{8} \times 16 \times (3)^2$

$4M_B + 14M_C = -64 - 54 = -118$ --- --- --- (ii)

Solving equations (i) and (ii)

$M_B = -4.84$ KN -m and $M_C = -7.04$

Support Reactions

Taking moment about B offorces to the left of B

$$R_A \times 3 - 12 \times 2 = M_B = -4.84 \quad \text{or} \quad R_A = 6.38\text{KN}$$

Taking moments about C of all forces to the left of C

$$R_A\,(3+4) + R_B \times 4 - 12 \times 6 - 4 \times 4 \times \frac{4}{2} = M_C = -7.04$$

$$7R_A + 4R_B - 72 - 32 = -7.04$$

$$4R_B = -7.04 + 72 + 32 - 7 \times R_A$$

$$= -7.04 + 104 - 7 \times 6.38$$

$$= 104 - 51.7 = 52.3$$

$$R_B = 13.05 \text{ KN}$$

Taking moment about C of all forces to the right C of

$$R_D \times 3 = 16 \times 1.5$$

$$R_D = 8 \text{ KN}$$

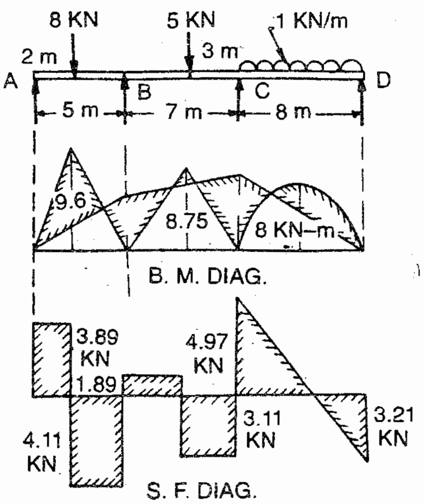
Now $R_A + R_B + R_C + R_D = 12 + 16 + 16 = 44$

$$6.38 + 13.075 + R_C + 8 = 44$$

$$R_C = 16.545 \text{ KN}$$

Example 9.31

A continuous beam ABCD, 20m long rests on supports at its e... ud is propped at the same level at 5m and 12m from left end A . It carri... two point loads of 8 KN and 5 KN at a distance of 2m and 9m respectively ... m end. A and a u. d. l. of 1 KN/m over the span CD. Draw the B. M. and F. diagrams. (J.M.I.)



S. F. Diagram
Fig. 9.48

Solution

$M_A = M_D = 0$

Span AB, Free moment

$$M_{max} = \frac{Wab}{l} = \frac{8 \times 2 \times 3}{5} = 9.6 \text{ KN-m}$$

Span BC.

$$M_{max} = \frac{Wab}{l} = \frac{5 \times 4 \times 3}{7} = 8.57 \text{ KN}$$

$$\text{Span } CD,\ M_{max} = \frac{wl^2}{8} = \frac{1 \times (8)^2}{8} = 8 \text{ KN-m}$$

Applying three moment theorm on span AB and BC

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2$$

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = -\frac{W_1 a_1}{l_1}(l_1^2 - a_1^2) - \frac{W_2 a_2}{l_2}(l_2^2 - a_2^2)$$

$$\text{or,}\quad 2MB\ (5 + 7) + MC \times 7 = \frac{-8 \times 2}{5}(5^2 - 2^2) - \frac{5 \times 3}{7}(7^2 - 3^2)$$

$$24 M_B + 7M_C = \frac{-16}{5} \times 21 \frac{-15}{7} \times 40$$

$$24 M_B + 7M_C = -67.3 - 85.7 = -153 \qquad \text{---} \quad \text{---} \quad \text{---} \qquad (i)$$

Again applying 3 - moment theorm on span BC and CD.

$$M_B \times l_2 + 2M_C (l_2 + l_3) + M_D l_3 = -\frac{W_2 a_2 (l_2^2 - a_2^2)}{l_2} - \frac{W l_3^3}{4}$$

$$7M_B + 2M_C (7 + 8) + 0 = \frac{-5 \times (4)\ (7^2 - 4^2)}{7} - \frac{1 \times (8)^3}{4}$$

$$7M_B + 30 M_C + 0 = -94.3 - 128 = -222.3 \qquad \text{---} \quad \text{---} \quad \text{---} \qquad (ii)$$

Solving equations (i) and (ii) we get

$M_B = 4.54$ KN -m

$M_C = 6.35$ KN -m

Support Reactions

Taking moments about B

$R_A \times 5 - 8 \times 3 = -M_B = -4.54$

$5_A = 24 - 4.54 = 19.46$ or, $R_A = 3.89$ KN

Taking moments about C of forces to the left of C

$R_A (5 + 7) + R_B\ 7 - 8\ (3 + 7) - 5 \times 3 = -M_C = -6.35$

$3.89\ (12) + 7\ R_B - 180 - 15 = -6.35$

$7R_B = 180 + 15 - 6.35 - 46.68$

or, $R_B = 6.0$ KN

Again taking moment about C of forces to the right of C

$$R_D \times 8 - 1 \times 8 \times \frac{8}{2} = -6.35$$

or, $8R_D = -6.35 + 32$ or, $R_D = 3.21$ KN

Now,

$$R_A + R_B + R_C + R_D = 8 + 5 + 8 = 21$$

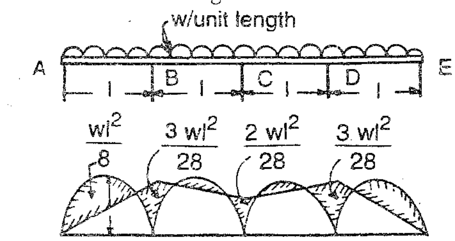
or, $3.89 + 6.0 + R_C + 3.21 = 21$

or, $R_C = 21 - 13.10 = 7.90$ KN

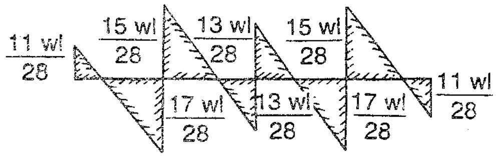
The *B.M* and *S. F* diagrams are shown in figure 9.48.

Example 9.32

A continuous beam of four equal spans l each carries a uniformly distributed load of w per unit length on all the spans. Determine the moments at the supports and draw the bending moment and shear force diagram. The beam has a constant section throughout.



B.M. Diagram



S.F. Diagram Fig. 9.49

From the symmetry of loading and spans

$$M_B = M_D \text{ and end moments } M_A = M_E = 0$$

Applying 3 - moments theorm on span AB and BC.

$$M_A\, l + 2M_B\,(l + l) + M_C l = \frac{-6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

$$4M_B.l + M_C l = \frac{-wl^3}{4} - \frac{wl^3}{4}$$

$$4M_B + M_C = -\frac{wl^2}{2} \qquad \text{---} \quad \text{(i)}$$

Applying three moments theorm on span BC and CD.

$$M_B.l + 2M_C\,(l + l) + M_D.l = \frac{-wl^3}{4} - \frac{wl^3}{4}$$

or, $$2M_B + 4M_C = -\frac{wl^2}{2} \qquad \text{---} \quad \text{(ii)}$$

Solving equations (i) and (ii)

$$M_B = \frac{-3}{28} wl^2 \qquad \text{and} \qquad M_C = \frac{-2wl^2}{28}$$

Since $\quad M_A = M_E = 0$

$$M_B = M_D \doteq \frac{-3}{28} wl^2 \qquad \text{and} \qquad M_C = \frac{-2wl^2}{28}$$

Support Reaction.

Taking moments about B

$$R_A \times l - \frac{wl^2}{2} = -M_B \qquad \text{or,} \qquad R_A = \frac{11wl}{28} = R_E$$

Taking moment about C

$$R_A \times 2l + R_B \times l - 2wl.\, l = \frac{-2wl^2}{28}$$

$$R_B = \frac{32wl}{28}$$

Now $R_A + R_B + R_C + R_D + R_E = 4wl$

$$\frac{11wl}{28} + \frac{32}{28} wl + \frac{32wl}{28} + R_C + \frac{11wl}{28} Wl = 4wl$$

or $R_C = \dfrac{112wl - 86wl}{28} = \dfrac{26wl}{28}$

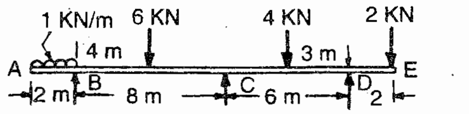
The B.M and S.F. diagrams can now be drawn as shown in fig. 9.49

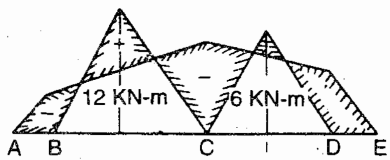
Beams with overhanging ends

In continuous beams with overhangs on one side or on both sides, the overhang portions are treated as cantilevers. Three moments theorem is applied on the rest of the protions to determine support moments.

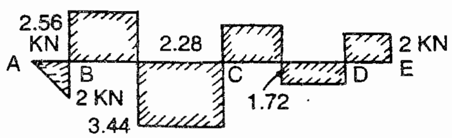
Example 9.33

Draw the bending moment and shear force diagrams for the beam shown in figure. 9.50





B. M. Diagram



S. F. Diagram Fig. 9.50

Solution.

Free moments

Span AB, $M_B = -\frac{wl^2}{2} = \frac{1 \times (2)^2}{2} = -2$ KN-m

Span BC, $M_{max} = \frac{Wl}{4} = \frac{6 \times 8}{4} = 12$ KN-m

Span CD $M_{max} = \frac{Wl}{4} = \frac{4 \times 6}{4} = 6$ KN-m

Moment at D, $M_D = 2 \times 2 = -4$ KN-m

Applying three - moments theorem on spans BC and CD

$$M_B\, l_2 + 2M_C\,(l_2 + l_3) + M_D\, l_3 = \frac{-6A_2\, x_2}{l_2} - \frac{6A_3\, x_3}{l_3}$$

$$-2 \times 8 + 2M_C\,(8+6) + (-4)\,6 = \frac{-3}{8} W_2 l_2^2 - \frac{3}{8} W_3 l_3^2$$

$$-16 + 28\, M_C - 24 = \frac{-3}{8} \times 6\,(8)^2 \; \frac{-3}{8}\; 4\,(6)^2$$

$$28\, M_C - 40 = -144 - 54 = -198$$

$$28\, M_C = -198 + 40 = -158$$

$$M_C = \frac{-158}{28} = -5.64 \text{ KN-m}$$

End moments $M_A = M_E = 0$

$\therefore$ $M_A = 0$, $M_B = -2$ KN-m, $M_C = -5.64$, $M_D = -4$, $M_E = 0$

Bending moment diagram can now be drawn as shown in figure9.50.

Support Reactions

Taking moments about C.

$$R_B \times 8 - (1 \times 2)\left(\frac{2}{2} + 8\right) - 6 \times 4 = M_C = -5.64$$

$$8R_B - 18 - 24 = -5.64$$

$$8R_B = 18 + 24 - 5.64 = 42 - 5.64 = 36.36$$

$$R_B = \frac{36.36}{8} = 4.56 \text{ KN}$$

Taking moments about D

$$R_B \times (8+6) - 1 \times 2\left(\frac{2}{2} + 8 + 6\right) - 6\,(4+6) + R_C \times 6 - 4 \times 3 = M_D = -4$$

$$63.63 - 30 - 60 + 6\,R_C - 12 = -4$$

$$6\,R_C = 102 - 4 - 63.63 = 34.37$$

$$R_C = \frac{34.37}{6} = 5.72 = 5.72$$

Now $R_B + R_C + R_D = 2 + 6 + 4 + 2 = 14$

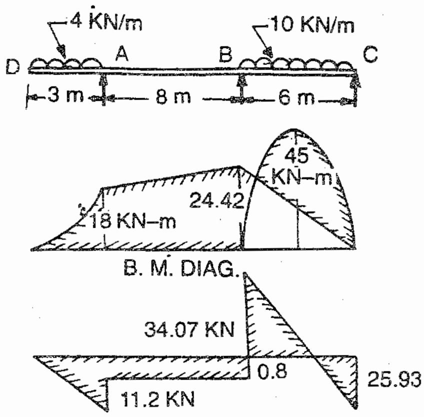
$$4.560 + 5.72 + R_D = 14$$

$$R_D = 3.72 \text{ KN}$$

Shear force diagram can now be drawn as shown in figure 9.50.

Example 34

Draw B.M and S.F. diagrams for the beam shown in fig 9.51.



S. F. Diagram Fig. 9.51

Solution

End moments $M_D = M_C = 0$

Free moment ordinates for

$$M_A = \frac{wl^2}{2} = \frac{4 \times (3)^2}{2} = 18 \text{ KN-m}$$

Span AB,

Free $B.M.$ diagram on span AB is a straight line

Span BC

$$M_{max} = \frac{wl^2}{8} = \frac{10(6)^2}{8} = 45 \text{ KN-m}$$

Applying three - moments theorem on spans AB and BC

$$M_A\, l_2 + 2M_B\,(l_2 + l_3) + M_C\, l_3 = -\frac{6A_2 x_2}{l_2} - \frac{6A_3\, x_3}{l_3}$$

$$M_A \times 8 + 2M_B\,(8+6) + M_C \times 6 = 0 \; \frac{-w_3\, l_3^3}{4}$$

$$18 \times 8 + 28\, M_B + 0 = \frac{-10(6)^3}{4}$$

$$144 + 28\, M_B = -540$$

or $\quad M_B = 24.42$ KN-m

Support reactions

Taking moment about of B to the left of B

$$R_A \times 8 - 4\times 3\left(\frac{3}{2}+8\right) = M_B = -24.42$$

or $R_A = 11.19$ KN say 11.2

Taking moments about B of all forces to the right of B

$$R_C \times 6 - 10\times 6\times \frac{6}{2} = -24.42$$

$$R_C = 25.93 \text{ KN}$$

$R_A + R_B + R_C = 12 + 60 = 72$

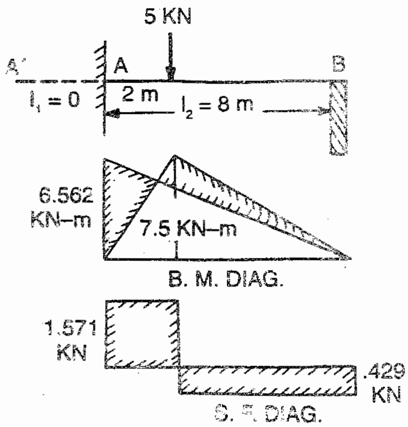
$11.19 + RB + 25.93 = 72$ or $R_B = 34.87$

Application of theorem of three moments to beams having fixed ends.

When a beam is fixed at one end and freely supported at the other, the theorem of three moments may be applied by imagining a zero span and moment of inertia α on the side of the fixed end.

Example 9.35

A rolled steel joist is firmly built-in at one end and rests freely on the top of a cast iron column. The span of the joist is 8 metres and it carries a point load of 5 KN at distance of 2 metres from the fixed end. Determine the reaction on the column and draw B.M. and S.F. diagrams.



S. F. Diagram
Fig. 9.52

Solution

Imagine a span AA' of length $l_1 = 0$ to the left of fixed end A. Now applying three moments theorm on span $A'A$ and AB

$$M_A'\,l_1 + 2M_A\,(l_1 + l_2) + M_B\,l_2 \;= \frac{-6A_1\,x_1}{l_1} - \frac{6A_2\,x_2}{l_2}$$

$$M_A' \times 0 + 2M_A\,(0 + 8) + 8\,M_B \;= \frac{-Wa_2}{l_2}\,(l_2^2 - a_2^2) - 0$$

Since end B is freely supported $M_B = 0$

$$\text{or} \quad 2M_A\,(8) = \frac{-5 \times 6}{8}\,(8^2 - 6^2) \;= \frac{-30}{8}\;28$$

$$\text{or,} \quad M_A \;= -\frac{30 \times 28}{8 \times 16} \;= -\,6.5625 \text{ KN-m}$$

Maximum central ordinate for the free moment diagram

$$= \frac{Wab}{l} = \frac{5 \times 2 \times 6}{8} = 7.5 \text{ KN-m}$$

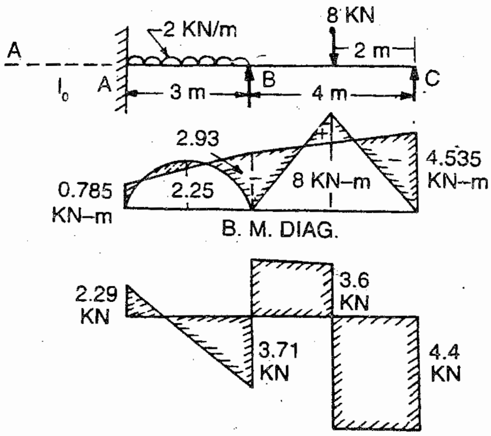
For support reactions, $R_B \times 8 - 5 \times 2 = -\,6.5625$

$$R_B = \frac{-6.5625 + 10}{8} = \frac{3.4375}{8} = .429 \text{ KN}, \quad R_A = 5 - .429 = 4.571 \text{ KN}$$

B.M. and *S.F.* diagrams have been drawn as shown in fig 9.52

Example 9.36

A cantilever ABC of uniform section 7 metres long, is fixed at A and freeely supported at B and C to the same level as the fixed end. The span AB is 3 metres and carries a udl of 2KN/m. Span BC is 4 metres long and carries a point load of 8 KN at its centre. Draw the B.M. and S.F. diagrams.



S. F. Diagram Fig. 9.53

Solution

Assume a span $A'A$ of length $l_1 = 0$ to the left of the fixed end A. Now applying 3 - moments theorm on span $A'A$ and AB.

$$M_A'\,l_1 + 2M_A\,(l_1 + l_2) + M_B l_2 \;= \frac{-\,wl^3}{4}$$

$$M_A' \times 0 + 2M_A(0+3) + 3M_B = \frac{-2(3)^3}{4}$$

$$6M_A + 3M_B = -13.5$$

Applying 3 - moments theorm on span AB and BC.

$$M_A \times 3 + 2M_B(3+4) + \quad M_C \times 4 = \frac{-wl^3}{4} - \frac{3}{8}wl^2$$

$$3M_A + 14M_B + 4M_C = \frac{-2(3)^3}{4} - \frac{3}{8} \times 8 \times (4)^2 = -13.5 - 48$$

Since $M_C = 0$

$$3M_A + 14M_B = 61.5 \qquad --- \qquad --- \qquad --- \qquad \text{(ii)}$$

Solving equation (i) and (ii) we get

$M_B = -4.38$ KN-m $\qquad M_A = -.06$ KN-m

Support reactions

Taking moments about B of forces to the left of B

$$R_A \times 3 - 2 \times 3 \times \frac{3}{2} = -4.38 \quad \text{or,} \quad R_C = 1.56 \text{ KN}$$

Taking moments about B of forces to the right of B.

$$R_C \times 4 - 8 \times 2 = -4.38 \quad \text{or,} \quad R_C = 2.9 \text{ KN}$$

Now $R_A + R_B + R_C = 2 \times 3 + 8 = 14$KN

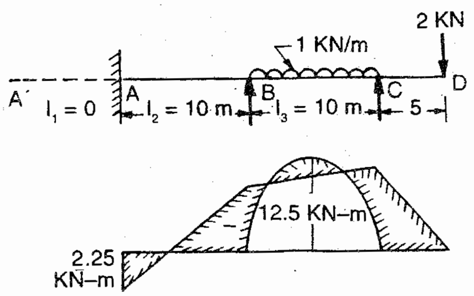
$$1.56 + R_B + 2.9 = 14$$

$$R_B = 14 - 1.56 - 2.9 = 9.54 \text{ KN}$$

The *B.M.* and *S.F.* diagrams are shown in figure. 9 .53

Example 9.37

A cantilever ABCD of uniform section 25m long is encastred at A and supported at B and C all supports being at the same level. Spans AB and BC are 10 metres each and beam overhangs C by 5 metres and supports a load of 2KN at the free end. A uniformly distributed load of 1KN/m acts on span BC. Calculate the support moments.



B.M. Diagram
Fig. 9.54

Solution

Imagine a span $A'A$ of length $l_1 = 0$ to the left of the fixed end A.

Applying three moments theorms on span $A'A$ and AB.

$$M_{A'}\, l_1 + 2M_A\,(l_1 + l_2) + M_B l_2 = \frac{-6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

$$M_{A'} \times 0 + 2M_A\,(0 + 10) + M_B \times 10 = 0$$

$$\text{or,}\quad 20M_A + 10M_B = 0 \quad \text{---} \quad \text{---} \qquad \text{(i)}$$

Applying 3 - moments theorm on span AB and BC

$$M_A \times 10 + 2M_B\,(10 + 10) + M_C \times 10 = \frac{-6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

$$= 0 - \frac{1}{4} wl^3$$

$$10\,M_A + 40\,M_B + 10M_C = \frac{1}{4} \times (1) \times (10)^3 = \frac{1000}{4} = 250$$

Moment at $C = 2 \times 5 = -10$ KN-m

$\therefore\ 10\,M_A + 40\,M_B - 10 \times 10 = -250$

$10\,M_A + 40\,M_B = -250 + 100 = -150$

$$\text{or,}\quad M_A + 4\,M_B = -15 \quad \text{---} \quad \text{---} \quad \text{---} \qquad \text{(ii)}$$

Solving (i) and (ii) we get

$M_A = +2.15$ KN-m

$M_B = -4.30$ KN-m

SUMMARY

1. In case of propped cantilevers determine y_1 the down word deflection at the propped place. If the prop is rigid then equate it to y_2, the upward deflection caused by the prop reaction. This shall give the prop reaction.
2. A cantilever with a point W at mid span and supported on a rigid prop at the free end

   Prop raction $R = \dfrac{5}{16} W$
3. u.d.l. on the entire span of the cantilever and propped at the free end

   Prop reaction R $= \dfrac{3}{8} Wl.$

   $$y_{max} = \frac{0.005415\, Wl^4}{EI}$$
4. Fixed beam with a point load at mid span

   $$M_A = M_B = \frac{-Wl}{8}$$

5. Fixed beam with a u.d.l. over entire span

$$M_A = M_B = \frac{-wl^2}{12}$$

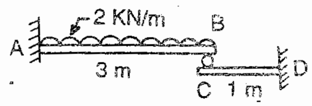


Fig. 9.55

$$y_{max} = -\frac{wl^4}{384EI}$$

6. Fixed beam with a point load not at the mid span

$$M_A = -\frac{Wab^2}{l^2} \quad \text{and} \quad M_B = -\frac{Wa^2b}{l^2}$$

$$y_C = \frac{Wa^3b^3}{3l^3EI}$$

7. Three moment theorem on span AB and BC of a continuous beam

$$\frac{M_A l_1}{E_1 I_1} + 2M_B\left(\frac{l_1}{E_1 I_1} + \frac{l_2}{E_2 I_2}\right) + M_C \frac{l_2}{E_2 I_2} = \frac{-6A_1 x_1}{E_1 l_1 I_1} - \frac{6A_2 x_2}{E_2 l_2 I_2}$$

When both the spans are of same material and Cross-Section then $E_1 = E_2 = E$ and $I_1 = I_2 = I$, the theorem may be written in a simplified form as

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = \frac{-6A_1 x_1}{l_1} - \frac{6A_2 x_2}{l_2}$$

8. When a Continuous beam is fixed at its one or both ends, then an imaginary span is taken and then three moment theorem is applied considering the zero span as the first span of the beam.

EXERCISES

1. A cantilever 8 metres long carries a uniformly distributed load of 12 KN per metre run over the entire span. A rigid prop is provided at 6 metres from the fixed end level with the support. Calculate the reaction at the prop. (56.8 KN)
2. A cantilever of span 6 metres carries at concentrated load of 20 KN at the free end. It is propped at a distance of 1.5 metres from the free end. Determine the prop reaction. (30 KN)
3. A timber cantilever of length L is propped at its free end. The cantilever carries a uniformly distributed load of w perunit length over the whole span. If the prop sinks by an amount δ, find the reaction at the prop.

$$R = \frac{3\,EI}{L^3}\left(\frac{Wl^4}{8EI} - \delta\right)$$

4. A cantilever 4 metres long is propped at its free end. It carries a u. d.l. of 6 KN/metre over the whole length. Find by how much above the level of the fixed end the level of the prop must be fixed so that the load may be equally shared by the supports. (10 mm)
5. A cantilever AB 3 metres long carries a u.d.l of 12 KN/m rests on an other cantilever CD of 1 metre span as shown in figure 9.55 calculate the reaction at C (13.01 KN)

6. A cantilever is propped at a distance L from the fixed end and carries a uniformly distributed load w KN/m run. The cantilever projects a distance of $\frac{L}{4}$ beyond the prop and on this length there is a uniformly distnibuted load of 2 w KN/m run. If the prop is rigid and holds its point of application on the horizonal, find what proportion of the total load W is taken by the prop.

$$\left(\frac{31}{48}w\right)$$

7. A fixed beam of span 4 metres carries a point load of 12 KN at mid span. Determine the support moments at the fixed ends. Also calculate the maximum deflection.
$I = 20\times 10^4$ mm^4 and $E = 210$ KN/mm^2 ($M_A = M_B = -6$ KN-m, $y_C = 15.87$ mm)
8. An encastre beam AB of span 3 metres carries a uniformly distributed load of 4 KN/m over its entire span and a concentrated load of 10 KN at its centre. Calculate the fixing moments at A and B and draw the S.F. and bending moment diagrams. ($M_A = M_B = -6.75$ KN-m)
9. A built in beam of span 6 mètres supports a concentrated load of 10 KN at 1.5 metres from the right hand support. Determine the fixed end moments and the reactions at the supports. Also calculate the position of the points of contraflexure.
$M_A = -8.4375$ KN-m, $R_A = 1.5625$ KN, $x_1 = 1.8$ m from A
$M_B = -2.812$ KN-m, $R_B = 8.4375$ KN, $x_2 = 1$m from B
10. A fixed beam AB of span 4 metres carries two concentrated loads of 4 KN each at a distance of one metre from the fixed ends. Calculate the fixing moments and the points of contraflexure.
($M_A = M_B = -3$ KN-m and $x = 0.75$ m from either end.)
11. A built in beam of span 7 metres carries a uniformly distributed load of 1.5 KN/m run over the left half of the span. Calculate the support moments and the reactions at the supports ($M_A = -4.20$ KN-m, $M_B = 1.93$ KN-m
$R_A = -3.722$ KN and $R_B = 0.988$ KN)
12. An encastre beam AB of span 6 metres carries a uniformly varying load whose intensity varies from zero at A to 10 KN/m at the fixed end B. Find the fixed end moments at A and B.
($M_A = -12$ KN-m and $M_B = -18$ KN-m)
13. A fixed beam AB 4 metres long supports a uniformly varying load whose intensity varies from zero at fixed ends A and B to a maximum of 10 KN/m run at the mid span C. Determine the fixed end moments at A and B.
($M_A = M_B = -8.33$ KN)
14. A beam AB of uniform section and span 6 metres is built-in at the ends. A uniformly distributed load of 3 KN/m runs over the left half of the span. It also supports a concentrated load of 4 KN at 1.5 metres from the other end. Determine th fixed end moments at A and B and the support reactions at the two ends. Draw the shearing force and bending moment diagrams for the beam.
($M_A = -7.3$ KN-m, $M_B = -6.2$ KN-m
$R_A = 7.93$ KN, $R_B = 5.07$ KN)
15. A beam of span 6 metres is fixed at both ends. When a uniformly distributed load of 2 KN/m is placed on the beam, the level of right hand support sinks 10 mm below that of the left hand one. Find The support moments. Take $E = 200$ KN/mm^2 and $I = 90\times 10^6$ mm^4. ($M_A = -36$ KN-m and $M_B = +24$ KN-m)

16. A continuous beam 15 metre long is supported at A,B and C, the supports being on the same level span AB is 8 metres long and carries a u.d.l of 1.5 KN/m and the rate of loading on the second span is 1 KN/m. Calculate the support moments. Draw the B.M and S .F. diagrams and locate the points of inflexion. The beam has uniform thickness throughout.

[M_B = 9.26 KN-m, x = 6.45 from A]
x = 4.36 m from C

17 A beam ABC 30 metres long is fixed in a wall at A and simply supported at B and C. AB = 18 m carries a point load of 6 KN at 12 m from A and BC = 12 m carries a poin t load of 4 KN at 24 m from A. Draw the B.M. and S. F. diagrams. Take moment of inertia of AB twice that of BC. Alos locate the points of inflexion.

{M_A = 10.25 KN-m x = 5.31, 15.17 m (J.M.I)
M_B = – 11.5 KN-m and 21.89 m from A

18. Draw the B.M. and S.F. diagrams for the two span continuous beam shown in figure 9.56. The beam is simply supported at A and C and is continuous over support B (J.M.I.)

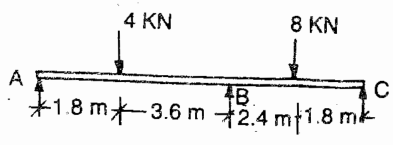


Fig. 9.56

19. A girder 15 m long carrying a uniformly distributed load of 6 KN/m covers three spans AB = CD = 4.5 m each and BC = 6 metres. Draw the B.M. diagrams and calculate the position of points of contraflexure.

(17.06 KN-m, 9.94 KN-m 3.24m and 5.67m from ends.

20. A continuous girder of 2 spans, 20 metres and 10 m has an overhang of 5m from the smaller span. It carries a u.d.l of 0.5 KN/m run and an isolated load of 1.5 KN at the free end. Find the support moments and draw the B.M. and S.F. diagrams. (– 17.5 KN-m and – 7.5 KN-m)

21. A continuous beam consists of two spans. The left span is twice as long as the second span. The beam is uniformly loaded from one end to the other. If the length of the beam is $3l$ and the weight per unit length is w, Find the reactions and support moments. ($M = \frac{3}{8}\ wl^2$

Reactions = $\frac{33}{16}\,wl,\ \frac{2}{16}\,wl,\ \frac{13}{16}\,wl,$)

22. A Continuous beam $ABCD$ is hinged at A and simply supported at B and C, all the points being at the same level. AB = 3m, BC = 4m and CD = 2m. The beam carries a u.d.l of 1.5 KN/m on the whole span and a point load of 10 KN at mid point of BC. Draw the B.M and S.F. diagrams.

23. A continuous beam $ABCD$ is supported at B and C and is fixed at D. A point load of 16 KN acts at A and a total. u.d.l of 10 KN on span CD. Assuming the beam being of uniform section and span AB = 10m, BC = 8 m and CD = 12m. Draw the B.M. and S.F. diagrams and locate the points of inflexion

M_A = 0, M_B = –16 KN-m, M_C = – 1.53 KN-m
M_D = – 14.24 KN-m, R_B = 17.81, R_C = 2.12
R_D = 6.66

□ □ □

10

Combined Direct And Bending Stresses

Structural members subjected to direct stresses and bending stresses separately have been discussed in previous chapters.

There are instances when a body is subjected both to direct and bending stresses simultaneously.

Eccentric Loading

A load whose line of action is parallel to vertical axis passing through the C. G. of the section is called eccentric load. Eccentric load induces both direct as well as bending stresses in the section . Hence at any point in the section of a body the cumulative effect of eccentric loading is the algebraic sum of the direct and bending stresses.

Dams, retaining walls, chimneys, hooks and certain machine parts have to with stand both direct and bending stresses. In this chapter you will analyse the stresses in these structures

Analysis of stresses due to eccentric loading on a Short Column.

Consider a short column subjected to a load W acting at a distance e from the vertical axis passing through the C. G. of the section. Now apply two equal and opposite forces each equal to W along the vertical axis. This will reduce the system to

(i) An axial force W and (ii) A couple $M = W.e$

A section which is at a distance y from the geometric axis will thus experience

(a) A direct stress $\sigma_d = \dfrac{W}{A}$, where A = area of x-section and (b) A bending stress $\sigma_b = \dfrac{M\,.\,y}{I}$

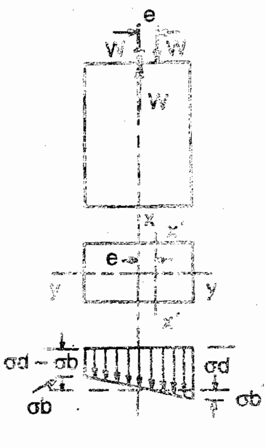


Fig. 10.1

Where I = Moment of inertia of the section

M = Bending Moment = $W.e$

Hence total stress at the point

$$= \sigma_d \pm \sigma_b$$

$$= \frac{W}{A} \pm \frac{M.y}{I} = \frac{W}{A} \pm \frac{M}{Z}$$

Where Z is the section modulus, the sign depending upon its position.

The maximum stress at a section will be

$$\sigma_{max} = \sigma_d + \sigma_b$$

and the minimum stress $\sigma_{min} = \sigma_d - \sigma_b$

The nature of the resultant stress σ will therefore depend on the nature and magnitude of direct stress σ_d and bending stress σ_b

(i) If $\sigma_b < \sigma_d$ the combined stress will be of the same sign

(ii)If $\sigma_b > \sigma_d$ the combined stress will change sign being partly compressive and partly tensile.

(iii) If $\sigma_b = \sigma_d$ the combined stress will be of the same sign.

The three possible distribution of stresses are shown in figure 10.2

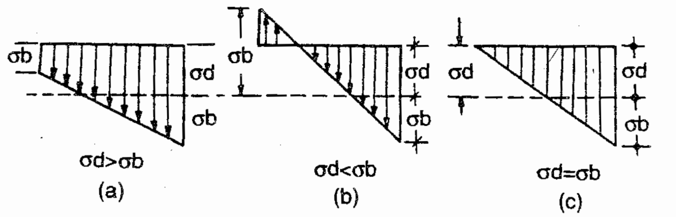


Fig. 10.2

Limit of eccentricity

The above diagrams are theoretical representations only. From practical considerations the stress should not be allowed to change its sign. Hence in no case the bending stress σ_b should be greater than the direct stress σ_d. At the most σ_b should be less or equal to σ_d. For the stress to be of the same sign.

$$\sigma_b \leq \sigma_d$$

or $$\frac{M}{Z} \leq \frac{W}{A}$$

or $$\frac{We.d}{2I} \leq \frac{W}{A}$$ (For symmetrical section $Z = I / \frac{d}{2}$

or $$\frac{w.e.d}{2AK^2} \leq \frac{W}{A}$$ (Where K is the radius of gyration of the section

$$\therefore \; e \leq \frac{2k^2}{d},$$ (Where d is the depth of the section.)

The above equation gives the limit of eccentricty.

Eccentric Limit for Various Sections

With the help of the above equation we can find out a certain region where we can apply a load and remain sure that stress will not change its sign.

(a) Rectangular section of breadth b and depth d.

$I = \frac{d\,b^3}{12}$, if the load line is in the vertical plane bisecting d, then

$$\sigma_d = \frac{W}{A} \text{ and } \sigma_b = \frac{M}{Z}$$

$$= \frac{W}{bd} \text{ and } \sigma_b = \frac{\frac{W.e}{\frac{db^2}{6}}}{} = \frac{6We}{db^2}$$

If $\quad \sigma_b < \sigma_d$

or $\quad \frac{6We}{db^2} < \frac{W}{b\,d}$

or $\quad e < \frac{1}{6} b$

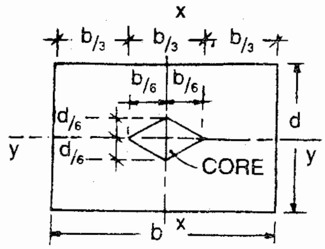


Fig. 10.3

Therefore with respect to centre the eccentric limit goes upto $\frac{b}{6}$ on either side along y--axis and $\frac{d}{6}$ on either side along x-axis. This creates a middle third region or zone in the form of a rhombus with diagonal equal to $\frac{b}{3}$ and $\frac{d}{3}$ on the respective principal axis. This rhombus is known as the "CORE" or KERNEL of the section.

(b) Circular Section

Let D be the diameter of a circular section. Let W be the force acting along the diameter $x-x$ at an eccentricity of e from the centre Fig 10.4

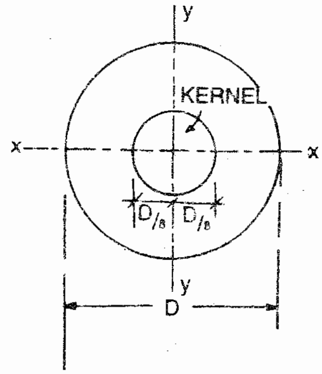


Fig. 10.4

Direct stress $\sigma_d = \frac{W}{A} = \frac{W}{\frac{\pi}{4}(D)^2}$

Bending Stress $\sigma_b = \frac{M}{Z} = \frac{W.e}{I/y}$

$$\sigma_b = \frac{W.e}{\frac{\pi}{32}D^3} = \frac{32We}{\pi D^3}$$

For no tension

$$\sigma_d = \sigma_b$$

$$\frac{4W}{\pi D^2} = \frac{32\,W\,e}{\pi D^3}$$

or $\quad e = \frac{D}{8}$

Load Eccentric to both Axes

Let the load W be at a distance e_x and e_y from the principal axes oy and $o\,x$ as shown in the figure 10.5

We may consider the eccentric load W to be equivalent of a central load W together with a bending moment $W.e_x$ about y axis and a bending

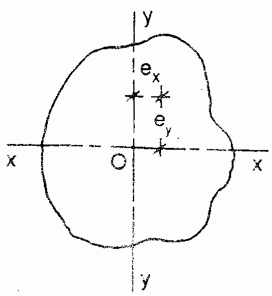


Fig. 10.5

moment $W.e_y$ about x-axis.

The stress at any point in the section defined by the Coordinates x , y is made up of three parts.

$$\sigma = \frac{W}{A} + \frac{W.e_y.x}{I_{y-y}} + \frac{W.e_x.y}{I_{x-x}}$$

Where x and y are to be reckoned positive when on the same side of their respective axis oy and ox as the load W

Therefore the maximum stress occurs at a point in the same quadrant as the load and the minimum stress in the opposite quadrant.

Example 10.1

A short column of solid circular section diameter D is to carry a vertical compressive load offset from the centre of the section. Deterine the maximum allowable offset if there is to be no tension induced in the column.

Solution

Let W be the Compressive load

Let A be the cross-sectional area

then $\sigma_d = \frac{W}{A} = \frac{W}{\frac{\pi}{4}D^2}$

Let e be the offset from the centre line of the column

Then bending moment at the column base $= M = W.e$

Section modulus $Z = \frac{I}{y} = \frac{\frac{\pi}{64}D^4}{D/2} = \frac{\pi}{32}D^3$

$\therefore \quad \sigma_b = \frac{M}{Z} = \frac{W.e}{\frac{\pi}{32}D^3}$

For no tension at the base

$$\sigma_d = \sigma_b$$

or $\frac{W.}{\frac{\pi}{4}D^2} = \frac{W.e}{\frac{\pi}{32}D^3}$

or $\frac{4W}{\pi D^2} = \frac{32We}{\pi D^3}$ or $1 = \frac{8e}{D}$

or $e = \frac{D}{8} = .125\,D$ **Answer**

Example 10.2

A short column of I-section is built-up of 200 × 20 mm flanges and 300 × 20 mm web plates. A vertical load of 600 KN is applied on the web at a distance of 90 mm from the centre. Calculate the maximum and minimum intensities of stresses developed in the section

Solution

Area of the section $A = 2 \times (200 \times 20) + (300 \times 20)$

$$A = 8000 + 6000 = 14000 \text{ mm}^2$$

Moment of inertia of the section

$$I_{xx} = \frac{200 \times (340)^3}{12} - \frac{130 \times (300)^3}{12}$$

$$= 25\,007 \times 10^4 \text{ mm}^4$$

$$\text{Direct stress } \sigma_d = \frac{600 \times 10^3}{14 \times 10^3} = 42.85 \text{ MPa (Comp.)}$$

Bending moment $= M = W \times e$

$$M = 600 \times 10^3 \times 90 = 54 \times 10^6 \text{ N–mm}$$

$$Z = \frac{I}{y}$$

$$= \frac{25007 \times 10^4}{100} = 25007 \times 10^2 \text{ mm}^3$$

$$\text{Bending stress } \sigma_b = \frac{M}{Z} = \frac{54 \times 10^6}{25007 \times 10^2} = 21.59 \text{ MPa}$$

$$\sigma_{man} = \sigma_d + \sigma_b = 42.85 + 21.59 = 64.44 \text{ MPa (Comp)}$$

$$\sigma_{min} = \sigma_d - \sigma_b = 42.85 - 21.59 = 21.26 \text{ MPa (Comp)}$$

Example 10.3

A hollow circular column has a projecting bracket on which a load of 30 KN rests The centre line of this load is 500 mm from the centre of the column. Determine the maximum and minimum stress intensities if the external diameter is 250 mm and internal diameter is 200 mm. *(J.M.I)*

Solution

Area of cross-section of the column

$$A = \frac{\pi}{4} D^2 - d^2)$$

$$= \frac{\pi}{4} (250^2 - 200^2)$$

$$= 176.78 \times 10^2 \text{ mm}^2$$

Moment of inertia along y-axis

$$I_{yy} = \frac{\pi}{64} (D^4 - d^4)$$

$$= \frac{\pi}{64} (250^4 - 200^4)$$

$$= 11325.33 \times 10^4 \text{ mm}^4$$

Section modulus $Z = \frac{I}{y}$

$$Z = \frac{11325.33 \times 10^4}{250/2} = 90.60 \times 10^4 \text{ mm}^3$$

Bending moment $M = We$

$$M = 30 \times 10^3 \times 500 = 15 \times 10^6 \text{ N-mm}$$

Direct stress $\sigma_d = \frac{W}{A} = \frac{30 \times 10^3}{176.78 \times 10^2} = 1.69$ MPa (Comp)

Bending stress $\sigma_b = \frac{M}{Z} = \frac{15 \times 10^6}{90.6 \times 10^4} = 16.55$ MPa

Maximum stresses

$$\sigma_{max} = \sigma_d + \sigma_b = 1.69 + 16.55 = 18.24 \text{ MPa (Comp)}$$

$$\sigma_{mim} = \sigma_d - \sigma_b = 1.69 - 16.55 = -14.86 \text{ MPa (Tensile)}$$

Example 10.4

A pillar 1000 mm × 600 mm in section carries an axial load of 250 KN. The maximum moment of inertia of the section is 224 × 10^6 mm^4 and the area is 123.6 × 10^2 mm^2. A bracket is bolted to the flange of the pillar and supports a vertical load of 60 KN which acts in the plane of the major axis of the section at a distance of 400 mm from the face of the flange. Calculate the maximum and minimum intensities of stress in the section

Solution

Bending moment due to eccentric loading

$$M = 60\ (500 + 400) = 54000 \text{ KN-mm} = 54 \times 10^6 \text{ N-mm}$$

Section modulus $Z = \frac{I}{y} = \frac{224 \times 10^6}{500} = 44.8 \times 10^4 \text{ mm}^3$

Resultant stress = Direct stress ± Bending stress

Direct stress $= \frac{W}{A} = \frac{(250 + 60) \times 10^3}{123.6 \times 10^2} = 25.08$ MPa

Bending stress $= \pm \frac{M}{Z} = \frac{54 \times 10^6}{44.8 \times 10^4} = 120.5$ MPa

$$\therefore\ \sigma_{max} = \frac{W}{A} + \frac{M}{Z} = 25.08 + 120.5$$

$$= 145.58 \text{ MPa (Comp)}$$

$$\sigma_{min} = 25.08 - 120.50$$

$$= -95.42 \text{ MPa (Tensile)}$$

Example 10.5

A short masonry pier 0.5 m × 1 metre in section is subjected to a compressive load of 600 KN at A and a bending moment of 40 KN-m causing tension above the section x-x Fig. 10.6. Determine the maximum and minimum stress intensities across the section.

Solution

Direct stress $= \dfrac{600}{0.5 \times 1} = 1200\ \text{KN/m}^2$

Bending moment M

$$M = (600 \times 0.5 - 40)$$
$$= 260\ \text{KN-m}$$

Section modulus

$$Z = \frac{bd^2}{6} = \frac{0.5 \times (1)^2}{6}$$

Bending stress $\sigma_b = \dfrac{260}{0.5/6}$

$$\sigma_b = 3120\ \text{KN/m}^2$$

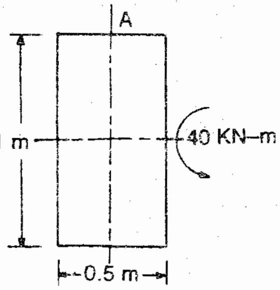


Fig. 10.6

Maximum stress

$$\sigma_{max} = \sigma_d + \sigma_b$$
$$= 1200 + 3120$$
$$= 4320\ \text{KN/m}^2$$

Minimum stress

$$\sigma_{mim} = 1200 - 3120$$
$$= -1920\ \text{KN/m}^2$$

Answer

Example 10.6

Determine the maximum tensile and compressive stresses on the section x-x of the clamp shown in fig 10.7. When a force of 2 KN is exerted by the screw. The section of the screw is 24 mm × 10 mm.

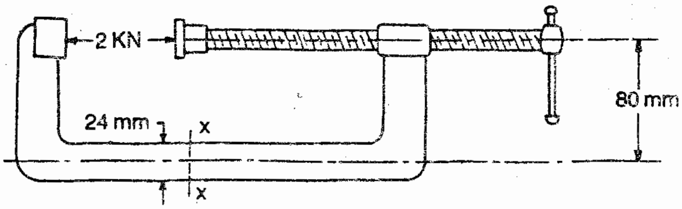


Fig. 10.7

Solution

The section $x-x$ is subjected to a tensile force of 2 KN and a bending moment of $2 \times 10^3 \times 80$ N–mm

Section area $= 24 \times 10 = 240\ \text{mm}^2$

Direct stress $\sigma_d = \dfrac{2 \times 10^3}{240} = 8.33\ \text{N/mm}^2 = 8.33\ \text{MPa}$

Maximum stress due to *B.M.*

$$\sigma_b = \frac{M}{z} = \frac{2 \times 10^3 \times 30}{960} = 166.66 \qquad z = \frac{bd^2}{6} = \frac{10(24)^2}{6} = 960$$

Maximum stress in the section

$$\sigma_{max} = \sigma_d + \sigma_b$$
$$= 8.33 + 166.66 = 172.99 \text{ MPa (tensile)}$$

Minimum stress in the section

$$\sigma_{min} = \sigma_d - \sigma_b$$
$$= 8.33 - 166.66 = -158.33 \text{ (Comp)}$$

Example 10.7

A bent up bar ABCD has a diameter of 120 mm. If a tensile load of 80 KN is applied at the free end of the bar as shown in figure 10.8. Determine the maximum and minimum stresses induced in the section of portion BC of the bar.

Solution

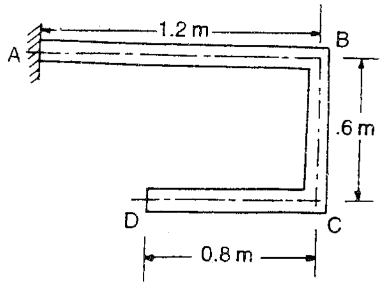


Fig. 10.8

Area of cross-section of the bar

$$= \frac{\pi}{4}(120)^2 = 3600\,\pi \text{ mm}^2$$

The portion BC of the bar will be subjected to a direct stress as well as bending stress due to the load of 80 KN

$$\text{Direct stress } \sigma_d = \frac{80 \times 1000}{3600\,\pi}$$
$$= 7.07 \text{ MPa}$$

Bending moment

$$M = (80 \times 1000)\,800$$
$$= 64 \times 10^6 \text{ N-mm}$$

$$\text{Section modulus } Z = \frac{\pi}{32}(120)^3 = 169.64 \times 10^3 \text{ mm}^3$$

$$\text{Bending stress } \sigma_b = +\frac{M}{Z} = \frac{64 \times 10^6}{169.64 \times 10^3} = 377.2 \text{ MPa}$$

Maximum stress

$$\sigma_{max} = 7.07 + 377.2 = 384.27 \text{ MPa Tensile}$$
$$\sigma_{min} = 7.07 - 377.2 = -370.13 \text{ MPa Tensile}$$

Example 10.8

A bar of rectangular section 60 mm × 40 mm is subjected to an axial compressive load of 70 KN. By how much can the width of the section be reduced by removing material from one edge only if there is to be no tensile stress in the bar and the axis of the bar is exchanged ? For this condition calculate the maximum compressive stress in the bar.

Solution

Suppose a portion of thickness t mm be removed from the width of the bar as shown in figure. 10.9

The applied load will now act at $\frac{t}{2}$ mm from the vertical centre line of the remaining section.

In the limiting case of zero resultant stress at the right hand edge, the eccentricity will be $\frac{1}{6}^{th}$ of the new width of the section as per the middle third rule

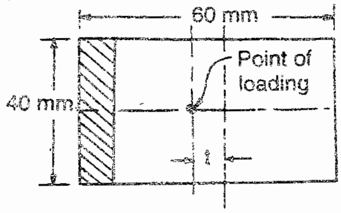


Fig. 10.9

$$\therefore \quad \frac{t}{2} = \frac{1}{6}(60 - t) \text{ or } \frac{t}{2} + \frac{t}{6} = \frac{60}{6}$$

or $t = 15$ mm

$$\text{Hence direct stress} = \frac{\text{Load}}{\text{Area}} = \frac{70 \times 10^3}{(60 - 15) \times 40} = 38.3 \text{ MPa}$$

Answer

Example 10.9

A clamp is shown in figure 10.10 Determine the thickness of the section at x–x if the pressure exerted by the screw is 4 KN and the maximum permissible stress is not to exceed 160 MPa.

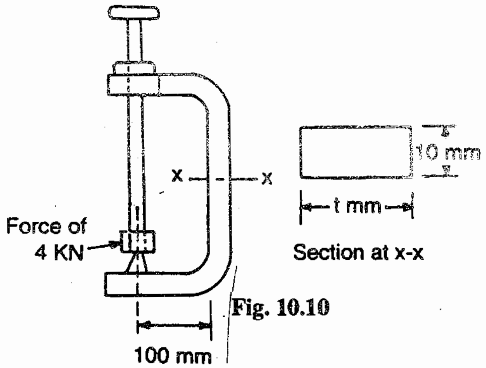


Fig. 10.10

Solution

Let t be the thickness of the section at $x - x$

$$\text{Direct stress } \sigma_d = \frac{4 \times 10^3}{t \times 10} = \frac{400}{t} \text{ N/mm}^2$$

$$\text{Moment of inertia of the section } I = \frac{1}{12} \times 10\, t^3$$

$$\text{Section modulus Z} = \frac{I}{y} = \frac{1}{12}\frac{10\, t^3}{t/2} = \frac{10}{6}\, t^2$$

$$\text{Bending moment} = 4 \times 10^3 \times 100 = 4 \times 10^5 \text{ N–mm}$$

$$\text{Bending stress } \sigma_b = \pm\frac{M}{z} = \frac{4 \times 10^5}{\frac{10}{6} t^2} = \frac{240000}{t^2}$$

Now Permissible stress = 160 MPa

$$\therefore \quad 160 = \sigma_d + \sigma_b$$

$$160 = \frac{400}{t} + \frac{240000}{t^2} \quad \text{or} \quad 1 = \frac{2.5}{t} + \frac{1500}{t^2}$$

or $\quad t^2 - 2.5\,t - 1500 = 0 \quad$ or $\quad t = \dfrac{+\,2.5 \pm \sqrt{(2.5)^2 - 4\,(-1500)}}{2}$

or $\quad t = \dfrac{+\,2.5 + \sqrt{6.25 + 6000}}{2} \quad$ or $t = \dfrac{2.5 + 77.2}{2} = 40$ mm

Walls And Chimneys Subjected To Wind Pressure.

Wind pressure on walls and chimney cause bending moment. at the base of these structures. Therefore at any point in the base, stress induced will be the sum of (i) direct stress induced due to self weight and (ii) Bending stress induced due to wind pressure.

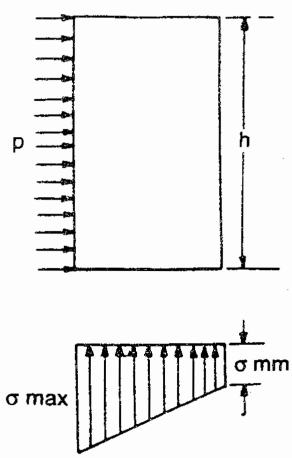


Fig. 10.11

Let W be the self Wt. of the wall

A = Area of cross-section at the base.

h = height of the wall

ρ = density of masonry

then

$$W = \rho . A . h$$

and Direct stress $\sigma_d = \dfrac{W}{A} = \dfrac{\rho . A . h}{A} = \rho . h$

Let p = intensity of wind pressure

Let P = total horizontal force on the area exposed to wind.

Bending moment at the base $M = P\,\dfrac{h}{2}$

Bending stress $\sigma b = \pm\,\dfrac{M}{Z}$

$$\sigma_{man} = \sigma_d + \sigma_b$$
$$\sigma_{min} = \sigma_d - \sigma_b$$

In case of circular sections the total horizontal wind thrust $P = c.p.$Area exposed to wind, where C = Coef ficient of wind resistance C = 0.66

Example 10.10

A masonry wall is 6 metres high and 1.5 metre thick and 4 metres wide. It is subjected to a wind pressure of 1.5 KN/m^2 acting on the 4 metres side. Determine the maximum and minimum stress intensities at the base of the wall. Masonry weighs 20 KN/m^3.

Solution

Self weight of the wall

$= $ Volume $\times$ density

$= 6 \times 1.5 \times 4 \times 20 = 720$ KN

$$\text{Direct stress} = \frac{\text{Wt. of the wall}}{\text{Area of cross-section of the wall}}$$

$$\sigma_d = \frac{720}{4 \times 1.5} = 120 \text{ KN/m}^2$$

Total horizontal thrust due to wind $= p.\, h \times L$

$$P = 1.5 \times 6 \times 4 = 36 \text{ KN}$$

$$M = P \times \frac{h}{2} = 36 \times \frac{6}{2} = 108 \text{ KN-m}$$

$$\text{Section Modulus} = Z = \frac{I}{y} = \frac{\frac{1}{12}(4)(1.5)^3}{1.5/2} = 1.5$$

$$\therefore \sigma_b = \frac{M}{Z} = \frac{108}{1.5} = 72 \text{ KN/m}^2$$

$$\sigma_{max} = \sigma_d + \sigma_b = 120 + 72 = 192 \text{ KN/m}^2 \quad \text{(Comp)}$$

$$\sigma_{min} = \sigma_d - \sigma_b = 120 - 72 = 48 \text{ KN/m}^2 \quad \text{(Comp)}$$

Example 10.11

A masonry climney 25 metres high is of uniform circular section 5 metres external diameter and 0.5 m thickness throughout. The chimney has to with stand a horizontal wind pressure of 2.5 KN/m² on projected area. Determine the maximum and minimum stress intensities at the base if the masonry weighs 20 KN/m³.

Solution

Direct stress at the base

$$\sigma_d = \rho.h$$

$$= 20 \times 25 = 500 \text{ KN/m}^2$$

Total wind pressure.

$$P = p.d.h$$

$$= (2.5)(5) \times 25$$

$$= 312.5 \text{ KN}$$

$$M = \frac{P\,h}{2}$$

$$= \frac{312.5 \times 25}{2} = 3906.25 \text{ KN-m}$$

$$\text{Section modulus} = \frac{\pi}{32}\frac{(D^4 - d^4)}{D}$$

$$Z = \frac{\pi}{32}\frac{[5^4 - 4^4]}{5} = 7.245 \text{ m}^3$$

$$\text{Bending stress } \sigma_b = \pm\frac{M}{Z} = \frac{3906.25}{7.245} = 539.16 \text{ KN/m}^2$$

$$\sigma_{max} = \sigma_d + \sigma_b$$

$$= 500 + 539.16 = 10.39.16 \text{ KN/m}^2$$

$$\sigma_{min} = 500 - 639.16 = -39.16 \text{ KN/m}^2$$ **Answer.**

Example 10.12

A hollow masonry chimney of square section 2.5 m × 2.5 m has an opening 2 m × 2 m. It has to withstand a uniform wind pressure of 2 KN/m². Determine the height of chimney if no tension is allowed to develop at the base. Take weight of masonry = 20 KN/m³.

Solution

Direct stress $\sigma_d = \rho . h = 20 \times h \text{ KN/m}^2$

Bending stress $\sigma_b = \pm \frac{M}{Z}$

Bending moment at the base $M = P . \frac{h}{2}$

$$M = p\,(3 \times h) \times \frac{h}{2} = \frac{2 \times 3 \times h^2}{2} = 3h^2$$

$$Z = \frac{I}{y} = \left[\frac{\frac{1}{12}(2.5 \times 2.5^3 - 2 \times 2^3)}{2.5/2}\right] = \frac{3.255}{2.5/2}$$

$$= 2.604$$

$$\therefore \quad \sigma_b = z \pm \frac{M}{Z} = \frac{3h^2}{2.604} = 1.152\, h^2$$

For no tension at base

$$\sigma_d = \sigma_b$$

or $\quad 20\, h = 1.152\, h^2$

$\therefore \quad$ or $h = 17.36$ metres **Answer**

Example 10.13

A hollow square masonry chimney is to have an internal bore 500 mm × 500 mm for its entire height of 22 metres. The thickness of masory is uniform throughout. If the chimney has to with stand a wind press of 1.40 KN/m² on one of its face determine the wall thickness of the chimney. Take Weight of masonry as .22 KN/m³. *(Roorkee Univ.)*

Solution

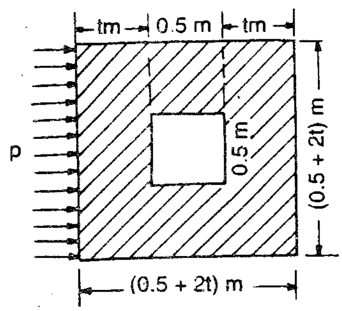


Fig. 10.12

Let t be the thickness of masonry in metres.

Direct stress due to weight of masonry $\sigma_d = \rho . h = 22 \times 22 = 484$ KN.

Total horizontal wind pressure

$$P = (0.5 + 2t) \times 22 \times 1.4$$
$$= 29.8\,(0.5 + 2t) \text{ KN.}$$

Bending moment at the base

$$M = P \times \frac{h}{2} = 29.8\,(0.5 + 2t) \times \frac{22}{2}$$
$$= 29.8 \times 11\,(0.5 + 2t)$$

Moment of inertia of the section

$$I = \frac{1}{12}\ [\ (0.5 + 2t)^4\ -\ (0.5)^4\ ]\ \mathrm{m}^4$$

Maximum distance of extreme fibre

$$y = \left(\frac{0.5 + 2\,t}{2}\right)$$

Hence maximum bending stress

$$\sigma_b = +\frac{M}{I}\ .\ y\ = \frac{29.8 \times\ 11\ (0.5 + 2\,t) \times\ (0.5 +\ 2\,t)}{\frac{1}{12}\ [(0.5 + 2\,t)^4 - (0.5)^4] \times\ 2]}$$

For no tension at base

$$\sigma_d =\ \sigma_b$$

$$\text{or}\quad 484 = \frac{29.8 \times\ 11 \times\ 6\,(0.5 + 2\,t)^2}{[(0.5 +\ 2\,t)^4 -\ (0.5)^4\ ]}$$

$$\text{or}\quad [\ (0.5 + 2\ t)^4 - (0.5)^4] = \frac{29.8 \times\ 66}{486}\ (0.5 + 2\,t)^2$$

$$(0.5 + 2\,t\,)^4 - (0.5)^4 =\ 4.06\ (0.5 + 2t)^2$$

Now Put $(0.5 + 2t) =\ x$ then

$$x^4 -\ (0.5)^4\ -\ 4.06\ (x)^2\ =\ 0$$

$$\text{or}\quad x^4 -\ 4.06\,x^2 -\ (0.5)^4\ \ = 0$$

$$x^2 = \frac{+\,4.06 \pm \sqrt{(4.06)^2 - 4\ (0.5)^4}}{2}$$

$$= \frac{+\,4.06 \pm \sqrt{16.48 - 4 \times .0625}}{2}$$

$$= \frac{+\,4.06 \pm \sqrt{16.48 - .25}}{2}$$

$$x^2\ = \frac{+\,4.06 \pm \sqrt{16.23}}{2} = \frac{+\ 4.06 \pm\ 4.02}{2}$$

$$x^2\ = \frac{8.08}{2}\ =\ 4.04$$

But $x\ = (0.5 + 2t)$

$$\therefore\ \ (0.5 +\ 2\,t)^2\ =\ 4.04$$

$$\text{or}\quad 0.5 + 2\,t\ = 2.009$$

$$\text{or}\quad 2t\ = 1.509\quad \text{or}\quad t\ = .754\ \text{meter}$$

Required thickness of brick masonry is 0.754 metres **Answer.**

Example 10.14

A masonry chimney has 2 metres diameter at the base and one metre diameter at the top. the thickness of wall at the base is 0.5 metre fig 10.18. If the weight of the chimney is 200 KN, determine the uniform horizontal *wind pressure that may act per unit projected area of the chimney to avoid any tension. The height of the chimney may be taken as 24 metres.*

Solution

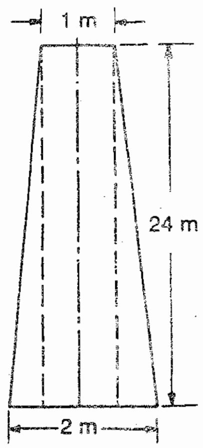


Fig. 10.18

Area of the base

$$A = \frac{\pi}{4}(2^2 - 1^2)$$

$$= 0.75\,\pi \text{ sq.m.}$$

Moment of inertia of the base section about a

$$\text{Diameter} = \frac{\pi}{64}(2^4 - 1^4)$$

$$= \frac{15\,\pi}{64}\ \text{m}^4$$

Section modulus of the base section

$$Z = \frac{\pi}{32}\frac{(D^4 - d^4)}{D}$$

$$= \frac{\pi}{32}\frac{(2^4 - 1^4)}{2}$$

$$= \frac{15\,\pi}{64}\ \text{m}^3. = .735$$

Direct stress due to the weight of the chimney $= \sigma_d = \dfrac{2\,000}{0.75\pi}$

$$= 843.2\ \text{KN/m}^2$$

Let the uniform intensity of wind pressure be p KN/m^2 of the projected area of the chimney

Projected area of the chimney = Area of the trapezium *ABCD*

$$= \frac{24}{2}(2 + 1) = 36 \text{ Sq. metres}$$

Total wind pressure $P = 36\,p$ KN

This resultant pressure acts at the level of the centroid of the trapezium *ABCD* Height of centroid of the trapezium *ABCD* above the base

$$\bar{y} = \left(\frac{2 + 2\times 1}{2+1}\right)\times\frac{24}{3}$$

$$= \frac{4\times 24}{9} = 10.66 \text{ metres}$$

Moment due to wind pressure

$$M = P.\,\bar{y} = 36\,p \times 10.66 \text{ KN–m}$$

$$\text{Bending stress } \sigma_b = \pm\,\frac{M}{Z} = \frac{36\,p\times 10.66}{.735}\ \text{KN/m}^2$$

For no tension at the base

$$\sigma_d = \sigma_b$$

$$\text{or} \qquad 843.22 = \frac{36 \times p \times 10.66}{.735}$$

$$\text{or} \qquad p = \frac{843.22 \times .735}{36 \times 10.66}$$

$$= 1.614 \text{ KN/m}^2$$

Masonry Dams

Structures constructed to store large quantity of water are known as dams. These structures are subjected to water, wind and wave pressures acting horizonfally and forces due to self weight acting vertically down wards. These forces induce both direct and bending stresses in the dam section. They are designed in such a manner that only compressive stresses are allowed to develop in masonry. The design criteria for such structures are

(1) Tensile stress should not be allowed to develop at any point in the cross-section of the masonry structures.

(2) The maximum compressive stress induced should be less than the permissible or working stress in the masonry.

(3) The shearing forces must not be greater than the frictional forces between the masonry.

Analysis of stresses in a trapezoidal dam section with a vertical water face.

Refering to the figure 10.11

Let $a =$ top width of the dam in metres

$b =$ width of base in metres

$H =$ Height of the dam in metres

$h =$ depth of water

ρ = density of masonry

$w =$ density of water.

Considering one metre length of the dam

$$\text{Weight of the dam } W = \frac{(a+b)}{2} \times H \times \rho$$

The weight of the dam acts vertically at a distance of $\bar{x}$ from the vertical face AB

$$\bar{x} = \frac{a^2 + ab\ \ b^2}{3\,(a+b)}$$

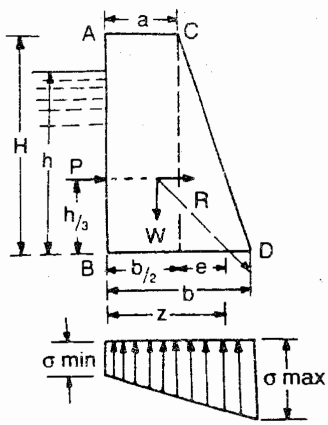


Fig. 10.11

Total horizontal water pressure.

$$P = \frac{w\,h^2}{2} \text{ acting at } \frac{h}{3} \text{ from the base of the dam.}$$

Let the resultant R of W and P cut the base at a distance Z from the vertical face.

For stability of the dam the base must offer a reaction equal and opposite to R. The vertical and horizontal component of R will be W and P.

Taking moments about B

$$W.\bar{x} + P.\frac{h}{3} = \text{Moment of } R \text{ about } B$$

$$= \text{Moment of vertical and horizontal components of } R \text{ about } B$$

$$= W.Z + P \times 0$$

or $\quad Z = \bar{x} + \frac{P}{W} \cdot \frac{h}{3}$

Let e be the distance of the vertical component of R from the centre of the base $B\,D$ then $Z = \left(\frac{b}{2} + e\right)$

$$e = Z - \frac{b}{2}$$

The normal stresses set up at the base $B\,D$ will therefore be due to an axial load W and a bending moment $W.e$

$$\text{Direct stress} = \frac{W}{A} = \frac{W}{b \times 1}$$

$$\text{Bending stress} = \frac{M}{I}.y = \frac{W.e.\frac{b}{2}}{\frac{1}{12}(b)^3(1)} = \frac{6\,W\,e}{b^2}$$

$$\sigma_{max} = \sigma_d + \sigma_b$$

$$= \frac{W}{b} + \frac{6\,W\,e}{b^2}$$

$$= \frac{W}{b}\left(1 + \frac{6\,e}{b}\right)$$

$$\sigma_{min} = \sigma_d - \sigma_b$$

$$= \frac{W}{b} - \frac{6\,W\,e}{b^2}$$

$$= \frac{W}{b}\left(1 - \frac{6\,e}{b}\right)$$

Conditions of Stability

(i) For no tension at the base

$$\sigma_d \geq \sigma_b$$

$$\frac{W}{b} \geq \frac{6\,W\,e}{b^2}$$

$$1 \geq \frac{6e}{b}$$

$$\text{or} \quad e \leq \frac{b}{6}$$

Hence the resultant R must always be in the middle third portion of the base width b. Under worst conditions

$$Z = \frac{2b}{3}$$

(ii) Safety against sliding

If μ is the coefficient of friction the maximum frictional resistance set up is μW.

Hence the horizontal water pressure P must not exceed μW in order to prevent the section from sliding.

$$P \leq \mu W$$

$$\text{Factor of safety against sliding} = \frac{\mu W}{P}$$

Generally a factor of safety of 1.5 (minimum) should be provided

(iii) Safety against over turning

For the stability of the section against overturning, the restoring moment must be equal to the overturning moment about the toe of the dam.

$$P \times \frac{h}{3} = W\ (b - x)$$

Factor of safety against overturning

$$= \frac{W(b-x)}{P\,h/3}$$

It should be more than unity.

(iv) Safety against Crushing

To avoid crushing of masonry at the base the maximum compressive stress acting normal to the base must be less than the permissible compressive stress for masonry

$\sigma_{max} \leq$ Permissible compressive stress

or $\dfrac{W}{b}\left(1 + \dfrac{6e}{b}\right) \leq$ Permissible compressive stress.

Example 10.15

A trapezoidal masonry dam 8 metres high has a top width of 2 metres and a base width 5 metres, it retains water to its full depth with water face vertical. Determine the maximum and minimum stress intensities at the base masonry weighs 20.7 KN/m^3 and wt of water per cubie metre may be taken as 10 KN.

Solution

Consider 1 metre length of the dam

Self wt of the dam

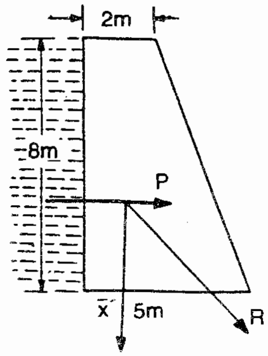


Fig. 10.12

$$W = \left(\frac{a+b}{2}\right) . \text{H. } \rho$$

$$= \frac{(2-5)}{2} \times 8 \times 20.7 = 580 \text{ KN}$$

Line of action of W from the vertical face.

$$\bar{x} = \frac{a^2 + a\,b + b^2}{3\,(a+b)}$$

$$= \frac{(2)^2 + (2)\,(5) + (5)^2}{3\,(2+5)} = 1.85 \text{ m}$$

Horizontal thrust of water

$$P = \frac{w\,h^2}{2} = \frac{10\,(8)^2}{2} = 320 \text{ KN}$$

Line of action of P from base = $h/3 = \frac{8}{3}$

$$Z = \bar{x} + \frac{P}{W} \cdot \frac{h}{3} = 1.85 + \frac{320}{580} \times \frac{8}{3} = 1.85 + 1.47$$

$$Z = 3.32 \text{ m} \quad \text{and} \quad e = Z - \frac{b}{2} = 3.32 - 2.50 = .82\ m$$

$$\sigma_{max} = \frac{W}{b}\left(1 + \frac{6\,e}{b}\right) = \frac{580}{5}\left(1 + \frac{6 \times .82}{5}\right) = 230.14 \text{ KN/m}^2$$

$$\sigma_{min} = \frac{W}{b}\left(1 - \frac{6\,e}{b}\right) = \frac{580}{5}\left(1 - \frac{6 \times .82}{5}\right) = 1.85 \text{ KN/m}^2 \quad \textbf{Answer}$$

Example 10.16

A trapezoidal dam with one face vertical is 12 m high. The top width is 4 metres and the base of the dam is 7 metres wide. It retains water upto a height of 10 metres. If masonry weighs 20 KN/m^3, determine the maximum and minimum intensities of stresses at the base.

Solution

Consider one meter length of the dam

Self weight of the dam

$$W = \left(\frac{a+b}{2}\right) \times H \times \rho$$

$$= \left(\frac{4+7}{2}\right) 12 \times 20 = 1320 \text{ KN}$$

Line of action of W from the vertical face

$$\bar{x} = \frac{a^2 + ab + b^2}{3\,(a+b)} = \frac{4^2 + 4 \times 7 + 7^2}{3\,(4+7)}$$

$$= 2.818 \text{ m}$$

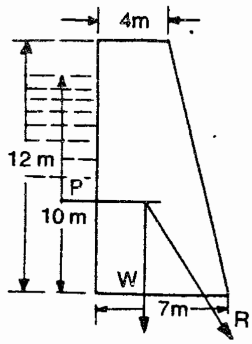


Fig. 10.13

Horizontal thrust of water $P = \frac{w\,h^2}{2} = \frac{10\,(10)^2}{2} = 500$ KN

Line of action of P from the base $= \frac{h}{3} = \frac{10}{3}$

$$Z = \bar{x} + \frac{P}{W} \cdot \frac{h}{3}$$

$$= 2.818 + \frac{500}{1320} \times \frac{10}{3} = 2.818 + 1.266 = 4.084$$

$$e = Z - \frac{b}{2} = 4.084 - 3.50 = 0.584$$

$$\sigma_{max} = \frac{w}{b}\left(1 + \frac{6\,e}{b}\right) = \frac{1320}{7 \times 1}\left(1 + \frac{6 \times 0.584}{7}\right)$$

$$= \frac{1320}{7}\ (1 + 0.500) = 188.57\ (1.5) = 282.8\ \text{KN/m}^2$$

$$\sigma_{min} = \frac{W}{b}\left(1 - \frac{6\,e}{b}\right)$$

$$= \frac{1320}{7}\left(1 - \frac{6 \times 0.584}{7}\right) = \frac{1320}{7}\ (1 - 0.5)$$

$$= 188.57\ (0.5) = 94.285\ \text{KN/m}^2.$$

Example 10.17

A concrete dam of trapezoidal section is 10 m high, 2 metres wide at the top with water face vertical. It retains water upto the top level of the dam. Find the minimum width at the base to avoid tension in masonry. What is the maximum Compressive stress ? Take weight of concrete as 24 KN per cubic metre. *J.M.I. 1995*

Solution

Consider one metre length of the dam

Self weight of the dam

$$W = \left(\frac{a+b}{2}\right) \times H \times \rho$$

$$= \frac{(2+b)}{2} \times 10 \times 24 = 120\,(2+b)\ \text{KN}$$

$$\bar{x} = \frac{(a^2 + ab + b^2)}{3\,(a+b)}$$

$$= \frac{(2)^2 + 2b + b^2}{3\,(2+b)}$$

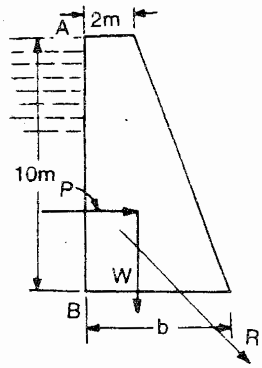


Fig. 10.14

Horizontal thrust of water $P = \frac{w\,h^2}{2}$

$$= \frac{10\,(10)^2}{2} = 500\ \text{KN}$$

$$P \text{ wil act at } \frac{h}{3} = \frac{10}{3} \text{ m from base}$$

For no tension at base the maximum value of $Z \leq \dfrac{2b}{3}$

$$Z = \bar{x} + \frac{P}{W} \cdot \frac{h}{3}$$

$$\frac{2b}{3} = \frac{4 + 2b + b^2}{3(2+b)} + \frac{500}{120(2+b)} \cdot \frac{10}{3}$$

$$\frac{2b}{3} = \frac{4 + 2b + b^2}{3(2+b)} + \frac{13.88}{(2+b)}$$

or $\quad 4 + 2\text{ b} + b^2 + 3\,(13.88) = \dfrac{2b}{8}\,(2+b)$

or $\quad 4 + 2b + b^2 + 41.6 - 4\,b - 2b^2 = 0$

or $\quad b^2 + 2b - 45.6 = 0$

Solving the quadratic equation

$$b = \frac{-2 \pm \sqrt{(2)^2 - 4\,(1)\,(-45.6)}}{2}$$

$$= \frac{-2 \pm 13.65}{2} = 5.82 \text{ metres.}$$

Therefore for no tension at base the minimum base width shoulc 5.82 metres.

$$\text{Now } Z = \frac{2b}{3} = \frac{2 \times 5.82}{3} = 3.88 \text{ m}$$

$$\text{and} \quad e = Z - \frac{b}{2} = (3.88 - 2.91) = .97 \text{ m}$$

Maximum Compressive stress

$$\sigma_{max} = \frac{W}{b}\left(1 + \frac{6e}{b}\right)$$

$$W = \frac{(a+b)}{2} \times H \times \rho = \frac{(2 + 5.82)}{2} \times 24 \times 10 = 938.4 \text{ KN}$$

$$\sigma_{max} = \frac{938.4}{1 \times 5.82}\left(1 + \frac{6 \times 0.97}{5.82}\right)$$

$$= 161.23\,(1 + 1) = 322.46 \text{ KN/m}^2$$

$$\sigma_{max} = 322.46 \text{ KN/m}^2$$

Example 10.18

A masonry dam trapezoidal in section is 2 metres wide at top and 5 metres wide at base. It retains water level with top against the vertical face. Calculate the height of the dam so that there is no tension at the base. Take Wt of masonry as 22 KN/m^3. (Madras)

Solution

Self Wt of the dam $= \dfrac{(a+b)}{2} \cdot \rho \cdot H$

$$W = \frac{(2+5)}{2} \times 22 \times H = 77\,H \text{ KN}$$

$$\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)} = \frac{(2)^2 + (2)(5) + (5)^2}{3(2+5)} = \frac{39}{21} = 1.85 \text{ m}$$

Horizontl water pressure

$$P = \frac{wH^2}{2} = \frac{10\,H^2}{2} = 5\,H^2 \text{ KN acting at } \frac{H}{3}$$

$$Z = \bar{x} + \frac{P}{W} \times \frac{H}{3}$$

For no tension at base $Z = \dfrac{2b}{2}$

$$\frac{2b}{3} = \bar{x} + \frac{P}{W} \times \frac{H}{3}$$

$$\frac{2 \times 5}{3} = 1.85 + \frac{5\,H^2}{77\,H} \times \frac{H}{3}$$

$$3.3 = 1.85 + \frac{5\,H^2}{231}$$

or $\dfrac{5}{231} H^2 = 3.33 - 1.85 = 1.48$

$$H^2 = \frac{1.48 \times 231}{5} = 68.37$$

$H = 8.26$ metres **Answer.**

Example 10.19

Show that the minimum base width required to avoid tension at the base is $\dfrac{H}{\sqrt{\gamma}}$ *whether the section is triangular or rectangular, where H is the height of the dam and* γ *is the sp. grairty of the aterial of the dam.* (A.M.I.E)

Solution

When the section of the dam is trapezoidal and water face is vertical

Self Wt of the dam $= W \;\; \dfrac{(a+b)}{2} \times H.\rho$

Line of action of W from vertical face

$$\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)}$$

Horizontal water pressure $P = \dfrac{wH^2}{2}$ acting at H/3

For the stability of the dam

$$Z = \bar{x} + \frac{P}{W} \cdot \frac{H}{3} \leq \frac{2b}{3}$$

$$\therefore \frac{a^2 + ab + b^2}{3(a+b)} + \frac{wH^2}{2} \times \frac{1}{\frac{(a+b)}{2} \times H.\rho} \cdot \frac{H}{3} \leq \frac{2b}{3}$$

$$\text{or} \quad a^2 + ab + b^2 + \frac{w}{\rho} . H^2 \leq \frac{2b}{3} \times 3(a+b) \leq 2b(a+b)$$

$$\text{or} \quad a^2 + ab + b^2 + \frac{w}{\rho} . H^2 \leq 2b + 2b^2$$

$$\text{or} \quad a^2 + ab + b^2 + H^2 \quad \text{---} \quad \text{---} \quad \text{---} \quad (i)$$

Hence base width can be calculated from equation no (i) with the resultant of P and W passing through the middle third of the base.

When the ection is triangular a = o, hence equation (i) beco

$$b^2 = \frac{w}{\rho} . H^2$$

$$= \frac{H^2}{\gamma} \quad \text{When } \gamma = \frac{\rho}{w} = \text{Specific gravity of the masonry}$$

$$\text{or } b = \frac{H}{\sqrt{r}}$$

When the section is rectangular, then a=b, hence equation no (i) become

$$b^2 + b^2 = b^2 + \frac{w}{\rho} . H^2$$

$$\text{or } b^2 = \frac{w}{\rho} . H^2$$

$$= \frac{H^2}{\gamma}$$

$$\text{or } b = \frac{H}{\sqrt{\gamma}}$$

Therefore the minimum base width to avoid tension at the base is b = $\frac{H}{\sqrt{\gamma}}$ when the section is traingular or rectangular.

Example 10.20

A masonry dam of trapezoidal section has a vertical water face and height 18 metres. Determine the widths at the top and bottom if the normal pressure on the base varies uniformly from Zero at one side to 500 KN/m^2 at the otherside. The depth of water impounded is 15 metres. Take weight of masonry as 22 KN/m^3 and that of water as 10 KN/m^3. (calcutta)

Solution

Consider one metre length of the dam

Let a and b be the top and bottom widths of the dam.

Self weight of dam

$$W = \frac{(a+b)}{2} \times 18 \times 22 = 198\,(a+b) \text{ KN}$$

over 2 times~H. rho

$$P = \frac{wh^2}{2} = \frac{10\,(15)^2}{2} = 1125 \text{ KN}$$

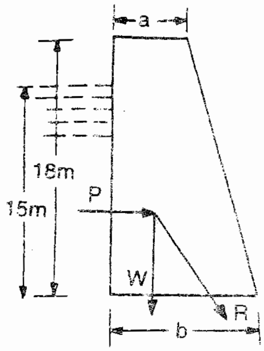


Fig. 10.18

As the intensity of pressure at the base section is given, therefore area of stress diagram at base $= \frac{1}{2}(0 + 500) \times b = 198\,(a + b)$

or $\quad 250\,b = 198\,a + 198\,b$

or $\quad (250\,b - 198\,b) = 198\,a \qquad$ or $\quad b = \frac{198}{50} = 3.807\,a$

$$\bar{x} = \frac{a^2 + ab + b^2}{3\,(a+b)} = \frac{a^2 + 3.807\,a^2 + (3.807\,a)^2}{3\,(a + 3.807\,a)} = 1.338\,a$$

$$Z = \bar{x} + \frac{P}{W} . h\ ove3$$

$$= 1.338\,a + \frac{1125}{198\,(a+b)} . \frac{15}{3}$$

$$= 1.338\,a + \frac{1125}{198\,(a + 3.807\,a)} \times \frac{15}{3}$$

$$= 1.338\,a + \frac{5.909}{a}$$

Since the intensity at the top is Zero there fore there is no tension,

hence $\quad Z = \frac{2b}{3}$

$$\frac{2b}{3} = 1.338\,a + \frac{5.909}{a}$$

or $\quad \frac{2\,(3.807a)}{3} = 1.388\,a + \frac{5.909}{a}$

or $\quad (2.538\,a - 1.338\,a) = \frac{5.909}{a}$

or $\quad (1.20)\,a = \frac{5.909}{a}$ or $a^2 = \frac{5.909}{1.20} = 4.924$

or $a = 2.21$ metres

Hence $b = 2.21 \times 3.807 = 8.447$ metres.

Top width $a = 2.21$ metres, Base width $b = 8.447$ metres **Answer**

Example 10.21

A trapezoidal masonry dam 2 metres wide at top 8 metres wide at its bottom is 12 metres high. The face exposed to water has a slope of 1 horizontal to 12 vertical fig. 10.16. Determine the maximum stress intensities, when water rises to the top level of the dam. Masonry weighs 24 KN/m^3. (ENGG. Services)

Solution

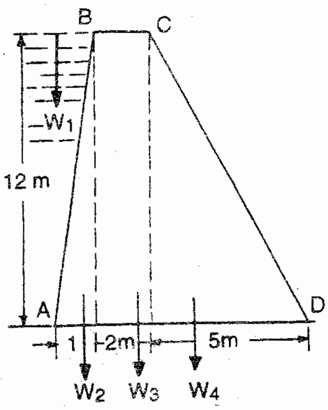


Fig. 10.19

Consider one metre length of the dam.

Sum of the vertical forces.

$W = W_1 + W_2 + W_3 + W_4$

$W_1 = \frac{1}{2} \times 1 \times 12 \times 10 = 60$ KN

Moment of W_1 about A,

$M_1 = 60 \times \frac{1}{3} \times 1 = 20$ KN–m

$W_2 = \left(\frac{1}{2} \times 1 \times 12\right) \times 24 = 120$ KN

Moment of W_2 about A,

$M_2 = 120 \times \frac{2}{3} \times 1 = 80$ KN–m

$W_3 = (2 \times 12) \times 24 = 576$ KN

Moment of W_3 about A $M_3 = 576\left(\frac{1+2}{2}\right) = 1152$ KN–m

$W_4 = \left(\frac{1}{2} \times 5 \times 12\right) \times 24 = 720$ KN

Moment of W_4 about A $M_4 = 720\left(1 + 2 + \frac{5}{3}\right) = 3360$ KN–m

Sum of all vertical forces

$$W = W_1 + W_2 + W_3 + W_4$$
$$= 60 + 144 + 576 + 720 = 1500 \text{ KN}$$

Horizontal water thrust $P = \dfrac{w\,h^2}{2}$

$$P = \frac{10\,(12)^2}{2} = 720 \text{ KN}$$

Moment of P about $A = P \times \dfrac{h}{3} = \dfrac{720 \times 12}{3}$

$$M_5 = 2880 \text{ KN–m}$$

Sum of the moments of all forces about A

$M = M_1 + M_2 + M_3 + M_4 + M_5$

$= 20 + 96 + 1152 + 3360 + 2880 = 7508$ KN–m

The distance at which the resultant strikes the base from A

$$Z = \frac{7508}{1500} = 5.005 \text{ m.}$$

$$e = Z - \frac{b}{2} = (5.005 - 4) = 1.005 \text{ m}$$

$$\sigma_{max} = \frac{W}{b}\left(1 + \frac{6\,e}{b}\right)$$

$$= \frac{1500}{1 \times 8}\left(1 + \frac{6 \times 1.005}{8}\right) = 328.6 \text{ KN/m}^2$$

$$\sigma_{min} = \frac{W}{b}\left(1 - \frac{6\,e}{b}\right)$$

$$= \frac{1500}{8}\left(1 - \frac{6 \times 1.005}{8}\right) = 46.90 \text{ KN/m}^2 \qquad \textbf{Answer.}$$

Example 10.22

A masonry dam is one metre wide at top 4 metre at the base and 8 metres high. It retains water up to 6 metres height. Test the stability of the dam against tension, compression sliding and overturning.Take weight of masonry 24 KN/m3 . Bearing capacity of soil 240 N/m2 and $\mu = 0.6$

Solution

Self weight of the dam

$$W = \frac{1}{2}(a + b)\,H.\rho = \frac{(2 + 4)}{2} \times 8 \times 24 = 480 \text{ KN.}$$

$$\bar{x} = \frac{a^2 + ab + b^2}{3(a + b)} = \frac{(1)^2 + (1)\,(4) + (4)^2}{3(1 + 4)} = \frac{21}{15} = 1.4 \text{ m}$$

Horizontal water pressure

$$P = \frac{w\,h^2}{2} = \frac{10\,(6)^2}{2} = 180 \text{ KN.acting at } \frac{6}{3} \text{ m}$$

$$Z = \bar{x} + \frac{P}{W} \cdot \frac{h}{3}$$

$$= 1.4 + \frac{180}{480} \times \frac{6}{3} = 1.4 + 0.75 = 2.15 \text{ metres}$$

$$e = Z - \frac{b}{2} = 2.15 - 2 = 0.15 \text{ m}$$

$$\sigma_{max} = \frac{W}{b}\left(1 + \frac{6e}{b}\right)$$

$$= \frac{480}{4}\left(1 + \frac{6 \times 0.15}{4}\right) = 120\,(1 + 0.225)$$

$$= 147 \text{ KN/m}^2.$$

(1) Since the eccentricity $e = 0.15$ m is less than

$$e < \frac{b}{6} = \frac{4}{6} = .66 \text{ m}$$

Therefore the section is safe against tension

(2) Since max. compressive stress $\sigma_{max} = 147$ KN/m^2 is less than the bearing capacity of 240 KN/m^2 hence the section is safe against compression.

(3) For safety against sliding

$$P < \mu W$$

$$\text{Factor of safety} = \frac{0.6 \times 480}{180} = 1.6 \text{ hence safe.}$$

(4) To be safe against overturning

Restoring moment > overturning moment

$$\text{Factor of safety} = \frac{480(4 - 1.4)}{180 \times \frac{6}{3}}$$

$$= \frac{480 \times 2.6}{180 \times 2} = 3.46$$

Hence the section is safe against all the four factors

Example 10.23

A trapezoidal masonry dam is 16 metres high with a top width of 4 metres. The Water face has a batter of 1 in 16. Determine the minimum base width so that no tension develops at the base of the dam. Take Wt. of masonry as 22 KN/m^3. Water stands upto the top level of the dam. (Cambridge)

Solution

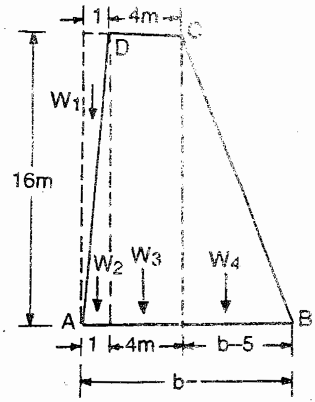


Fig. 10.20

Consider one metre length of the dam.

Sum of vertical forces.

$$W = W_1 + W_2 + W_3 + W_4$$

$$W_1 = \frac{1}{2}(1 \times 16) \times 10 = 80 \text{ KN}$$

Moment of W_1 about A

$$M_1 = 80 \times \frac{1}{3} \times 1 = 26.66 \text{ KN–m}$$

$$W_2 = \frac{1}{2}(1 \times 16) \times 22 = 176 \text{ KN}$$

Moment of W_2 about A

$$M_2 = 176 \times \frac{2}{3} \times 1 = 117.36 \text{ KN–m}$$

$$W_3 = (4 \times 16) \times 22 = 1408 \text{KN}$$

Moment of W_3 about A

$$M_3 = 1408\left(1 + \frac{4}{2}\right) = 4224 \text{ KN–m}$$

$$W_4 = \frac{1}{2}(b-5) \times 16 \times 22 = 176\,(b-5) \text{ KN}$$

Moment of W_4 about A

$$M_4 = \left[176\,(b-5)\left\{1+4+\frac{1}{3}(b-5)\right\}\right] \text{KN–m}$$

Sum of all vertical forces

$$W = W_1 + W_2 + W_3 + W_4$$
$$= [\,80 + 176 + 1408 + 176\,(b-5)\,] \text{ KN}$$
$$= [\,1664 + 176\,(b-5)\,]$$

Horizontal Water thrust

$$P = \frac{wh^2}{2} = \frac{10\,(16)^2}{2} = 1280 \text{ KN acting at } \frac{16}{3}$$

Moment of P about A

$$M_5 = 1280 \times \frac{16}{3} = 6826.66 \text{ KN–m}$$

Sum of all the moments

$$M = M_1 + M_2 + M_3 + M_4 + M_5$$

$$M = 26.66 + 117.3 + 422.4 + \left[176\,(b-5)\left\{5+\frac{1}{3}(b-5)\right\}\right] + 6826.66$$

$$= 11194.62 + \left[176\,(b-5)\left\{5+\frac{1}{3}(b-5)\right\}\right]$$

Distance of the point of application of the resultant on the base fromA

$$Z = \frac{\text{Total Moment about A}}{\text{Total Vertical Load}}$$

For no tension at the base $Z = \dfrac{2b}{3}$

$$Z = \frac{2b}{3} = \frac{11194.62 + \left[176(b-5) \times \left\{5+\frac{1}{3}(b-5)\right\}\right]}{1664 + 176\,(b-5)}$$

$$\frac{2b}{3}\,[1664 + 176\,(b-5) = 11194.62 + 58.08\,b^2 + 295.68\,b - 2930.4]$$

or $\quad 512.7\,b + 1173\,b^2 = 8264.22 + 58.08\,b^2 + 295.68\,b$

or $\quad 58.22\,b^2 + 217.02\,\text{b} - 8264.22$

or $\quad b^2 + 3.72\,b - 141.94 = 0$

or $b = \dfrac{-3.72 \pm \sqrt{(3.72)^2 + 4\,(141.94)}}{2}$

or $b = 10.18$ metres **Ans.**

Retaining Walls

A retaining wall has to withstand pressure due to earth which it retains. This pressure depends upon the weight of earth and the angle of repose. Just as in the case of water, earth pressure increases uniformly with the depth, giving a straight line pressure variation diagram. It varies linearly from Zero at the top to maximum at the base. The resultant thrust will act at one third the height of earth retained, from the bottom of the retaining wall.

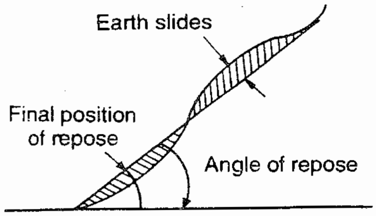


Fig. 10.21

Angle of repose

When a heap of earth is allowed to rest freely, it will crumble down under the action of weather and finally it will take a certain definite position. The angle which the inclined surface makes with the horizontal in this condition is termed as angle of repose for a particular granular material. This angle of repose may be considered to be the angle of friction for one portion of the material tending to slide over the other. In the case of water, in which no friction exists, the angle of repose is zero.

Retaining walls may be with or without surcharge. We shall discuss retaining walls without surcharge only when the top of the earth retained is horizontal.

Rankine's Formula

Horizontal Pressure per metre length of the wall

$$P = \frac{w\,h^2}{2}\,\frac{(1-\sin\theta)}{(1+\sin\theta)}$$

Where h = height of earth retained

w = density of earth

θ = angle of repose

P will act at h/3 from the base of the wall. The rest of the analysis is similar to dams.

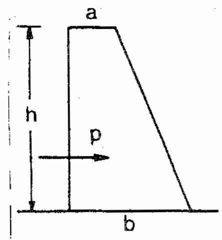


Fig. 10.22

Example 10.24

A masonry retaining wall trapezoidal in section is 10 metres high, 6 metres wide at base has one face vertical and the other battered 1 in 5. It retains earth level with the top. Calculate how far, the resultant will Strike from the centre of the base. Find the maximum and minimum stress *intensities at the base. Earth weighs 16 KN/m^3 and masonry weighs 24 KN/m^3. Angle of repose is 30°.*

Solution

Batter of sloping face is 1 in 5

$\therefore$ Top width = 6 – 2 = 4 metres

Consider 1 metre length of the retaining wall.

Self weight $W = \frac{(a+b)}{2} \times H \times \rho$

$= \frac{(4+6)}{2} \times 10 \times 24$

$= 1200$ KN

$\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)} = \frac{(4)^2 + (4)(6) + (6)^2}{3(4+6)}$

$= 2.533$ m

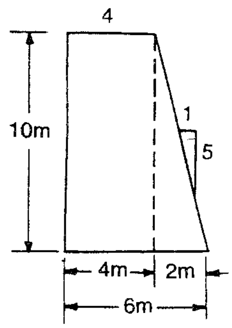


Fig. 10.23

Horizontal earth pressure

$P = \frac{wh^2}{2} \frac{(1-\sin\theta)}{(1+\sin\theta)}$

$= \frac{16(10)^2}{2} \frac{(1-0.5)}{(1+0.5)} = 266.7$ KN

P will act at $\frac{h}{3}$ from the base, $\frac{10}{3}$ m

$Z = \bar{x} + \frac{P}{W} \frac{h}{3}$

$= 2.533 + \frac{2667}{1200} \times \frac{10}{3} = 2.533 + 0.740 = 3.27$ m

$e = Z - \frac{b}{2} = 3.27 - 3 = 0.27$ m

Hence the resultant will strike the base at 0.27 metres from the centre towards the toe of the wall.

$\sigma_{max} = \frac{W}{b}\left(1 + \frac{6e}{b}\right)$

$= \frac{1200}{6 \times 1}\left(1 + \frac{6 \times 0.27}{6}\right) = 200\,(1 + 0.27)$

$= 200\,(1.27) = 254$ KN/m^2.

$\sigma_{min} = \frac{1200}{6 \times 1}\left(1 - \frac{6e}{b}\right)$

$= 200\,(1 - 0.27) = 200 \times 0.73$

$= 146$ KN/m^2 **Answer**

Example 10.25

Design a retaining wall for a height of 6 metres. The face in contact with earth is to be vertical and earth level with the top. Take the wts. of earth and masonry as 18 KN and 21 KN per cubic metre. respectively. Maximum Compressive stress for masonry may be taken as 200 KN/m^2.

The angle of repose 30 ° and coefficient of friction is 0.5.

Solution

Assume the top width as 1 metre and base width b metres then

$$W = \frac{(a+b)}{2} \times H \ddot{o}\ r = \frac{(1+b)}{2} \times 6 \times 21 = 63\ (1+b)\ \text{KN}$$

Horizontal earth pressure

$$P = \frac{1}{2}\ wh^2\ \frac{(1-\sin\theta)}{1+\sin\theta}$$

$$= \frac{1}{2} \times 18 \times 6^2 \left(\frac{1-0.5}{1+0.5}\right) = 108\ \text{KN}$$

P will act at $\frac{h}{3} = \frac{6}{3}$ metres from the base

$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)} = 1 + b + b^2 ver3(1+b)›P$ will act at h/3 from the base of the wall. The rest of the analysis is similar to dams.

$$Z = \bar{x} + \frac{P}{W} \cdot \frac{h}{3}$$

$$= \frac{1+b+b^2}{3(1+b)} + \frac{108}{63(1+b)} \times \frac{6}{3}$$

For no tension at base $Z = \frac{2b}{3}$

$$\frac{2b}{3} = \frac{1+b+b^2}{3(1+b)} + \frac{108 \times 6}{63\ (1+b) \times 3}$$

or $\quad 2b\ \ (1+b) = (1+b+b^2) + 10.28$

or $\quad b^2 + b - 11.28 = 0$

Solving the quadratie equation we get

$$b = \frac{-1 \pm \sqrt{(1)^2 - 4\ (1)\ (-11.28)}}{2}$$

$b = 2\ .895$ metres.

Now $\quad e = Z - \frac{b}{2} = \frac{2b}{3} - \frac{b}{2} = \frac{b}{6}$

$$= \frac{2.895}{6} = .48 \text{ metres}$$

G. SivaRama Krishnan

(i) Check against sliding

$P = 108$ KN and $\mu W = 0.5 \times 63\ (1 + 2.89)$

$= 122.69$

As μW is more than P, the section is safe against sliding

(ii) Check against Crushing

$$\sigma_{max} = \frac{W}{b}\left(1 + \frac{6e}{b}\right)$$

Here $W = 63\ (\ 1 + b) = 63\ (1 + 2.89) = 245.35$

$$\therefore\ \sigma_{max} = \frac{245.35}{2.89}\left(1 + \frac{6 \times .48}{2.89}\right) = 169.77\ \text{KN/m}^2$$

Since σ_{max} is less than the permissible compressive stress of 200 KN/m^2, the section is safe against crushing.

Safety against overturning

$$\text{overturning moment} = P \times \frac{h}{3} = 108 \times \frac{6}{3} = 216 \text{ KN–m}$$

$$\text{Balancing moment} = W\ (b - \bar{x})$$
$$= 245.35\ (2.895 - \bar{x})$$

$$\text{Where } \bar{x} = \frac{a^2 + ab + b^2}{3(a+b)} = \frac{1 + (2.895) + (2.895)^2}{3(1 + 2.895)}$$
$$= 1.05 \text{ m}$$

$$\text{Balancing moment} = 245.35\ (2.895 - 1.05)$$
$$= 452.67 \text{ KN–m}$$

As the balancing moment is more than overturning moment, the section is safe.

$$\text{Factor of safety} = \frac{452.67}{216} = 2.09$$

Example 10.26

A masonry retaining wall 10 metres high is stepped as shown in figure. 10.21 If the weight of earth filling is 12 KN/m^3 and that of masonry 16 KN/m^3, determine the stress intensities at the base. The angle of repose is 30° and μ = 0.6. Check the safety against sliding. (Baroda)

Solution

Consider one metre length of the retaining wall

Total vertical load $W = W_1 + W_2 + W_3 + W_4 + W_5$

$W_1 = (1 \times 8) \times 12 + (1 \times 2) \times 16$
$= 96 + 32 = 128$ KN

$W_2 = (1 \times 6)12 + (1 \times 4)\ 16$
$= 72 + 64 = 136$ KN

$W_3 = (1 \times 4)\ 12 + (1 \times 6) \times 16$
$= 48 + 96 = 144$ KN

$W_4 = (1 \times 2)\ 12 + (1 \times 8)\ 16$
$= 24 + 128 = 152$ KN

$W_5 = (1 \times 10) \times 16 = 160$ KN

Sum of all vertical forces

$W = 128 + 136 + 144 + 152 + 160$
$= 720$ KN

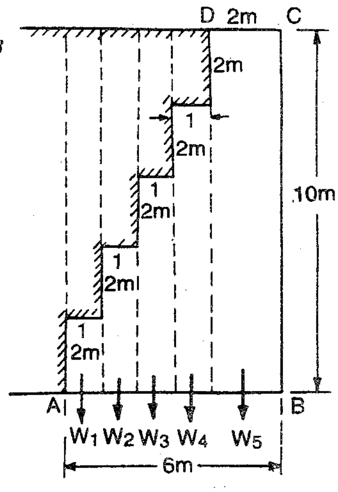


Fig. 10.24

Moment of W_1 about A

$$= M_1 = W_1 \times 0.5 = 128 \times 0.5$$
$$= 64 \text{ KN–m}$$

Moment of W_2 about A, M_2 = 136 (1 + 0.5) = 254 KN–m
Moment of W_3 about A, M_3 = 144 (2.5) = 360 KN–m
Moment of W_4 about A, M_4 = 152 × 3.5 = 456 KN–m
Moment of W_5 about A, $M_5 = 160\left(4+\frac{2}{2}\right) = 800$ KN–m

Horizontal thrust of earth

$$P = \frac{1}{2}\, wh^2\, \frac{(1-\sin\theta)}{(1+\sin\theta)}$$

$$= \frac{1}{12}\,(12)\,(10)^2\,\frac{(1-0.5)}{(1+0.5)} = 200 \text{ KN}$$

Moment of P about $A = 200 \times \frac{10}{3} = 666.3$ KN–m

Sum of the moments of W and P about A

$$= 64 + 254 + 360 + 456 + 800 + 666.3 = 2600.3 \text{ KN–m}$$

$$Z = \frac{2600.33}{720} = 3.6 \text{ metres}$$

$$e = Z - \frac{b}{2} = 3.6 - 3 = 0.6 \text{ m}$$

$$\sigma_{max} = \frac{W}{b}\left(1+\frac{6e}{b}\right) = \frac{720}{6}\left(1+\frac{6\times 0.6}{6}\right)$$

$$= 120\,(1+0.6) = 192 \text{ KN/m}^2$$

$$\sigma_{min} = 120\,(1-0.6) = 48 \text{ KN/m}^2.$$

Check against sliding

$$P < \mu W$$

$$200 < 0.6 \times 720$$

Since P is less than μ W there fore retaining wall is safe against sliding

Factor of safety $= \frac{\mu W}{P} = \frac{0.66 \times 720}{200} = \frac{432}{200} = 2.16$ **Answer.**

SUMMARY

1. Direct stress $\sigma_d = W/A$

   Where W is the vertical load and A us the area of Cross-Section.

2. Bending stress $\sigma_b = \frac{M}{Z} = \frac{W.e}{Z}$

   When e is the eccentricity of at which W is acting and Z is the section modulus

3. $\sigma_{max} = \sigma_d + \sigma_b$

   $\sigma_{min} = \sigma_d - \sigma_b$

   If σ_{min} is negative the stress is tensile.

4. For no tension $e \leq \dfrac{Z}{A}$

   For rectangular sections $e \leq \dfrac{b}{6}$

   and For circular sections $e \leq \dfrac{d}{8}$

5. In case walls and chimneys subjected to lateral loads.

$$\sigma_d = P.\,h$$

$$\sigma_b = \pm\frac{M}{Z}$$

6. In case of retaining wall, always analyse for 1 metre length

$$W = \text{Area} \times \text{ length} \times \text{ density of masonry}$$

$$= \frac{(a+b)}{2}\,.\,H\,.\,\rho$$

$$P = \frac{wh^2}{2} \text{ Where } w \text{ is the density of water}$$

$$\bar{x} = \frac{a^2 + ab + b^2}{3\,(a+b)}$$

$$Z = \bar{x} + \frac{P}{W}\cdot\frac{h}{3}$$

$$e = \left(Z - \frac{b}{2}\right)$$

At the base of width b

direct stress $\sigma_d = \dfrac{W}{b}$

Bending stress $\sigma_b = \dfrac{6\,w\,e}{b^2}$

$$\sigma_{max} = \sigma_d + \sigma_b$$

$$= \frac{W}{b}\left(1 + \frac{6e}{b}\right)$$

$$\sigma_{min} = \sigma_d - \sigma_b$$

$$= \frac{W}{b}\left(1 - \frac{6e}{b}\right)$$

Rankine's formula $P = \dfrac{wh^2}{2}\,\dfrac{(1-\sin\theta)}{(1+\sin\theta)}$

Conditions of stability

(i) For no tension $e \leq \dfrac{b}{6}$

(ii) Against sliding $\dfrac{\mu W}{p} > 1$

(iii) Against overturning, $\frac{3W(b-x)}{P.h} > 1$

Against crushing σ_{max} <safe bearing capacity of soil or safe compressive stress for the masonry.

QUESTIONS

(1) What is meant by eccentric loading ? Explain the effect of eccentric loading on a short column.

(2) What do you understand by middle third rule ? Show that for no tension in the base of a dam the line of action of the resultant must pass through the middle third portion of the base.

(3) What are the various conditions for the stability of a dam ? Explain them

(4) Explain angle of repose. What is the effect of earth pressure on retaining wall ?

EXERCISES

(5) A steel flat 200 mm wide and 18 mm thick is subjected to a compressive load of 20 KN at an eccentricity of 30 mm from the geometrical axis of the flat. Determine the maximum and minimum stress intensities induced in the section

(σ_{max} = 60 MPa, σ_{min} = 50 MPa)

(6) In a tension specimen 25 mm in diameter the line of pull is parallel to the axis of the specimen. Determine the eccentricity of the load when the maximum stress is 20 percent grater than the average stress on a section normal to the axis

(e = 0.9 mm)

(7) A masonry wall 2.4 metre wide is exposed to a wind pressure of 1.4 KN/m^2. Find the maximum height of the wall so that there is no tension at the base of the wall. Take weight of masonry as 20 KN/m^3.

(27.43 metres)

(8) A masonry chimney 20 metres high has a uniform circular section. The external and internal diameters are 4 m and 3 m respectively. The chimney has to with stand a horizontal wind pressure of 1.6 KN/m^2 of projected area. Determine the maximum and minimum stress intensities at the base if the weight of masonry per cubic metre is 20 KN. (698 KN/m^2 and 102 KN/m^2)

(9) A square chimney 25 metres high has an opening 1.2 m × 1.2 m inside. Find the necessary thickness at the base if the maximum permissible stress in brick masonry is 750 KN/m^2 and the intensity of horizontal wind pressure is 1.4 KN/m^2. take weight of masonry as 21 KN/m^3. (1.12 metres)

(10) A square chimney 20 metres high has an opening of 1 m × 1 m and wall thickness 0.30 metres. Calculate the maximum stress in masonry if the horizontal wind pressure is 2 KN/m^2 and weight of masonry 20 KN/m^3 (150.7 KN/m^2)

(11) A masonry dam is 8 metres high, 2 metres wide at top and 5 metres wide at bottom it retains water on the vertical face to the full height of the dam. Determine the stresses developed at the base. take weight of masonry as 21 KN/m^3 and that of water as 10 KN/m^3. (217 KN/m^2, 18.8 KN/m^2)

(12) A masonry dam 9 metre high is one metre wide at top and 3 metre wide at base has water on the vertical face up to 8.4 m. Calculate the maximum and minimum stresses at the base. Take weight of masonry 20 KN/m^3

(677.8 KN/m^2, 437.8 KN/m^2)

(13) A masonry dam of trapezoidal section with vertical water face is 12 metres high and 1.5 metre wide at the top. It retains water upto the full height of the dam. Find the necessary base. Width for no tension. Take wt. of masonry = 2 KN/m^3

(b = 7.175 metres)

(14) A retaining wall 6 metres high has to support earth level with the top on its vertical face. The batter of the sloping side is 1 in 3. Determine the top and bottom width if the angle of repose is 30 °. Take weight of earth = 18 KN/m^3 and wt. of masonry = 22 KN/m^3 Ans. (a = 1 m and b = 3 m)

(15) A masonry retaining wall trapezoidal is cross-section 12 m. high, has one face vertical and the other batter 1 in 6 and retains earth at its vertical face, level with the top. Calculate its base width for no tension at base. Earth weighs 16 KN/m^3 and masonry weighs 24 KN/m^3, angle of ropse of earth is 30° JMI.

□□□

11

Torsion Of Shafts

Torsion

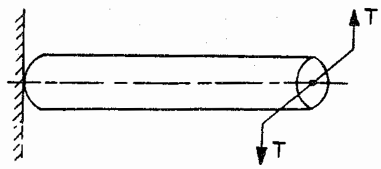


Fig. 11.1

When a shaft is rigidly fixed at one end and twisted at the other by a torque applied in a plane perpendicular to the longitudinal axis of the shaft as shown in figure 11.1, the shaft is said to be in a state of torsion.

The applied torque produces the following effects

(i) It imparts an angular displacement of one end cross-section with respect to the other end

(ii) It sets up shearing stresses on any cross-section of the shaft perendicular to its axis.

Twisting Moment

Twisting moment at any section along the shaft is the algebraic sum of the moments of the applied couples that lie to one side of the section under consideration.

Shearing Stress Due To Torsion

Shear stress produced due to the applied torque T at a distance r from the centre of the shaft is given by τ. This is also called torsional shear

$$\tau = \frac{T \, . \, r}{J}$$

Where J represents the polar moment of inertia of the shaft section.

Shearing Strain Due To Torsion

The angular displacement of one surface of the shaft from its original position due to the applied torque is called shearing strain at the surface and measured in radians.

Modulus Of Rigidity

The ratio of the shear stress and shear strain is called shear modulus or modulus of rigidity.

Assumptions

The torsion equation is based on the following assumptions.

1. A plane section of the shaft normal to its axis remains plane after the torques have been applied.
2. All diameters in the section which were straight before torque was applied remain straight

3. The twist along the length of the shaft is uniform throughout
4. The material of the shaft is uniform throughout.
5. Maximum shear stress induced in the shaft due to applied torque does not exceed its elastic limit value.

Relation between torsional stress, strain and angle of twist.

Consider a cylindrical shaft of length l and radius R as shown in figure 11.2

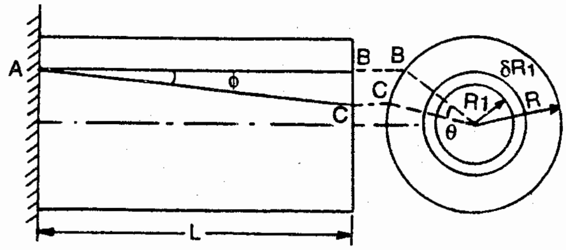


Fig. 11.2

A couple of magnitude T is applied at one end and the other end of the shaft is held by a balancing couple of equal magnitude. Because of the applied torque there is a relative twist of the two end cross-sections.

Since one end is fixed the line AB on the surface of the shaft moves to the position AC after strain. The angular displacement ϕ of the line AB to the helix AC is the shear strain at this surface and since ϕ is very small

$$\therefore B_C = L\phi \text{ or } \phi = \frac{BC}{L} \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

But $\quad \phi = \dfrac{\tau}{G} \quad$ or $\tau = \phi.G$

Where τ is the shear stress in the material at the surface of the shaft and G is the modulus of rigidity of the material. Let the angle of twist BOC be the angular movement of the radius OB due to the strain in the length L of the shaft.

Hence $\tau = \phi.G$ and $BC = R.\theta$

$$\tau = \frac{Bc}{L} \times G = \frac{R.\theta}{L}.G$$

or $$\frac{\tau}{G} = \frac{R.\theta}{L} \text{ or } \frac{\tau}{R} = \frac{G\theta}{L} \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Put $\dfrac{G.\theta}{L} = K$, a constant then

$$\tau_1 = R.K$$

Similarly if τ_1 is the shear stress at a radius R_1, then it follows that

We therefore deduce that the intensity of shear stress at any point in

$$\frac{\tau}{R} = \frac{\tau_1}{R_1} = \frac{\tau_2}{R_2} = K$$

, the cross-section of a circular shaft is proportional to its distance from the axis of the shaft.. It varies from zero at the axis to a maximum at the surface of the shaft.

Relation between twisting couple and shear stress.

Let us consider an elementry annular ring of radius R_l and thickness δR_l . Let τ_1 be the shear stress acting on it then,

The total force acting on the ring = $\tau_1 . 2\pi \ R_l \ \delta R_1$ and the moment of this force about the axis of the shaft = $\tau_1 . 2\pi R_l \delta \ R_l . R_l$

$$= \tau_1 .2\pi R_1^2.\delta R_l$$

$$= \frac{\tau}{R} . 2\pi . R_1^3.\delta R_l$$

When the ring is infinitely thin

The total resisting moment of the section

$$= 2\pi \ \frac{\tau}{R} \int_0^R \ R_1^3.\delta R_l = \frac{\tau.\pi R^3}{2}$$

But the total resisting moment of the section is equal to the couple T applied on it, then

$$T = \tau . \frac{\pi R^3}{2}$$

or $$T = \frac{\tau}{R} . \frac{\pi R^4}{2}$$

We know that the polar moment of inertia of the section $J = \dfrac{\pi R^4}{2}$

$$\therefore \quad T = \frac{\tau}{R} . J \quad \text{or} \quad \frac{T}{J} = \frac{\tau}{R}$$

Hence from equations (i) and (ii) we can write

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$$

this is known as torsion equation.

Units of measurement of these quantities are

T = Torque or Twisting moment in N–mm
J = Polar moment of inertia in mm^4
τ = Shear stress in MPa
G = Modulus of rigidity in KN/mm^2 or GN/m^2
R = Radius of shaft in mm
L = Length of shaft in metres or mm.
θ = Angle of twist in radians.

Torsional Rigidity

Torsional rigidity is the torque that produces a twist of one radian in a shaft of unit length

Polar Modulus

For a given shaft J and R are constants. The ratio $\frac{J}{R}$ is also a constant and called polar modulus of the section.

$$\text{Polar modulus} = \frac{\text{Polar moment of inertia}}{\text{Maximum radius}}$$

Angle Of Twist

when a torque T is applied on a circular shaft a line $A\,B$ on the surface of the shaft moves to the position AB' producing a shearing strain ϕ, and simultaneously the radius O B moves through an angle θ to the corresponding position OB'. Since this is caused by the twisting moment hence this angle θ is called angle of twist. Fig. 11.3

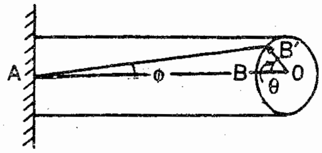


Fig. 11.3

Strength Of A Solid Shaft

The maximum torque or power transmitted by a solid shaft is known as the strength of the solid shaft.

From the torsion equation we know that

$$\frac{T}{J} = \frac{\tau}{R}$$

or $$T = J \times \frac{\tau}{R}$$

Maximum torque will be transmitted when maximum shear stress is produced at the top surface of the shaft of radius R

$$\therefore \quad T = \frac{\pi}{32} D^4 \frac{\tau}{D/2} = \frac{\pi}{16} \tau \times D^3$$

Hence the strength of a solid shaft is given by

$$T = \frac{\pi}{16} \tau D^3$$

Strength Of A Hollow Shaft

The maximum torque transmitted by a hollow shaft of external diameter D and internal diameter d will be

$$T = \frac{\pi}{16} \tau \frac{(D^4 - d^4)}{D}$$

Example 11.1

Find the maximum torque that can be applied safely to a shaft of 300 mm diameter. The permissible angle of twist is 1.5 degree in a length of 7.5 metres and shear stress is not to exceed 42 MPa.

Take $G = 84.4$ KN/mm^2 J. M. J

Solution

Torque that can be applied from the consideration of permissible angle of twist .

$$\frac{T}{J} = \frac{G\theta}{L}$$

Now $\quad J = \frac{\pi}{32}(30)^4 \text{ mm}^4 = 795.2 \times 10^6 \text{ mm}^4$

or $\quad T = \frac{J \times G\theta}{L} = \frac{795.2 \times 10^6 \times 84.4 \times 10^3 \times 1.5}{7.5 \times 10^3} \frac{\pi}{180}$

$= 234.6$ KN–m

Torque from shear stress consideration.

$$\frac{T}{J} = \frac{\tau_s}{R}$$

or $\quad T = \frac{\pi}{16}\tau_s \,.\, D^3 = \frac{\pi}{16} \times 42 \times (300)^3$

$= 222.7$ KN–m

The smaller value of the torque ie 222.7 KN–m is the maximum torque that can be safely applied.

Example 11.2

A specimen metallic bar 300 mm diameter and 300 mm long stretches 1.25 mm when a tensile force of 60 KN is applied. The same specimen when tested under torsion twisted .030 radian under an applied torque of 500 N–m. Determine the poisson's ratio and the values of elastic constants E, G, and K.

Area of the bar $= \frac{\pi}{4}(30)^2$

Applied tensile force $= 60 \times 10^3$ N

Tensile stress $= \sigma = \frac{P}{A} = \frac{60 \times 10^3}{\frac{\pi}{4}(30)^2} = 84.88$ MPa

Strain $\varepsilon = \frac{.1025}{300}$

Modulus of elasticity $= E = \frac{\sigma}{\varepsilon} = \frac{84.88}{.1025/300} = 248.4$ KN/mm^2

From torsion equation we know that

$$\frac{T}{J} = \frac{G.\theta}{L}$$

or $\quad G = \frac{TL}{J.\theta} = \frac{500 \times 10^3 \times 300}{\frac{\pi}{32}(30)^4 \times .03} = 62.8$ KN/mm^2

Now using relation $E = 2G(1 + \mu)$

$248.4 = 2 \times 62.8(1 + \mu)$

or $\quad \mu = .318 \quad$ or $\quad$ Poission's Ratio $= .318$

Again using the relation

$$E = 2\,K\,(1 - 2\,\mu)$$

$$K = \frac{E}{2(1-2\,\mu)} = \frac{248.4}{2(1-2\times.318)} = 227.4\ \text{KN/mm}^2$$

Example 11.3

Determine the maximum shearing stress in a 100 mm diameter solid shaft carrying a torque of 25 KN–m. What is the angle of twist per unit length of the shaft. Take G = 85 GN/m².

Solution

Applied Torque = 25 KN–m = 25×10^6 N–mm

Diameter of the shaft = 100 mm

Modulus of rigidity = 85 GN/m² $= \dfrac{85 \times 10^9}{10^6}$ N/mm²

$= 85 \times 10^3\ \text{N/mm}^2$

Applying torsion equation

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G.\theta}{L}$$

$$T = \frac{J.\tau}{R} = \tau \,.\, \frac{\pi}{32}\,\frac{D^4}{\frac{D}{2}} = \frac{\pi}{16}\,\tau.D^3$$

$$25 \times 10^6 = \frac{\pi}{16}\,\tau\,(100)^3 \quad \text{or} \quad \tau = \frac{25 \times 10^6 \times 16}{\pi\,(100)^3}$$

$$\tau = 127.3\ \text{MPa}$$

For angle of twist per meter length

$$\frac{T}{J} = \frac{G.\theta}{L}$$

$$\text{or } \theta = \frac{T.L}{J \times G} = \frac{25 \times 10^6 \times 1000}{\frac{\pi}{32}(100)^4 \times 85 \times 10^3}$$

$\theta = 0.0299$ radian per metre.

Example 11.4

A mild steel shaft 50 mm in diameter and 0.5 metre long is tested in a tension testing machine untill one end rotates through an angle of 0.6 degrees with respect to the other end. For this angle of twist the torque measured was 1135-N–m. Find the value of shear modulus and shear stress.

Solution

Polar moment of inertia

$$J = \frac{\pi}{32}(d)^4 = \frac{\pi}{32}(50)^4 = 61.35 \times 10^4\ \text{mm}^4$$

Using the relation $\dfrac{T}{J} = \dfrac{G\,\theta}{L}$

$$G = \frac{T}{J} \times \frac{L}{\theta} = \frac{1135 \times 10^3 \times 0.5 \times 10^3}{61.35 \times 10^4 \times 0.6 \times \frac{\pi}{180}}$$

$$G = 88.32 \times 10^3 \text{ N/mm}^2 = 88.32 \text{ KN/mm}^2$$

For shear stress

$$\tau = \frac{T}{J} \times R = \frac{135 \times 10^3 \times 25}{61.35 \times 10^4} = 46.24 \text{ MPa}$$

$\tau = 46.24$ MPa **Answer.**

Power transmitted through shaft

Let

T = Average torque applied in N–m

N = Number of revolutions per minute

P = Power transmitted in Kilo watts

Then

Power transmitted = Av. torque $\times$ angle turned per second

$$P = T \cdot \frac{N}{60} \times 2\pi \text{ watts}$$

$$P = \frac{2\pi N T}{60} \text{ watts}$$

$$P = \frac{2\pi NT}{60{,}000} \text{ Kilo watts}$$

Example 11.5

Determine the Power transmitted by a solid shaft of diameter 100 mm running at 120 rpm if the angle of twist per metre length of the shaft is 0.5 degree.

Take modulus of rigidity G = 80 GN/m^2.

Solution

Diameter of Shaft = 100 mm

$$\text{Modulus of rigidity} = 80 \text{ GN/m}^2 = \frac{80 \times 10^9}{10^6} \text{ N/mm}^2$$

$$= 80 \times 10^3 \text{ N/mm}^2$$

Angle of twist = 0.5 degree/metre

Number of revolutions/minute = 120

From Torsionequation we know that

$$\frac{T}{J} = \frac{G.\theta}{L}, \qquad \text{Now } \theta = 0.5° = \frac{.5 \times \pi}{180} \text{ radian}$$

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (100)^4 \text{ m.}$$

$$\therefore \quad T = \frac{J \times G \times \theta}{L} = \frac{\pi}{32} \frac{(100)^4 \times 80 \times 10^3 \times 0.5 \times \pi}{1000 \times 180} \text{ N–mm}$$

$= 6850$ N–m

$$\text{Power transmitted} = \frac{2\pi N T}{60{,}000}$$

$$= \frac{2\pi \times 120 \times 6850}{60{,}000} = 86.07 \text{ KW} \qquad \textbf{Answer}$$

Example 11.6

A solid shaft of 100 mm diameter transmits 140 KW at 200 rpm. Determine the maximum intensity of shear stress and the angle of twist for a length of 8 metres. Take G = 80 GN/m².

Solution

Power transmitted = 140 KW

Speed = 200 r pm, Length = 8 m = 8000 mm

$$\text{Modulus of rigidity} = \frac{80 \times 10^9}{10^6} = 80 \text{ KN/mm}^2$$

$$P = \frac{2\pi NT}{60{,}000} \quad \text{or} \quad T = \frac{140 \times 60{,}0000}{2\pi \times 200} = 6.684 \times 10^3 \text{ N–m}$$

Applying torsion equation

$$\frac{T}{J} = \frac{\tau}{R} \quad \text{or} \quad \tau = \frac{T.R}{J}$$

$$T = \frac{6.684 \times 10^3 \times 10^3 \times 50}{\frac{\pi}{32}(100)^4} = 34.04 \text{ MPa}$$

Again for angle of twist

$$\frac{T}{J} = \frac{G.\theta}{L} \quad \text{or} \quad \theta = \frac{T}{J} \times \frac{L}{G}$$

$$\theta = \frac{6.684 \times 10^6 \times 8000}{\frac{\pi}{32}(100)^4 \times 80 \times 10^3} = 0.068 \text{ radian}$$

θ = 3.89 degrees **Answer.**

Example 11.7

Determine the diameter of a solid steel shaft which will transmit 112.5 KW at 200 rpm. Also determine the length of the shaft if the twist must not exceed 1.5 ° over the entire length. The maximum shear stress is limited to 55 N/mm². Take the value of modulus of rigidity = 8 × 10⁴ N/mm²

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Solution

Power transmitted

$$P = \frac{2\pi NT}{60{,}000}$$

$$112.5 = \frac{2\pi \times 200 \times T}{60{,}000}$$

$$\text{or } T = \frac{112.5 \times 60{,}000}{2\pi \times 200} \text{ N–m} = 5371 \text{ N–m}$$

$$= 5371 \times 10^3 \text{ N–mm}$$

Now

$$T = \frac{\pi}{16} \tau_s \,.\, D^3$$

$$5371 \times 10^3 = \frac{\pi}{16} \times 55 \times D^3 \quad \text{or } D = 79.2 \text{ mm say } 80 \text{ mm}$$

$$\text{Again} \quad T = \frac{G.\theta}{L} \times J$$

$$\text{or } L = \frac{G.\,\theta\,.J}{T}$$

$$L = 8 \times 10^4 \times \frac{1.5 \times \pi}{180} \times \frac{\pi}{32} \,.\, (79.2)^4 \text{ mm}$$

$$= 1.5 \text{ meters} \qquad \textbf{Answer}$$

Example 11.8

A hollow steel shaft has to transmit 6000 KW at 110 rpm. If the allowable shear stress is 60 MPa and inside diameter is $\frac{3}{5}$ th of the outside diameter, determine the diameters of the shaft. Also find the angle of twist in a length of 3 metres. Take $G = 80$ KN/mm^2.

Solution

$$\text{Power transmitted } P = \frac{2\pi NT}{60{,}000}$$

$$6000 = \frac{2\pi NT}{60{,}000}$$

$$\text{or} \qquad T = \frac{6000 \times 60{,}000}{2\pi \times 110} \text{ N–m} = 5\,20.8 \times 10^6 \text{ N–mm}$$

Applying torsion equation

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\text{or} \quad T = \tau \,.\, \frac{J}{R} = \frac{\pi}{16} \tau \frac{(D^4 - d^4)}{D}$$

$$\text{or} \quad \left(\frac{D^4 - d^4}{D}\right) = \frac{T \times 16}{\pi \times \tau} = \frac{520.8 \times 10^6 \times 16}{\pi \times 60}$$

$$\left[\frac{(D^4 - \left(\frac{3}{5} D\right)^4}{D}\right] = 44212.8$$

$$\text{or} \left(\frac{625 - 81}{625}\right) D^3 = 44212.8$$

$$D^3 = \frac{44212.8 \times 625}{544} = 50795.97$$

$$\therefore \; D = 370 \text{ mm and } d = 370 \times \frac{3}{5} = 222 \text{ mm}$$

Hence external diameter of the shaft = 370 mm

and internal diameter = 222 mm

Angle of twist

From torsion equation

$$\frac{\tau}{R} = \frac{G\theta}{L}$$

$$\theta = \frac{\tau \times L}{G \times R} = \frac{60 \times 3 \times 10^3}{80 \times 10^3 \times \frac{370}{2}} = 0.012 \text{ radian}$$

$\theta = 0.012$ radian **Answer.**

Example 11.9

A hollow shaft of diameter ratio 3/5 is to transmit 600 KW at 110 rpm, the maximum torque being 12% greater than the mean. If the shear stress is not to exceed 60 MPa and the twist in length of 3 metres not to exceed 1 °, determine the minimum external diameter satisfying these conditions. Take G = 80 KN/mm^2. (Bombay Univ.)

Solution

$$\text{Average Power transmitted } P = \frac{2\pi N T}{60{,}000}$$

$$\text{or } 600 = \frac{2\pi 110 T}{60{,}000} \text{ or } T_{mean} = 52.08 \times 10^3 \text{ N–m}$$

$$\therefore \; T_{max} = (T_{mean} + 12\% \text{ of } T_{mean})$$
$$= (1.12 \times 52.08 \times 10^3) = 58.33 \times 10^3 \text{ N–m}$$
$$= 58.33 \times 10^6 \text{ N–mm}$$

$$T = \frac{\pi}{16} \tau \frac{(D^4 - d^4)}{D} \text{ or } \frac{D^4 - d^4}{D} = \frac{16T}{\pi\tau}$$

$$\text{or} \left[\frac{\left(D^4 - \left(\frac{3}{5}D\right)^4\right)}{D}\right] = \frac{16 \times 58.33 \times 10^6}{\pi \times 60} = 4951.8 \times 10^3$$

$$\text{or } \frac{D^4}{D}\left(1 - \frac{81}{625}\right) = 4951.8 \times 10^3 \text{ or } D^3 = \frac{4951.8 \times 10^3 \times 625}{544}$$

$$\text{or } D = 178.5 \text{ mm}$$

From shear stress consideration

$$\frac{\tau}{R} = \frac{G\theta}{L} \text{ or } R = \frac{\tau \times L}{G \times \theta} = \frac{60 \times 3 \times 1000}{80 \times 10^3 \times \frac{\pi}{180}}$$

or $R = 128.9$ mm or $D = 257.8$ mm

Adopt the larger value of D, Hence dia of shaft = 258 mm **Answer**

Example 11.10

A hollow circular shaft of 80 mm external diameter and 70 mm internal diameter is subjected to a torque of 600 N–m and an angle of twist of 0.3 degrees was observed over a length of 1.25 metres. Determine the deflection at the centre of the shaft when placed horizontally over supports 1.25 m apart. The seft weight of the shaft may be taken as 200 N/metre and poisson's ratio $\mu = 0.25$

Solution

Polar moment of inertia $J = \frac{\pi}{32}[(80)^4 - (70)^4]$

$$J = 166.4 \times 10^4 \text{ mm}^4$$

From the relation $\frac{T}{J} = \frac{G.\theta}{L}$

$$G = \frac{T \times L}{J \times \theta} = \frac{600 \times 10^3 \times 1.25 \times 10^3}{166.4 \times 10^4 \times \frac{\pi}{180} \times 0.3}$$

$$= 86 \text{ KN/mm}^2.$$

Using the relation $E = 2\,G\,(1 + \mu)$

$$E = 2 \times 86 \times 10^3\ (1 + 0.25) = 215 \text{ KN/mm}^2$$

Moment of inertia $I = \frac{J}{2} = \frac{166.4 \times 10^4}{2} = 83.2 \times 10^4 \text{ mm}^4$

Total weight of the shaft = $(w.l) = 200 \times 1.25 = 250$ N

Deflection at the centre of the shaft

$$y_c = \frac{5wl^4}{384EI} = \frac{5\,W\,l^3}{384\,E\,I}$$

$$= \frac{5 \times (200 \times 1.5)\,(1.25 \times 10^3)}{384 \times 215 \times 10^3 \times 83.2 \times 10^4}$$

$y_c = 0.0354$ mm **Answer.**

Example 11.11

The power transmitted by a hollow shaft at 90 rpm is 360 K watts. If the shear stress is not to exceed 60 MPa and the diameter ratio is 07, find the external and internal diameters of the shaft. Assume that the maximum torque is 30% greater than the mean torque. *(Mysore Univ.)*

Solution

Diameter ratio $= 0.7 = \frac{d}{D}$

or $d = 0.7\,D$

Power transmitted $= \frac{2\pi N T}{60{,}000} = 360$ KW

$$\therefore \quad T_{mean} = \frac{360 \times 60{,}000}{2\pi \times 90} = 38197.18 \text{ N–m}$$

$$T_{\max} = (38197.18) \times 1.3 = 49656.34 \text{ N–m}$$

Now $\quad \frac{T}{J} = \frac{\tau}{R}$

or $\quad T = \frac{\tau}{R} \times J$

Polar modulus $= \frac{\pi}{16} \frac{(D^4 - d^4)}{D}$

$$Z_p = \frac{\pi}{16D} [D^4 - (0.7\,D)^4] = .1011\,D^3$$

$$T_{max} = \tau \times .1011\,D^3 = 60 \times .1011\,D^3$$

or $\quad 49656.34 \times 10^3 = 60 \times .1011\,D^3$

or $\quad D^3 = \frac{49656.34 \times 10^3}{60 \times .1011}$

or $\quad D = 202$ mm

$d = .7 \times 202 = 141.4$ mm $\quad$ **Answer**

Comparison Between Solid And Hollow Shafts

(i) Comparison By Strength

Let us consider two shafts made of same material equal in weight and length and same maximum shear stress.

Let

d = internal diameter of the hollow shaft

D = external diameter of the hollow shaft

D_1 = diameter of the solid shaft

A_s = cross-sectional area of solid shaft

A_H = cross-sectional area of hollow shaft.

Now

$$T_s = \frac{\pi}{16} \tau_s . D_1^{\,3} \quad \text{and} \quad T_H = \frac{\pi}{16} \tau_s \frac{(D^4 - d^4)}{D}$$

$$\frac{T_H}{T_S} = \frac{D^4 - d^4}{D D_1^3} \qquad \text{Let } \frac{D}{d}\ n \text{ or } D = n\,d$$

Then $\quad \frac{T_H}{T_s} = \frac{n^4 d^4 - d^4}{nd \,.\, D_1^{\,3}} = \frac{d^3 (n^4 - 1)}{n D_1^{\,3}} \quad \ldots\ldots \quad \ldots\ldots \quad$ (i)

Since cross-sectional areas are same

$$A_H = A_S$$

$$\therefore \frac{\pi}{4} D_1^{\,2} = \frac{\pi}{4} (D^2 - d^2)$$

$$\text{or} \quad D_1 = \sqrt{(D^2-d^2)} \quad \text{or} \quad D_1^{\,3} = (D^2-d^2)\sqrt{(D^2-d^2)}$$

$$\text{or} \quad D_1^{\,3} = (n^2d^2-d^2)\sqrt{(n^2d^2-d^2)} = d^3\,(n^2-1)\sqrt{n^2-1}$$

Substituting this value of D_{13} in equation (i) we get

$$\frac{T_H}{T_S} = \frac{d^3(n^4-1)}{n\,d^3(n^2-1)\sqrt{n^2-1}} = \frac{(n^2-1)(n^2+1)}{n\,(n^2-1)\sqrt{(n^2-1)}}$$

$$\frac{T_H}{T_S} = \frac{n^2+1}{n\sqrt{n^2-1}}$$

Now if $\frac{D}{d} = 2$ ie $n = 2$, then we get

$$\frac{T_H}{T_S} = \frac{2^2+1}{2\sqrt{2^2-1}} = \frac{5}{2\sqrt{3}} = 1.442$$

This shows that the torque transmitted by a hollow shaft is 1.442 times more than the torque transmitted by a solid shaft. The hollow shaft is 1.442 times stronger than the solid shaft hence for heavy torques hollow shafts are preferred.

Comparison By Weight

Assume that both the shafts are made of same material and same length. Let the applied torque to both the shafts be same. The maximum shear stress will also be same in both hollow and the solid shafts.

Let W_H = Weight of the hollow shaft

W_S = Weight of the solid shaft.

As the length and material of both the shafts are same, therefore the weight of each shaft will be equal to its cross-sectional area

$$\therefore \quad W_H = A_H = \frac{\pi}{4}(D^2-d^2)$$

$$W_S = A_S = \frac{\pi}{4}D_1^{\,2}$$

$$\text{or} \quad \frac{W_H}{W_S} = \frac{D^2-d^2}{D_1^{\,2}}, \quad \text{Let } \frac{D}{d} = n \quad \text{or } D = n\,d$$

$$\text{or} \quad \frac{W_H}{W_S} = \frac{n^2d^2-d^2}{D_1^{\,2}} = \frac{(n^2-1)\,d^2}{D_1^{\,2}} \quad \ldots \quad \ldots \quad \text{(i)}$$

Since the applied torque $T_H = T_S$

$$\therefore \quad \frac{\pi}{16}\tau_s\frac{(D^4-d^4)}{D} = \frac{\pi}{16}\tau_s D_1^{\,3}$$

$$\text{or} \quad D_1^3 = \frac{D^4-d^4}{D} = \frac{d^3(n^4-1)}{n}$$

or $\quad D_1 = d\left(\frac{n^4-1}{n}\right)^{1/3}$ and $D_1{}^2 = d^2\left(\frac{n^4-1}{n}\right)^{2/3}$

Substituting in equation (i) we get

$$\frac{W_H}{W_S} = \frac{(n^2-1)n^{2/3}}{(n^4-1)^{2/3}}, \text{ Now if } \frac{D}{d} = n = 2$$

Then $\frac{W_H}{W_S} = \frac{(2^2-1)\,2^{2/3}}{(2^4-1)^{2/3}} = 0.78$

Hence hollow shafts are lighter in weight than the solid shafts.

Example 11.12

What percentage of strength of a solid circular steel shaft 120 mm diameter is lost by boring 60 mm axial hole in it ? Determine the loss of strength in the two cases. (J.M.I.)

Solution

Strength of the solid shaft $T_s = \frac{\pi}{16}\,\tau \times (120)^3$

$$T_s = 1728 \times 10^3 \times \frac{\pi}{16}\tau$$

Strength of the hollow shaft $T_H = \frac{\pi}{16}\tau\left[\frac{(120^4 - 60^4)}{120}\right]$

$$T_H = 153.66 \times 10^4 \times \frac{\pi}{16}\,\tau$$

Loss of Strength $= \frac{\pi}{16}\tau\left[1728 \times 10^3 - 1536.6 \times 10^3\right]$

$$= \frac{\pi}{16} \times \tau \times 192.2 \times 10^3$$

Percentage loss of strength

$$= \frac{\frac{\pi}{16} \times \tau \times 192.2 \times 10^3}{\frac{\pi}{16} \times \tau \times 1728 \times 10^3} \times 100$$

$$= \frac{192.2}{1728} \times 100 = 11.12\%$$

Example 11.13

A solid circular shaft 125 mm in diameter has the same cross-sectional area as a hollow shaft of the same material with an internal diameter of 100 mm find (a) the ratio of the power transmitter by the two shafts at the same angular velocity.

(b) Compare the angle of twist in equal lengths of these shafts when subjected to the same intensity of shear stress

Solution

Area of solid shaft $= \frac{\pi}{4}(125)^2$

Area of hollow shaft $= \frac{\pi}{4}(D^2 - 100^2)$

Equating the two areas we get

$$\frac{\pi}{4}(125)^2 = \frac{\pi}{4}(D^2 - 100^2)$$

or $D = 160$ mm

Torque transmitted by solid shaft

$$T_s = \frac{\pi}{16}\tau.\, D^3 = \frac{\pi}{16}\,\tau.\,(125)^3$$

Torque transmitted by hollow shaft

$$T_H = \frac{\pi}{16}\,\tau\,\frac{(D^4 - d^4)}{D} = \frac{\pi}{16}\,\tau\,\frac{(160^4 - 100^4)}{160}$$

$$\frac{\textit{Power transmitted by solid shaft}}{\textit{Power transmitted by hollow shaft}} = \frac{\textit{Torque transmitted by solid shaft}}{\textit{Torque transmitted by hollow shaft}}$$

$$\frac{T_S}{T_H} = \frac{\frac{\pi}{16}\tau.\,(125)^3}{\frac{\pi}{16}\tau.\frac{(160^4 - 100^4)}{160}} = \frac{125^3 \times 160}{160^4 - 100^4}$$

$$= \frac{125^3 \times 160}{65536 \times 10^4 - 100^4} = \frac{125^3 \times 160}{55536 \times 10^4}$$

$= 0.56$

(b) From the torsion equation we know

$$\frac{\tau}{R} = \frac{G.\theta}{L}$$

or $\theta = \frac{T.L}{G.R}$

For the same length and same intensity of shear stress the modulus of rigidity will be same

$$\therefore \frac{\text{Angle of twist of solid shaft}}{\text{Angle of twist of hollow shaft}} = \frac{\theta_s}{\theta_H} = \frac{\tau_s .l}{Rs.G}\ \frac{\tau_w .l}{R_H.G}$$

$$\frac{\theta_s}{\theta_H} = \frac{(R_H)\,(hollow)}{(R_s)\,(solid)} = \frac{160/2}{125/2} = 1.28 \qquad \textbf{Answer}$$

Replacing of Shaft

When a solid shaft is to be replaced by a hollow shaft or vice versa, then the power transmitted by the new shaft should always be equal to the power transmitted by the shaft to be replaced.

Example 11.14

A solid shaft of 200 mm diameter is replaced by a hollow shaft of external diameter 280 mm. Determine the thickness of the hollow shaft if the same power is transmitted at the same maximum shear stress and at the same rotational speed by both the shafts *(Engineering services)*

Solution

Power transmitted by solid shaft

= Power transmitted by hollow shaft

$P_{solid} = P_{hollow}$

$$\frac{2\pi N T_s}{60,000} = \frac{2\pi N T_H}{60,000} \quad \text{Since N is same for both the shaft}$$

$\therefore Ts = T_{hollow}$

$$\text{or} \quad \frac{\pi}{16}\tau_s \,.\, D_s^3 = \frac{\frac{\pi}{16}\tau_H\,(D_H^4 - d_H^4)}{D_H}$$

$$\therefore \quad D_s^3 = \frac{D_H^4 - d_H^4}{D_H}$$

$$(200)^3 = \frac{280^4 - d_H^4}{280} \quad \text{or} \quad d_H^4 = (280)^4 - 280\,(200)^3$$

$$d_H^4 = 614656 \times 10^4 - 2240000 = 390656 \times 10^4$$

$$d_H^2 = 62500 \quad or \quad d_H = 250 \text{ mm}$$

Internal diameter of the shaft = 250 mm

External diameter of the shaft = 280 mm

Thickness of hollow shaft $= \dfrac{280 - 250}{2} = 15$ mm **Answer**

Example. 11.15

A solid shaft 180 mm diameter is to be replaced by a hollow steel shaft whose internal diameter is 60% of the external diameter Determine the internal and external diameters and saving in the material. The value of maximum shear stress may be assumed as same for both the shafts.

Solution

Since shear stress for both shafts is same

$$\tau_S = \tau_{hollow}$$

Using the formula

$$T_{solid} = \frac{\pi}{16}\tau_s \,.(D_s)^3 = \frac{\pi}{16}\tau_s \,.\,(180)^3 \quad --- \quad --- \quad --- \quad (i)$$

$$T_{\text{hollow}} = \frac{\frac{\pi}{16}\tau_H\,[D_H^4 - (0.6D_H)^4]}{D_H} \qquad \text{---} \quad \text{---} \quad \text{---} \qquad \text{(ii)}$$

Equating (i) and (ii) we get

$$\frac{\pi}{16}\,\tau_s\,(180)^3 = \frac{\pi}{16}\,\frac{\tau_H\,[D_H^4 - (0.6D_H)^4]}{D_H}$$

or $(180)^3 = 0.8704\ D_H^3$ or $D_H^3 = \frac{(180)^3}{0.8704}$

or $D_H = 188.67$ mm

$\therefore$ $d_H = 0.6\,D_H = 0.6 \times 188.67 = 113.2$ mm

Net saving in the material $= \frac{A_s - A_H}{A_s}$

$$A_S = \frac{\pi}{4}\,(180)^2 = 25446.9\ \text{mm}^2$$

$$A_H = \frac{\pi}{4}\left[(188.67)^2 - (113.2)^2\right] = 17893.04\text{mm}^2$$

$\therefore$ Percentage saving $= \frac{A_S - A_H}{A_S} \times 100$

$$= \frac{25446.9 - 17893.04}{25446.9} \times 100 = 29.68$$

Example. 11.16

A hollow steel shaft is made to replace a solid wrought iron shaft of the same internal diameter, the material being 30% stronger than wrought iron. Find what fraction of external diameter of the shaft would be the internal diameter. (J.M.I. 1990)

Solution

For wrought iron solid shaft let D be the diameter, then

$$T_{\text{solid}} = \frac{\tau_w}{R}\,J = \frac{\pi}{16}\,\tau_w\,D^3$$

For hollow steel shaft

Let D be the external diameter and d the internal diameter since shear stress in steel is 30% more than in wrought iron shaft. Therefore allowable stress in steel shaft = 1.3 τ_w

$$\tau_S = 1.3\,\tau_w$$

$$T_{\text{hollow}} = \frac{\pi}{16} \times 1.3\,\tau_w \times \frac{(D^4 - d^4)}{D}$$

Torque transmitted by both the shaft is same

$\therefore$ $T_{\text{solid}} = T_{\text{hollow}}$

$$\frac{\pi}{16}\,\tau_w\,D^3 = \frac{\pi}{16} \times 1.3\,\tau_w\,\frac{(D^4 - d^4)}{D}$$

or $$D^4 = 1.3\,(D^4 - d^4)$$
$$= 1.3\,D^4 - 1.3\,d^4$$

or $$1.3\,d^4 = 0.3\,D^4$$

or $$\frac{d^4}{D^4} = \frac{0.3}{1.3} = 0.23076$$

or $$\frac{d}{D} = \sqrt[1/4]{0.23076} = 0.693$$

The internal diameter = 0.693 D **Answer**

Composite Shaft

When two shafts of same or different lengths, cross-sections or materials are connected together to form a single shaft it is known as a composite shaft.

Shafts in Series

When a Composite shaft connected in series is subjected to a torque then torque transmitted by each individual shaft is same. Torque applied at one end of the Composite shaft is equal to the resisting torque at the other end.

Total angle of twist at the fixed end or the resisting end of the shaft is the sum of the angles of twist of the two shafts. If θ_1 and θ_2 are the angles of twist of first and second shaft the total angle of twist θ will be

$$\theta = \theta_1 + \theta_2$$
$$= \frac{TL_1}{J_1G_1} + \frac{TL_2}{J_2G_2}$$

or $$\theta = T\left\{\frac{L_1}{J_1G_1} + \frac{L_2}{J_2G_2}\right\}$$

When both shafts are of same material then $G_1 = G_2 = G$, the total angle of twist will be

$$\theta = \frac{T}{G}\left(\frac{L_1}{J_1} + \frac{L_2}{J_2}\right)$$

If both shafts have same length and cross section *ie.*

$$L_1 = L_2 = \frac{L}{2} \text{ and } J_1 = J_2 = J$$

then $$\theta = \frac{T}{G}\left(\frac{L}{2J} + \frac{L}{2J}\right)$$

$$\theta = \frac{T.L}{GJ}$$

Shafts in Parallel

When the driving torque is applied at the junction of two connected shafts they are said to be connected in parallel. Resisting torques develop at both the ends. Torque transmitted by each shaft is different but the angle of twist for both the shafts is same

$$\theta_1 = \theta_2$$

or $$\frac{T_1 L_1}{J_1 G_1} = \frac{T_2 L_2}{J_2 G_2}$$

Total torque $T = T_1 + T_2$

If the shafts are of same material

$$\frac{T_1 L_1}{J_1} = \frac{T_2 L_2}{J_2} \quad \text{or} \quad \frac{T_1}{T_2} = \frac{J_1 L_2}{J_2 L_1}$$

If the shafts have same cross section

or $$\frac{T_1}{T_2} = \frac{L_2}{L_1}$$

Example. 11.17

A Solid Steel Shaft 60 mm diameter is fixed rigidly and Co-axially inside a bronze sleeve 90 mm diameter. Calculate the angle of twist in a 2 metre length of the composite shaft when subjected to a pure torque of 1000 N -m. Take the modulus of rigidety of steel as 80 KN/mm^2 and of bronze as 42 KN/mm^2.

Solution

$$J_S = \frac{\pi}{32} d^4 = \frac{\pi}{32} (60)^4 = 127.2 \times 10^4 \text{ mm}^4$$

$$J_b = \frac{\pi}{32} (90^4 - 60^4) = 517 \times 10^4 \text{ mm}^4$$

Since the two shafts are connected in parallel therefore total torque

$$T = T_s + T_b$$

$$= \frac{G_s . J_s . \theta}{L} + \frac{G_b J_b . \theta}{L}$$

or $$10^6 = (80 \times 10^3 \times 127.2 \times 10^4 + 42 \times 10^3 \times 517 \times 10^4) \frac{\theta}{2000}$$

or $$\theta = \left(\frac{2000 \times 10^6}{3189 \times 10^8} \right) \times \frac{180}{\pi}$$

$= .359$ degree **Answer.**

Example 11.18

A Composite shaft consists of a solid aluminium alloy shaft of diameter 50 mm enclosed in a hollow circular steel shaft 60 mm external diameter and 5 mm thick. The two metals are rigidly connected at their juncture. If the composite shaft is loaded by a twisting moment of 2 KN -m, Calculate the shearing stress at the outer fibres of steel and aluminium, if both the shafts have equal lengths and welded to a plate at each end, so that their twists are equal. Take G_A = 30 KN/mm^2 and G_s = 85 KN/mm^2

Solution

Let T_1 = torque carried by aluminium shaft

T_2 = torque carried by steel shaft

Then $T_1 + T_2 = T = 2$ KN -m

Since twist in both shafts are equal and $L_1 = L_2$

Hence $\theta_1 = \theta_2$

or $\dfrac{T_1 L_1}{J_1 G_1} = \dfrac{T_2 L_2}{J_2 G_2}$ or $\dfrac{T_1}{J_1 G_1} = \dfrac{T_2}{J_2 G_2}$

$$T_1 = \frac{T_2 \cdot J_1 G_1}{J_2 G_2} = T_2 \left[\frac{\frac{\pi}{32}(50)^4 \times 30 \times 10^3}{\frac{\pi}{32}(60^4 - 50^4) \times 85 \times 10^3}\right]$$

$$T_1 = T_2 \times \frac{625 \times 10^4 \times 30 \times 10^3}{671 \times 10^4 \times 85 \times 10^3} = .328\, T_2$$

Now $T_1 + T_2 = 2$ KN -m

$$.328\, T_2 + T_2 = 2 \quad \text{or} \quad T_2 = \frac{2}{1.328} = 1.506 \text{ KN -m}$$

$\therefore \quad T_1 = (2 - 1.506) = 0.494$ KN -m

The shearing stress at the extreme fibre of the steel shaft

$$\tau_2 = \frac{1.506 \times 10^3 \times 30 \times 10^3}{\frac{\pi}{32}(60^4 - 50^4)} = \frac{1.506 \times 30 \times 10^6}{\frac{\pi}{32} \times 671 \times 10^4}$$

$$= 68.5 \text{ MPa}$$

$$\tau_1 = \frac{0.494 \times 10^3 \times 25 \times 0^3}{\frac{\pi}{32}(50)^4} = 20.12 \text{ MPa}$$

Example. 11.19

A shaft of 30 mm diameter and 1 metre length is subjected to a torque of 0.3 KN-m at one end. The other end of the shaft is fixed and a hole is drilled earlier in a part of the shaft. If the maximum permissible shear stress is 80 MPa and allowable angle of twist is 0.3 degree, calculate the diameter and length of the hole. Take G = 80 KN/mm²

Solution

Due to drilled hole, the shaft has two sections and torque is applied at one end. The shaft is connected in series. Since the area of cross-section of hollow region is less it is subjeceted to maximum shear stress. Let l be the length of drilled portion and d be the diameter of the hole.

$$\text{Torque } T = \frac{\pi}{16}\, \tau\, \frac{(D^4 - d^4)}{D}$$

$$0.3 \times 10^6 = \frac{\pi}{16} \times 80 \, \frac{(30^4 - d^4)}{30}$$

$$\text{or} \quad (30^4 - d^4) = \frac{0.3 \times 10^6 \times 16 \times 30}{\pi \times 80}$$

$$d = 22 \text{ mm}$$

Total angle of twist $\theta = \theta_h + \theta_s$

$$\theta = \frac{T}{G}\left\{\frac{l}{J_h} - \frac{(1000 - l)}{J_s}\right\}$$

$$J_h = \frac{\pi}{32}(D^4 - d^4)$$

$$= \frac{\pi}{32}(30^4 - 22.00^4) = \frac{\pi}{32}(81000 - 234256)$$

$$= 56523.5 \text{ mm}^4$$

$$J_s = \frac{\pi}{32}(D)^4 = \frac{\pi}{32}(30)^4 = 79521.56$$

$$\therefore \quad \theta = \frac{300000}{80{,}000}\left[\frac{l}{56523.5} - \frac{(1000 - l)}{79521.5}\right]$$

$$\frac{0.3 \times \pi}{180} = \mathbf{3.75}\,[1.76\times 10^{-5} l - .0125 + 1.25\times 10^{-5} l\,]$$

or $l = 461.6$mm **Answer**

Example. 11.20

A steel shaft 30 mm diameter and 4 metres long is rigidly fixed at ends as shown in figure 11.4. A twisting moment of 200 N-m is applied at a distance of 1 metre from one end. Calculate the fixing couples at the ends, the maximun shear stress induced and the angle of twist of the section where the twisting moment is applied. Take G = 84 GN/m^2. *(Camb. Univ.)*

Solution

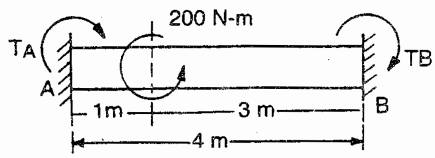


Fig. 11.4

Let T_A and T_B be the fixing couples at A and B, then $T_A + T_B$ = 200 N-m

From the consideration of consistent deformation the angle of twist in each portion is same $\theta_A = \theta_B$

$$\frac{T_A L_A}{JG} = \frac{T_B L_B}{JG} \quad \text{or} \quad T_A L_A = T_B . L_B$$

$$\text{or} \quad T_A = T_B \cdot \frac{L_B}{L_A} \quad \text{or} \quad T_B \cdot \frac{3}{1} = 3\,T_B$$

Now $T_A \times T_B = 200$ putting $T_A = 3\,T_B$, we get

$3T_B + T_B = 200$ or $T_B = 50$ N -m and $T_A = 150$ N -m

For Maximum shear stress in segment A

$$T_A = \frac{\pi}{16}\tau\,(d)^3 \quad \text{or} \quad \tau = \frac{150 \times 10^3 \times 16}{\pi\,(30)^3} = 28.3 \text{ MPa}$$

For Angle of twist $\theta = \dfrac{\tau}{R}\dfrac{L_A}{G} = \dfrac{28.3 \times 1 \times 10^3}{15 \times 84 \times 10^3}$

$\theta = 0.022$ radian **Answer**

Strain energy stored in a shaft subjected to a torque T.

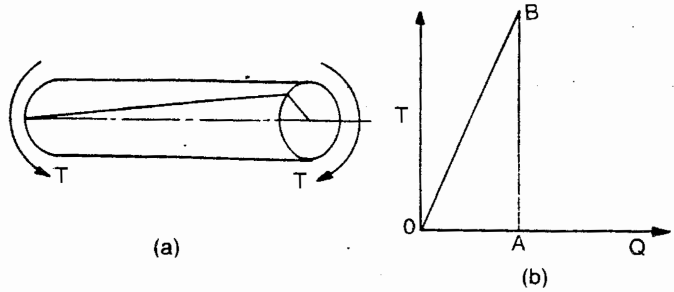


Fig. 11.5

When a shaft is subjected to a torque T, the angle of twist θ is given by the relation

$$\theta = \frac{T.L}{G.J}$$

If the torque T and angle of twist θ are represented along the vertical and horizontal axis as shown in figure 11.5 (b) and point B represents the applied torque T the amount of work done on the shaft is stored as internal energy in the shaft and represented by the Δ OAB.

$$U = \frac{1}{2} T.\theta$$

$$= \frac{1}{2} T. \frac{T.L}{G.J}$$

or $$U = \frac{T^2.L}{2GJ} = \frac{\tau^2}{4G} \times \text{ Volume of the shaft}$$

Where L is the length of shaft, G the modulus of rigidity and J is the polar moment of inertia of the shaft section.

For hollow shafts of outer radius R and inner radius r

$$\text{Strain energy} = \frac{1}{2} T.\theta$$

$$= \frac{1}{2} \frac{T^2}{J} \cdot \frac{L}{G}$$

$$= \frac{1}{2} \frac{\left[\frac{\pi}{2} \tau \left(\frac{R^4 - r^4}{R}\right)\right]^2}{\frac{\pi}{2}(R^4 - r^4)} \times \frac{L}{G}$$

$$= \frac{1}{4} \frac{\tau^2}{G} \cdot \frac{R^2 + r^2}{R^2} [\pi \ (R^2 - r^2) \ L]$$

$$= \frac{1}{4} \frac{\tau^2}{G} \cdot \frac{R^2 + r^2}{R^2} \times \text{ Volume of hollow shaft.}$$

$$= \frac{\tau^2}{4G} \cdot \frac{(D^2 + d^2)}{D^2} \times \text{Volume of the hollow shaft.}$$

Example 11.21

A solid shaft 100 mm diameter is to be replaced by a hollow shaft of the same material, weight and length. Calculate the diameter of this shaft if its strain energy is to be 15% more than that of the solid shaft when transmitting torque at the same maximum shear stress.

Solution

Let D and d be the external and internal diameters of the hollow shaft strain energy of the hollow shaft

Strain energy of the hollow shaft

$$= 1.15 \times \text{ strain energy of solid shaft}$$

$$= 1.15 \times \frac{\tau^2}{4G} \times \text{ Volume of solid shaft}$$

or $$\frac{\tau^2}{4G} \cdot \frac{(D^2 + d^2)}{D^2} \times \text{ Volume} = 1.15 \times \frac{\tau^2}{4G} \times \text{ Volume}$$

or $$D^2 + d^2 = 1.15 . D^2$$

$$.15\, D^2 = d^2 \qquad --- \qquad --- \qquad --- \qquad \text{(i)}$$

Since for both shafts, length weight and materials are same therefore cross-sectional areas of the two must be equal.

$\therefore$ $$\frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(100)^2$$

or $$D^2 - d^2 = (100)^2$$

$\therefore$ $$D^2 - .15\, D^2 = (100)^2 \quad \text{or} \quad D = 108.46 \text{ mm}$$

$\therefore$ $$d = 42. \text{ mm}$$

Example 11.22

A solid circular shaft is required to transmit 220 KW at 100 rpm. If the shear stress is not to exceed 50 MPa, calculate the diameter of the shaft and the strain energy stored per metre length. Take $G = 80\ GN/m^2$.

Solution

$$\text{Power transmitted} = \frac{2\,\pi\, NT}{60{,}000}$$

$$220 = \frac{2\,\pi\, NT}{60{,}000} \quad \text{or} \quad T = \frac{60{,}000 \times 220}{2\pi \times 100}$$

$$T = 21.008 \times 10^3 \text{ N-m} \qquad = 21.008 \times 10^6 \text{ N-mm}$$

$$T = \frac{\pi}{16}\tau\,(d)^3 \quad \text{or} \quad d^3 = \frac{16T}{\pi.\tau} = \frac{21.0 \times 10^6 \times 16}{\pi \times 50}$$

$$d = 128.8 \text{ mm}$$

Strain energy per metre length

$$U = \frac{\tau^2}{4G} \times \text{ Volume of the shaft}$$

$$= \frac{(50)^2 \times 10^6}{4 \times 80 \times 10^9} \times \frac{\pi}{4} (128.8)^2 \times 1000 = 101.79 \text{ KN-mm}$$

Answer

Keys and Flanged Couplings

A key is inserted between two machine parts to prevent relative motion between them. A key is necessery for connecting a shaft and the surrounding hub as shown in figure 11.6

Let l_k = length of the key

b_k = width of the key

τ_k = safe shearing stress in the key

then resistance set up by the key = $\tau_K . l_k . b_k$

Let d = diameter of the shaft then the moment that can be transmitted by the Key = $\tau_k . l_k . b_k . \frac{d}{2}$

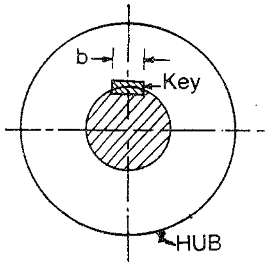


Fig. 11.6

Maximum Torque $T = \tau_s . \frac{\pi}{16} d^3$

The moment transmitted by the key must be equal to the torsion on the shaft

$$T = \tau_s . \frac{\pi}{16} d^3 = \tau_k . l_k . b_k . \frac{d}{2}$$

Coupling

When shafts of required lengths are not available, then two shafts are connected by coupling. The coupling surrounds the two shafts and connection between each shaft and coupling is provided by the key. The two parts of the coupling are held together by bolts as shown in fig. 11.7 the bolts

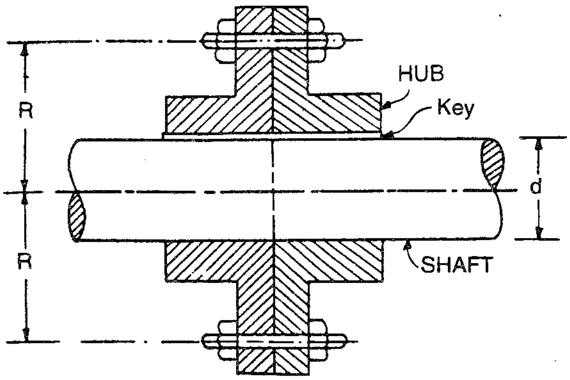


Fig. 11.7

are arranged along a circle known as bolt circle. The bolts are subjected to shear stress when the torque is being transmitted.

Let d_b = diameter of the bolt

n = number of bolts provided on the bolt circle of radius R_b

τ_b = Safe shear stress in the bolt

Resistance of one bolt = $\tau_b . \frac{\pi}{4} d_b^2$

Total moment transmitted by n bolts

$$T = n. \ \tau_b . \frac{\pi}{4} d_b^2 \times R$$

Equating maximum torque on the shaft to the moment transmitted by the bolts

$$T = \tau_s . \frac{\pi}{16} d^3 = n. \tau_b . \frac{\pi}{4} d_b{}^2 . R_b$$

Example 11.23

Two 100 mm diameter shafts are connected by means of two flanges with 20 mm dia. bolts equalty spaced on a circle of diameter 240 mm. If the maximum shear stress in the shafts due to the torque is not to exceed 120 MPa and the average shear stress in the bolts is not to exceed 80 MPa for the same torque, determine the number of bolts required.

Solution

Torque transmitted by the shaft

$$T = \frac{\pi}{16} \tau D^3 = \frac{\pi}{16} \times 120 (100)^3 = 23.56 \times 10^6 \text{ N-mm}$$

Torque transmitted by bolts

$$T = \frac{\pi}{4} d_b^2 . \tau_b . n . R_b$$

$$= \frac{\pi}{4} (20)^2 \times 80 \times n \times 120 = 301.59 \times 10^4 . n$$

$$= 301.59 \times 10^4 \times n \quad \text{N-mm}$$

Since the torque transmetted is the same

$\therefore (301.59) \times 10^4 \text{ n} = 235.56 \times 10^6$

or $\quad n = \dfrac{235.56 \times 10^6}{301.59 \times 10^4} = 7.8$

Number of bolts required = 8 **Answer**

Example 11.24

A 60 mm dia. shaft transmits 120 KW at 100 rpm. A flanged coupling is keyed to the shaft by means of a key 120 mm long and 30 mm wide. The coupling has 6 bolts of 20 mm diameter symmetrically arranged along a bolt circle of 240 mm diameter. Determine the shear stresses in the shaft the key and the bolts of the coupling.

Solution

$$\text{Power transmitted } P = \frac{2\pi NT}{60{,}000}$$

$$120 = \frac{2\pi \times 100T}{60{,}000} \quad \text{or} \quad T = 11.45 \times 10^3 \text{ N-m}$$

$$T = 11.45 \times 10^6 \text{ N-mm}$$

(i) shear stress in the shaft

$$T = \frac{\pi}{16} \tau_s . d^3$$

$$11.45 \times 10^6 = \frac{\pi}{16} \times \tau_s (60)^3$$

$$\tau_s = \frac{11.45 \times 10^6 \times 16}{\pi \times (60)^3} = 270\, MPa$$

(ii) Shear stress in the Key

$$T = \tau_k . l_k . b_k (\frac{d}{2})$$

$$11.45 \times 10^6 = \tau_k . 120 \times 30 \times 30$$

or $$\tau_k = \frac{11.45 \times 10^6}{120 \times 30 \times 30} = 106 \text{ MPa}$$

(iii) Shear stress in bolts

$$T = \tau_b . n . \frac{\pi}{4} d_b^2 \times R_b =$$

$$11.45 \times 10^6 = \tau_b \times 6 \times \frac{\pi}{4} (20)^2 (120)$$

$$\tau_b = \frac{11.45 \times 10^6 \times 4}{\pi \times 6 \times 400 \times 120} = 25.33 \text{ MPa}$$ **Answer**

SUMMARY

1. Torsion equation

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$$

2. $J = \dfrac{\pi D^4}{32}$ for solid circular shaft

$J = \dfrac{\pi}{32} (D^4 - d^4)$ for hollow circular shafts

3. $T = \dfrac{\pi}{16} \tau D^3$ for solid shafts.

$= \dfrac{\pi}{16} \tau \dfrac{(D^4 - d^4)}{D}$ for hollow shafts.

4. Power transmitted

$$P = \frac{2\pi NT}{60} Watts$$

$$= \frac{2\pi NT}{60{,}000} Kilo\ Watts$$

5. Comparision by strength

$$\frac{T_H}{T_S} = \frac{n^2+1}{n\sqrt{n^2-1}} \text{ Where } n = \frac{D}{d}$$

6. Comparision by weight

$$\frac{W_H}{W_S} = \frac{(n^2-1)\, n^{2/3}}{(n^4-1)^{2/3}}$$

7. Strain energy due to torsion

$$U = \frac{\tau^2}{4G} \times \text{ Volume of shaft}$$

8. For hollow shafts

$$U = \frac{1}{4}\frac{\tau^2}{G}\frac{R^2+r^2}{R^2} \times \text{ Volume of hollow shaft}$$

QUESTIONS

(1) State the assumptions made in the theory of torsion of shafts.

(2) Establish the relationship

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G.\theta}{L}$$

(3) Explain the following terms

(a) Angle of twist

(b) Polar section modulus

(c) Torsional rigidity

EXERCISES

(4) A solid circular shaft 80 mm diameter runs at 120 rpm. Determine the power transmitted by the shaft if the maximum permissible shear stress is limited to 64 MPa.

Ans. (80.85 KW)

(5) A solid shaft 100 mm diameter transmits 160 KW at 200 r.p.m. Determine the maximum intensity of shear stress induced and the angle of twist for a length of 3 metres. Take $G = 80$ KN/mm^2

Ans. (38.9 MPa, $1° - 36'$)

(6) A hollow cylindrical shaft transmits 500 KW at 125 r.p.m. Find the external diameter of the shaft if the internal diameter is 80% of the external diameter and the permissible shear stress is 60 MPa (140 mm)

(7) A hollow circular shaft of steel is made to replace a solid wrought iron shaft of

the same internal diameter, the material being 40% stronger than wrought iron. Find what fraction of the external diameter, would be the internal diameter of the shaft ? ($\frac{d}{D} = 0.731$)

(8) A solid steel shaft has to transmit 75 KW at 200 r.p.m. Find a suitable diameter of the shaft if the maximum torque transmitted exceeds the mean by 25% Also find the outer diameter of a hollow shaft to replace the solid if the diameter ratio is 0.6. Allowable shear stress is 60 MPa (112.5 mm, 120.7 mm, 72.4 mm)

(9) The outside diameter of a shaft is double the inside diameter for a hollow circular steel shaft which is to transmit a power of 500 KW at an average speed of 100 r.p.m. If the maximum shear stress is limited to 75 N/mm^2. calculate the dimensions of the shaft. (D = 151 mm, d = 75.5 mm)

(10) A hollow circular shaft 12 meters long is required to transmit 11000 KW at a speed of 200 r.p.m. If the maximum permissible shear stress is 80 MPa and the diameter ratio is 3/4, find the external diameter of the shaft and the angle of twist of one end relative to the other. Take $G = 8.5 \times 10^4$ N/mm^2 (169.7 mm, 0.066 radian) J.M.I, AMIE

(11) Design a hollow shaft 2m long with diameter ratio as 2/3 to transmit 200 K.W at 150 r.p.m Allowable shear stress is 60 MPa and the angle of twist not to exceed 1° per metre. Take modulus of rigidity for shaft material as 80 KN/mm^2

(44.4 mm, 29.6 mm) (J.M.I 1984)

(12) A solid circular shaft is to be replaced by a hollow circular shaft whose inside diameter is 3/4 of the outside. Compare the weights of equal lengths of these two shafts required to transmitt the same torque, if the max. permissible shear stress in both shafts is equal. $\left(\frac{W_H}{W_S} = 0.563\right)$

(13) A propeller shaft is 350 mm in diameter. An axial hole of 175 mm is bored throughout its length. If the allowable shear stress is 50 MPa and the angle of twist is not to exceed 1° in a length of 15 diameters. Determine the maximum torque when the hole was not bored.
By what percentage the torque is reduced after the hole has been bored ? By what percentage is the weight of the shaft reduced. (416 KN-m, 6% and 25%)

(14) A shaft 5 metres long and 60 mm diameter. is fixed at both ends. If a twisting moment of 20 KN-m is applied at a distance of 2 meters from one end, determine the twisting moment induced at the two ends of the shaft.

(8 KN-m and 12 KN-m)

(15) A compound shaft consists of a copper rod of 40 mm diameter enclosed in a steel tube of 50 mm diameter 5 mm thickness. If a twisting moment of 6000 N-m is to be transmitted, determine the shearing stresses developed in the two materials if both shafts have equal lengths and welded to a plate at each end so that their twists are equal Take $G_s = 2\,G_C$ (τ_s = 307 MPa, τ_c = 122 MPa)

(16) A hollow shaft is to transmit 338 KW at 100 r.p.m. If the shear stress is not to exceed 65 N/mm^2 and internal diameter is 0.6 of the external dia. Find the external and internal diameters, assuming that the maximum torque is 1.3 times the mean. (AMU. 1993)

□ □ □

12

Springs

Springs are devices meant to store energy or absorb excess energy. They are elastic bodies or resilient members which get distorted when loaded and recover their original shape when the distorting force is removed. Springs are used in clockwork to store energy which is used to run the watch. A carriage spring is used to absorb shocks in railway carriages etc. A spring which can absorb maximum amount of energy for a given stress is supposed to be the best spring.

Classification of Springs

Springs may be classified into the following types.

1. Bending spring 2. Torsion spring.

Bending springs.

A bending spring is subjected to bending only and resilience is mainly due to bending. Laminated springs or leaf springs are examples of bending springs.

Laminated springs are of two types

(a) Semi – elliptical type

(b) Quarter – elliptical type

Torsion Springs

A spring which is subjected to twisting moment and resilience is mainly due to torsion is called a torsion spring. Helical springs are examples of torsion springs.

Helical Spring

When a length of a wire is wound into a helix, it is called a helical spring.

Close-coiled helical spring

In close coiled helical springs the wire is wound quite closely so that the distance between the turns is very small.

Open coiled helical springs

In these springs the pitch or the distance between the turns is large as compared to the pitch in case of close-coiled helical springs. An open coiled helical spring falls under both the categories.

Stiffness – The load required to produce unit deflection is called stiffness of a spring

Proof Load – The maximum load which a spring can carry without suffering any permanent distortion is called proof load.

Proof-stress – It is the maximum stress that develops in a spring when subjected to the proof load.

Proof resilience

The strain energy stored in the spring when subjected to the proof load is called proof resilience.

Spring Constant

The stiffness of a spring is also called spring constant.

Laminated spring or leaf spring

(Semi-elliptical type)

These springs are also called carriage springs. Semi-elliptical type carriages springs are widely used in railway carriages, trucks, and other vehicles to absorb shocks.

Laminated springs are made of a number of laminations or strips of a metal of uniform section and varying lengths bent into a semi circular arc and placed one over the other as shown in fig 12.1. The plates are secured together at the centre with a bolt. They are also provided with clamps at distances to secure compactness. These springs rest on the axle of the vehicle and are pin- jointed to the chesis through two horns provided at the ends of the top plate.

When the spring is loaded to the designed load, all the plates become straight and the central deflection disappears.

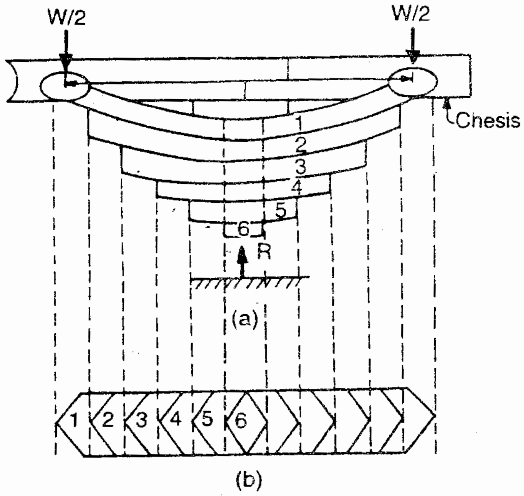


Fig. 12.1

Let W be the load acting on the spring

l = length of the spring

b = *breadth of the plates*

t = thickness of the plates

n = number of plates

δ = original deflection of the top spring

$\sigma =$ Maximum bending stress in the strips and

R = radius of the spring then

Maximum bending moment at the centre $= \dfrac{WL}{4}$

Moment of resistance of one plate

$$Mr = \frac{\sigma}{y} \cdot I = \frac{\sigma}{y} \cdot \frac{bt^3}{12} = \frac{\sigma}{t/2} \cdot \frac{bt^3}{12} = \frac{\sigma bt^2}{6}$$

Moment resisted by n plates $= \dfrac{\sigma.b.n.t^2}{6}$

The maximum bending moment will be equal to the total resisting moment of n plates

$$\therefore \quad \frac{WL}{4} = \frac{\sigma.b.n.t^2}{6}$$

$$\text{or} \quad \sigma = \frac{3WL}{2nbt^2}$$

Deflection at the centre $\delta = \dfrac{l^2}{8R}$,

$$\text{When } R = \frac{E.y}{\sigma} = \frac{E}{\sigma} \times \frac{t}{2}$$

$$\therefore \; \delta = \frac{\sigma l^2}{4Et}, \text{ putting } \sigma = \frac{3}{2}\frac{Wl}{nbt^2} \text{ we get}$$

$$\delta = \frac{3Wl^3}{8En.bt^3} = \frac{Wl^3}{32EI.n} \text{ Where } I = \frac{bt^3}{12}$$

Strain energy or Resilience $= \dfrac{\sigma^2}{6E} \times bt.l$

$$U = \frac{\sigma^2}{6E} \text{ (volume of spring).}$$

Example 12.1

A carriage spring is built up of 9 plates 75 mm wide and 6.5 mm thick. Find the length of the spring so that it may carry a central load of 4 KN, the stress is limited to 160 MPa. Also find the deflection at the centre of the spring. Take E = 200 KN/mm^2.

Solution

For length of the spring

$$\sigma = \frac{3Wl}{2nbt^2}$$

$$160 = \frac{3 \times 4 \times 10^3 \times l}{2 \times 9 \times 75\,(6.5)^2} \quad \text{or} \quad l = \frac{160 \times 2 \times 9 \times 75\,(6.5)^2}{3 \times 4 \times 10^3}$$

$$l = 760.5 \text{ mm}$$

For deflection at the centre

$$\delta = \frac{\sigma l^2}{4Et}$$

$$= \frac{160(760.5)^2}{4 \times 200 \times 10^3 \times 6.5}$$

$$= 17.795 \text{ mm} \qquad \textbf{Ans.}$$

Example 12.2

A Laminated spring 0.8 metres long is required to carry a central proof load of 7.5 KN. If the central deflection is not to exceed 20 mm and bending stress is not to exceed 200 MPa, determine the thickness width and number of plates. Assume width of plate equal to 10 times the thickness. Also find the radius to which the plates should be curved. Take E = 200 KN/mm².

Solution

Thickness of plates

Using the relation $\delta = \dfrac{\sigma l^2}{4Et}$

$$\text{or } t = \frac{\sigma l^2}{4\delta E} = \frac{200 \times (0.8 \times 1000)^2}{4 \times 20 \times 200 \times 10^3} = 8 \text{ mm}$$

Width of the plate $b = 10 \times t = 80$ mm

Number of plates, using the relation

$$\sigma = \frac{3Wl}{2 \times nbt^2}$$

$$\text{or } n = \frac{3Wl}{2\sigma bt^2}$$

$$= \frac{3 \times 7.5 \times 10^3 \times 0.8 \times 10^3}{2 \times 200 \times 80 \times (8)^2} = 8.75$$

$$n = 9 \text{ plates}$$

Radius of curvature, using the relation

$$\delta = \frac{l^2}{8R}$$

$$\text{or} \quad R = \frac{l^2}{8\delta} = \frac{(800)^2}{8 \times 20} = 4000 \text{ mm}$$

$$= 4 \text{ metres} \qquad \textbf{Answer.}$$

Example 12.3

A leaf spring 1 metre long is made up of steel plates with width equal to 6 times the thickness. Design the spring for a load of 15 KN when the maximum permissible stress is 160 MPa and deflection is not to exceed 16 mm. Take E = 200 KN/mm².

Solution

Let n be the number of laminations and b and t be the breadth and thickness in mm.

$$\text{Max. B. M.} = \frac{Wl}{4} = \frac{15 \times 1}{4} \text{ KN–m}$$

$$\text{Resisting moment of each plates} = \frac{15}{4.n} \text{ KN–m}$$

$$\text{Applying bending equation } \frac{M}{I} = \frac{\sigma}{y}$$

$$M = \frac{\sigma}{y} . I = \frac{\sigma}{t/2} \times \frac{bt^3}{12} = \frac{\sigma b t^2}{6}$$

or $$\frac{15}{4n} \times 10^6 \text{ N–mm} = \frac{160 \times b.t^2}{6} = \frac{160 \times (6t)(t^2)}{6}$$

or $$\frac{15 \times 10^6}{4n} = 160\, t^3 \quad \text{or} \quad n\, t^3 = \frac{15 \times 10^6}{4 \times 160} = 2.34 \times 10^4$$

$$\text{Maximum deflection } \delta = \frac{Wl^3}{32EI.n}$$

$$16 = \frac{15 \times 10^3 \times (1000)^3 \times t^3}{32 \times 200 \times 10^3 \times 0.5t^4 \times 2.34 \times 10^4} \text{ where}$$

$$I = \left[\frac{bt^3}{12} = \frac{6t^4}{12}\right] = 0.5t^4$$

$$or \quad t = \frac{15 \times 10^{12}}{32 \times 2 \times 0.5 \times 2.34 \times 10^9 \times 16}$$

$t = 12.5$ mm

Hence $b = 6 \times t = 75$ mm

$$n = \frac{2.34 \times 10^4}{t^3} = \frac{2.34 \times 10^4}{(12.5)^3} = 12$$

Breadth $b = 75$ mm, thickness = 12.5 mm, and $n = 12$ **Answer.**

Example 12 .4

A leaf spring of semi elliptical type has 10 plates each of 75 mm width and 10 mm thickness. The length of the spring is 1.2 metres. The plates are made up of steel having proof stress of 600 MPa. To what curvature the plates can be initially bent ? From what height should a load of 500 N fall on the centre of the spring if the maximum stress produced is to be one half of the proof stress.

Take $E = 200$ KN/mm^2 .

Solution

The leaf spring should initially bend to such a radius that under proof load, the spring may straighten up.

Applying bending equation to one plate

$$\frac{\sigma}{y} = \frac{E}{R} \quad \text{or} \quad R = \frac{E.y}{\sigma} = \frac{E.t}{2\sigma}$$

or $R = \frac{200 \times 10^3 \times 10}{2 \times 600} = 1.66$ metres

$\therefore$ Initial radius of curvature = 1.66 metres

Let W_p be the proof load, then

Maximum $B.M.$ due to proof load $= \frac{W_p.l}{4}$

$$B.M. \frac{W_p\,(1200)}{4} = 300\,W_p$$

Resisting moment of each leaf

$$M_r = \frac{300W_p}{10} = 30\,W_p$$

substituting in the bending equation

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$I = \frac{bt^3}{12} = \frac{75\,(10)^3}{12} = 6.25 \times 10^3 \text{ mm}^4$$

$$y = \frac{t}{2} = \frac{10}{2} = 5 \text{ mm}, \quad \sigma = \frac{600}{2} = 300$$

$$\therefore \quad \frac{30W_p}{6.25 \times 10^3} = \frac{300}{5}$$

or $\quad W_p = 12.5 \times 10^3$ Newton = 12.5 KN

The maximum deflection produced by the proof load

$$\delta = \frac{W_p.l^3}{32.EIn} = \frac{12.5 \times 10^3 \times (1200)^3}{32 \times 200 \times 10^3 \times 6.25 \times 10^3 \times 10}$$

$$\delta = 54 \text{ mm.}$$

The deflection produced by the falling load of 500 N will also be 54 mm and the work done by it will be equal to the work done by the gradually applied proof load Wp

$$500\,(h + \delta) = \frac{W_p}{2}\,.\,\delta = \frac{12500}{2}\,.\,\delta$$

$$(h + \delta) = \frac{12500}{2 \times 500}\,\delta = 12.5\,\delta$$

or $\quad h = (12.5\,\delta - \delta) = 11.5 \times \delta$

Hence $\quad h = (11.5 \times 54) = 621$ mm

Height from which a load of 500 N should fall is 621 mm. **Answer.**

Quarter elliptical springs

Quarter elliptical springs are cantilever type with a number of strips of same width and cross-section but different lengths, fixed at one end as shown in fig 12.2 Effective length is taken as the projecting part of the spring. All the plates are initially bent to the same radius and are free to slide

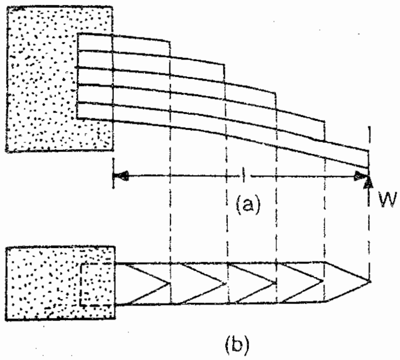


Fig. 12.2

one over the other. Quarter elliptical springs are half of the semi-elliptical springs. It can be imagined that the maximum stress and deflection in this case will be the same as that in a semi elliptical spring of length $2\ l$ acted upon by a load $2\ W$ at the centre and a reaction W at each end.

Let W = load acting at the free end of spring of length l, width b and thickness t of the plates. Let n be the number of plates and δ be the original deflection of the spring.

Then

Maximum bending moment at the fixed end of the leaf

$$M = W \cdot l$$

Moment resisted by one plate

$$M = \sigma \cdot I/y$$

Total moment resisted by n plates

$$M = \frac{n.\sigma.bt^2}{6} \quad \text{Where} \left[\frac{I}{y} = \frac{bt^3}{12} / \frac{t}{2} = \frac{bt^2}{6}\right]$$

Equating the maximum bending moment to the total resisting moment, we get

$$W \cdot l = \frac{n.\sigma.bt^2}{6}$$

$$\text{or } \sigma = \frac{6W.l}{nbt^2}$$

Deflection

$$\delta = \frac{l^2}{2R} \quad \text{Where} \quad \left[R = \frac{E.y}{\sigma} = \frac{Et}{2\sigma}\right]$$

$$\text{or } \delta = \left(\frac{l^2}{2 \times Et/2\sigma}\right) = \frac{\sigma l^2}{Et}$$

$$\text{Now put } \sigma = \frac{6W.l}{nbt^2}$$

$$\therefore \quad \delta = \frac{6Wl^3}{n.Eb.t^3}$$

Example 12.5

A cantilever leaf spring 600 mm long is composed of 12 leaves each of 60 mm wide and 7 mm thick. If the allowable flexural stress is 500 MPa, determine the allowable load at the free end.

Solution

Flexural stress of the spring

$$\sigma = \frac{6Wl}{bnt^2}$$

$$\text{or} \quad 500 = \frac{6W \times 600}{60 \times 12(7)^2}$$

$$\text{or} \quad W = 4900 \text{ N}$$

$$\therefore \quad W = 4.9 \text{ KN}$$ **Answer.**

Example 12.6

A quarter elliptical spring has a length of 600 mm and consists of plates each 50 mm wide and 9 mm thick. Calculate the minimum number of plates which can be used if the deflection under gradually applied load of 5 KN is not to exceed 70 mm. Take E = 200 KN/mm²

Solution

Let n be the number of plates

$$\delta = \frac{6Wl^3}{nEbt^3}$$

$$70 = \frac{6 \times 5 \times 10^3 \times 600^3}{n \times 200 \times 10^3 \times 50 \times 9^3}$$

$$\text{or} \quad n = \frac{6 \times 5 \times 10^3 \times 600^3}{70 \times 200 \times 10^3 \times 50 \times 9^3}$$

$$= 12.69 \quad \text{say } 13$$ **Answer.**

Example 12.7

A carriage spring quarter elliptical type is one metre long, 60 mm wide and 50 mm thick. If modulus of elasticity is 200 KN/mm² and the number of leaves is 10, what load at the free end will produce an extension of 20 mm. If the allowable flexural stress is 800 MPa, determine the stiffness of the spring.

Solution

$$\delta = \frac{6Wl^3}{E.n.b.t^3}$$

$$20 = \frac{6W(1000)^3}{200 \times 10^3 \times 10 \times 60 \times 50} = 20 \text{ mm}$$

$$\text{or} \quad W = 50 \text{ KN.}$$

When the permissible stress is 800 MPa

$$\delta = \frac{\sigma l^2}{4Et}$$

$$\delta = \frac{800(1000)^2}{4 \times 200 \times 10^3 \times 50} = 20 \text{ mm}$$

$$\text{Stiffness} = \frac{W}{\delta} = \frac{50}{20} = 2.5$$

$S = 2.5$ KN/metre.

Close-Coiled Helical spring subjected to axial load

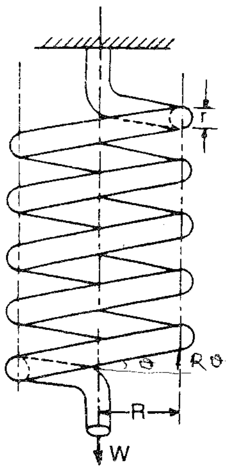


Fig. 12.3

A close-coiled helical spring with a load W acting axially is shown in fig. 12.3 . In these springs the wire is so closely wound that each turn is practically a plane at right angles to the axis of the helix. Each cross-section of the spring is subjected to a twisting moment as well as bending moment which tends to alter the curvature of the coils. Since the coils are closely wound the bending stress induced is very small as compared to the torsional stresses and hence neglected. A direct stress also acts on the cross-section but this being exceedingly small is also ignored.

Therefore while analysing a close coiled helical spring carrying an axial load only shear stress due to torsion is considered.

Let r = radius of the wire of which the spring is made

n = the number of turns of coils

R = The mean radius of the coils

Then

Length of the wire $l = 2\pi R.n$

Twisting moment due to axial at load $T = W.R$

Let θ be the angle of twist and δ the axial deflection

$$\text{Resilience of the spring} = \frac{1}{2} T.\theta \quad .... \quad .... \quad .... \quad \text{(i)}$$

$$\text{Work done by the load} = \frac{1}{2} W.\delta \quad .... \quad .... \quad .... \quad \text{(ii)}$$

Equating (i) and (ii) we get

$$\frac{1}{2} W.\delta = \frac{1}{2} T.\theta$$

$$\text{or} \quad \delta = \frac{T}{W}.\theta$$

Now from the torsion equation we have

$$\frac{T}{J} = \frac{G.\theta}{l} \quad \text{or} \quad \theta = \frac{T}{J} \times \frac{l}{G}$$

$$\text{or} \quad \theta = \frac{W.R}{\frac{\pi}{2}r^4} \cdot \frac{2\pi R.n}{G} = \frac{4WR^2.n}{G.r^4}$$

$$\text{Hence} \quad \delta = \frac{T}{W}.\theta = \frac{W.R}{W} \cdot \frac{4WR^2.n}{Gr^4} = \frac{4WR^3.n}{Gr^4}$$

or $\delta = \dfrac{64W.R^3.n}{Gd^4}$ Where d is the diameter of the wire $d = 2\,r$

Stiffness of the spring, Which is the force per unit deflection

$$S = \delta = \frac{Gd^4}{64WR^3.n}$$

Springs of square section wire

Let x be the side of the square section of the wire of the spring then

$$\delta = \frac{T}{W} \cdot \frac{T.l}{G} \cdot \frac{42J}{A^4}$$

$$= R \,.\, \frac{W.R.l}{G} \cdot \frac{42.x^4}{6.x^8}$$

$$= \frac{7WR^2.l}{Gx^4}$$

Strain energy stored in the spring

If U is the strain energy storedin the spring

$$U = \frac{1}{2} \,.\, T \,.\, \theta$$

Using torsion equation

$$\theta = \frac{\tau}{r} \cdot \frac{l}{G} \quad \text{and } T = \frac{\tau}{r} \,.\, J$$

$$\therefore\; U = \frac{1}{2}\left(\frac{\tau}{r} : J\right)\left(\frac{\tau}{r} \cdot \frac{l}{G}\right)$$

$$= \frac{1}{2}\,\frac{\tau^2}{r^2} \cdot \frac{l}{G} \cdot \frac{\pi}{2}\, r^4 = \frac{1}{4}\,\frac{\tau^2}{G} \,.\, \pi\, r^2 .\, l$$

$$U = \frac{1}{4}\,\frac{\tau^2}{G} \text{ (volume of the spring wire)}$$

Example 12.8

A close coiled helical spring is to absorb 40 KN–mm of energy. If the diameter of the coil is 10 times the diameter of the wire and the extension observed is 100 mm, determine the mean diameter of the helix, diameter of the wire and the number of turns, if the shear stress is not to exceed 160 MPa. Take G = 80 KN/mm^2.

Solution

Strain energy absorbed = Work done

$$40 \times 10^3 = \frac{W}{2} \times 100 \quad \text{or} \quad W = 800 \text{ Newtons}$$

$$\text{Torque } T = J \times \frac{\tau}{r} = \frac{\pi}{2}\, r^4 \,.\, \frac{\tau}{r} =$$

$$\text{or} \quad r^3 = \frac{2T}{\pi\,\tau} = \frac{2W.R}{\pi\,\tau} \quad \text{Taking R} = 10\,r$$

$$r^3 = \frac{2W\,(10r)}{\pi \times 160} \quad \text{or} \quad r^2 = \frac{2 \times 800 \times 10}{\pi \times 160} = \frac{100}{\pi}$$

or $r = 5.64$ mm or say 6 mm $\therefore\ d = 12$ mm

Mean diameter of helix $= R = 10 \times 6 = 60$ mm

Deflection

$$\delta = \frac{64WR^3.n}{G.d^4} = 100 \text{ mm}$$

$$\therefore\ n = \frac{8 \times 10^3\,(12)^4 \times 100}{64 \times 800 \times (60)^3} = 15$$

Number of turns = 15 Answer.

Example 12.9

A close-coiled helical spring consists of 16 coils each of 100 mm mean diameter and 13 mm dia. wire. If it is subjected to an axial load of 1 KN find (a) the maximum shear stress in the wire (b) the extension suffered by it. Take G = 80 KN/mm^2.

Solution

Twisting moment due to axial load

$$T = W.R = 1000 \times \frac{100}{2} = 50{,}000 \text{ N–mm}$$

Shear stress $\tau = \dfrac{T}{J} \times r$

$$\tau = \frac{T}{\frac{\pi}{2}r^4}.r = \frac{2T}{\pi r^3} = \frac{2 \times 50{,}000}{\pi\,(6.5)^3} = 115.90$$

$\tau = 115.90$ MPa

Deflection $\delta = \dfrac{64WR^3.n}{G(d)^4}$

$$\delta = \frac{64 \times 1000\,(50)^3 \times 16}{80 \times 10^3\,(13)^4} = 56 \text{ mm}$$

Example 12.10

A close coiled helical spring of 20 mm diameter wire has 20 coils each of mean diameter 80 mm. Determine the height from which a weight of one KN should fall on the spring so that it is compressed by 40 mm. Take G = 80 KN/mm^2.

Solution

Let h be the height of drop

Let W be the equivalent gradually applied load to produce the same compression.

Then

$$\delta = \frac{64WR^3.n}{Gd^4}$$

$$40 = \frac{64W(40)^3 \times 20}{8 \times 10^3 (20)^4}$$

$$\text{or} \quad W = \frac{40 \times 8 \times 10^3 \times (20)^4}{64(40)^3 \times 20} \quad \text{or} \quad W = 6.25 \text{ KN}$$

Equating the energy supplied by the impact load to the energy stored

$$P(h + \delta) = \frac{1}{2} W.\delta$$

$$1000(h + 40) = \frac{1}{2} \times 6.25 \times 10^3 \times 40$$

$$(h + 40) = 125$$

$$\text{or} \quad h = (125 - 40) = 85 \text{ mm}$$ **Answer.**

Strain energy stored within an elastic bar subjected to a pure bending moment

When an elastic bar is subjected to a pure bending moment M it deforms into a circular are of radius of curvature R. We have already studied in the chapter on bending stresses

That

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

The length of the bar L is equal to the product of central angle θ Subtended by the circular are of radius R. Thus we can write $L = R.\theta$

$$\text{or} \quad \frac{I}{R} = \frac{\theta}{L}$$

That

$$\frac{M}{IE} = \frac{1}{R} = \frac{\theta}{L}$$

$$\text{or} \quad \theta = \frac{M.L}{IE}$$

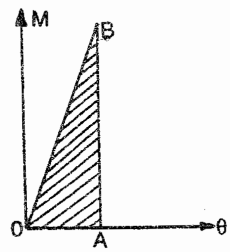


Fig. 12.4

From the above equation it can be said that the relation between moment and the subtended angle is a linear one.

If now a graph is plotted between a specific value of M on the vertical axis and θ on the horizontal axis as shown in figure 12.5. The work done by the moment M is given by the area of the shaded portion $OAB = \frac{1}{2} M.\theta$

This is the amount of internal energy stored in the bar

$$U = \frac{1}{2} M.\theta = \frac{1}{2} \frac{M^2 L}{EI}$$

$$\text{or} \quad U = \frac{1}{2} \frac{M^2 L}{EI}$$

Close-Coiled helical springs subjected to axial twist

When a close-coiled helical spring is subjected to an axial twist it produces a constant bending moment on the coils. The magnitude of the bending moment is always equal to the applied torque. As a result of this torque the curvature of the coils increases or decreases depending upon the direction or sense of the bending moment induced. The number of turns of the coil increases when mean radius of the coil decreases and vice-versa. If L is the effective length of the wire of the spring then.

$$L = 2\pi\ R n = 2\pi\ \ R_1\ n_1$$

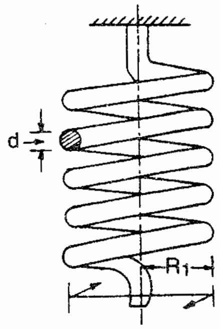


Fig. 12.5

Where Let R = initial mean radius of the coils

R_1 = final mean radius of the coils

n = initial number of turns

n_1 = final number of turns

Let θ be the angle of twist in radians due to the applied torque.

Depending upon the direction of the applied torque the final number of turns n_1 will be, more than or less than n by a factor $\frac{\theta}{2\pi}$ turns. When the spring tends to close then

$$n_1 = n + \frac{\theta}{2\pi}$$

and when the spring tends to open the final number of turns

$$n_1 = n - \frac{\theta}{2\pi}$$

Assuming each coil as a beam of large curvature

Energy stored in the spring

$$= \frac{1}{2} M . \theta = \frac{M^2 L}{2EI}$$

$$\text{or} \quad \theta = \frac{M L}{EI}$$

$$\text{or} \quad \theta = \frac{M.2\pi R.n}{E.\frac{\pi}{2}(r)^4} = \frac{8MR.n}{Er^4}$$

$$\text{or} \quad \theta = \frac{128MRn}{Ed^4}$$

$$\text{Resilience } U = \frac{1}{2} M .\theta = \frac{1}{2} . \frac{M.^2 L}{EI} \text{ putting } M^2 = \frac{\sigma^2 I^2}{r^2}$$

$$U = \frac{\sigma^2 . I^2}{r^2} . \frac{L}{2IE} = \frac{\sigma^2}{2E} \times \frac{\pi r^2 L}{4}$$

$$U = \frac{1}{8} \frac{\sigma^2}{E} \text{ Volume of the wire}$$

For wire of square section of side x

$$\theta = \frac{24\pi RMn}{Ex^2}$$

Resilience $U = \frac{1}{6}\frac{\sigma^2}{E}$ (Volume of wire)

Example 12.11

A close coiled helical spring is made up of 10 mm diameter wire having 12 coils with 120 mm mean diameter. If a twisting moment of 12 N–m is applied axially determine.

(i) The maximun bending stress in the wire

(ii) The angle of twist

(iii) Strain energy and (iv) The number of turns.

The torque is applied in such a way that the spring tends to close. Take $E = 200\ KN/mm^2$.

Solution

(i) $\sigma = \frac{M}{I} \cdot y = \frac{12 \times 10^3}{\frac{\pi}{4}(5)^4} \times \frac{10}{2} = 122.2$ MPa

(ii) $\theta = \frac{ML}{EI} =$ $\quad L = 2\pi R n = 2\pi \times 60 \times 12 = 144\pi$

$I = \frac{\pi}{4}(5)^4 = 156.25$ p

$$= \frac{12 \times 10^3 \times 144\pi}{200 \times 10^3 \times 156.5\pi}$$

$= 0.55$ radian

(iii) Strain energy $U = \frac{\sigma^2}{8E}$ (Volume of wire)

$$U = \frac{(122.2)^2}{8 \times 200 \times 10^3} \times (2\pi \times 60 \times 12)$$

$= 2579.1$ N–mm

(iv) Since the spring tends to close

$$n_1 = n + \frac{\theta}{2\pi}$$

$$= 12 + \frac{0.55}{2\pi} = 12 + 08 = 12.08$$

Say 13 turns

Open-Coiled Helical Spring Subjected to axial Load

Let

$R =$ mean radius of the spring

$d =$ diameter of the wire

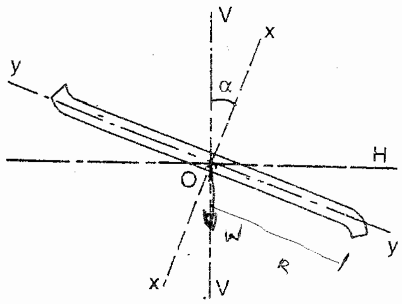


Fig. 12.6

n = number of turns

δ = deflection of the spring caused by the axial load W.

α = Angle of helix

Moment due to the axial load W about $OH = W.R$ this moment can be resolved into two components

(a) A moment T along plane xx causing twisting (b) a moment M along y–y Causing bending.

Twisting moment $T = WR \cos\alpha$

Bending moment $M = WR \sin\alpha$

Let θ be the angle of twist and ϕ the angle of bend due to the bending moment

Then from torsion equation

$$\frac{T}{J} = \frac{G\theta}{L} \quad \text{or} \quad \theta = \frac{T \times L}{G \times J} = \frac{WR \cos\alpha . L}{JG}$$

The angle of bend due to bending moment

$$\phi = \frac{ML}{EI} = \frac{WR \sin\alpha . L}{EI}$$

Work done by the load W in causing a deflection δ of the spring is equal to the strain energy of the spring

$$\frac{1}{2} W . \delta = \frac{1}{2} T . \theta + \frac{1}{2} M . \phi$$

$$\text{or} \quad \frac{1}{2} W.\delta = \frac{1}{2} W R \cos\alpha \left(\frac{WR \cos\alpha . l}{JG}\right) + \frac{1}{2} WR \sin\alpha \left(\frac{WR \sin\alpha . l)}{EI}\right)$$

Putting the values of $J = \frac{\pi}{32} d^4$ and $I = \frac{\pi}{64} d^4$

and $L = 2\pi R.n . \sec\alpha$, we get

$$\delta = \frac{64WR^3 .n. \sec\alpha}{d^4} \left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

For open coiled helical spring subjected to axial torque T,

$$\delta = \frac{64TR . n . \sin\alpha}{d^4} \left(\frac{1}{G} - \frac{2}{E}\right)$$

Example 12.12

An open coiled helical spring is made out of 10 mm diameter steel rod having 10 turns and a mean diameter 80 mm, the angle of helix being 15°. Calculate the deflection under an axial load of 250 Newtons. Take E = 210 KN/mm² and G = 85 KN/mm².

Solution

The deflection of an open coiled helical spring

$$\delta = \frac{64WR^3 \,.\, n \sec\alpha}{d^4} \qquad \left(\frac{Cos^2\alpha}{G} + \frac{2Sin^2\alpha}{E}\right)$$

Angle of helix $\alpha = 15^\circ$

$$Sec\alpha = \frac{1}{0.965}, \quad Cos\,\alpha = 0.965, \quad Sin\,\alpha = 0.25$$

$n = 10$, $R = 40$ mm and $d = 10$ mm, putting these values in the above equation

$$\delta = \frac{64 \times 250 \times 40^3 \times 10 \times 1}{(10)^4 \quad 0.965}\left[\frac{(0.965)^2}{85 \times 10^3} + \frac{2 \times (0.25)^2}{210 \times 10^3}\right]$$

$$= \frac{1024 \times 10^3}{0.965} \times \frac{1}{10^3}(0.0115 + 0.00063)$$

$\delta = 1024 \times 0.0125 = 12.87$ mm **Answer.**

Compound Springs

(a) Springs in series

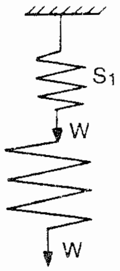


Fig. 12.7

When two springs are connected in series and a load W is applied then the total extension produced will be the sum of the extensions in each one of the springs

$$\delta = \delta_1 + \delta_2 \qquad \therefore \qquad \frac{W}{S} = \frac{W}{S_1} + \frac{W}{S_2}$$

and the stiffness of the composite spring will be

$$\frac{1}{S} = \frac{1}{S_1} + \frac{1}{S_2}$$

(b) Springs in Parallel

When springs are connected in parallel and a load W is applied

then $W = W_1 + W_2$ and $\delta.\, s = \delta S_1 + \delta S_2$

Hence the stiffness of the spring will be

$$S = S_1 + S_2$$

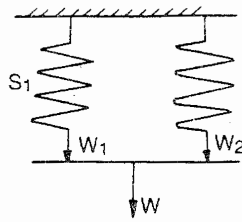


Fig. 12.8

Example 12 .13

Two Close-Coiled helical springs A and B made of the same wire show axial compression of 80 mm and 30 mm respectively, when subjected to the same axial load. The spring A has 9 coils of mean diameters 80 mm while the spring B has 8 coils. Determine the mean coil diameter of spring B.

Solution

The springs are connected in parallel therefore

$$\delta_A = \delta_B$$

spring A, Mean dia = 80 mm $\therefore R_A = 40$ mm

$\delta_A = 80$ mm, number of coils = 9, $d_A = d_B = 80$ mm

$$\delta_A = 80 \text{ mm} = \frac{64WR_A^3.n}{G(d_A)^4} = \frac{64W \times 40^3 \times 9}{G.(d_A)^4}$$

$$\text{or} \quad W = \frac{80 \times G \times (d_A)^4}{64\,(40)^3 \times 9} = \frac{80G\,(80)^4}{64(40)^3 \times 9}$$

For spring B, $\delta_B = 30$ mm, $n = 8$, $R_B = ?$

$$\delta_B = \frac{64W\,(R_B)^3.n}{G(d_B)^4} =$$

$$30 = \frac{64W\,(R_B)^3 \times 8}{G(80)^4} = \frac{64 \times 8\,(R_B)^3}{G(80)^4} \times W$$

$$30 = \frac{64 \times 8 \times (R_B)^3}{G(80)^4} \times \left[\frac{80 \times G \times (80)^4}{64 \times (40)^3 \times 9}\right]$$

$$30 = \frac{8R_B^3 \times 80}{(40)^3 \times 9} \quad \text{or} \quad (R_B)^3 = \frac{30 \times 40^3 \times 9}{8 \times 80}$$

$$R_{B3} = \frac{27}{64} \times 40^3 \quad \text{or} \quad R_B = \frac{3}{4} \times 40 = .75 \times 40 = 30$$

Mean coil diameter of spring B is 60 mm **Answer**

Example 12.14

Two Close Coiled springs are connected in series and the stiffness of the compound spring is 2.5 N/mm. If the wire diameter of spring A be 5 mm, determine the wire diameter of spring B. The number of coils in springs A and B are 20 and 15 repectively. Each spring has a mean coil diameter equal to 8 times of its wire diameter. What would be the safe load for the compound spring so that the shear stress in the wire does not exceed 250 MPa. Take $G = 30\ KN/mm^2$.

Solution

The springs are connected in series

Spring A,

$$\therefore \quad \frac{1}{S} = \frac{1}{S_A} + \frac{1}{S_B}$$

$$\frac{1}{S_A} = \frac{64R_A^3\,.\,n_A}{G.(d_A)^4} = \frac{64 \times \left(\frac{8 \times 5}{2}\right)^3 \times 20}{80 \times 10^3 (5)^4}$$

$$\frac{1}{S_B} = \frac{64R_B^3\,.\,n_B}{G(d_B)^4} = \frac{64 \times \left(\frac{8 \times d_B}{2}\right)^3 \times 15}{80 \times 10^3 (d_B)^4}$$

$$\text{or} \quad \frac{1}{2.5} = \frac{64 \times (20)^3 \times 20}{80 \times 10^3 (5)^4} + \frac{64 \times (4d_B)^3 \times 15}{80 \times 10^3\,(d_B)^4}$$

$$= \frac{64}{80 \times 10^3} \left\{ \frac{20^4}{5^4} + \frac{4^3 \times d_B{}^3 \times 15}{d_B{}^4} \right\}$$

$$\text{or} \quad .4 = .8 \times 10^{-3} \left\{ \frac{16 \times 10000}{25 \times 25} + \frac{64 \times 15}{d_B} \right\}$$

$$\text{or} \quad 0.5 \times 10^3 = \left\{ 256 + \frac{960}{d_B} \right\}$$

$$\text{or} \quad 500 - 256 = \frac{960}{d_B}$$

$$\text{or} \quad d_B = \frac{960}{244} = 3.93 \text{ mm}$$

Example 12.15

Two close-coiled helical springs A and B are connected in parallel and made up of the same material and number of coils. Coil diameter of spring A is 100 mm and that of spring B is 75 mm. The wire diameters are 9mm and 6mm for A and B respectively. If the applied load is 2 KN, determine the load taken by each spring and the maximum stresses induced.

Solution

Since the springs are connected in parallel

$$\delta_1 = \delta_2$$

$$\frac{64 W_A \,.\, R_A^3 \,.\, n}{G\, d_A^4} = \frac{64 W_B \,.\, R_B^3 \,.\, n}{G d_B^4}$$

$$\text{or} \quad \frac{W_A}{W_B} = \frac{R_B^3}{R_A^3} \times \frac{(d_A^4)}{(d_B^4)}$$

$$\frac{W_A}{W_B} = \left(\frac{75}{100}\right)^3 \times \left(\frac{9}{6}\right)^4 = 2.13$$

Applied load will be shared by the two springs

$$W_A + W_B = 2000 \text{ Newton}$$

$$\text{or} \quad 2.13\, W_B + W_B = 2000$$

$$W_B = 2000/3.13 = 638.96 \text{ Newton}$$

$$W_A = 2.13\, W_B = 1360.98 \text{ Newton}$$

$$\tau_A = \frac{16 W_A R_A}{\pi (d_A)^3} = \frac{16 \times 1360.98 \times 100/2}{\pi\, (9)^3} = 475.38 \text{ MPa}$$

$$\tau_B = \frac{16 W_B R_B}{\pi (d_B)^3} = \frac{16 \times 638.96 \times 75/2}{\pi (6)^3} = 564\ .96 \text{ MPa}$$

Let W be the maximum axial load which causes maximum shear stress of 250 MPa

For spring A $\tau_A = \dfrac{16 W_A\, R_A}{\pi\, d_A^3}$

or $$W_A = \frac{\tau_A \,.\, \pi \,.\, d_A^3}{16\, R_A} = \frac{250 \times \pi \times (5)^3}{16 \times \frac{8 \times 5}{2}}$$

$$W_A = \frac{250 \times \pi \times 125}{16 \times 20} = 306\,.97 \text{ Newton}$$

For spring B

$$W_B = \frac{250 \times \pi \times (3.93)^3}{16 \times 8 \times \frac{3.93}{2}}$$

$$W_3 = \frac{250 \times \pi \times (3.93)^2}{16 \times 4} = 189.53 \text{ Newton}$$

Hence safe load for the compound spring is lesser of the two values of W

$\therefore \quad W = 189.53$ Newton

SUMMARY

1. Leaf spring or Laminated spring (semi elliptical type)

$$\sigma = \frac{3Wl}{2nbt^2} \qquad \text{where } \sigma \text{ is the bending stress}$$

$$\delta = \frac{3}{8}\frac{Wl^3}{nEbt^3}$$

Strain energy $U = \frac{\sigma^2}{6E}$ (Volume of spring)

Stiffness of the spring $\delta = \frac{8}{3}\frac{nEbt^3}{L^3}$

2. Quarter elliptical type

$$\sigma = \frac{6Wl}{bnt^2}$$

$$\delta = \frac{6Wl^3}{Ebnt^3}$$

3. Close-coiled helical spring subjected to axial load

Max^m. shear stress $\tau_{max} = \frac{16WR}{\pi\, d^3}$

Angle of twist $\theta = \frac{4WR^3.n}{Gr^4}$

Deflection $\delta = \frac{64WR^3n}{Gd^4}$

Strain energy $u = \frac{32W^2R^3n}{Gd^4} = \frac{1}{4}\frac{\tau^2}{G}(\pi\, r^2 \,.\, l\,)$

$$\text{Stiffness } s = \frac{Gd^4}{64R^3.n}$$

4. Close-coiled helical spring subjected to axial twist

$$\theta = \frac{128MRn}{Ed^4}, \; U = \frac{\sigma^2}{6} \text{(Volume of wire)}$$

5. Open coiled helical spring subjected to axial load.

$$T = WR \; Cos\alpha$$

$$M = WR \; Sin\alpha$$

$$f = \frac{ML}{EI} = \frac{WR \; Sin \; \alpha \; . \; l}{EI}$$

$$\delta = \frac{64WR^3.n.sec \; \alpha}{d^4}\left(\frac{Cos^2\alpha}{G} + \frac{2Sin^2 \; \alpha}{E}\right)$$

6 . For open coiled helical spring subjected to axial torque T

$$\delta = \frac{64TR \; n \; . \; Sin \; \alpha}{d^4}\left(\frac{1}{G} - \frac{9}{E}\right)$$

7. Compound Springs

(i) Springs in series

$$\frac{1}{S} = \frac{1}{S_1} + \frac{1}{S_2}$$

(ii) Springs in parallel

$$S = S_1 + S_2$$

QUESTIONS

(1) What is the function of a spring ?

When are they used ? How would you classify springs ? In which catagory would you place an open coiled helical spring.

(2) Distinguish between the terms Proof load, Proof stress and Proof resilience.

(3) What do you understand by the term spring constant? Closely coiled helical spring is subjected to an axial load W, derive a formula for the energy stored in the spring in terms of max. shear stress volume of the spring wire and the shear modulus of elasticity.

(4) What are helical springs ? Derive an expression for deflection of an open coiled helical spring.

EXERCISES

(5) A laminated spring one metre long 60 mm wide and 6 mm thick plates is to support a load of 240 N. It the permissible bending stress is not to exceed 140 MPa, find the number of turns required. (12 turns)

(6) A leaf spring is made up of a plates 600 mm long and 100 mm wide. The spring is to carry a load of 5.5 KN. If the deflection is limited to 20 mm, claculate the

maximum stress and thickness of plates. Take $E = 200/\text{mm}^2$.

(t = 5 mm, stress = 220 MPa)

(7) A leaf spring 750 mm long is required to carry a central proof load of 800 N. If the central deflection is not to exceed 20 mm and the bending stress is not grater than 200 MPa. Determine the width and thickness of plates. Assume width of plate as 12 times thickness (84 . 36 mm; 7 . 03 mm) Take E = 200 KN/mm^2.

(8) A close coiled helical spring is made of 12 mm steel wire the coils having 10 complete turns and a mean diameter of 100 mm. Calculate the increase in the number of turns and bending stress induced in the section if its is subjected to an axial twist of 15000 N–m.

Take E = 200 KN/mm^2. (.0346 turns; 1130 N–mm/degree)

(9) A close coiled helical spring is required to carry an axial load of 1 KN. The spring is to have a mean diameter of 50 mm. If the maximum shearing stress is not to exceed 30 MPa, determine the diameter of the wire used. (d = 7.5 mm)

(10) A weight of 2500 N is dropped on a closely coiled helical spring of 16 turns. Find the height from which the weight may be dropped before striking the spring so that the spring may be compressed by 220 mm. Mean dia. of the coils may be taken as 120 mm and the dia. of the wire as 30 mm. Take G = 90 KN/mm^2.

(h = 176 . 8 mm)

(11) Compare the resistance of a close coiled helical spring of square section wire with that of a circular section if the volume of both the springs is same.

(12) Two close coiled helical springs of wire diameter 12 mm and core radii 120 mm and 80 mm are compressed between rigid plates at their ends. Calculate the maximum stress induced in each spring if the applied load is 600 Newtons

(τ_1 = 163.6 MPa, τ_2 = 32.49 MPa)

□ □ □

13

Columns And Struts

Vertical members of a building supporting compressive loads are called columns. Columns may be axially loaded or eccentrically loaded. Sometimes they are also called pillars or stanction.

Struts are members subjected to compressive stresses. They may be vertical inclined or horizontal.

The aim of this chapter is to discuss the behaviour of columns under various types of loadings, slendrness ratio and end conditions.

Mode of failure of columns

Under the action of axial compressive forces columns may fail due to (i) Crushing (ii) Buckling and (iii) Combined effect of crushing as well as buckling.

Classification of columns

Depending upon the mode of failure columns may be classified into the following catagories.

(a) Short Columns

(b) Long Columns

(c) Medium Columns.

(a) Short columns

In short columns failure occurs purely due to crushing. The ratio $\frac{l}{d}$ is less than 8 and $\frac{l}{K}$ is less than 32

Where l = Effective length of the column

d = Least lateral dimension

K = Least radius of gyration.

(b) Long columns

Failure occurs due to buckling only. These columns fail due to lateral bending before the compressive stress reaches crushing value. The direct stress induced is insignificant as compared to bending stress. For long columns.

$$\frac{l}{d} > 30 \text{ and } \frac{l}{k} > 120$$

(C) Medium columns

Such columns fail due to combined effect of both the direct as well as bending stresses. For medium columns.

$$\frac{l}{d} > 8 \text{ and} < 30$$

$$\frac{l}{k} > 32 \text{ and} < 120$$

Buckling of columns

The lateral bending of a compression member under axial loading is called buckling. Buckling occurs in a direction perpendicular to the axis about which the radius of gyration is minimum.

Buckling load

The axial load at which lateral bending starts is called buckling load. Buckling of column depends upon its effective length and least lateral dimension

Effective length or equivalent length of a column

The length of a compression member that bends as if the ends are hinged is called effective length or equivalent length of a column. Depending upon end conditions a column may have different effective lengths.

End conditions.

(i) Both ends hinged

(ii) One end fixed and the other end free

(iii) One end fixed and the other end hinged

(iv) Both ends fixed.

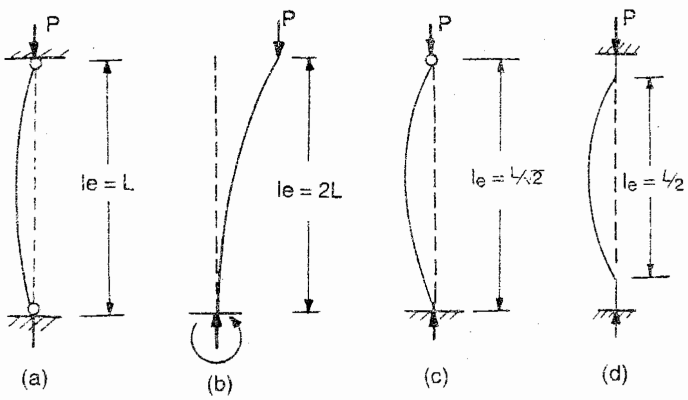


Fig. 13.1

Both ends hinged | One end fixed and other end free | One end fixed and other end hinged | Both ends fixed

Radius of gyration :–

It is the geometrical property of a section and is denoted by

$$K = \sqrt{I/A}$$

Where $K =$ Radius of gyration

$I =$ Moment of inertia of the section

$A =$ Area of cross-section

Slenderness Ratio

It is the ratio of the effective length of a column and its least radius of gyration.

$$\text{Slenderness ratio} = \frac{l}{K} = \frac{\text{Effective length}}{\text{Least radius of gyration}}$$

Load Carrying Capacity of Columns

The strength or load carrying capacity of a column is its capacity to support the maximum load till its failure. The load carrying capacity of a column depends upon.

(i) Cross-sectional dimension

(ii) Length of the column

(iii) Its end conditions

(iv) Its initial curvature ie whether it is perfectly straight or imperfectly straight before loading.

Crushing load :–

The ultimate load beyond which the column fails due to crushing stresses is called crushing load.

Buckling load

The laod at which the column just buckles is called buckling load or crippling load or critical load.

Euler's theory for long columns

The first rational attempt in the study of columns was made by Euler in 1757. The following assumptions are made in Euler's theory.

(i) Initially the column is perfectly straight and load acts truly axially.

(ii) The material of the column is perfectly elastic, isotropic and homogenous and obeys Hooke's law.

(iii) The length of the column is very large as compared to its cross-sectional dimensions.

(iv) The shortening of the column due to direct compression is neglected.

(v) The failure of long column occurs due to buckling alone.

(vi) The self weight of the column is neglected.

(vii) The Cross-section of the column is uniform throughout

Proof of Euler's Formula

Case I -- Both ends hinged

Consider a column AB hinged at both ends and subject to a critical load P as shown in figure. 13.2

Consider a section at a distance x from end A. Let 'y' be the deflection at this section from the centre line B. M at this section

$$EI\frac{d^2y}{dx^2} = M_x = -P_y$$

$$\text{or} \quad EI\frac{d^2y}{dx^2} + P_y = 0$$

$$\text{or} \quad \frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

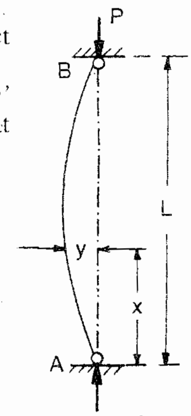


Fig. 13.2

The general solution of this differential equation is

$$y = C_1 \; Cos\, x \sqrt{\frac{P}{EI}} + C_2 \, Sin\, x \sqrt{\frac{P}{EI}}$$

Where C_1 and C_2 are the constants of integration.

Now applying end conditions

at $A, x = 0, \quad y = 0 \quad \therefore C_1 = 0$

at $B, x = l, \quad y = 0 \quad \therefore 0 = C_2 \, Sin \quad l\sqrt{\frac{P}{EI}}$

This is possible if C_2 is Zero in which case, the column has not bent at all or Sin $l\sqrt{\frac{P}{EI}} = 0$

$$\therefore \quad l\sqrt{\frac{P}{EI}} = 0, \pi, 2\pi \; ----$$

Taking the least significant value we get

$$P_{cr} = \frac{\pi^2 . EI}{l^2}$$

Where P_{cr} is the critical load.

Case II

Columns with one end fixed and the other end free.

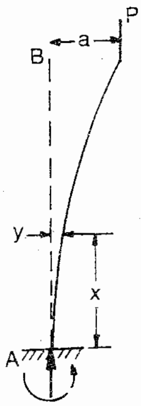


Fig. 13.3

A column AB fixed at A and free at end B is shown in the figure. Let a be the deflection of the free end under a critical load P.

Now consider a section at a distance x from A Let y be the deflection at this section.

Bending moment at the section $= P\,(a - y)$

$$\text{Hence} \quad EI\,\frac{d^2y}{dx^2} = P\,(a - y)$$

$$\text{or} \quad \frac{d^2y}{dx^2} + \frac{P}{EI}\,y = \frac{Pa}{EI}$$

The solution of the above differential equation is

$$y = C_1 \, Cos\, x\sqrt{\frac{P}{EI}} + C_2 \, Sin\, x\sqrt{\frac{P}{EI}} + a$$

At $A, x = 0, \quad y = 0 \quad \therefore C_1 = -a$

$$\frac{dy}{dx} = -a\sqrt{\frac{P}{EI}}\; Sin\, x\sqrt{\frac{P}{EI}} + C_2\sqrt{\frac{P}{EI}}\; Cos\, x\sqrt{\frac{P}{EI}}$$

Also at $A, x = 0, \; \frac{dy}{dx} = 0$

$$\therefore \quad C_2\sqrt{\frac{P}{EI}} = 0$$

Hence $C_2 = 0$, since P is not Zero

Substituting the values of C_1 and C_2 we get

$$y = -a\, Cos\, x\sqrt{\frac{P}{EI}} + a = a\left(1 - Cos\, x\sqrt{\frac{P}{EI}}\right)$$

Again at B, $x = l$ $y = a$

$$\therefore \quad a = a\left(1 - Cos\, l\sqrt{\frac{P}{EI}}\right)$$

since a = 0 $\therefore$ $$Cos\, l\sqrt{\frac{P}{EI}} = 0$$

Hence $$l\sqrt{\frac{P}{EI}} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

Taking the least significant value we get

$$l\sqrt{\frac{P}{EI}} = \frac{\pi}{2}$$

or $$P_{cr} = \frac{\pi^2 EI}{4l^2}$$

Hence the effective length of a column with one end fixed and the other end free is $2\,l$.

Columns with both ends fixed

A column AB with both ends fixed in position as well as in direction is shown in figure 13.4. Consider a section at a distance x from end A, then

$$EI\,\frac{d^2y}{dx^2} = MA - P.y$$

Where MA is the fixing moment at A

$$\frac{d^2y}{dx^2} + \frac{P}{EI}.\,y = \frac{MA}{EI}$$

The solution of this differential equation is

$$y = C_1\, Cosx\sqrt{\frac{P}{EI}} + C_2\, Sinx\sqrt{\frac{P}{EI}} + \frac{MA}{P} \quad \text{.... .... (i)}$$

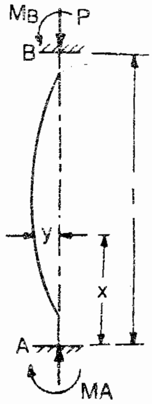


Fig. 13.4

Differentiating

$$\frac{d_y}{d_x} = -C_1\sqrt{\frac{P}{EI}} \times Sin\, x\sqrt{\frac{P}{EI}} + C_2\sqrt{\frac{P}{EI}}\, Cos\, x\sqrt{\frac{P}{EI}} \quad \text{.... .... (ii)}$$

At $x = 0$, $\frac{d_y}{d_x} = 0 \;\therefore\; C_2\sqrt{\frac{P}{EI}} = 0 \;\therefore\; C_2 = 0$

Also when $a = 0, y = 0,$ or $0 = C_1 + \frac{M_A}{P}$ or $C_1 = \frac{-M_A}{P}$

$$\therefore y = \frac{-M_A}{P}\left[Cos x\sqrt{\frac{P}{EI}} - 1\right] \quad .... \quad .... \quad .... \quad \text{(iii)}$$

When $x = l, \quad y = o$

$$\therefore Cosl\sqrt{\frac{P}{EI}} = 1, \text{or } l\sqrt{\frac{P}{EI}} = 0, 2\pi, 4\pi \quad .... \quad .... \quad .... \quad \text{(iv)}$$

When $x = l$, $\frac{d_y}{d_x}$ is also Zero,

$$\therefore -C_1\sqrt{\frac{P}{EI}}\,Sin\, l\sqrt{\frac{P}{EI}} = 0$$

Since C_1, *and* P are not Zero $\therefore$ Sin $l\sqrt{\frac{P}{EI}} = 0$

$$\therefore l\sqrt{\frac{P}{EI}} = 0, \pi, 2\pi \quad .... \quad .... \quad .... \quad \text{(iv)}$$

The minimum significant value consistent with equation (iv) and (v) is 2π

$$\therefore P_{cr} = \frac{4\pi^2 EI}{l^2}$$

Columns With One End Fixed And The Other End Hinged

Consider a column with end A fixed in position as well as direction and the end B hinged. Since end B is free to rotate a bending moment M will be induced only at end A. Let R be the horizontal force required to keep $A\,B$ in static equilibrium as shown in figure 13.5 Now consider a section at a distance x from A, then

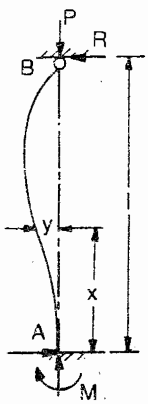


Fig. 13.5

$$EI\frac{d^2y}{dx^2} = -P.y + R(l - x)$$

$$\text{or} \quad \frac{d^2y}{dx^2} + \frac{p}{EI}.y = \frac{R}{EI}(l - x)$$

The solution to this differential equation is

$$y = C_1\, Cos\sqrt{\frac{P}{EI}}\,x + C_2\, Sin\sqrt{\frac{P}{EI}}\,.x + \frac{R}{P}(l - x)$$

At $x = 0, \quad y = 0 \quad \therefore C_1 = \frac{-Rl}{P}$

$$\frac{dy}{dx} = +\frac{Rl}{p}\sqrt{\frac{P}{EI}}\,Sin\sqrt{\frac{P}{EI}}\,x + C_2\sqrt{\frac{P}{EI}}\,Cos\sqrt{\frac{P}{EI}}\,.x - \frac{R}{P}$$

At $x = 0, \frac{dy}{dx} = 0 \quad \therefore C_2 = \frac{R}{P}\sqrt{\frac{EI}{P}}$

$$\therefore y = \frac{-Rl}{P} Cos \sqrt{\frac{P}{EI}} \,.\, x + \frac{R}{P}\sqrt{\frac{EI}{P}} \,.\, Sin \sqrt{\frac{P}{EI}} \,.\, x + \frac{R}{P}(l - x)$$

At B, $y = o$ When $x = l$

or $\quad \frac{-Rl}{P} Cos\, l\sqrt{\frac{P}{EI}} + \frac{R}{P}\sqrt{\frac{EI}{P}} \,.\, \text{Sin}\, l\sqrt{\frac{p}{EI}} = 0$

$$\therefore \tan l\sqrt{\frac{P}{EI}} = l\sqrt{\frac{P}{EI}}$$

(The tangent of the angle = angle itself)

The smallest root of the above equation is

$$l\sqrt{\frac{P}{EI}} = 4.49 \text{ radian}$$

$$l^2 \frac{P}{EI} = 20 = 2\pi^2$$

or $\quad P_{cr} = \frac{2\pi^2 EI}{l^2}$

Limitations of Euler's formula

Euler's formula may be used for long columns when slenderness ratio exceeds 100. If the value of slenderness ratio is less than 100 Euler's equation can not be used as such and has to be modified keeping in view the passing of the material into plastic stage. The Euler's formula is not applicable for crippling stress beyond 264 MPa

Equivalent lengths for various end conditions

Table – 13 .1

	End Conditions	*Equivalent Lengh l*
1	Both ends hinged	$l = L$
2	One End fixed and the other end free	$l = 2L$
3	Both ends fixed	$l = L/2$
4	One end fixed and the other end hinged	$l = \frac{l}{\sqrt{2}}$

Example 13 .1

A mild steel tube 25 mm external diameter and 2 .5 mm thick is 3 metre long. It is used as a column with both ends hinged. Calculate the collapsing load by Euler's formula. Take E = 200 KN/mm^2.

Solution

$$P_{cr} = \frac{\pi^2 EI}{l^2}$$

External diameter = 25 mm

Thickness = 2.5 mm

Internal diameter = (25 – 2 × 25) = 20 mm

Moment of Inertia

$$I = \frac{\pi}{64}(25^4 - 20^4) = 11320.77 \text{ mm}^4$$

Since both ends are hinged $l = L = 3000$ mm

$$P_{cr} = \frac{\pi^2 \times 200 \times 10^3 \times 11320.77}{(3000)^2}$$

$$= 2.483 \text{ KN}$$

Example 13 .2

Calculate the safe compressive load on a hollow cast iron column (one end rigidly fixed and the other hinged) of 100 mm external diameter and 70 mm internal dia and 8 metres in length. Use Euler's formula with a factor of safety of 4 and E = 96 KN/mm² *M. U.*

Solution

Moment of inertia of the column section

$$I = \frac{\pi}{64}(100^4 - 70^4)$$

$$= 373 \times 10^4 \text{ mm}^4$$

Since one end of the column is fixed and the other is hinged

$$\text{Effective length } l = \frac{L}{\sqrt{2}} = \frac{8 \times 10^3}{\sqrt{2}} = 5657.70 \text{ mm}$$

$$\text{Euler's crippling load } P_{cr} = \frac{2\pi^2 EI}{L^2}$$

$$P_{cr} = \frac{2.\pi^2 \times 96 \times 10^3 \times 373 \times 10^4}{(5657.70)^2}$$

$$= 220.8 \text{ KN}$$

$$\text{Safe load} = \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{220.8}{4}$$

$$P_{cr} = 55.2 \text{ KN}$$

Example 13.3

An alloy tube 5 metres long extends 6.4 mm under a tensile load of 60 KN. Calculate the Euler's buckling load, when used as a strut with pin jointed ends. The tube diameters are 40 mm and 25 mm. J. M. I.

Solution

Area of Cross-Section

$$A = \frac{\pi}{4}(40^2 - 25^2) = 765.7 \text{ mm}^2$$

Moment of inertia of the section

$$I = \frac{\pi}{64}(40^4 - 25^2) = 10.64 \times 10^4 \text{mm}^4$$

Stress induced due to a load of 60 KN

$$\sigma = \frac{\text{Load}}{\text{Area of Cross-section}}$$

$$= \frac{60 \times 10^3}{765.7} = 78.3 \text{ MPa}$$

Strain produced in the tube =

$$\frac{6.4}{5 \times 1000} = 1.28 \times 10^{-3}$$

Therefore modulus of elasticity

$$E = \frac{78.3}{1.28 \times 10^{-3}} = 61.17 \times 10^3 \text{ N/mm}^2$$

$$= 61.17 \text{ KN/mm}^2$$

Since both ends are pin-jointed $L = l$

$l = 5000$ mm

Eulers buckling load $P_{cr} = \dfrac{\pi^2 EI}{l^2}$

$$P_{cr} = \frac{\pi^2 \times 61.17 \textit{ time } 10^3 \times 10.64 \times 10^4}{(5000)^2}$$

$= 2.56$ KN **Answer.**

Example 13 .4

An I-section R. S. J 200 mm × 160 mm with flanges 15 mm thick and web 10 mm thick is used as a column with one end fixed and the other end entirely free. Determine the Euler's crippling load if the length of the column is 6 metres. Take E = 200 KN/mm²

Solution

$$I_{xx} = \frac{160\,(200)^3}{12} - 2 \times \frac{(75)\,(170)^3}{12}$$

$$= 10666.7 \times 10^4 - 6141.25 \times 104$$

$$= 4525.5 \times 10^4 \text{ mm}^4$$

$$I_{yy} = \frac{2 \times 15\,(160)^3}{12} + \frac{170\,(10)^3}{12}$$

$$= 1024 \times 10^4 + 1.41 \times 10^4$$

$$= 1025.4 \times 10^4 \text{ mm}^4$$

$\therefore$ I Lest = 1025.4×10^4 mm^4

Equivalent length $l = 2L$

$$\therefore P_{cr} = \frac{\pi^2 EI}{4L^2}$$

$$= \frac{\pi^2 \times 200 \times 10^3 \times 1025.4 \times 10^4}{4 \times (6000)^2}$$

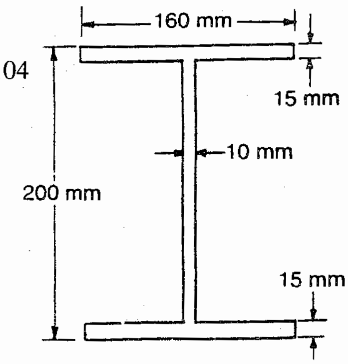


Fig. 13.6

$$= \frac{\pi^2 \times 200 \times 1025.4 \times 10}{4 \times 36} \text{ Newtons}$$

$$= 140500 \text{ N} = 140.5 \text{ KN} \qquad \textbf{Answer.}$$

Example 13 .5

A steel bar of rectangular section 30 mm × 60 mm is used as a column with both end hinged and subjected to an axial compression. If the critical stress developed is 240 MPa and modulus of elasticity is 200 GN/m^2 . Determine the minimum length for which Euler's Equation may be used. If the length of the colum is 2 metres, determine the safe load with a factor of safety of 4.

Solution

Minimum moment of inertia of the section

$$I_{yy} = \frac{1}{12}(d)(b)^3 = \frac{1}{12}(60)(30)^3 = 135 \times 10^3 \text{ mm}^4$$

Least radius of gyration $K = \sqrt{I/A}$

$$K^2 = \frac{I}{A} = \frac{135 \times 10^3}{30 \times 60} = 75$$

Crippling load $P_{cr} = \dfrac{\pi^2 EI}{l^2}$

Critical Stress $= \dfrac{P_{cr}}{A} = \dfrac{\pi^2 EAk^2}{Al^2} = \dfrac{\pi^2 EK^2}{l^2}$

$$240 = \frac{\pi^2 \times 200 \times 20^9 \times 75}{l^2 \times 10^6}$$

$$l^2 = \frac{\pi^2 \times 200 \times 10^9 \times 75}{240 \times 10^6} = 61.84 \times 10^4$$

$l \;= 785$ mm.

When length is two meters

$$P_{cr} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 200 \times 10^9 \times 135 \times 10^3}{(2000)^2 \times 10^6}$$

$P_{cr} = 66620$ Newton = 66 .62 KN

Safe load $P_w = \dfrac{\text{Crippling Load}}{\text{Factor of Safety}} = \dfrac{66.62}{4}$

$= 16.65$ KN

Example 13.6

A cast iron cylindrical column 4 meters long when hinged at both ends supports a buckling load of P Newtons. When both ends are fixed the critical load rises to (P + 250 KN) newtons. If the ratio of external diameter to internal diameter is 1.25 and E = 100 KN/mm^2. Determine the external diameter of the column. *(J.M.I)*

Solution

Let D = External diameter

d = Internal diameter

Diameter ratio $\frac{D}{d} = 1.25$ or $D = 1.25\,d$

When both ends of the column are hinged

$$l = L = 4000 \text{ mm.}$$

$$P_{cr} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 100 \times 10^3 \times 1}{(4000)^2}$$

$$\text{or} \quad P_{cr} = \frac{\pi^2 I}{160} \qquad \dots \qquad \dots \qquad \dots \qquad \text{(i)}$$

When both ends are fixed, $l = \frac{1}{2}$

$$P + 250000 = \frac{\pi^2 EI}{(L/2)^2} = \frac{4\pi^2 EI}{L^2} \qquad \dots \qquad \dots \qquad \dots \qquad \text{(ii)}$$

$$= \frac{4\pi^2 \times 100 \times 10^3 \times 1}{(4000)^2} = \frac{\pi^2}{40} . I$$

$$\text{or} \quad \frac{\pi^2}{160} I + 250000 = \frac{\pi^2}{40} . I$$

$$\text{or} \quad 250000 = \frac{\pi^2}{40} I - \frac{\pi^2 I}{160} \quad \text{or} \quad \frac{\pi^2 I}{40}\left(1 - \frac{1}{4}\right) = \frac{.75\pi^2 I}{40}$$

$$\text{or} \quad I = \frac{250000 \times 40}{.75 \times \pi^2} = 135.09 \times 10^6 \text{ mm}^4$$

$$\frac{\pi}{64}(D^4 - d^4) = 135.09 \times 10^6$$

$$(D^4 - d^4) = \frac{135.09 \times 10^6 \times 64}{\pi}$$

$$[(I.25\,d)^4 - d^4] = 2752.03 \times 10^6$$

$$\text{or} \quad [(2.44\,d^4) - d^4] - 2752.03 \times 10^6$$

$$1.44\,d^4 = 2752.03 \times 10^6$$

$$d^4 = \frac{2752.03}{1.44} \times 191173 \times 10^4, \; d = 209 \text{ mm}$$

Hence external diameter = 261.3 mm **Answer.**

Example 13.7

Determine the ratio of the strengths of a solid steel column to that of a hollow column of the same material and having the same cross-sectional area. The internal diameter of hollow column is $\frac{1}{2}$ of the external diameter. Both the column are of the same length and are pinned at both ends.

(Bangalore University)

Let P_S = Crippling load supported by solid column

D_S = Diameter of solid column.

P_H = Crippling load supported by hollow column

Let D_H and d_H be outer and inner diameter of the hollow column

$$\frac{d_H}{D_H} = \frac{1}{2} \text{ or } d_H = 0.5\,DH$$

Since both ends are hinged

L effective = L actual

$$P_S = \frac{\pi^2 EI_s}{l^2} \text{ and } P_H = \frac{\pi^2 EI_H}{l^2}$$

or $$P_S = \frac{\pi^2 E\,(AK_S^2)}{l^2} \text{ and } P_H = \frac{\pi^2 E(AKH^2)}{l^2}$$

or $$\frac{\text{P hollow}}{\text{P solid}} = \left(\frac{K_H}{K_S}\right)^2$$

Now for solid section radius of gyration

$$K_S = \sqrt{\frac{I_S}{A_S}} \quad \text{or} \quad K_S^2 = \frac{I}{A_S} = \frac{\frac{\pi}{64} D_S^4}{\frac{\pi}{4} D_s^2} = \frac{1}{16} \mathrm{D}^2_{\mathrm{s}}$$

$\therefore$ For Hollow Section

$$K_H^2 = \frac{I_H}{A_H} = \frac{\frac{\pi}{64}(D_H^4 - d_H^4)}{\frac{\pi}{4}(D_H^2 - d_H^2)} = \frac{1}{16}\,(D_H^2 + d_H^2)$$

$$\therefore \quad \frac{P_H}{P_S} = \frac{K_H^2}{K_S^2} = \frac{D_H^2 + d_H^2}{D_S^2} = \left[\frac{D_H^2 + (0.5D_H)^2}{D_S^2}\right]$$

$$= \frac{D_H^2 + .25 D_H^2}{D_S^2} = \frac{1.25 D_H^2}{D_S^2}$$

Since the cross-sectional areas of the columns are equal

$$A_S = A_H$$

$$\frac{\pi}{4} D_S^2 = \frac{\pi}{4}\,[D_H^2 - (0.5\,D_H)^2] = \frac{\pi}{4}\,[D_H^2 - 0.25\,D_H^2]$$

$$D_S^2 = 0.75\,D_H^2$$

Hence $$\frac{P_H}{P_S} = \frac{1.25 D_H^2}{D_S^2} = \frac{1.25 D_H^2}{0.75 D_H^2} = \frac{5}{3}$$

$$\frac{P_H}{P_S} = \frac{5}{3}$$ **Answer**

Example 13.8

A Load of 150 N produced a deflection of 15 mm when placed at the center of a bar of length 3 metres. Determine the Euler's bucking load that the same bar can support if used as a column with both ends restrained in position but not in direction.

Solution

Load = 150 Newtons

Deflection produced at the centre = 15 mm

Span = 3 meters = 3000 mm.

$$y_c = \frac{W l^3}{48\,EI}$$

$$15 = \frac{150\,(3000)^3}{48\,EI}$$

or $$EI = \frac{150\,(3000)^3}{48 \times 15} = 5625 \times 10^6$$

When used as a column with both ends hinged $l = L$

$$\text{Buckling Load } P_{cr} = \frac{\pi^2\,EI}{l^2}$$

$$P_{cr} = \frac{\pi^2 \times 5625 \times 10^6}{3000^2} = 6168 \text{ N}$$

$$P_{cr} = 6.168 \text{ KN}$$

Example 13.9

A straight length of steel bar 1.5 m long and 20 mm × 5 mm section is compressed longitudinally untill it buckles. Assuming Euler's formula to apply to this case, estimate the maximum central deflection before the steel passes the yield point at 320 MPa. Take E = 210 KN/mm² (AMIE)

Solution

Moment of inertia of the section

$$I = \frac{20\,(5)^3}{12} = 208.33 \text{ mm}^4$$

Euler's Crippling Load, $l = L = 1500$ mm

$$P_{cr} = \frac{\pi^2\,E\,I}{l^2} = \frac{\pi^2 \times 210 \times 10^3 \times 208.33}{(1500)^2}$$

$$= 191.9 \text{ Newton}$$

Let the central deflection of the strut be δ

$$M_{max} = P_{cr} \times \delta$$

$$M = 191.9\,\delta$$

$$\text{Direct Stress } \sigma_d = \frac{P_{cr}}{A} = \frac{191.9\delta}{20 \times 5} = 1.91\delta \text{ MPa}$$

$$\text{Bending stress } \sigma_b = \frac{M}{Z} = \frac{191.9\,\delta}{20 \times \frac{5^2}{6}} = 2.302\,\delta \text{ MPa}$$

$\therefore$ Resultant maximum stress

$$\sigma = \sigma_d + \sigma_b$$

$$320 = 1.91\,\delta + 2.302\,\delta$$

or $$\delta = \frac{320}{4.212} = 75.99 \text{ mm} \qquad \textbf{Answer}$$

Example 13.10

A bar of length 4 metres when used as a simply supported beam and subjected to a uniformly distributed load of 3 KN per meter run over the whole span, deflects 15 mm at the centre. Determine the crippling load when it is used as a column with following ends condition.

(i) Both ends pin jointed

(ii) one end fixed and other hinged.

(iii) Both ends fixed (AMIE)

Solution

Load = 3 KN/m

Deflection at mid span = 15 mm

$$y_c = \frac{5wl^4}{384\,EI}$$

$$15 = \frac{5 \times (3 \times 10^3) \times (4 \times 1000)^4}{384\,EI}$$

or $$EI = \frac{5 \times 3 \times 10^3 \times 256 \times 10^{12}}{384 \times 15} = 0.666 \times 10^{15}$$

(1) When both ends are pin Jointed $l = L = 4 \times 1000$ mm

$$\text{Crippling Load } P_{cr} = \frac{\pi^2 EI}{l^2}$$

$$P_{cr} = \frac{\pi^2 \times 0.666 \times 10^{15}}{(4000)^2} = 4.112 \text{ KN}$$

(ii) When one end fixed and other end hinged $l = \dfrac{L}{\sqrt{2}}$

$$P_{cr} = \frac{\pi^2 EI}{l^2} = \frac{2\,\pi^2 EI}{L^2} = \frac{2\,\pi^2 \times 0.666}{(4000)^2} = 0.224 \text{ KN}$$

(iii) When both ends fixed $l = L/2$

$$P_{cr} = \frac{4\,\pi^2 EI}{L^2} = \frac{4\,\pi^2 \times 0.666 \times 10^{15}}{(4000)^2}$$

$$= 16.448 \text{ KN} \qquad \textbf{Answer}$$

Emperical Formula

Rankine's formula

$$P = \frac{\sigma_c . A}{1 + a\left(\frac{l}{k}\right)^2}$$

Where A = Area of cross - section of the column

σ_c = Ultimate stress for column material

l = Effective length of column

k = Least radius of gyration

a = Rankine's constant

The Valucs of Rankine's constant (a) and (σ_c) are given in the following table. These Valucs are only for a column with both ends hinged or pinjointed. For other end conditions the proper effective lengths should be used.

Values of σ_c and a are given in the follwing table

Table 13.2

S. No.	*Material*	*σ_c in MPa*	*Value of a*
1.	Wrought iron	2500	1/9000
2.	Cast iron	5500	1/1600
3.	Mild steel	3200	1/7500
4.	Timber	500	1/750

Johnson's Straight Line Formula

$$P = A\left[\sigma_c - n\left(\frac{l}{k}\right)\right]$$

Where σ_c = allowable stress in the material

n = a constant depending upon the material

If $\frac{P}{A}$ is plotted against l/k then a straight line is obtained, hence it is called straight line formula.

The values of σ and n are given in the following table.

Table 13.3

S.No.	*Material*	*σ_c in MPa*	*n*
1.	Mild steel	3200	0.0053
2.	Wrought iron	2500	0.0053
3.	Cast iron	500	0.008

Johnson's Parabolic Formula

$$P = A\left[\sigma_c - r\left(\frac{l}{k}\right)^2\right]$$

Where P = Safe load on the column

A = cross-sectional area of the column

σ_c = Allwable stress in the material

r=A Constant whose value depends upon material of the column

$\frac{l}{k}$ = Slenderness ratio

The following table gives the values of σ_c and r.

Table 13.4

S. No.	*Material*	σ_c *MPA*	*r*
1.	Mild Steel	3200	0.000057
2.	Wrought iron	2500	0.000039
3.	Cast iron	5500	0.00016

Example 13.11

A cast iron hollow column is 3 meters long and both ends are fixed. The external diameter is 80 mm and the internal diameter is 60 mm. Determine the crippling load using Rankine's formula.

Take the value of σ_c = *550 MPa and* $a = \frac{1}{1600}$ (Aligarh University)

Solution

Area of cross-section $A = \frac{\pi}{4}(80^2 - 60^2) = 700\,\pi \text{ mm}^2$

Moment of inertia $I = \frac{\pi}{64}(80^4 - 60^4) = 625 \times 700\,\pi \text{ mm}^4$

Least radius of gyration $k = \sqrt{I/A}$

$$k = \sqrt{\frac{625 \times 700\pi}{700\pi}} = \sqrt{625} = 25 \text{ mm}$$

Since both ends are fixed $l = \frac{L}{2} = \frac{3000}{2} = 1500$ mm

$$\therefore \frac{l}{k} = \frac{1500}{25} = 60$$

Crippling Load

$$P = \frac{\sigma_c A}{1 + a\,(l/k)^2} = \frac{550 \times 700\,\pi}{1 + \frac{1}{1600}(60)^2}$$

P = 372.15 KN **Answer**

Example. 13.12

A hollow cylindrical cast iron column 5 metres long has both ends fixed. Determine the miximum diameter of the column if it has to carry a safe load of 250 KN with a factor of safety of 4. Take the internal diameter as 0.8 times the external diameter. Take σ_c = 550 MPa and $a = \frac{1}{1600}$ in Ramkine's formula.

Solution :

Let D be the exteral diameter then the internal diameter $d = .8D$ since both ends are fixed, the effective length

$$l = \frac{L}{2} = 2.5 \text{ metres} = 2500 \text{ mm}$$

Sectional area of the columm

$$A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}\,[D^2 - (.8D)^2] = .09\,\pi\,D^2 \text{ mm}^2$$

Moment of inertia of the column section

$$I = \frac{\pi}{64}(D^4 - d^4)$$

and
$$K^2 = \frac{I}{A} = \frac{\frac{\pi}{64}(D^4 - d^4)}{\frac{\pi}{4}(D^2 - d^2)} = \frac{1}{16}(D^2 + d^2)$$

$$= \frac{1}{16}[D^2 + (.8D)^2] = \frac{1.64D^2}{16} = .1025\,D^2$$

Crippling load = safe load × factor of safety

$$= (250 \times 4) = 10{,}00 \text{ KN} = 10^6 \text{ N}$$

$$P = \frac{\sigma_C . A}{1 + a(l/K)^2}$$

$$10^6 = \frac{550 \times .09\,\pi D^2}{1 + \frac{1}{1600}\left[\frac{2500 \times 2500}{0.1025 D^2}\right]} = \frac{155.5 D^2}{1 + \frac{3.81 \times 10^4}{D^4}}$$

or
$$10^6 = \frac{155.5 D^4}{D^2 + 3.81 \times 10^4}$$

or
$$155.5\,D^4 - 10^6\,D^2 - 3.81 \times 10^{10} = 0$$

Solving the quadiratic equation

$$D^4 - 6430\,D^2 - 2.45 \times 10^8 = 0$$

$$D^2 = \frac{6430 \pm \sqrt{(6430)^2 + 4 \times 2.45 \times 10^8}}{2}$$

$$D^2 = \frac{6430 \pm 31990}{2} = \frac{38420}{2} = 19210$$

or $D = 138.6$ mm

and $d = 138.6 \times .8 = 110.88$ mm **Answer.**

Example 13.13

A mild steel rod has a cross-section of 60 m × 30 mm and is one metre between centres. Assume that it is a pin ended strut for bending in a plane parallel to 60 mm side and fixed ended for bending in a plane perpendicular to 60 mm side, calculate the maximum pressure that can be allowed on a 300 mm diametre piston. Assume that the crank is at top dead centre and take a factor of safety of 4 Take for mild steel σ_c = 3250 MPa and for pin jointed ends $a = \frac{1}{7500}$ (Engg. Services)

Solution.

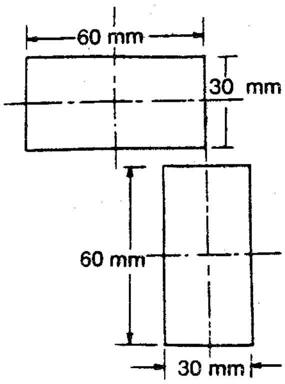


Fig. 13.7

For buckling in a plane parallel to 60mm side effective

Length $l = L = 1000$ mm

Moment of inertia $I = \frac{1}{12}(60)(30)^3$

$= 13.5 \times 10^4$ mm^4

$$K = \sqrt{I/A} = \sqrt{\frac{135000}{60 \times 30}} = 8.66 \text{ mm}$$

$$\frac{l}{K} = \frac{1000}{8.66} = 115.8$$

For buckling in a plane parallel to 30 mm side

$$I = \frac{1}{12} \times 30 \times (60)^3 = 54 \times 10^4 \text{ mm}^4$$

The ends are fixed hence $l = \frac{L}{2} = 500$ mm

$$K = \sqrt{I/A} = \sqrt{\frac{54 \times 10^4}{60 \times 30}} = 17.3 \text{ mm.}$$

$$\frac{l}{K} = \frac{500}{17.3} = 28.9$$

The maximum slenderness ratio = 115.8

Hence crippling load.

$$P = \frac{\sigma_c . A}{1 + a(l/K)^2} = \frac{3250 \times 1800}{1 + \frac{1}{7500}(115.8)^2} = 2100 \text{ KN}$$

$$\text{Allowable load} = \frac{\text{cripplingg load}}{\text{factor of safety}} = \frac{2100}{4} = 525 \text{ KN}$$

$$\text{Maximum Pressure} = \frac{525 \times 10^3}{\frac{\pi}{4}(300)^2} = 7.42 \text{ MPa} \qquad \textbf{Answer}$$

Example 13.14

A mild steel strut is built of 4 angles each 100 mm × 100 mm × 12 mm size forming a square section of side 350 mm over all as shown in figure 13.8. If the length of the strut is 10 metres and the ends are hinged. Calculate the safe axial load using Rankine's constants and a factor of safety of 3. properties of angle section are (i) $I_{xx} = I_{yy} = 207 \times 10^4$ mm^4
$(C_{x\text{-}x} = C_{y\text{-}y}) = 29.2$ mm.

Solution -

Area of the angle = 2259 mm^2

Moment of inertia of the composite section

$= 4\,[\,207 \times 10^4 + 2259\,(145.8)^2]$

$= 4 \times 5009.1 \times 10^4$

Total area of the compound section

4×2259 mm^2

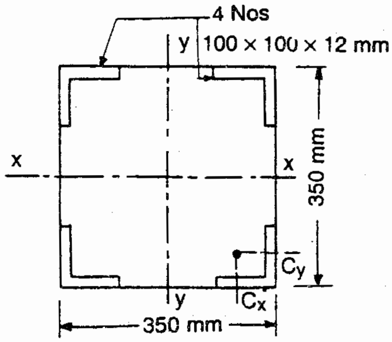


Fig. 13.8

$$K = \sqrt{I/A} = \sqrt{\frac{4 \times 5009.1 \times 10^4}{4 \times 2259}}$$

$= 148.90$

$$\left(\frac{l}{K}\right) = \frac{10 \times 1000}{148.90} = 67.15$$

$$P = \frac{\sigma_c \cdot A}{1 + a\,(l/K)^2} = \frac{3200 \times 4 \times 2259}{1 + \frac{1}{7500}(67.16)^2} \quad \text{Newtons}$$

$$= \frac{3200 \times 4 \times 2259}{1 + .601} \times \frac{1}{1000} \text{ KN} = 18060.7$$

$$\text{Safe Load} = \frac{18060}{3} = 602 \text{ KN} \qquad \textbf{Answer}$$

$P_w = 60.2$ KN

Example 13.15

The section of a compound column is shown in the figure. The column is 3 meter long and both ends are hinged. Using Rankine's Formula. Determine the safe load the column can take if factor of safety is 4. Take $\sigma_c = 3200$, $a = \frac{1}{7500}$

Solution

Area of the section = (1200 + 1200 + 1150 + 1150)

= 4700 mm^2

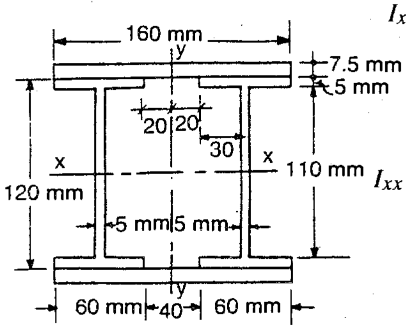


Fig. 13.9

I_{xx} for joists

$$= 2\left[\frac{60 \times 120^3}{12} - \frac{55 \times 110^3}{12}\right]$$

$$= 508 \times 10^4 \text{ mm}^4$$

I_{xx} for plates

$$= \left[\frac{160 \times 135^3}{12} - \frac{160 \times 120^3}{12}\right]$$

$$= 976 \times 10^4 \text{ mm}^4$$

$\therefore$ I_{xx} for the compound section $= 1484 \times 10^4$ mm^4

I_{yy} for joists

$$= 2\left[\frac{10 \times 60^3}{12} - \frac{110 \times 5^3}{12} + 1150 \times 50^2\right]$$

$$= 612 \times 10^4 \text{ mm}^4$$

$$I_{yy} \text{ for plates } = \frac{15 \times 160^3}{12} = 521 \times 10^4 \text{ mm}^4$$

I_{yy} for the compound section $= 1124 \times 10^4$ mm^4

Hence least value of $K^2 = \dfrac{I_{Least}}{A}$

$$K^2 = \frac{1124 \times 10^4}{47 \times 100} = 2400$$

$$\text{Crippling Load} = \frac{\sigma_c . A}{1 + a(l/K)^2}$$

$$P = \frac{3200 \times 4700}{1 + \dfrac{1}{7500} \times \dfrac{3000 \times 3000}{2400}} = \frac{3200 \times 4700}{1.5}$$

$P = 1002.6 \times 10^4$ Newtons.

$$\text{Safe Load} = \frac{1002.6 \times 10^4}{4}$$

$P_w = 2506$ KN **Answer**

I. S. Code Formula

The maximum permissible axial compressive load P is given by the formula

$$P = \sigma_{ac} . A$$

Where P = Axial compressive load

σ_{ac} = Permissible stress in axial compression

A = Effective Cross-sectional area of the member (Gross Sectional area minus deductions for any hole not filled completely by rivets or bolts)

As per Is – 800 – 1984 the following formula is used for calculating σ_{ac}

$$\sigma_{ac} = 0.6 \times \frac{f_{ce} + f_y}{[f_{ce}^n + f_y^n]^{1/n}}$$

Where σ_{ac} = Permissible stress in axial compression

f_y = Yield stress of steel

f_{ec} = Elastic critical stress in compression

$$f_{ec} = \frac{\pi^2 E}{\lambda}$$

Where λ = Slenderness ratio $\frac{l}{r}$

E = Modulus of elasticity 2×10^5 MPa

n = a factor assumed as 1.4

Values of are given in the table for convinience corresponding to various values of σ_{ac} yield stress σ_y and slenderness ratio $\frac{l}{r}$

Example 13.15

Determine the safe axial load an a strut built up of 100 mm × 100 mm × 10 mm angles to from the shape of a square as shown in figure 13 .10 The column is 5 metres long and hinged at both ends take yield stress of steel as 250 MPa

Solution

Elective length = 5 meters

Properties of 100 mm × 100 mm × 10 mm Angle from steel table

$I_{xx} = I_{yy} = 177 \times 10^4$ mm^4

$C_{xx} = C_{yy} = 29.4$ mm, a = 1903 mm^2

Since the section is symmetrical about x–x and y–y axis

$\therefore\ I_{xx} = I_{yy} = 4\,[I_{xx} + a\,C_{xx}^2]$

$= 4\ [177 \times 10^4 + 1903\,(29.4)^2]$

$= 1367.6 \times 10^4$

Gross area $A = 4 \times 1903$

$= 7912$ mm^2

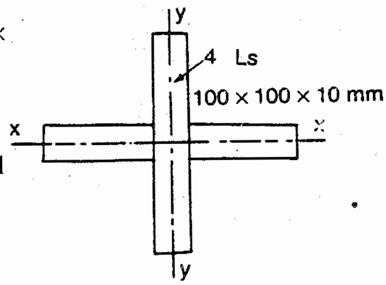


Fig. 13.10

Radius of gyration $= \sqrt{I_{xx}/A} = \frac{1367.6 \times 10^4}{7612} = 14.3$ mm

Slenderness ratio $\frac{l}{r} = \frac{5000}{42.3} = 118.2$

TABLE. 13.4

Permissible Compressive Stresses in Compression as per IS 800—1984.
Permissible Stress σ_{ac} (MPa) in Axial Compression for Steels with various Yield Stresses (f_y) and Slenderness ratio (λ) (Clause 5.5.1 IS : 800—1984

λ	λ_y = *yield stress in MPa* = N/mm^2													
	220	230	240	250	260	280	300	320	340	360	380	400	420	450
	σ_{ac} = *permissible compressive stress in MPa* = N/mm^2													
10	132	138	144	150	156	168	180	192	204	215	227	239	251	269
20	131	137	142	148	154	166	177	189	201	212	224	235	146	263
30	128	134	140	145	151	162	172	183	194	204	215	225	136	251
40	124	129	134	139	145	154	164	174	183	192	201	210	218	231
50	118	123	127	132	136	145	153	161	168	176	183	190	197	207
60	111	115	118	122	126	133	139	146	152	158	163	168	173	180
70	102	106	109	112	115	120	125	130	135	139	142	147	150	155
80	93	96	98	101	103	107	111	115	118	121	124	127	129	133
90	85	87	88	90	92	95	98	101	103	105	108	109	111	114
100	76	78	79	80	82	84	86	88	90	92	93	94	96	97
110	68	69	71	72	73	74	76	77	79	80	81	82	83	84
120	61	62	63	64	64	66	67	67	69	70	71	71	72	73
130	55	55	56	57	57	58	59	60	61	61	62	62	63	63
140	49	50	50	51	51	52	53	53	54	54	54	55	55	56
150	44	45	45	45	46	46	47	47	48	48	48	49	49	49
160	40	40	41	41	41	42	42	42	43	43	43	43	43	44
170	36	36	37	37	36	37	38	38	38	38	39	39	39	39
180	33	33	33	33	33	34	34	34	34	35	35	35	35	35
190	30	30	30	30	30	30	31	31	31	31	31	31	32	32
200	27	27	28	28	28	28	28	28	28	28	28	28	28	28
210	25	25	25	25	25	25	26	26	26	26	26	26	26	26
220	23	23	23	23	23	23	23	24	24	24	24	24	24	24
230	21	21	21	21	21	21	22	22	22	22	22	22	22	22
240	20	20	20	20	20	20	20	20	20	20	20	20	20	20
250	18	18	18	18	18	18	18	18	18	19	19	19	19	19

Using yield stress = 250 MPa and l/r = 118.2

From table

for $l/r = 100$, $\sigma_{ac} = 72$

for $\frac{l}{r} = 120$, $\sigma_{ac} = 64$

By inter polation, for $l/r = 118.2$ $\sigma_{ac} = 65.44$

$\therefore\ P = \sigma_{ac} \times \text{Area}$

$= 65.44 \times 7612 = 498.12$ KN

Example 13.16

A single angle strut ISA 100× 100× 8 mm is 2 meters long. Determine the safe compressive load if the yield stress for steel is 250 MPa

Solution

Area of the section = 1539 mm^2

r mm = 19.5

Effective length = 2 meters = 2000 mm

Slenderness ratio $= \frac{2000}{195} = 102.5$

Allowable stress from tables taking

$\sigma_y = 250$, $\sigma_{ac} = 76.84$

But permissible value for single angle strut dis-continous member

$= 0.8\ \sigma_{ac} = 0.8 \times 76.84 = 61.4$ MPa

Hence safe load = 1539×61.4 Newtons

$P_w = 94.6$ KN **Answer.**

Eccentric loading on long columns

Rankine's formula

When a long column is subjected to eccentric loading, the reduction factor is modified taking into account the effect of eccentricity as well as buckling. Hence Rankine's formula becomes

$$P = \frac{\sigma_c A}{\left\{1 + a\left(\frac{l}{k}\right)^2\right\} + \left\{1 + \frac{e\,y_c}{k^2}\right\}}$$

When eccentricity is about both the axes the formula is further modified as under

$$P = \frac{\sigma_c A}{\left\{1 + a\left(\frac{l}{k}\right)^2\right\} + \left\{1 + \frac{e\,y_c}{k_x^2} + \frac{e' x_c}{k_y^2}\right\}}$$

The above formula is valid for columns with both ends hinged. Other cases with different end conditions may be solved accordingly.

The secant formula

Consider a column with both ends hinged with a load P acting at a distance e from the axis of the column Fig. 13.11

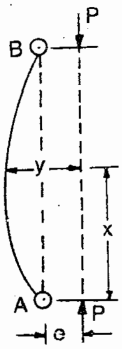


Fig. 13.11

Consider a section at a distance x from A. Let y be the deflection of the column from the line of action of the load P, then

Bending Moment $= -P.y$

$$EI\,\frac{d^2y}{dx^2} = -P.y$$

$$\therefore\ \frac{d^2y}{dx^2} + \frac{P}{EI}\,.\,y = 0$$

The Solution of this differential equation is

$$y = C_1 \operatorname{Sin} x\sqrt{\frac{P}{EI}} + C_2 \operatorname{Cos} x\sqrt{\frac{P}{EI}} \qquad \ldots\ldots \qquad \ldots\ldots \qquad \text{(i)}$$

at $x = 0,\quad y = e \quad \therefore\ C_2 = 0$

At the mid height of the column $\frac{dy}{dx} = 0$

and $x = \frac{l}{2}$

$$0 = C_1\sqrt{\frac{P}{EI}}\,.\operatorname{Cos}\frac{l}{2}\sqrt{\frac{P}{EI}} - e\sqrt{\frac{P}{EI}}\,.\operatorname{Sin}\frac{l}{2}\sqrt{\frac{P}{EI}}$$

$$\therefore\ C_1 = \frac{e\ \operatorname{Sin}\frac{l}{2}\sqrt{\frac{P}{EI}}}{\operatorname{Cos}\frac{l}{2}\sqrt{\frac{P}{EI}}}$$

Substiluting in equation (i) we get

$$y = e\left\{\frac{\operatorname{Sin}\frac{l}{2}\sqrt{\frac{P}{EI}}}{\operatorname{Cos}\frac{l}{2}\sqrt{\frac{P}{EI}}}\,.\operatorname{Sin} x\sqrt{\frac{P}{EI}} + \operatorname{Cos} x\sqrt{\frac{P}{EI}}\right\}$$

At $x = \frac{l}{2}$

$$y_{max} = e\left\{\frac{\operatorname{Sin}^2\frac{l}{2}\sqrt{\frac{P}{EI}}}{\operatorname{Cos}\frac{l}{2}\sqrt{\frac{P}{EI}}} + \operatorname{Cos}\frac{l}{2}\sqrt{\frac{P}{EI}}\right\}$$

$$= e \sec\frac{l}{2}\sqrt{\frac{P}{EI}}$$

The maximum $B.\ M.$ will occur when $x = \frac{l}{2}$ where y is the maximum

$$M_{max} = P \ . \ y_{\max}$$

$$= P.e.\ \text{Sec}\ \frac{l}{2}\sqrt{\frac{P}{EI}}$$

The maximum bending stress

$$\sigma_{max} = \frac{P}{A} + \frac{M.y}{I}$$

$$= \frac{P}{A} + \frac{M}{Z}$$

$$= \frac{P}{A} + \frac{P \ . \ e \ . \ Sec\frac{l}{2}\sqrt{\frac{P}{EI}}}{Z}$$

$$\text{or} \quad \sigma_{max} = \frac{P}{A}\left\{1 + \frac{y_c}{K^2} \ . \ e \ . \ Sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right\}$$

Where y_c is the distance of the extreme compression fibre from the neutral axis

The term $\frac{y_c.e}{K^2}$ is called the eccentricity ratio and l is the effective length of the column.

Example 13.17

A hollow circular column of length 4 metres, external diameter 150 mm and internal diameter 100 mm is hinged at both ends. It supports an eccentric load of 250 KN at an eccentricity of 10 mm from the vertical axis of the column. Determine the maximum stress induced in the column. Take $E = 200\ KN/mm^2$.

Solution

$$\text{Direct stress} = \frac{250 \times 10^3}{\frac{\pi}{4}(150^2 - 100^2)} = \frac{250 \times 10^3 \times 4}{\pi \times 12500} = 25.46 \text{ MPa}$$

Moment of inertia of the circular column

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}[(150)^4 - (100)^4] = 19.941 \times 10^6 \text{ mm}^4$$

Since both ends are hinged equivalent length = 4000 mm

$$\text{Bending stress} = \left(P.e.Sec.\frac{l}{2}\sqrt{\frac{P}{EI}}\right) \times \frac{1}{Z}$$

$$\text{Now Calculate Sec}\ . \ \frac{l}{2}\sqrt{\frac{P}{EI}}$$

$$= \text{Sec}\ .\ \frac{4000}{2}\sqrt{\frac{250\times10^3}{200\times10^3\times19.941\times10^6}}$$

$$= \text{Sec}\ .\ \frac{2000}{10^3}\sqrt{\frac{1.25}{19.941}}$$

$= \text{Sec. } 2\sqrt{0.6268}$ $\quad = \text{Sec } 2\times .250$ $\quad = \text{Sec . } 0.50$ radian

$= \text{Sec . } 28°.68$ $\quad = 0\ .\ 8772$

$$\text{Bending stress} = \frac{250\times10^3\times10\times .8772}{19.941\times10^6}\times75$$

$= 8.20$ MPa

Maximum stress developed

σ_{max} = Direct stress + Bending stress

$= 25.46 + 8.20 = 33.66$ MPa **Answer**

Columns with initial curvature

Consider a column AB of length l and having an initial curvature such that the maximum central deflection is e Fig. 13.12

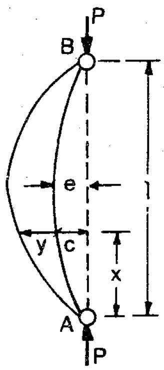


Fig. 13.12

Let the initial deflection at a distance x from A be c

$C = 0$ when $x = 0$

and also when $x = l$

$C = \text{e}$ when $x = \frac{l}{2}$

Assume that the initial shape of the column is governed by the relation

$C = e \text{ Sin } \frac{\pi x}{l}$ and this satisfies the above stated conditions.

On application of the load P let there be a further deflection of y at x from A

$$EI\ \frac{d^2y}{dx^2} = -P\ (y + c)$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}\ y + \frac{P}{EI}\ e \text{ Sin } \frac{\pi x}{l} = 0$$

The solution of the above differential equation is

$$y = C_1 \text{ Cos } x\sqrt{\frac{P}{EI}} + C_2 \text{ Sin } x\sqrt{\frac{P}{EI}} + \frac{\frac{P}{EI}.e.Sin\frac{\pi x}{l}}{\frac{\pi^2}{l^2} - \frac{P}{EI}}$$

When $x = 0$, $y = 0$

$$\therefore \quad 0 = C_1 \text{ Cos } 0 + C_2 \text{ Sin } 0 + \frac{\frac{P}{EI}.e.Sin\,0}{\frac{\pi^2}{l^2} - \frac{P}{EI}}$$

$$\therefore \quad C_1 = 0$$

When $x = l, \quad y = 0$

$$\therefore \quad 0 = C_2 \text{ Sin } l \sqrt{\frac{P}{EI}} + \frac{\frac{P}{EI}.e.Sin\,\pi}{\frac{\pi^2}{l^2} - \frac{P}{EI}}$$

$$= C_2 \text{ Sin } l \sqrt{\frac{P}{EI}}$$

Either $C_2 = 0$ or $\text{Sin } l \sqrt{\frac{P}{EI}} = 0$

The later is the Euler's solution for a two hinged straight column. Hence $C_2 = 0$

$$\therefore \quad y = \frac{\frac{P}{EI}.e.Sin\frac{\pi x}{l}}{\frac{\pi^2}{l^2} - \frac{P}{EI}}$$

Total eccentricity at any point is $y + c$

$$= \frac{\frac{P}{EI}.e.Sin\frac{\pi x}{l}}{\frac{\pi^2}{l^2} - \frac{P}{EI}} + \text{e Sin}\frac{\pi x}{l}$$

$$= e \text{ Sin}\frac{\pi x}{l}\left[\frac{P}{\frac{\pi^2 EI}{l^2} - P} + 1\right]$$

Since $\frac{\pi^2 EI}{l^2} = P_{cr}$ (Euler's load)

$$\therefore \text{ Eccentricity at any point} = e \text{ Sin}\frac{\pi x}{l}\left(\frac{P}{P_{cr} - P} + 1\right)$$

$$= e \text{ Sin}\frac{\pi x}{l}\left(\frac{P_{cr}}{P_{cr} - P}\right)$$

Maximum deflection occurs at the centre when $x = \frac{l}{2}$

$$\therefore \text{ Maximum deflection} = e \text{ Sin } \frac{\pi}{l} \times \frac{l}{2} \left[\frac{P_{cr}}{P_{cr} - P} \right]$$

$$= \frac{eP_{cr}}{P_{cr} - P}$$

Maximum Bending moment at centre = Load × Max. deflection

$$= \frac{P.e\,P_{cr}}{P_{cr} - P}$$

Maximum stress = direct stress + Bending stress

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max} \cdot y_c}{I}$$

Where y_c is the distance of the extreme fibre in compression from the neutral axis

$$\sigma_{max} = \frac{P}{A} + \frac{P_{cr} \cdot P \cdot e \cdot y_c}{(P_{cr} - P) \cdot I}$$

$$= \frac{P}{A} \left[1 + \frac{e\,P_{cr} . y_c}{(P_{cr} - P) \cdot k^2} \right]$$

$$= \sigma_0 \left[1 + \frac{e\,\sigma_c}{\sigma_c - \sigma_0} \cdot \frac{y_e}{k^2} \right]$$

Where $\sigma_0 = \dfrac{P}{A}$

and σ = Euler's buckling stress $= \dfrac{P_{cr}}{A}$

$$\frac{\sigma_{max}}{\sigma_0} - 1 = \frac{e y_c}{k^2} \times \frac{\sigma_c}{\sigma_\varepsilon - \sigma_0}$$

$$\text{or} \quad \frac{e\,y_c}{k^2} = \left(\frac{\sigma_{max} - \sigma_0}{\sigma_0} \right) \left(\frac{\sigma_c - \sigma_0}{\sigma_\chi} \right)$$

Struts With Transverse Loading

Strut with a point load at mid span

Let a strut $A\ B$ of length L and hinged at both the ends be subjected to an axial thrust P and a central load W as shown in figure 13.11

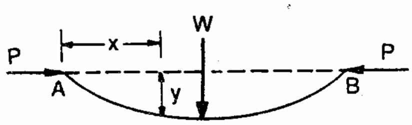


Fig. 13.13

Taking origin at A, the bending moment at a distance x from A is

$$EI \frac{d^2y}{dx^2} = -\frac{W}{2} \cdot x - P.y$$

or $$\frac{d^2y}{dx^2} + \frac{Py}{EI} = -\frac{W.x}{2EI}$$

The solution of the above differential equation is

$$y = C_1 \text{ Sin } Kx + C_2 \text{Cos } Kx - \frac{Wx}{2P}$$

Where $K = \sqrt{\frac{P}{EI}}$

$$\frac{dy}{dx} = C_1k \text{ Cos } kx - C_2k \text{ Sin } kx \frac{-W}{2P}$$

At $x = 0,\ y = 0\ \therefore C_2 = 0$

Also when $x = \frac{l}{2},\ \frac{dy}{dx} = 0$

$$\therefore 0 = C_1 k \text{ Cos } k\frac{l}{2} - \frac{W}{2P}$$

or $$C_1 = \frac{W}{2KP\ Cos\ K\frac{l}{2}}$$

$$\therefore\ y = \frac{W}{2KP\ Cos\ k\frac{l}{2}} \times \text{Sin } kx - \frac{Wx}{2P}$$

At $x = \frac{l}{2}$,

$$y = y_{max} = \frac{W}{2P}\sqrt{\frac{EI}{P}} \tan\sqrt{\frac{P}{EI}}\cdot\frac{l}{2} - \frac{Wl}{4P}$$

At $x = \frac{l}{2}$

$$B.\ M_{max} = -P\left(\frac{W}{2P}\sqrt{\frac{EI}{P}} \tan\sqrt{\frac{P}{EI}}\cdot\frac{l}{2} - \frac{WL}{4P}\right) - \frac{WL}{4}$$

$$M_{max} = \frac{-W}{2}\sqrt{\frac{EI}{P}} \tan\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}$$

$$y_{max} = \frac{W}{2P}\sqrt{\frac{EI}{P}} \tan\sqrt{\frac{P}{EI}}\cdot\frac{l}{2} - \frac{Wl}{4P}$$

Now Put $U^2 = \frac{PL^2}{4EI}$ then $U = \frac{l}{2}\sqrt{\frac{P}{EI}}$

$$\therefore y_{max} = \frac{W}{2P}\cdot\frac{l}{2u} \tan U - \frac{Wl}{4P}$$

$$= \frac{Wl}{4P}\left(\frac{\tan u}{u} - 1\right)$$

Now $P = \dfrac{4EIu^2}{l^2}$ from above

$$\therefore\ y_{max} = \frac{W.l.l^2}{4\times 4EIu^2}\left(\frac{\tan u - u}{u}\right)$$

$$= \frac{Wl^3}{16EI}\left(\frac{\tan u - u}{u^3}\right)$$

$$= \frac{Wl^3}{48EI}\times 3\left(\frac{\tan u - u}{u^3}\right)$$

Strut with an axial load P and a uniformly distributed load w per unit run over the whole length

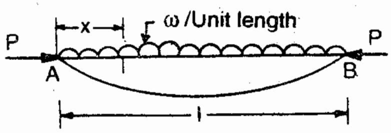


Fig. 13.14

Bending moment at a distance x

$$M_x = EI\frac{d^2y}{dx^2} = \frac{-wlx}{2} + \frac{wx^2}{2} - P.y$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}.y = -\frac{wx\ (l-x)}{2EI}$$

The Solution of the above differential equation is

$$y = C_1 \text{ Cos } Kx + C_2 \sin Kx - \frac{w - x(l-x)}{2P} - \frac{WEI}{P^2}$$

$$\frac{dy}{dx} = -C_1 \text{ K} \sin Kx + C_2K \text{ Cos } Kx\ \frac{-w}{2P}(l-2\text{ x})$$

At $x = 0, y = 0$

$$\therefore\ C_1 - \frac{wEI}{P^2} = 0 \quad \text{or} \quad C_1 = \frac{wEI}{P^2}$$

At $x = \dfrac{l}{2}, \dfrac{dy}{dx} = 0$

$$\therefore\ 0 = -C_1\text{K} \sin K\frac{l}{2} + C_2K \text{ Cos } K\frac{l}{2} - \frac{w}{2P}\left(l - 2\frac{l}{2}\right)$$

$$\therefore\ C_2 = C_1 \tan K\frac{l}{2}$$

and $y = \dfrac{wEI}{P^2}\left[Cos\ Kx + \tan K\dfrac{l}{2}\sin Kx - \dfrac{wx(l-x)}{2P} - \dfrac{wEI}{P^2}\right]$

At $x = \dfrac{l}{2}$

$$y = y_{max} = \frac{wEI}{p^2}\left\{Cos\,K\frac{l}{2} + \tan K\frac{l}{2}\,Sin\,K\frac{l}{2}\right\} - \frac{wL}{4P}\frac{l}{2} - \frac{wEI}{p\,2}$$

$$y_{\max} = \frac{w\,E\,I}{P^2}\left\{Sec\,K\frac{l}{2} - 1\right\} - \frac{w\,l^2}{8\,P}$$

$$= \frac{w\,E\,I}{P^2}\left\{Sec\sqrt{\frac{P}{E\,I}}\cdot\frac{l}{2} - 1\right\} - \frac{w\,l^2}{8\,P}$$

At $x = \frac{l}{2}$

$$M_{max}\,.\,B.M = -\frac{w\,l^2}{8} - P\cdot y_{max}$$

$$M_{max} = -\frac{w\,E\,I}{P}\left(Sec\sqrt{\frac{P}{E\,I}}\cdot\frac{l}{2} - 1\right)$$

Maximum Compressive Stress = Direct Stress + Bending Stress

$$= \sigma_0 + \sigma_b$$

$$= \frac{P}{A} + \frac{w\,E\,I}{P}\left(Sec\frac{l}{2}\sqrt{\frac{P}{E\,I}} - 1\right)\cdot\frac{y_e}{I}$$

$$= \frac{P}{A} + \frac{w\,y_e\,E}{P}\left(Sec\frac{l}{2}\sqrt{\frac{P}{E\,I}} - 1\right)$$

SUMMARY

1. For Short Columns

$$\frac{l}{k} < 32 \quad \text{or} \quad \frac{l}{d} < 8$$

2. For Long Columns

$$\frac{l}{k} > 120 \quad \text{or} \quad \frac{l}{d} > 30$$

3. For medium Columns

$$\frac{l}{k} > 32 \text{ and } < 120$$

$$\frac{l}{d} > 8 \text{ and } < 30$$

4. Euler's crippling load or critical load a for long columns

(a) $P_{cr} = \frac{\pi^2 EI}{l^2}$ (When both ends are hinged)

5. (b) $P_{cr} = \frac{\pi^2 EI}{4l^2}$ (When one end fixed and the other end free)

(c) $P_{cr} = \frac{4\pi^2 EI}{l^2}$ (When both ends are fixed)

(d) $P_{cr} = \dfrac{2\pi^2 EI}{l^2}$ (When one end fixed and the other hinged)

5. Rankine's Crippling Load

$$P_{cr} = \frac{\sigma_{c.A}}{1 + a\,(l/k)^2}$$

Where σ_c = ultimate stress for column material

A = Area of cross-section of column

l = effective length of column

K = Least radius of gyration

a = Rankine's constant.

6. Johnson's straight line formula

$$P = A\left[\sigma_c \; n\left(\frac{l}{K}\right)\right]$$

Where σ_c = allowable stress in the material

n a constant depending upon the material

7. Johnson's Parabolic formula

$$P = A\left[\sigma_c - r\left(\frac{l}{K}\right)^2\right]$$

8. I. S. Code formula

$$P = \sigma_{ac} \,.\, A$$

Where P = axial compressive load

σ_{ac} = Permissible stress in axial compression

A = Effective cross-sectional area of the member.

9. Eccentrically loaded long columns

$$P = \frac{\sigma A}{\left\{1 + a\left(\frac{l}{K}\right)^2\right\} + \left\{1 + \frac{cy_e}{k_x^2} + \frac{ex_e}{ky^2}\right\}} +$$

10. The secant formula

$$\sigma_{max} = \frac{P}{A}\left\{1. + \frac{y_c}{k^2} .e. Sec \frac{l}{2}\frac{\sqrt{P}}{EI}\right\}$$

QUESTIONS

(1) (a) Explain the terms "Column" and "Strut"

(b) What do you understand by the effective length of a column? Write the effective lengths for various end conditions.

(2) What are the various modes of failure of the following types of columns

(a) Long columns

(b) Short columns

(c) Medium Sized columns.

(3) (a) What are the assumptions made in Euler's theory for long columns.

(b) What are the limitations of Euler's theory

(4) Deduce an expression for the crippling load for a column by Euler's theory

(5) Explain slenderness ratio. Depending on slenderness ratio how are columns classified ?

EXERCISES

(6) A mild steel bar of diameter 50 mm is used as a column with both ends hinged. If the safe allowable stress in steel is 210 MPa and the modulus of elasticity is 200 KN/mm^2, Determine the minimum length for which Euler's Formula is valid

Ans. (1.21 metres)

(7) Determine the critical load for a rectangular bar 250 mm deep when used as a column with pin jointed ends. The bar is 4 metres long and $I_{xx} = 44 \times 10^6$ mm^4 and $I_{yy} = 4 \times 10^6$ mm^4 Take $E = 200$ KN/mm^2

Ans. (493 KN)

(8) Calculate the crippling load for T-section show in figure 13.13. When used as a strut 4 m long an hinged at bothends.

Take $E = 200$ KN/mm ^{2}

Ans. (711.25 KN)

J.M.I 1995

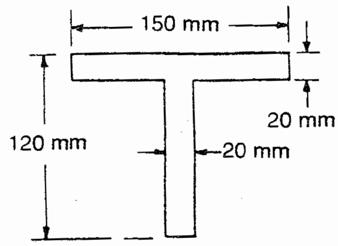


Fig. 13.15

(9) A uniform bar of span 2 metres deflects 6 mm under a central load of 150 newtons. Determine the Euler's buckling load when used as a column with bothends fixed

Ans. (41.6 KN)

(10) Find the Euler's crippling load for a hollow cylindrical steel column 30 mm external diameter and 2. mm thick. Take length of the column as 2.3 m and hinged at both ends. Take $E = 205$ KN/mm^2

Ans. (16.88 KN)

(11) A circular bar 5m long and 40 mm in diameter was found to extend 4.5 mm under a tensile load of 40 KN the bar is used as a strut with both ends hinged. Determine the buckling load for the bar and also the safe load. Taking factor of safety as 3.

(Aligarh Uni.)

Ans. (210 N, 70 N)

(12) Calculate the safe compressive load on hollow cast iron column (one end rigidly fixed and the other hinged) of 150 mm external diameter and 100 mm internal diameter and 10 m length. Use Euler's formula with a factor of safety 5 and $E =$ 95 KN/mm2 **Ans.** (748 KN)

(13) A solid cast iron column 5 m long and 150 mm in diameter is fixed in direction and position at the lower end and carries a load at the free upper end. Assuming a factor of safety 5, calculate the safe load. the column could carry. The value of 'a' in the Rankine's formula for cast iron may be taken as 1/1600 ands σ_c= 5500 KN/mm^2 **Ans.** (45 5KN, 91 KN) (AMIE)

(14) Determine the section of a cast iron hollow cylindrical column 5 meters long with ends firmly built in if it carries an axial load of 30 KN. The ratio of internal diameter to external diameter is 3/4 use factor of safety of 3
Take σ_c = 5500 and a = 1/1600 J.M.I. **Ans.** (166 mm, 125 mm)

(15) Find the Euler's crippling load for a hollow cylindrical cast iron column 150 mm external diameter and 20 mm thick. If it is 6 metre, long and hinged at both the ends. Compare this load with the crushing load as given by Rankine's Formula using constants. 620 MPa and 1/1600

Take E = 80 KN/mm^2 (Engg. Services)

Ans. (386.6 KN and 445 KN)

□□□

14

Analysis of Simple Trusses

Truss

Truss is a framework consisting of any number of bars forming triangles. The members are pin-jointed or riveted. All members in a truss are in axial tension or in axial compression.

Perfect frame

A perfect frame is one which has sufficient number of bars so as to keep the truss in static equilibrium under any system of load without distorting its geometrical shape. The forces in the members of a truss can be determined with the help of the equations of statics $\Sigma H = 0$, $\Sigma V = 0$ and $\Sigma M = 0$.

If n be the number of bars in a truss and j the number of joints, then a perfect frame or a statically determinate frame must satisfy the following equation.

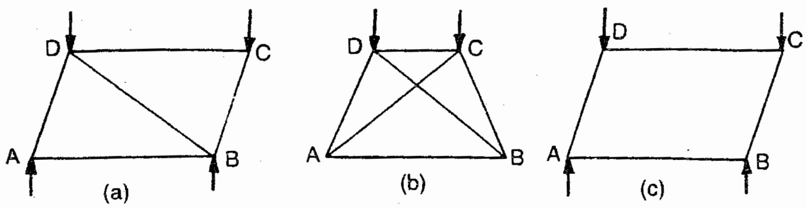
$$n = 2j - 3$$

Deficient frame

When the number of bars in a frame is less than the number required for a perfect frame, such a frame is called a deficient or impefect frame.

Redundant frame

When the number of bars is more than the one required for a perfect frame then the frame is called redundant frame or statically indeterminate. We shall confine our studies to perfect frame or statically determinate frames only.



Perfect frame | Redundant frame | Deficient frame

Fig. 14.1

Types of Supports.

Trusses are generally supported on the following type of supports.

(1) Roller or Free supports. These supports provide retraint in only one direction.

(2) Hinged or Pin-jointed – They provide restraint in two directions Vertical and horizontal movements are prevented

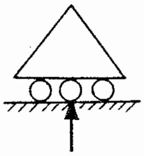


Fig. 14.2(a)

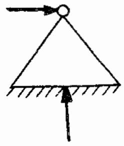


Fig. 14.2 (b)

Strut

A member of the truss in axial compression is called STRUT

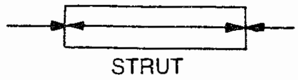


Fig. 14.3

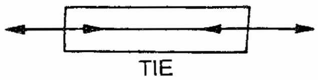


Fig. 14.4

Tie

A member in axial tension is called a Tie

Analysis of forces in perfect frames

The following methods are commonly used to determine the magnitude and nature of forces in members of framed structures.

(1) Method of Joints

(2) Method of Sections

(3) Graphical Method

Method of Joints

Since every joint in a perfect frame is in stable equilibrium, the sum of horizontal and vertical components of all forces acting on a joint must be equal to Zero. *i.e.* $\Sigma H = 0$ and $\Sigma V = 0$. After determing the support reactions a joint should be chosen where the number of unknown forces must not be more than two. Now resolve all the force on the joint into horizontal and vertical components and equate each equation to Zero. By solving these equations the two unknown forces can be determined. A suitable direction for the unknown forces should be assumed. If the magnitude of the force obtained is found to be negative, it means the assumed direction was wrong and the direction should be changed.

Example 14.1

Determine the magnitude and nature of forces in all the members of the truss shown in fig. 14.5

Solution

Number of members = 3

Number of joints = 3

Now $n = 2j - 3$

$n = 2 \times 3 - 3 = 3$

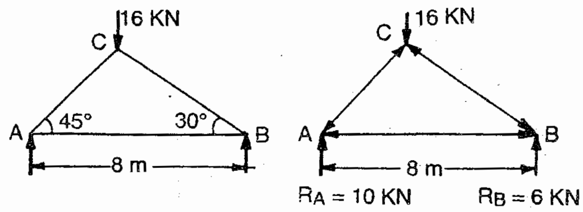


Fig. 14.5

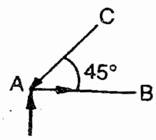
Hence $n = 3$, therefore the frame is statically determinate.
Taking moments about B

$$R_A \times 8 = 16 \times 5 \quad \text{or} \quad R_A = 10 \text{ KN}$$

and $R_B = 6$ KN.

Now consider the equilibrium of joint A

Resolving Vertically



$$\uparrow 10 - \downarrow f_{Ac} \operatorname{Sin}45° = 0 \quad \text{or} \quad f_{Ac} = \frac{10}{Sin\,45°}$$

$$f_{Ac} = \frac{10}{1/\sqrt{2}} \text{ KN} = 10\sqrt{2} \text{ KN (comp.)}$$

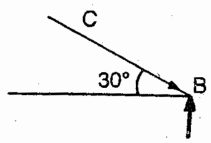
Resolving horizontally

$$\overset{\leftarrow}{f_{Ac}\operatorname{Cos}45°} - \overset{\rightarrow}{f_{AB}} = 0$$

$$\text{or} \quad f_{AB} = f_{Ac}\operatorname{Cos}45° = \frac{10}{\sqrt{2}}\sqrt{2} = 10 \text{ KN (Tension)}$$

Consider joint B

Resolving vertically we get



$$\uparrow 6 - f_{Bc} \operatorname{Sin}30° = 0 \quad \text{or} \quad f_{Bc} = \frac{6}{Sin\,30°}$$

or $f_{Bc} = 12$ KN (Comp.)

The magnitude and the nature of the forces are shown in the table

S. No.	*Member*	*Compression*	*Tension*
1	AB	$10\sqrt{2}$	10 KN
2	AC	12 KN	
3	BC		

Example 14.2

Determine the magnitude and nature of the forces in members of the truss as shown in figure. 14.6

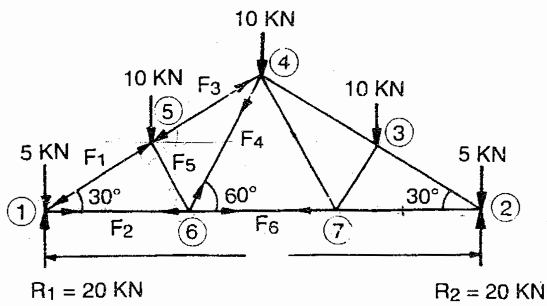


Fig. 14.6

Solution

Number members = 11

Number joints = 7

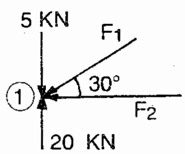
$$n = 2j - 3 = 2 \times 7 - 3 = 11$$

Hence it is a perfect frame

Since the loading is symmetrical

$$\therefore\ R_1 = R_2 = 20 \text{ KN}$$

Now consider joint No. (i)



Resolving vertically $\Sigma V = 0$

$\uparrow 20 - \downarrow 5 - F_1 \ \text{Sin}30° = 0$

$$F_1 = \frac{15}{Sin\,30°} = 30 \text{ KN (Comp.)}$$

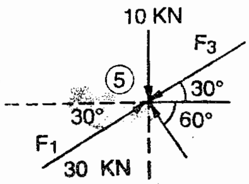
Since the result is positive, the direction assumed is correct

Resolving horizontally $\Sigma H = 0$

$$\overleftarrow{F_1 \text{Cos}30°} + \overleftarrow{F_2} = 0 \quad \text{or} \quad F_2 = -F_1 \text{Cos}30° \quad \text{or} \quad F_2 = \frac{-30 \times J\,3}{2}$$

$F_2 = -25.98$, Since the result is negative, the direction assumed is wrong. Therefore change the direction $\therefore F_2 = 25.98$ KN (Tension)

Consider joint No. 5



Resolving vertically

$\downarrow 10 - \uparrow F_1 \text{ Sin } 30° + \downarrow F_3 \text{ Sin } 30° - \uparrow F_5 \text{ Sin } 60° = 0$

$$10 - 30 \times \frac{1}{2} + F_3 \times \frac{1}{2} - F_5 \times \frac{\sqrt{3}}{2} = 0$$

$$10 - 15 + F_3 \times \frac{1}{2} - \frac{\sqrt{3}}{2}\ F_5 = 0$$

$$\frac{1}{2} F_3 - \frac{\sqrt{3}}{2} F_5 = 5 \quad \ldots\ldots \quad \ldots\ldots \quad \text{(i)}$$

Resolving horizontally

$$\overrightarrow{F_1 \text{Cos}}\ 30° - \overleftarrow{F_3 \text{Cos}}\ 30° - F_5 \text{Cos}\ \overrightarrow{60°} = 0$$

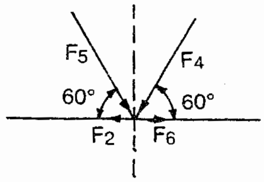
$$30\,\frac{\sqrt{3}}{2}\ \text{™}\ \frac{\sqrt{3}}{2}\,F_3 - \frac{1}{2}\,F_5 = 0$$

$$\frac{\sqrt{3}}{2}\,F_3 - \frac{1}{2}\,F_5 = 15\sqrt{3} \qquad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Solving equations (i) and (ii) we get

$F_3 = 25$ KN (Comp.) and $F_5 = 8.6$ KN (Comp.)

Consider joint No. 6



Resolving vertically $\Sigma V = 0$

$\downarrow F_5 \sin 60° + \downarrow F_4 \sin 60° = 0$

$F_4 = -F_5$

Result is negative Hence change the direction of arrohead. $F_4 =$ 8.6 (Tension)

Resolving horizontally

$$\overleftarrow{F_2} - \overrightarrow{F_5 \cos 60°} + \overleftarrow{F_4 \cos 60°} - \overrightarrow{F_6} = 0$$

$\therefore\ F_2 = F_6 = 25.98$ KN (Tension)

Forces in other members will be similarly Calculated.

Example 14.3

Figure 14.7 shows a pin jointed truss with a vertical force of 20 KN and a horizontal force of 10 KN acting at C. Determine the forces in all the members.

Solution

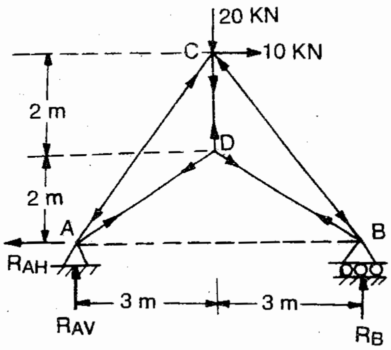


Fig. 14.7

Number of joints = 4

$n = 2j - 3$ or $2 \times 4 - 3 = 5$

The frame is perfect

Reactions

Taking moments about A

$R_B \times 6 = 20 \times 3 + 10 \times 4$

$= 60 + 40 = 100$

$$R_B = \frac{100}{6} = \frac{50}{3}\ \text{KN}$$

Taking moments about B

$R_{AV} \times 6 = 20 \times 3 - 10 \times 4$

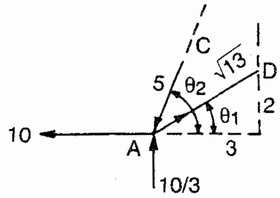
$= 60 - 40 = 20$

$$R_{AV} = \frac{20}{6} = \frac{10}{3}$$

Horizontal reaction at A $R_{AH} = 10$ KN

Joint *A*

$$\sin\theta_1 = \frac{2}{\sqrt{13}},\ \cos\theta_1 = \frac{3}{\sqrt{13}}$$



$\text{Sin } \theta_2 = \frac{4}{5}$, $\text{Cos } \theta_2$ = 3*oer*5 (Tensile)

Resolving vertically.

$$\uparrow \frac{10}{3} - \downarrow fA_c \text{ Sin } \theta_2 + \uparrow f_{AD} \text{ Sin } \theta_1 = 0$$

$$\frac{10}{3} - f_{\text{AC.}} \frac{4}{5} + f_{AD} \frac{2}{\sqrt{13}} = 0 \qquad \dots \qquad \dots \qquad \dots \qquad \text{(i)}$$

Resolving horizontally.

$\leftarrow$ ‹ $\rightarrow$

$$10 + f_{AC} \text{Cos}_{\theta 2} - f_{AD} \text{ Cos}_{\theta 1} = 0$$

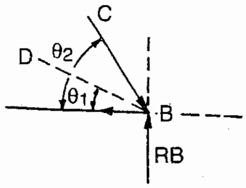
$$10 + f_{AC.} \frac{3}{5} - f_{AD} . \frac{3}{\sqrt{13}} = 0 \qquad \dots \qquad \dots \qquad \dots \qquad \text{(ii)}$$

Solving (i) and (ii) We get

$$f_{AD} = \frac{25}{3} \times \sqrt{13} \text{ (Tensile)} = \frac{25 \times \sqrt{13}}{3} \text{ KN}$$

$$f_{AC} = 25 \text{ KN (Compression)}$$

Joint B



Resolving vertically

$$\uparrow \frac{50}{3} - \downarrow f_{BC} \text{ Sin } \theta_2 + \uparrow f_{BD} \text{ Sin } \theta_1 = 0$$

$$\frac{50}{3} - f_{BC} \cdot \frac{4}{5} + f_{BD} \cdot \frac{2}{\sqrt{13}} = .0 \qquad \dots \qquad \dots \qquad \dots \qquad \text{(i)}$$

Resolving horizontally

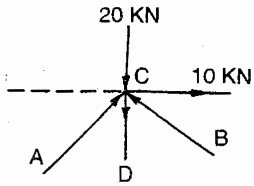
$$f_{\text{BD}} \text{ Cos}\theta_1 - f_{\text{BC}} \text{ Cos}\theta 2 = 0 \qquad \dots \qquad \dots \qquad \dots \qquad \text{(ii)}$$

Solving (i) and (ii) we get

(Tensile)

$$f_{BC} = \frac{125}{3} \text{ (Compressive)}$$

Joint *C*



Resolving Vertically

$$\downarrow 20 - {}_{Ac} \text{ Sin } \theta_2 - \uparrow f_{BC} \text{ Sin } \theta_2 + f_{CD} = 0$$

$$\downarrow 20 - 25 + \frac{4}{5} - \frac{125}{3} \times \frac{4}{5} + f_{CD} = 0$$

$$f_{CD} = \frac{100}{3} \text{ KN (Tensile)}$$

Example 14.4

Find the nature and magnitude o forces in the Pratt truss shown in the figure. 14.8

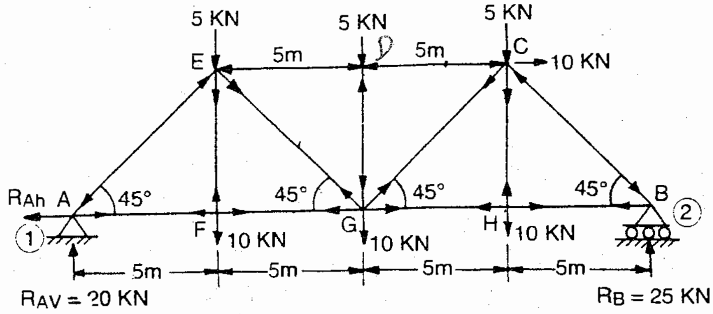


Fig.14.8

Solution

$n = 2j - 3$ or $n = 2 \times 8 - 3 = 13$

Hence the frame is perfect

Reaction at A shall have horizontal and vertical Components

$R_{Ah} = 10$ KN

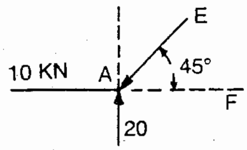
Taking moments about A

$R_B \times 20 = 10 \times 5 + 5 (15 + 10 + 5) + 10 (15 + 10 + 5)$

$= 500$ or $R_B = 25$ KN

$R_A = (45 - 25) = 20$ KN

Consider joint A



Resolving Vertically $\Sigma V = 0$

$\uparrow\ 20 - f_{AE} \text{ Sin } 45° = 0$

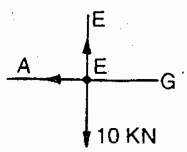
$$f_{AE} = \frac{20}{Sin\, 45°} = 28.28 \text{ KN (Comp)}$$

Resolving horizontally $\Sigma H = 0$

$$\overset{\leftarrow}{10} + \overset{\leftarrow}{f_{AE} \text{ Cos } 45°} - \overset{\rightarrow}{f_{AF}} = 0$$

$$f_{AF} = 10 + 28.28 \times \sqrt{\frac{1}{2}} = 30 \text{ KN (Tension)}$$

Joint F



Resolving Vertically $\Sigma V = 0$

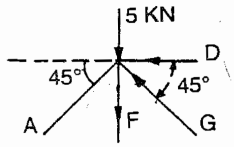
$\downarrow\ 10 - \uparrow f_{FE} = 0$

$f_{FE} = 10$ KN (Tension)

Resolving horizontally $\Sigma H = 0$

$$\overset{\leftarrow}{f_{FA}} - \overset{\rightarrow}{f_{FG}} = 0 \quad \text{or} \quad f_{FG} = 30 \text{ KN (Tension)}$$

Joint *E*



Resolving Vertical $\Sigma V = 0$

$\downarrow 5 + \downarrow f_{EF} - \uparrow f_{AE} \sin 45° - \uparrow f_{EG} \sin 45° = 0$

$$5 + 10 - 28.28 \times \frac{1}{\sqrt{2}} - \uparrow f_{EG} \sin 45° = 0$$

or $\quad f_{EG} \sin 45° = 5 + 10 - 20 = -5$

$$f_{EG} = \frac{-5}{\sin 45°} = -7.07 \text{ KN}$$

Since the value obtained is negative, change the direction of arrow head, hence

$f_{EG} = 7.07$ KN (Tension)

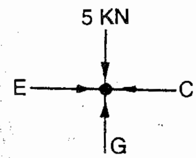
Resolving horizontally, $\Sigma H = 0$

$f_{AE} \cos 45° + f_{EG} \cos 45° - f_{ED} = 0$

$$f_{ED} = 28.28 \times \frac{1}{\sqrt{2}} + 7.07 \times \frac{1}{\sqrt{2}}$$

$= 25$ KN (Compression)

Joint *D*.



Resolving Vertically

$\Sigma V = 0$

$\uparrow 5 - \uparrow f_{DG} = 0$

or $f_{DG} = 5$ KN (Comp.)

Resolving horizontally

$\Sigma H = 0$

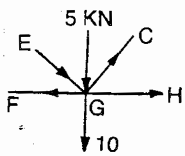
$\overrightarrow{f_{ED}} - \overleftarrow{f_{DC}} = 0$

or $\quad f_{DC} = f_{ED} = 25$ KN (Comp.)

Joint *G*

Resolving Vertically $\Sigma V = 0$

$\downarrow 5 + \downarrow 10 - \uparrow f_{EG} \sin 45° - \uparrow f_{CG} \sin 45° = 0$



$$5 + 10 - 7.07 \times \sqrt{\frac{1}{2}} - \uparrow f_{CG} \sin 45° = 0$$

$f_{CG} \sin 45° = 5 + 10 - 5 = 10$

$$f_{CG} = \frac{10}{\sin 45°} = \frac{10}{1/\sqrt{2}} = 14.14 \text{ (Tension)}$$

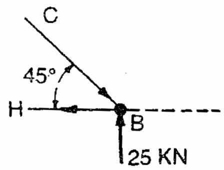
Resolving horizontally

$$\overleftarrow{f_{GF}} + \overleftarrow{f_{GE}} \cos 45° - \overleftarrow{f_{CG}} \cos 45° - f_{GH} = 0$$

$$f_{GF} + 7.07 \times \frac{1}{\sqrt{2}} - 14.14 \times \frac{1}{\sqrt{2}} - f_{GH} = 0$$

$f_{GH} = 30 + 5 - 10 = 25$ KN (Tension)

Joint *B*



Resolving Vertically

$\Sigma V = 0$

$\uparrow 25 - \downarrow f_{BC} \text{ Sin } 45° = 0$

$$f_{BC} = \frac{25}{Sin\, 45°} = 35.35 \text{ (Comp.)}$$

Resolving horizontally

$\Sigma H = 0$

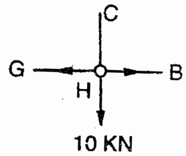
$\rightarrow \qquad \leftarrow$

$f_{BC} \text{ Cos} 45° - f_{HB} = 0$

or $f_{HB} = f_{BC} \text{ Cos } 45° = 35.35 \times \frac{1}{\sqrt{2}}$

$= 25$ KN (Tension)

Joint *H*



Resolving Vertically

$\Sigma V = 0$

$\downarrow 10 - \uparrow f\, H_C = 0$

or $fHc = 10$ KN (Tension)

Example 14.5

Determine the magnitude and nature of forces in all the members of the cantilever truss shown in the figure 14.9

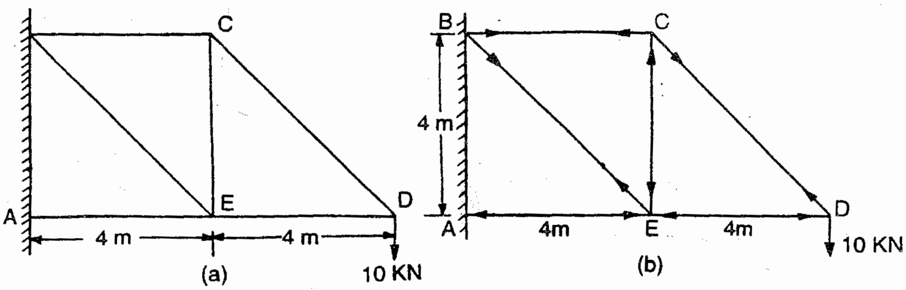


Fig. 14.9

Solution

Number of members = 7

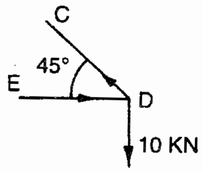
Number of Joints = 5

Now $n = 2j - 3 = 2 \times 5 - 3 = 7$

Hence the truss is statically determinate

$= 0$

$- f_{CD} \text{ Sin } 45° = 0$



$$\therefore f_{CD} = \frac{10}{Sin\,45^\circ} = 10\sqrt{2} \text{ KN (Tension)}$$

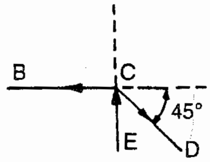
Resolving horizontally $\Sigma H = 0$

$$\leftarrow \qquad\qquad \rightarrow$$

$$f_{CD} \text{ Cos } 45^\circ - f_{DE} = 0$$

$$f_{DE} = f_{CD} \cos 45^\circ = 10\sqrt{2} \times \frac{1}{\sqrt{2}} = 10 \text{ KN) (Comp.)}$$

$$f_{DE} = 10 \text{ KN (Comp.)}$$



Jint *C*

Resolving Vertically

$$\downarrow f_{CD} \text{ Sin } 45^\circ - \uparrow f_{CE} = 0$$

$$f_{CE} = f_{CD} \text{ Sin } 45^\circ = 10\sqrt{2} \times \frac{1}{\sqrt{2}} = 10 \text{ KN}$$

(Comp.)

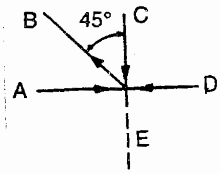
Resolving horizontally $\Sigma H = 0$

$$\rightarrow \qquad\qquad \leftarrow$$

$$f_{CD} \text{ Cos}45^\circ - f_{BC} = 0$$

$$\text{or } f_{BC} = f_{CD} \text{ Cos } 45^\circ = 10\sqrt{2} \times \frac{1}{\sqrt{2}} = 10 \text{ KN},$$

$$\therefore f_{BC} = 10 \text{ KN (Tensile)}$$



Joint *E*

Resolving Vertically $\Sigma V = 0$

$$\downarrow f_{CE} - \uparrow f_{BE} \text{ Sin } 45^\circ = 0$$

$$\text{or} \quad f_{BE} \text{ Sin}45^\circ = f_{CE} = 10 \text{ KN}$$

$$\therefore f_{BE} = \frac{10}{Sin\,45^\circ} = \frac{10}{1}\sqrt{2} - f_{AE} = 0$$

Resolving horizontally

$$\leftarrow \qquad \leftarrow \qquad \rightarrow$$

$$f_{DE} + f_{BE} \text{ Cos}45^\circ - f_{AE} = 0$$

$$10 + 10\sqrt{2} \times \frac{1}{\sqrt{2}} - f_{AE} = 0$$

$$\text{or} \quad f_{AE} = 10 + 10 = 20 \text{ KN (Compression)}$$

Member	Tension	Compression
BC	10 KN	
CD	$10\sqrt{2}$ KN	
DE		10 KN
BE	$10\sqrt{2}$ KN	
CE		10 KN
AE		20 KN

Example 14.6

Determine the magnitude and the nature of the forecs in all the members of the truss shown in figure 14.10. All inclined members are at 45° with the horizontal.

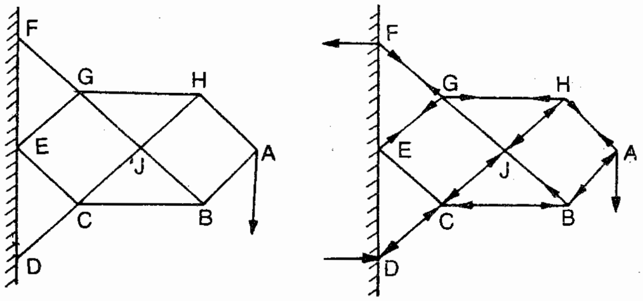
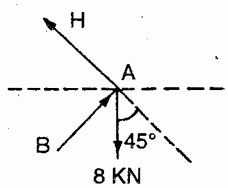


Fig. 14.10

Solution

Joint *A*



Resolving vertically

$\uparrow 8 - f_{AH}\,\text{Sin}45 - f_{AB}\,\text{Sin}45 = 0$

$$8 - f_{AH}\frac{1}{\sqrt{2}} - f_{AB}\frac{1}{\sqrt{2}} = 0 \quad \text{.... ....} \quad \text{(i)}$$

Resolving horizontally

$$f_{AH}\,\text{Cos } 45° = f_{AB}\,\text{Cos } 45° \text{ or } f_{AH} = f_{AB} \quad \text{--- --- ---} \quad \text{(ii)}$$

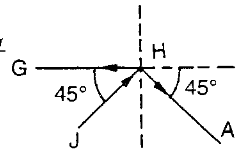
From equation (i)

$$8 - f_{AH}\frac{1}{\sqrt{2}} - f_{AH}\frac{1}{\sqrt{2}} = 0$$

$$8 - f_{A}\frac{2}{\sqrt{2}} = 0 \qquad \text{or} \qquad f_{AH}\frac{8}{2}\sqrt{2} = 4\sqrt{2} \text{ KN (Tensile)}$$

$\therefore f_{AB} = 4\sqrt{2}$ KN (Comp)

Joint *H*



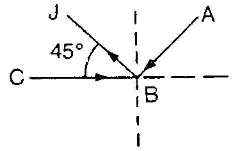
Since the vertical component of f_{AH} and f_{JH} should balance each other, hence $f_{HJ} = 4\sqrt{2}$ (Comp)

Resolving horizontally

or $f_{HG} = f_{HA} \text{ Cos } 45° + f_{HJ} \text{ Cos } 45°$

$$f_{HG} = 4\sqrt{2} \times \frac{1}{\sqrt{2}} + 4\sqrt{2} \times \frac{1}{\sqrt{2}} = 8\text{KN (Tensile)}$$

Joint *B*



The vertical components of the forces f_{BA} and f_{BJ} should balance each other

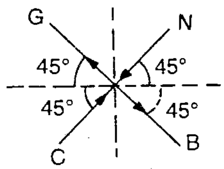
$$\therefore f_{BJ} = f_{BA} = 4\sqrt{2} \text{ (Tensile)}$$

Resolving horizontally

$f_{BC} = f_{BJ} \text{ Cos}45° + f_{BA} \text{ Cos}45°$

$$= 4\sqrt{2} \times \frac{1}{\sqrt{2}} + 4\sqrt{2} \times \frac{1}{\sqrt{2}} = 8 \text{ KN (Comp.)}$$

Joint *J*



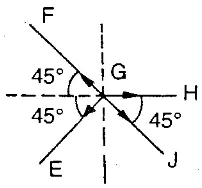
Resolving the forces in line With *HJC*, we have

$$f_{JC} = f_{JH} = 4\sqrt{2} \text{ KN (Comp.)}$$

And Resolving the forces in line with *GJB*, we get

$$f_{JG} = f_{JB} = 4\sqrt{2} \text{ (Tensile)}$$

Joint *G*



Resolving the forces in line with *GE*, we have

$$f_{GE} = f_{GH} \text{ Cos}45°$$

$$= 8 \times \frac{1}{\sqrt{2}} = 4\sqrt{2} \text{ (Tensile)}$$

Now resolving the forces in line with *FGJ*, we have

$$f_{GF} = f_{GE} + f_{GH} \text{ Cos } 45°$$

$$f_{GF} = 4\sqrt{2} + 8\frac{1}{\sqrt{2}} = 8\sqrt{2} \text{ (Tensile)}$$

Joint *C*

Resolving the forces in line with *CE*, we have

$$f_{CE} = f_{CB} \text{ Cos}45°$$

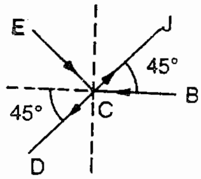
$$= 8 \text{ Cos}45°$$

$f_{CE} = 4\sqrt{2}$ KN (Comp.)

and resolving the forces in line with *DCJ*, we get

$$f_{CD} = f_{CJ} + f_{CD} \text{ Cos}45°$$

$$f_{CD} = 4\sqrt{2} + 8\frac{1}{\sqrt{2}} = 8\sqrt{2} \text{ (Comp.)}$$

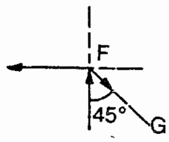


Support reactions

At *F*

Horizontal reaction = f_{FG} Cos45°

$$= 8\sqrt{2} \times \frac{1}{\sqrt{2}} = 8 \text{ KN} \leftarrow$$



Vertical reaction = f_{FG} Sin45° = 8 KN ↑

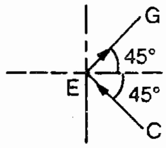
At *E*

Since the horizontal component of f_{EG} and f_{EC} will balance each other hence there will be no horizontal reaction.

Vertical reactions

$$= f_{EG} \text{ Sin } 45° + f_{EC} \text{ Sin}45°$$

$$= 2 \times 4\sqrt{2} \text{ Sin } 45° = 8 \text{ KN} \downarrow$$



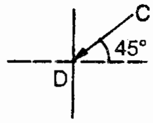
At *D*

Horizontal reaction

$$= f_{DC} \text{ Cos } 45° = 8\sqrt{2} \times \frac{1}{\sqrt{2}} = \overrightarrow{8 \text{ KN}}$$

Vertical reaction = f_{DC} Sin 45° = $8\sqrt{2} \times \frac{1}{\sqrt{2}}$

$$= \mathbf{8 \text{ KN}} \uparrow$$



Method of Sections

In this method the frame is divided into two portions by a section line passing through a few members. Generally the section should not cut more than three members including the one in which stress is required to be determined. Equili brium of one portion either to the left or to the right of the section is considered. Moments are taken at a suitable point where all forces except one meet.. Now with the help of the equations of statics, forces in various members are determined, by equating either

(i) Σ M = 0

or (ii) Σ H = 0 or Σ V = 0

The following examples will help in understanding the method.

Example. 14.7

Forthe truss shown in figure 14.11 determine the forces in members BC, BE and CE

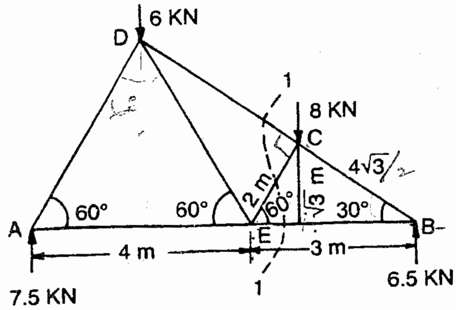


Fig. 14.11

Solution

Taking moments about B

$R_A \times 8 = 6 \times 6 + 8 \times 3 = 60$

$R_A = 7.5$ KN and

$R_B = 6.5$ KN

Consider section (1) - (1)

Taking moments about E

$f_{BC} \times 2 = -7.5 \times 4 + 6 \times 2$

$$f_{BC} = \frac{-30+12}{2} = -\frac{18}{2}$$

$= -9$ KN

$f_{BC} = 9$ KN (Compressive)

Taking moments about C

$f_{BE} \times \sqrt{3} = 6.5 \times 3$

or $f_{BE} = 6.5\sqrt{3}$ (Tensile)

Taking moments about B

$$f_{CE} \times 4\frac{\sqrt{3}}{2} = 6 \times 3$$

$$f_{CE} = \frac{18 \times 2}{\sqrt{3}} = \frac{9}{\sqrt{3}} \text{ KN (Comp.)}$$

Example. 14.8

The truss shown in figure 14.12 rests on supports A and D so that ABCD is horizontal. It carries a point load of 9 KN at B and 18 KN at C. Determine the magnitude and nature of forces in the members BC, FC and FE.

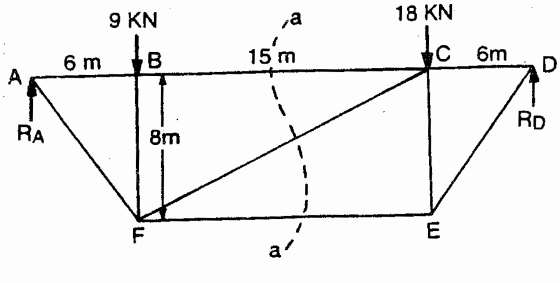


Fig. 14.12

Solution

$n = 2j - 3$

$= 2 \times 6 - 3 = 9$

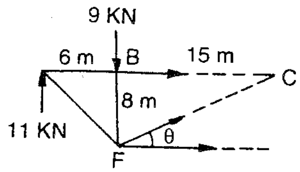
The frame is perfect

Taking moments about D

$R_A \times (27) = 9 \times 21 + 18 \times 6$

$R_A = 11$ KN and $R_B = 16$ KN

Draw a section a-a which cuts the members BC, FC and FE and divides the truss into two portions and consider the equilibrium of the portion to the left of the section. Assume that the member BC, FC and FE and all in tension.



Taking moment about F, the intersection of FC and FE

We have $11 \times 6 + f_{BC} \times 8 = 0$ or $f_{BC} = -8.25$ KN

Since the value obtained is negative, direction assumed is wrong

$\therefore \; f_{BC} = 8.25$ (Comp.)

Taking moments at the intersection of BC and FC.

$+ 11 \times (21) - 9\,(15) - f_{FE}\,(8) = 0$

$$\text{or } f_{FE} = \frac{231 - 135}{8} = \frac{96}{8} = 12 \text{ KN}$$

$f_{FE} = 12$ KN (Tension)

Resolving vertically $\Sigma V = 0$

$\uparrow 11 - \downarrow 9 + f_{FC} \sin\theta = 0$

$$f_{FC} = \frac{-2}{\sin\theta} = \frac{-2}{8/17} = \frac{-34}{8} = -4.25$$

Since the value obtained is negative the direction assumed is wrong.

Hence $f_{FC} = 4.25$ (Comp.)

Example. 14.9

A tower ABCDEF is loaded as shown in figure. 14.13 Determine the magnitude and nature of the forces in the members FE, FD and AC.

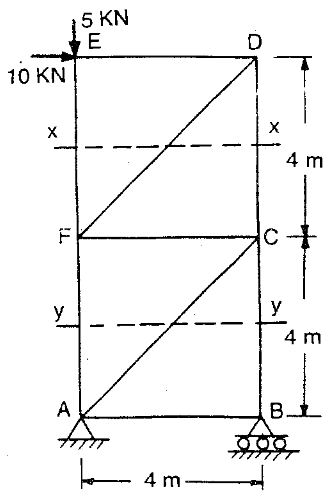


Fig. 14.13

Let section x-x cut members FE, FD and CD. Now consider the stability of the upper portion of the truss.

Taking moments about D

$f_{FE} \times 4 - 5 \times 4 = 0 \; \therefore \; f_{FE} = 5$ KN (Comp.)

Resolving horizontally $\Sigma H = 0$

$\leftarrow \qquad \rightarrow$

$- f_{FD} \cos 45° + 10 = 0$

$$\text{or} \quad f_{FD} = \frac{10}{\cos 45°} = 10\sqrt{2} \text{ KN (Tension)}$$

Consider the stability of the upper portion cut

The section y-y. Resolving horizontally $\Sigma H = 0$

$- f_{AC} \cos 45° + 10 = 0$

$$\text{or} \quad f_{AC} = \frac{10}{\cos 45°} = 10\sqrt{2} \text{ KN (Tension)}$$

Example. 14.10

Determine the magnitude and nature of forces in the members *DE, DH* and *HK* of the truss shown in figure 14.14

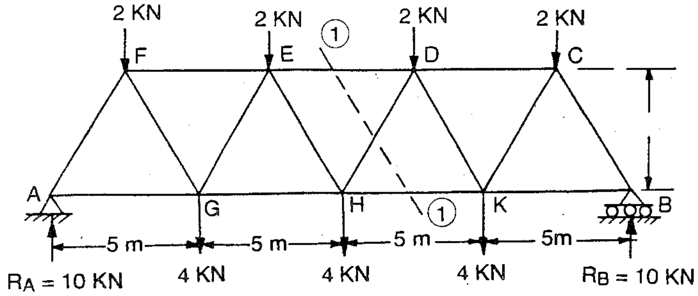


Fig. 14.14

Support reactions $R_A = R_B = \dfrac{20}{2} = 10KN$

Draw a section 1-1 which cuts the members *ED, DH* and *HK* and divides the truss into two portions. Now consider the equilibrium of the portion to the left of the section. Assume that all the three members are in tension.

Taking moments about *D*, the intersection of *ED* and *HD*, we have

$10 \times 12.5 - 2 \times 10 - 4 \times 7.5 - 2 \times 5 - 4 \times 2.5 - f_{HK} \times 4 = 0$

$125 - 20 - 30 - 10 - 10 - f_{HK} \times 4 = 0$

$4 f_{HK} = 125 - 70 = 55$

or $f_{HK} = \dfrac{55}{4} = 13.75$ (Tension)

Taking moments about *H*

$10 \times 10 - 2 \times 7.5 - 4 \times 5 - 2 \times 2.5 + f_{ED} \times 4 = 0$

$100 - 15 - 2\quad 0 - 5 + f_{ED} \times 4 = 0$

$60 + f_{ED} \times 4 = 0$ or $f_{ED} = \dfrac{-60}{4} = -15$

The negative value of f_{ED} shows that the direction assumed was wrong. Hence f_{ED} is in Compression.

$f_{ED} = 15$ KN (Compression)

Resolving Vertically

$\uparrow 10 - \downarrow 2 - \downarrow 4 - \downarrow - 2 - \downarrow\ 4 + \uparrow f_{HD} \sin\theta = 0$

or $10 - 12 + f_{HD} \sin\theta = 0$

or $f_{HD} \sin\theta = 2$

Now $\sin\theta = \dfrac{4}{4.716} = .848$

$f_{HD} = \dfrac{2}{\sin\theta} = \dfrac{2}{.848} = 2.53$ KN (Tension) **Answer**

Example. 14.11

Calculate the stresses in the members *BF, FG* and *GE* of the cantilever truss shown in figure. 14.15.

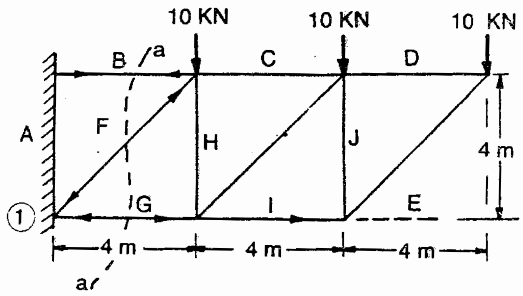


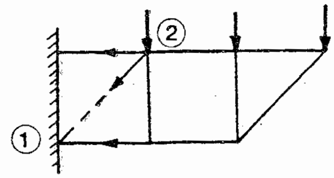
Fig. 14.15

Solution

Draw a section a-a which passes through the members *BF, FG* and *GE* and divides the truss in two portions. Consider th equilibrium of the portion to the right of the section. Assume all the members in tension.

Taking moments about joint at

The intersection of *FG* and *GE*



$$f_{BF} \times 4 = 10 \times 12 + 10 \times 8 + 10 \times 4$$

$$f_{BF} = 120 + 80 + 40 = \frac{240}{4}\ 60 \text{ KN}$$

(Tensile)

Since the resulting stress is positive, hence it will be tensile and the assumption is correct.

For stress in *GE*, take moments about joint No. 2 and assuming the stress in *GE* to be tensile.

$$+\ GE \times 4 + 10 \times 8 + 10 \times 4 = 0$$

$$GE = -\frac{80+40}{4} = \frac{-120}{4}$$

$$= -\ 30 \text{ KN}$$

Since the value is negative change the direction

Hence f_{GE} will be 30 KN (Comp.)

Now resolving vertically $\Sigma V = 0$

$$\downarrow f_{FG} \text{ Sin } 45° + \downarrow 10 + \downarrow 10 = 0$$

$$f_{FG} \text{ Sin } 45° = -30$$

$$f_{FG} = -\frac{-30}{Sin\ 45°} = \frac{30}{1/\sqrt{2}} = -30\sqrt{2}$$

Since the value obtained is negative the assumed direction is wrong

$$\therefore\ f_{FG} = 30\sqrt{2}\ (Comp.)$$
$$= 42.42 \text{ KN}$$

Graphical Method

Graphical method is the simplest of all the methods but accuracy in drawing and measurement is of utmost importance. It involves the following three steps.

(i) Drawing of space diagram to a suitable linear scale and denoting the forces by Bow's notation

(ii) Drawing of force diagram or vector diagram to some suitable load scale.

(iii) Presentation of the results in a tabular form showing the magnitude and nature of forces in various members of the truss.

Bow's Notation

According to Bow's notation each force in free body diagram or space diagram is denoted by two letters placed on either side of the force as shown in figure 14.16 (a) and the corresponding vector in the force diagram is labeled with the same letters placed one at each end in the vector diagram as shown in fig. 14.16 (b)

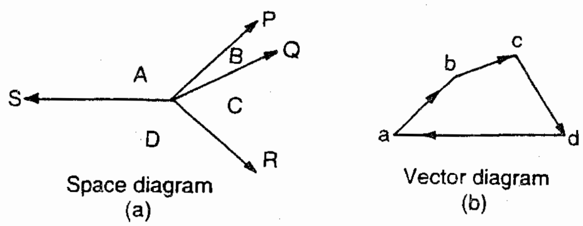


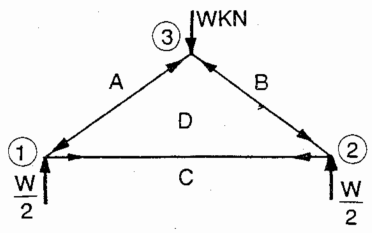
Fig. 14 .16

The force P in the space diagram is denoted by the letters A and B and force Q, by the letters B and C etc. If the point O is in stable equlibrium under the action of the forces AB, BC, CD and DA ,then these forces can be represented by ab, bc, cd and da in the vector diagram in which ab is drawn parallel to AB and bc is drawn parallel to BC etc. to a chosen load scale.

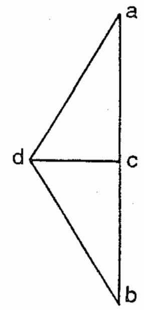
The vector ab means that the force is from a to b in directions. Similarly vector cd the force is from c to d in direction. The length of the side ab in the vector diagram gives the magnitude of the force AB. in the space diagram.

Space Diagram

Space diagram is constructed to show the actual shape and size of the framed structure along with the applied loads to a suitable linear scale. The support reactions are also shown in the diagram and forces are denoted by Bow's notations as shown in figure 14.17 (a)



(a) Space diagram



(a) Force diagram or Vector diagram

Fig. 14.17

Force Diagram Or Vector Diagram

All forces acting on the frame are shown in the vector diagram drawn to a suitable load scale as shown in fig. 14.17 (b)

To draw the vector diagram select a suitable point a and draw a vertical line parallel to AB to a suitable load scale say $W = 50$ mm.

(2) On this line mark 'bc' equal to force BC *i.e.* support reaction on $R_2 = \frac{W}{2} = 25\text{mm}$, then the line 'ca' represents the support reaction $R_1 = \frac{W}{2} = 25\text{mm}$

(3) Through c draw a line parallel to CD and from 'a' draw a line parallel to AD. These lines will intersect at *'d'*. Through *'b'* draw a line parallel to BD this will also meet the line through *'c'* at *'d.'* Thus we obtain the vector diagram

(4) Magnitude of the forces

From the vector diagram the length of the line *'ad'* will give the magnitude of the force in member AD on the space diagram. Similarly measure the lines *'bd'* and 'cd' obtain the magnitude of the forces in members BD and CD respectively.

(5) Nature of forces.

For joint ① draw the vector diagram separately. Showing the forces CA, AD and DC in a clockwise direction. Now follow the direction of the force CA and mark the arrowhead near the joint as shown in the fig. Put an

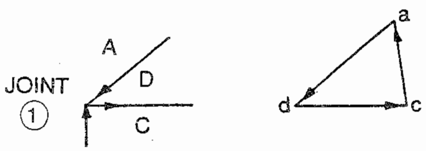
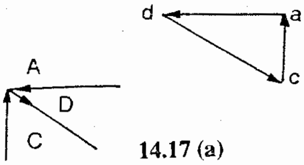


Fig. 14.

other arrow head at the other end of the member on the space diagram. Similarly draw seperate vector diagrams for each joint in order of the letters in the space diagram.



14.17 (a)

For joint ② the forces the forces are *BC*, *CD* and *DB*. Follow the direction of the force *BC* and mark the arrowhead near the joint. Similarly mark the direction of forces for joint ③

(6) The results are presented in the table as shown

S. No.	Name of Member	Magnitude	Nature
1	*AD*	*W* KN	Comp.
2	*BD*	*W* KN	Comp.
3	*CD*	0.866 KN	Tension

Example 14.12

Find graphically the forces in the members of the truss shown in fig. 14.18

Solution

Taking moments about joint ②

$R_1 \times 5 = 15 \times 2$ or $R_1 = 6$ KN and

Taking moments about joint ①

$R_2 \times 5 = 15 \times 3$ or $R_2 = 9$ KN

Choose a suitable load scale and draw a vertical line *ab* parallel to *AB*. Now mark *bc equal to force BC i.e.* $R2 = 9$ KN. Hence ca represents the support reaction $R_1 = 6$ KN. From 'a' draw a line parallel to*AD* and through '*C*' draw a line parallel to *CD*, these will intesect at '*d*' to give the vector diagram for joint ①. Similarly from '*C*' drawaline parallel to *CE* and from

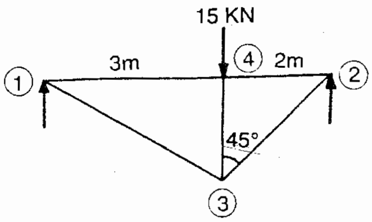


Fig. 14.18

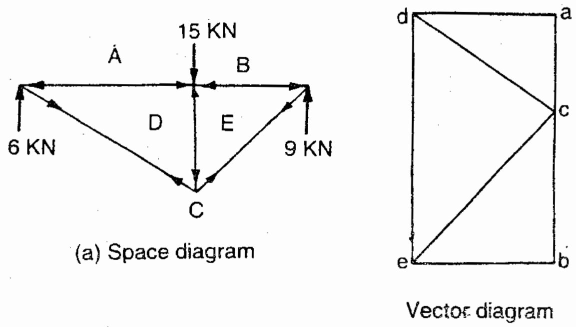
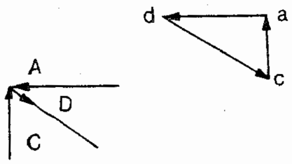


Fig. 14.18 (b)

'*b*' draw a line parallel to *BE*, these lines will meet at '*e*' to give the vector diagram for joint ②. Now join *de* which will represent the member *DE* of the truss. The complete vector diagram is shown in fig 14.18 (b)

Now for joint ① draw the vector diagram separately to know the nature of the forces. Start with known force *CA* and proceed in the direction of *CA* and mark the arrow heads near the joint as shown in figure 14.19



14.19

The magnitude and nature of forces in various members are shown in the table

S. No.	Members	Compression	Tension
1	*AD*	9 KN	
2	*BE*	9 KN	
3	*CD*		10.82 KN
4	*CE*		12.73 KN
5	*DE*	15 KN	

Example. 14.13

A two bay warren girder truss is loaded as shown in fig (14.20). Determine graphically or other wise the forces in all the members of the frame.

Solution

Calculate the support reactions by taking moments about joint ①

$$R_2 \times 4 = 30 \times 3 + 20 \times 1$$

$$R_2 = \frac{110}{4} = 27.5 \text{ KN}$$

and $R_1 = 22.5$ KN.

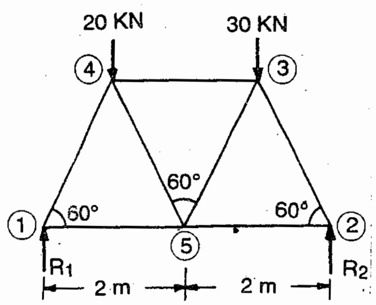
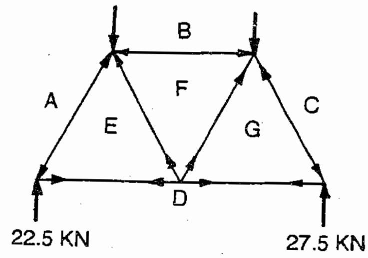
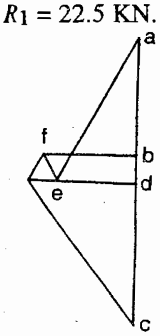


Fig. 14.20



(a) Space diagram



(b) Vector diagram

Fig. 14.20 (a)

Draw the space diagram to some suitable linear scale and name the member using Bow's notation. Draw a vertical line abc parallel to *AB* and *BC* to some suitable load scale. Mark *cd* equal to support reaction R_2 then *da* will represent the support reaction R_1. Through a draw a line paralled to *AE* and through *'d'* draw a line paralled to *DE*, these lines will intersect at *e*. Similarly draw parallel lines to *BF* and *EF* to get point *f*. Now complete the vector diagram as shown in figure.

Forces in various members are shown in the tabel

S. No.	Members	Compression	Tension
1	*AE*	26 KN	
2	*ED*		13 KN
3	*EF*		2.78 KN
4	*BF*	14.5 KN	
5	*CG*	32 KN	
6	*GD*		16 KN
7	*FG*	2.78 KN	

Example.14.14

For the truss shown in figure 1421 determine graphically the magnitude and nature of the forces in all the members.

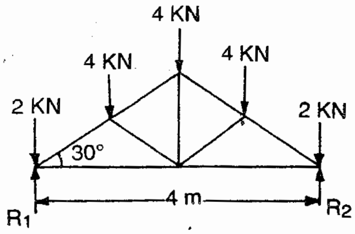


Fig. 14.21

Solution

The truss is symmetrically loaded hence $R_1 = R_2 = 8$ KN.

Draw the space diagram and name the members as shown. Select a point 'a' and draw a vertical line *abcdefga* representing all the loads and the support reactions. *fg* and *ga* represent the

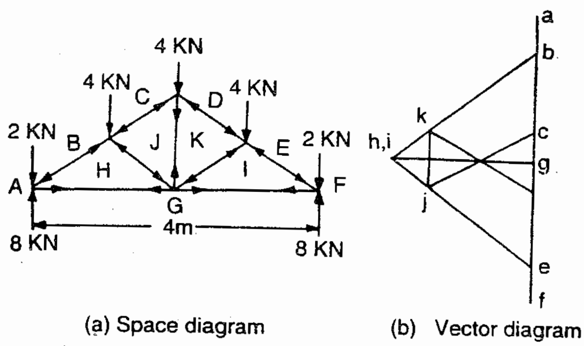


Fig. 14.22

support reactions acting vertically upwards. from g draw a line parallel to GI and GH. From 'b' draw a line parallel to BH meeting at i and h. Similarly draw lines parallel to CJ, from C and proceeding further complete the vector diagram as shown in fig. 14.22.

Magnitude and nature of forces are shown in the table.

S. N.	Members	Compression	Tension
1	*BH, EI*	12 KN	
2	*HG, IG*		10.4 KN
3	*CJ, DK*	8 KN	
4	*JH, KI*	4 KN	
5	*JK*		4 KN

Example. 14.15

A cantilever truss is shown in figure 14.23. Determine the magnitude and nature of forces in all the members.

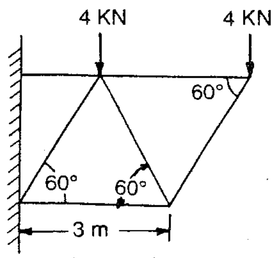
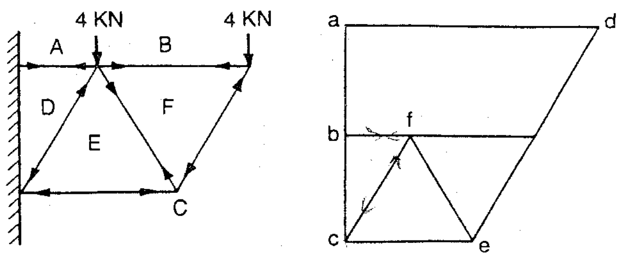


Fig. 14.23

Solution

Draw the space diagram to some suitable linear scale as shown. Vector diagram may be drawn starting from a vertical line abc paralle to the forces AB and BC to a suitable load scale.



(a) Space diagram (b) Vector diagram

Fig. 14.24

The table shows the magnitude and nature of forces in all the members

S. No.	Members	Compression	Tension
1	*BF*		2.4 KN
2	*CF*	4.8 KN	
3	*CE*	4.8 KN	
4	*DA*		9.6 KN
5	*DE*	9.6 KN	
6	EF		4.8 KN

Example 14.16

Find graphically or other wise the forces in te members of the truss shown in figure 14.25

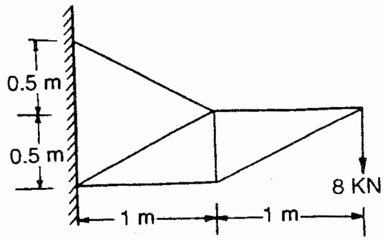
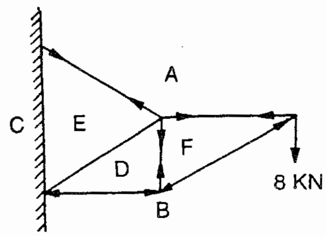


Fig. 14.25

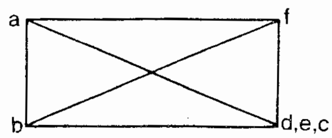
Solution

Choose a suitable load scale and draw *ab* to represent force *AB* of 8 KN. Now draw *bf* and *af* parallel to *BF* and *AF*, these lines will intersect at *f*. similarly draw *fd* and *bde* parallel to *FD* and *BD* which will meet at point *d*. Points *d*, *e* and *c* will Coincide as shown in the vector diagram



(a) Space diagram

Fig. 14.25 (a)



(b) Vector diagram

Fig. 14.25 (b)

Forces in all the members all shown in the table

S. No.	*Member*	*Compression*	*Tension*
1	AE		17.92 KN
2	AF		16.0 KN
3	BF	17.92 KN	
4	BD	16 KN	
5	ED	0	0
6	DF		8 KN

Example 14.17

The frame shown in the figure 14.26 is loaded at joint (2). A horizontal chain is a attached at joint (3) So that member 1-2 remains horizontal. Determine the pull on the chain and the forces in other members of the frame.

(AMIE)

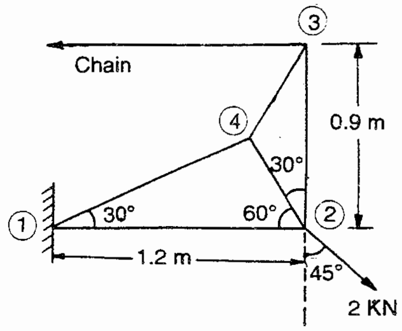


Fig. 14. 26

Solution

Draw the space diagram and use Bow's notation as shown in the fig. To find the pull in the chain take moments of all forces about joint (1)

$$f_{AE} \times 0.9 = 2 \text{ Cos } 45° \times 1.2$$

or $f_{AE} = 1.885$ KN

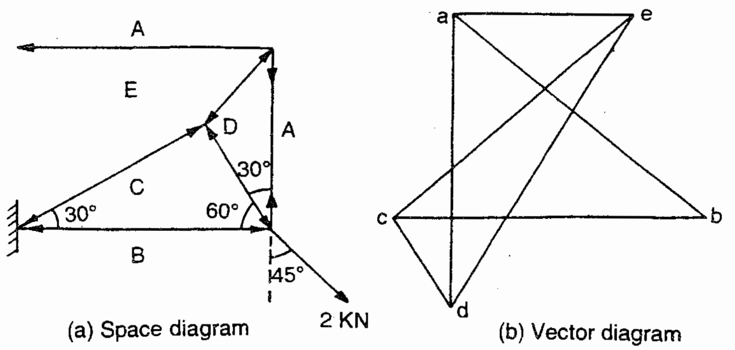


Fig. 14.27

Now to draw the vector diagram draw a line '*ad*' parallel to *AD* to a suitable load scale. Through '*a*' draw a line parallel to *AE*. Through *d* draw a line parallel to *DE*. These lines will intersect at *e* to give vector diagram for joint (3). Now consider joints (2) and (4) and complete the vector diagram for the frame. Forces in various members are shown in the table.

S. No.	*Members*	*Compression*	*Tension*
1	*AD*		2.38 KN
2	*BC*		1.96 KN
3	*CD*	1.12 KN	
4	*CE*	2.82 KN	
5	*DE*	3.03 KN	

SUMMARY

1. A perfect frame must satisfy the equation

   $n = 2j - 3$

   Where n is the number of members and j is the number of joints
2. In case of roller supports the reactions will be always normal to the plane on which the rollers rest
3. For determining support reaction, moments should be taken about one of the supports. If one support is a hinge then moment should be taken about the hinge.
4. In case of method of joints. select a joint where the number of unknown forces must not the more than two.
5. Resolve all the forces vertically and horizontally and apply the equations of static equilibrium$\Sigma V = 0$ and $\Sigma H = 0$
6. In case of method of sections, the section line should not cut more than three such members in which forces are not known
7. Select the point about which moments are to be taken in such a way that all except one cut member passes through it. In this method only the stafic equation $\Sigma M = 0$ is used.
8. In Graphical method represent all the forces by Bow's Notation in the space diagram to a suitable linear scale.
9. Draw the forces diagram or the stress diagram by choosing a suitable load scale. Choose a suitable point O and draw a vertical line representing all the vertical forces and the support reactions. Now complte the vector diagram by drawing lines parallel to various members in the space diagram.
10. For determining the nature of forces start from each joint and move in a clock wise direction.
11. A tension member is known as Tie
12. A member in compression is known as strut.

QUESTIONS

1. How would you classify framed structures into
   (a) Perfect frame or determinate frame
   (b) Imperfect frame or Indeterminate frame
   (c) Redundant frame.
2. Which equation should be satisfied when the frame is perfect ?
3. Which joint would you select while analysing a frame by the method of joints ?
4. What are the two conditions of static equilibrium which should be satisfied in the method of joints ?
5. How many members should be cut by a section, in which forces should be un known?
6. How many restrains are offered by
   (a) Hinged support

(b) Roller support

(c) Fixed support

7. What is a strut ?
8. Which member of a frame is called Tie ?

EXCERCISES

9. Find the nature and magnitude of the forces in the frame shown in figure 14.28

$F_1 = F_7 = 5.3$ KN

Comp. (T)

F_3 (T) = F_5 (c) = 1.8 KN

F_2 (T) = 3.75 KN

F_4 (C) = 4.25 KN

F_6 (C) = 6.25 KN

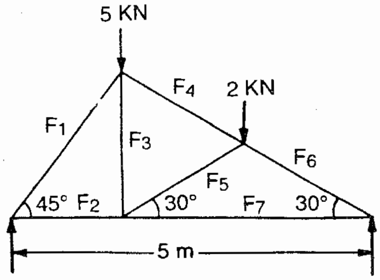


Fig. 14.28

10. Determine the magnitude and nature of forces in the members of the truss shown in figure. 14.29 (AMIE)

$AB = 5.215$ KN(Comp),
$CE = 2.20$ KN (Tension)
$AE = 2.885$ KN (Tension),
$CD = 4.78$ KN (Comp)
$BE = 2.82$ KN (Tension),
$ED = 3.06$ KN (Tension)
$BC = 4.45$ KN (Comp),

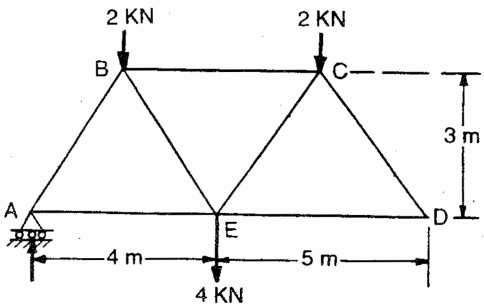


Fig. 14.29

11. The load at the crane head in figure 14.30 is 4 KN. Determine the stresses in various members.

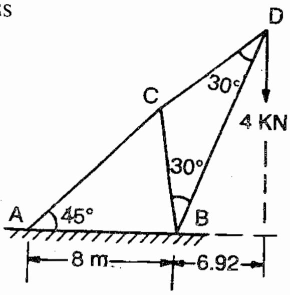
$AB = 3.5$ KN (Comp),

$BC = 1.5$ KN (Comp)

$AC = 5.0$ KN (Tension),

CD = 4.10 KN (Tension)

$BD = 7.05$ KN (Comp)



Fiig. 14.30

12. Determine the forces AB, BF and AF members of the truss shown in figure 14.31
    $AB = 12$ KN (Tension),
    $BF = 12.53$ KN (Comp)
    $AF = 6\sqrt{3}$ KN (Comp)

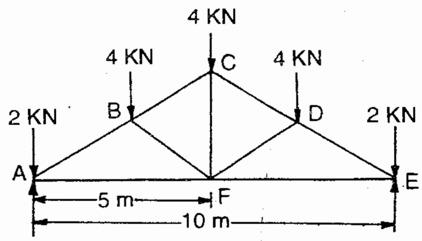


Fig. 14.31

13. Find the forces in the members AB, AC, CD and BD of the truss shown in the figure 14.29 by the method of sections.

(Roorkee Uriv.)

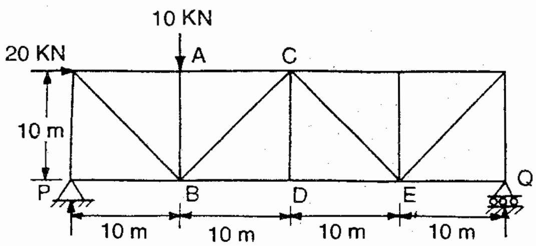


Fig. 14.32

$AB =$ 10 KN (Comp). $CD =$ Zero.
$AC =$ 22.5 KN (Comp). BD = 15 KN (Tension)

(14) Find the forces in the members of the truss.

(J.M.I)

$BC = 13.3$ (T)
$CD = 13.3$ KN (T)
$DE = 16.6$ (Comp.)
$CE = 10$ KN (Comp.)
$BE = 16.6$ KN (Comp.)
$AE = 16.6$ KN (Comp.)

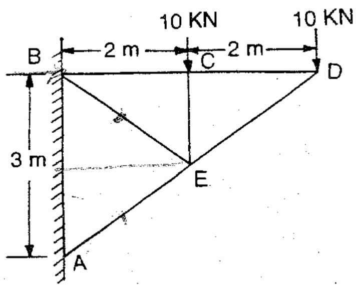


Fig. 14.33

15. A frame as shown in figure 14.31 carries a vertical load of 9KN at point A and equivalent horizontal thrust due to wind of 4.5 KN at B. Determine stresses in the inclined members of the truss.

(A.M.I.E)

$AC = BC = 12.72$ KN (T)
$DH = EG = 6.36$ KN (T)

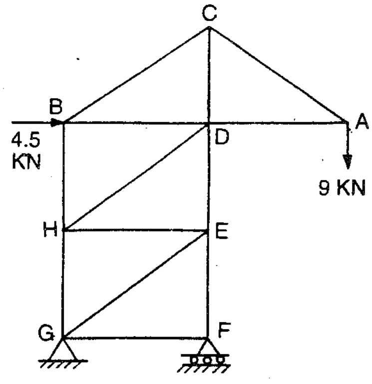


Fig. 14.34

16. A pin jointed frame is shown in figure. 14.35 . It is hinged at A and loaded at D. A horizontal chain is attached to C and pulled so that AD is horizontal. Determine the pull in the chain and also the forces in each member stating whether it is in tension or compression.

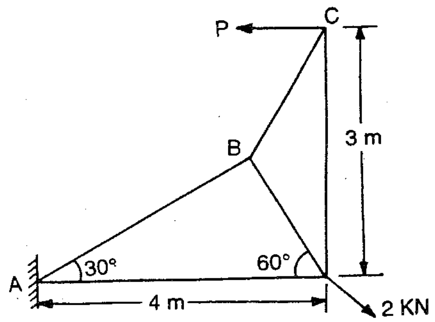


Fig. 14.35

$P = 1.885$ KN
$AB = 2.83$ KN (Comp)
$BC = 3.04$ KN (Comp)
$CD = 2.39$ KN (Ten sion)
$DA = 1.98$ KN (Ten sion)
$DB = 1.14$ KN (Comp)

□ □ □

Appendix -I
Prefixes

Prefix	*Symbol*	*Multiplication factor*
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	K	10^{3}
hecto	h	10^{2}
deca	da	10^{1}
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}

Appendix - II
Conversion Table

Multiply by	*To convert*	*To*	
2.54	Inches	Centimeters	0.3937
30.48	Feet	Centimeters	0.3228
9.14	Yards	Meters	1.094
1609.3	Mile	Meters	0.000621
1853.27	Nautical miles	Meters	0.000539
6.450	Sq.inches	cm^2	0.155
0.093	Sq.feet	m^2	10.764
16.390	cu.inch	cm^3	0.061
28.3	ft^3	litres	0.0353
0.0283	ft^3	m^3	35.34
746	H.P.	W	0.00134
70.3	Pound per sq. inch (psi)	gm/cm^2	0.0142
10.0	kg	N	0.1
0.1	kg/cm^2	N/mm^2	10.0
100	kg-cm	N-mm	0.01
1000	tonne	kg	0.001
100	quintal	kg	0.01
0.3732	pounds (Troy)	kg	2.68
0.4536	pounds (Avoir)	kg	2.2046
10^{-7}	erg	Joule	10^7
4.186	calorie	Joule	0.239
1.356	foot-pound	Joule	0.737
10^{-5}	dyne	N	10^5
	To obtain	From	Multiply by above

APPENDIX III

Typical properties of common materials

Materials	Density kg/m^3	Poisson's ratio μ	Coefficient of thermal expansion α 10^{-6} /°C	Proportional limit N/mm^2			Elastic limit in tension N/mm^2	Ultimate strength N/mm^2			Elastic c[illegible] $\times$ 10 ^{5}[illegible]	
				Ten.	Comp.	Shear		Ten.	Comp.	Shear	E	[illegible]
Mild steel	7860	0.288	11.7	245	245	145	250	400	630	315	1.9.2.1	0.7[illegible] 8[illegible]
Cast iron grey	7200	0.270	12.1	42	175	—	43	170	650	240	1.05	0.[illegible]
Cast iron malleable	7300	0.270	12.1	150	150	160	160	340	620	330	1.75	0.[illegible]
Wrought iron	7870	0.278	—	210	210	126	—	250–500	280	250	1.9	0.[illegible]
Aluminium	2710	0.330	23.6	25–140	25–140	—	50–400	100–420	80–240	70	0.7	0.[illegible]
Brass	8100	0.340	20.0	140	140	105	170	200	177	170	0.97	0.[illegible]
Bronze	8800	0.350	18.0	30	30	—	35	170	250	250	1.0	0[illegible]
Copper	8900	0.355	17.0	56	56	160	150	150	250	200	0.7	0[illegible]
Timber*	440–600	—	3.0–4.5	—	—	7.9.2	—	—	50	—	0.125	[illegible]
Concrete	2320	0.200	10.0	—	—	—	—	—	28	—	0.25	[illegible]

* Loaded parallel to grain. # For ductile materials, values f[illegible]mpression are same as given for tension.

Member	*Tension*	*Compression*
BC	10 KN	
CD	$10\sqrt{2}$ KN	
DE		10 KN
BE	$10\sqrt{2}\,KN$	
CE		10 KN
AE		20 KN

Example 14.6

Determine the magnitude and the nature of the forecs in all the members of the truss shown in figure 14.10. All inclined members are at 45° with the horizontal.

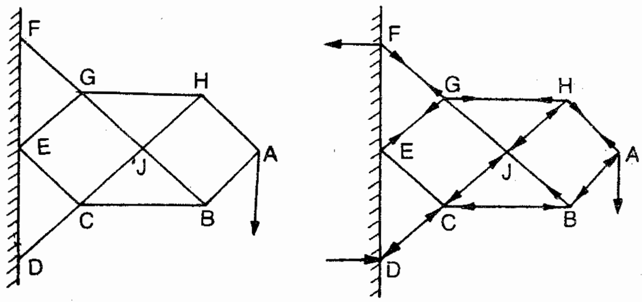
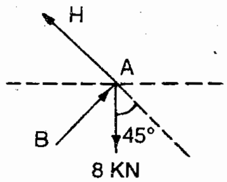


Fig. 14.10

Solution

Joint *A*



Resolving vertically

$\uparrow 8 - f_{AH}\,\text{Sin}45 - f_{AB}\,\text{Sin}45 = 0$

$8 - f_{AH}\frac{1}{\sqrt{2}} - f_{AB}\,\frac{1}{\sqrt{2}} = 0$

.... (i)

Resolving horizontally

$f_{AH}\,\text{Cos}\,45° = f_{AB}\,\text{Cos}\,45°$ or $f_{AH} = f_{AB}$

--- --- --- (ii)

From equation (i)

$8 - f_{AH}\frac{1}{\sqrt{2}} - f_{AH}\frac{1}{\sqrt{2}} = 0$

$8 - f_A\frac{2}{\sqrt{2}} = 0$ or $f_{AH}\frac{8}{2}\sqrt{2} = 4\sqrt{2}$ KN (Tensile)

$\therefore f_{AB} = 4\sqrt{2}$ KN (Comp)

INDEX

S.F BM Deflection Torsion Principal
10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22
23, 24, 25, 26, 27, 28

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